N=2 Supersymmetric Gauge Theories with Nonpolynomial Interactions

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Abstract

In this thesis the central charge of the vector-tensor multiplet is gauged, giving rise to $N = 2$ supersymmetric models in four dimensions which involve nonpolynomial yet local couplings of 1-form gauge potentials to an antisymmetric tensor field.
Following an introduction to the $N = 2$ supersymmetry algebra with a local central charge and a discussion of the massive Fayet-Sohnius hypermultiplet as a simple realisation, we investigate deformations of the superfield constraints that determine the vector-tensor multiplet. The supersymmetry and central charge transformations of its tensor components as well as the Bianchi identities for the field strengths are given for arbitrary consistent deformations, which facilitates the formulation of a particular model. To verify the validity of a given constraint, we supply a set of consistency conditions.
We then focus on the coupling to an abelian vector multiplet that gauges the central charge. The consistency conditions yield a system of partial differential equations, and two classes of solutions are presented which provide superfield constraints for both the linear and the self-interacting vector-tensor multiplet.
With these as the foundation, we first consider the linear case. It is shown how the particular structure of the Bianchi identities is responsible for the nonpolynomial central charge transformations of the vector and antisymmetric tensor. From a general prescription for the construction of invariant actions by means of a linear superfield we derive the Lagrangian, whose nonpolynomial vector-tensor interactions turn out to fit into the framework of new (nonsupersymmetric) gauge field theories found recently by Henneaux and Knaepen, to which we provide an introduction.
The nonlinear version of the vector-tensor multiplet is investigated in the last chapter. We explain in detail how the superfield constraints give rise to couplings of the antisymmetric tensor to Chern-Simons forms of both the vector and the central charge gauge field. We are unable, however, to construct a corresponding Henneaux-Knaepen model.

Keywords: Supersymmetry, Gauge Theories, Central Charge

N=2 supersymmetrische Eichtheorien mit nichtpolynomialen Wechselwirkungen

Zusammenfassung

In der vorliegenden Arbeit eichen wir die zentrale Ladung des Vektor-Tensor Multipletts, was auf $N = 2$ supersymmetrische Modelle in vier Dimensionen führt, welche nichtpolynomiale, jedoch lokale, Wechselwirkungen zwischen Eins-Form Eichfeldern und einem antisymmetrischen Tensor beinhalten.
Es wird zunächst die $N = 2$ Supersymmetrie-Algebra mit zentralen Ladungen vorgestellt. Als ein einfaches Beispiel für ein Modell mit lokaler zentraler Ladung diskutieren wir das massive Hypermultiplet nach Fayet und Sohnius. Anschließend untersuchen wir Deformationen der dem Vektor-Tensor Multiplett zugrunde liegenden Superfeld-Constraints. Die Supersymmetrie- und die von der zentralen Ladung erzeugten Transformationen der Tensor-Komponenten sowie die Bianchi-Identitäten der Feldstärken werden, soweit als möglich, für beliebige konsistente Deformationen bestimmt, was eine spätere Spezialisierung auf bestimmte Modelle erleichtert. Eine wesentliche Hilfestellung für das Auffinden möglicher Constraints bieten eine Reihe von Konsistenzbedingungen, welche wir aus der Supersymmetrie-Algebra ableiten.
Danach konzentrieren wir uns auf die Kopplung an ein abelsches Vektor-Multiplett, welches das Eichfeld für die zentrale Ladung bereitstellt. Die Konsistenzbedingungen lassen sich in ein System partieller Differentialgleichungen übersetzen, für das zwei Klassen von Lösungen gewonnen werden. Die entsprechenden Superfeld-Constraints beschreiben das lineare sowie das selbstwechselwirkende Vektor-Tensor Multiplett.
Wir betrachten zunächst den linearen Fall. Wir zeigen auf, wie die spezielle Struktur der Bianchi-Identitäten die nichtpolynomialen zentralen Ladungs-Transformationen des Vektors und des antisymmetrischen Tensors hervorruft. Mittels einer allgemeinen Vorschrift für die Konstruktion invarianter Wirkungen vermöge des sogenannten linearen Superfelds bestimmen wir die Lagrange-Dichte, deren nichtpolynomiale Vektor-Tensor Wechselwirkungen sich einordnen lassen in eine neue Art von (nicht supersymmetrischer) Eichtheorie, welche erst kürzlich von Henneaux und Knaepen gefunden wurde. Zu dieser geben wir eine kurze Einführung.
Im letzten Kapitel untersuchen wir dann die nichtlineare Version des Vektor-Tensor Multipletts. Detailliert wird gezeigt, wie die Superfeld-Constraints Kopplungen des antisymmetrischen Tensors an Chern-Simons Formen sowohl des Vektors wie auch des Eichfelds der zentralen Ladung hervorrufen. Wir sehen uns allerdings außerstande, auch diese auf ein Henneaux-Knaepen Modell zurückzuführen.

Schlagworte: Supersymmetrie, Eichtheorien, Zentrale Ladung

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Contents

Introduction

Despite the lack of experimental hints, supersymmetry counts among the most popular and promising concepts in theoretical high energy physics. It features prominently both in quantum theories of point particles and of extended objects; in particular it is a prerequisite to the formulation of realistic string theories, which are assumed to unify the standard model of strong and electroweak forces with Einstein's gravity.

Although less attractive from a phenomenological point of view, models with extended, i.e. more than one, supersymmetry have provided much insight into nonperturbative phenomena of quantum field theories as well as into various (mostly conjectured) dualities between different superstring theories. $N=2$ supersymmetry in four dimensions in particular has received great attention lately due to the seminal work of Seiberg and Witten on $N=2$ supersymmetric Yang-Mills theories. While these are usually formulated in terms of vector multiplets, there exists another multiplet describing the same kind of physical states, which trades one scalar for an antisymmetric tensor field. Such multiplets with 2-form gauge potentials occur universally in string theories, and the so-called vector-tensor multiplet especially has recently been shown to be part of the massless spectrum of four-dimensional $N = 2$ supersymmetric heterotic string vacua. It was this discovery that has renewed interest in the long known, yet largely ignored, vector-tensor multiplet and its possible interactions, and in the present thesis we offer a novel derivation of the most important results obtained on this subject in the last three years.

An off-shell formulation of the multiplet requires the presence of a central charge in the supersymmetry algebra, at least when only a finite number of components is desired. This central charge generates an on-shell nontrivial global symmetry of a rather unusual kind. It can be promoted to a local symmetry by coupling the vector-tensor multiplet to an abelian vector multiplet that provides the gauge field for the local central charge transformations. These and the couplings of the vector-tensor components in the invariant action share the peculiar property of being nonpolynomial in the gauge field. What at first had been considered a completely new type of gauge theory, turned out to fit into a larger class of models found somewhat earlier by Henneaux and Knaepen outside the framework of supersymmetry. In four dimensions, these bosonic models describe consistent interactions of 1-form and 2-form gauge potentials, which in general are nonpolynomial in both kinds of fields. While recently an $N = 1$ supersymmetric formulation of all Henneaux-Knaepen models has been given by Brandt and the author, so far all attempts to go beyond the vector-tensor multiplet in order to construct more general HK models with two supersymmetries have been unsuccessful. On the other hand, we are going to show that the vector-tensor multiplet itself suggests a possible

generalization, for we find gauge couplings that do not conform to the models originally formulated by Henneaux and Knaepen.
The thesis is organized as follows: In the first chapter we give an introduction to rigid $N = 2$ supersymmetry with central charges. We review how to incorporate gauge symmetries into the algebra, with special attention paid to local central charge transformations. A general prescription for invariant actions is then derived from the so-called linear multiplet, and as a demonstration of the previous results we gauge the central charge of the massive Fayet-Sohnius hypermultiplet.
In the second chapter the free vector-tensor multiplet is introduced. We then consider deformations of the corresponding superfield constraints and employ the supersymmetry algebra to derive consistency conditions that impose severe restrictions on the possible couplings of the vector-tensor multiplet to itself and to other multiplets. Focussing on the coupling to an abelian vector multiplet that gauges the central charge, we make an Ansatz for the constraints and translate a certain subset of the consistency conditions into a system of differential equations on the coefficient functions. An ensuing analysis shows that there are essentially two classes of solutions; one of which generalizes the free vector-tensor multiplet, while the other one will turn out to describe additional self-interactions.
In chapter 3 the first solution is discussed in detail. We demonstrate how the nonpolynomial central charge transformations of the vector-tensor complex arise as a result of a coupling between the Bianchi identities the field strengths are required to satisfy. Then, by means of the prescription derived earlier, the supersymmetric and gauge invariant action is constructed, which is also found to be nonpolynomial in the central charge gauge field. After extending the model by couplings to further nonabelian vector multiplets, which introduces, among other things, an interaction of the antisymmetric tensor with Chern-Simons forms of the additional gauge potentials, we discuss four-dimensional bosonic Henneaux-Knaepen models and their relevance to the vector-tensor multiplet.
The last chapter is devoted to the self-interacting vector-tensor multiplet. We present the nonlinear superfield constraints that underlie the construction and give a detailed derivation of the Bianchi identities, their solutions and the invariant action, which again displays the typical nonpolynomial dependence on the central charge gauge field.
Following the conclusions, we compile some useful formulae and list our conventions in an appendix.

Chapter 1
Gauging the Central Charge

There exist basically two approaches to theories with $N = 2$ supersymmetry. While without doubt the more sophisticated harmonic superspace [1] offers some advantages over ordinary superspace, in this thesis we shall nevertheless employ the latter only, which makes it easier to switch back and forth between superfields and components. For a treatment of theories with gauged central charge within the framework of harmonic superspace we refer to [2], where several results presented here have already been published.
The reader might want to have a look at the appendix first to become acquainted with our conventions concerning Lorentz and spinor indices.

1.1 The N=2 Supersymmetry Algebra

Extended supersymmetry algebras in four spacetime dimensions involve in addition to the Poincaré generators P_μ and $M_{\mu\nu}$ two-component Weyl spinor charges Q^i_α and their hermitian conjugates $Q^\dagger_{\dot\alpha i}$, which are Grassmann-odd and generate supersymmetry transformations. The index i belongs to a representation of an internal symmetry group and runs from 1 to some number N that counts the supersymmetries. In [3] Haag et al. have determined the most general supersymmetry algebra compatible with reasonable requirements on relativistic quantum field theories. It contains an invariant subalgebra that is spanned by the generators of translations and supersymmetry transformations, and for $N > 1$ additional bosonic generators, denoted by $\mathcal{Z}^{ij}$, may also occur. These must commute with every element of the supersymmetry algebra and for this reason are called *central charges*. The odd part of the subalgebra reads

$$\{Q^i_\alpha\,,\,Q^\dagger_{\dot\alpha j}\} = \delta^i_j \sigma^\mu_{\alpha\dot\alpha} P_\mu\ , \quad \{Q^i_\alpha\,,\,Q^j_\beta\} = \varepsilon_{\alpha\beta}\mathcal{Z}^{ij}\ , \quad \{Q^\dagger_{\dot\alpha i}\,,\,Q^\dagger_{\dot\beta j}\} = \varepsilon_{\dot\alpha\dot\beta}\mathcal{Z}^\dagger_{ij}\ , \tag{1.1}$$

while all commutators vanish. It is evident that the central charges $\mathcal{Z}^{ij}$ must be antisymmetric in the pair ij. For $N = 2$ this implies that there are at most two hermitian central charges,

$$N = 2 \quad \Rightarrow \quad \mathcal{Z}^{ij} = \varepsilon^{ij}(Z_1 + \mathrm{i}Z_2)\ , \quad \mathcal{Z}^\dagger_{ij} = -\varepsilon_{ij}(Z_1 - \mathrm{i}Z_2)\ . \tag{1.2}$$

When central charges are absent the above algebra is, among others, invariant under unitary transformations

$$(Q^i_\alpha)' = U^i{}_j Q^j_\alpha\ , \quad (Q^\dagger_{\dot\alpha i})' = U^*{}_i{}^j Q^\dagger_{\dot\alpha j}\ , \quad P'_\mu = P_\mu\ , \quad U \in \mathrm{U}(N)\ . \tag{1.3}$$

In the $N=2$ case the presence of central charges reduces this symmetry from U(2) to SU(2), under which ε^{ij} is an invariant tensor.

The algebra (1.1) with $N=2$ can be represented on a so-called central charge superspace [4] with coordinates $x^\mu, \theta_i^\alpha, \bar\theta^{\dot\alpha i}$ and a further bosonic complex variable z. On superfields infinitesimal supersymmetry transformations are generated by differential operators

$$Q_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} - \tfrac{\mathrm{i}}{2}(\sigma^\mu\bar\theta^i)_\alpha\partial_\mu + \tfrac{\mathrm{i}}{2}\theta_\alpha^i\partial_z \ , \quad \bar Q_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} + \tfrac{\mathrm{i}}{2}(\theta_i\sigma^\mu)_\alpha\partial_\mu + \tfrac{\mathrm{i}}{2}\bar\theta_{\dot\alpha i}\partial_{\bar z} \tag{1.4}$$

with commutation relations

$$\begin{gathered}\{Q_\alpha^i\,,\,\bar Q_{\dot\alpha j}\} = \mathrm{i}\delta_j^i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\\ \{Q_\alpha^i\,,\,Q_\beta^j\} = \mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\partial_z \qquad \{\bar Q_{\dot\alpha i}\,,\,\bar Q_{\dot\beta j}\} = -\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\partial_{\bar z}\ .\end{gathered} \tag{1.5}$$

The commutator of two rigid supersymmetry transformations

$$\Delta(\xi) = \xi_i^\alpha Q_\alpha^i + \bar\xi_{\dot\alpha}^i\bar Q_i^{\dot\alpha} \ , \quad (\xi_i^\alpha)^* = \bar\xi_{\dot\alpha}^i \ , \tag{1.6}$$

yields global translations in the bosonic directions,

$$[\,\Delta(\xi)\,,\,\Delta(\zeta)\,] = \mathrm{i}(\xi_i\sigma^\mu\bar\zeta^i - \zeta_i\sigma^\mu\bar\xi^i)\partial_\mu + \mathrm{i}\xi^i\zeta_i\partial_z + \mathrm{i}\bar\xi_i\bar\zeta^i\partial_{\bar z}\ . \tag{1.7}$$

Supercovariant spinor derivatives

$$D_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} + \tfrac{\mathrm{i}}{2}(\sigma^\mu\bar\theta^i)_\alpha\partial_\mu - \tfrac{\mathrm{i}}{2}\theta_\alpha^i\partial_z \ , \quad \bar D_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - \tfrac{\mathrm{i}}{2}(\theta_i\sigma^\mu)_\alpha\partial_\mu - \tfrac{\mathrm{i}}{2}\bar\theta_{\dot\alpha i}\partial_{\bar z} \tag{1.8}$$

anticommute with Q_α^i and $\bar Q_{\dot\alpha i}$ and therefore map superfields into superfields. Their algebra involves a minus sign relative to the algebra of the Q's,

$$\begin{gathered}\{D_\alpha^i\,,\,\bar D_{\dot\alpha j}\} = -\mathrm{i}\delta_j^i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\\ \{D_\alpha^i\,,\,D_\beta^j\} = -\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\partial_z \qquad \{\bar D_{\dot\alpha i}\,,\,\bar D_{\dot\beta j}\} = \mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\partial_{\bar z}\ .\end{gathered} \tag{1.9}$$

The coefficient functions in the θ-expansion of a superfield constitute a supersymmetry multiplet. Their supersymmetry transformations are generated by differential operators $\mathcal{D}_\alpha^i$ and $\bar{\mathcal{D}}_{\dot\alpha i}$, whose action can be read off from the relation

$$\mathcal{D}_\alpha^i\,\Phi(x,\theta,\bar\theta,z) = Q_\alpha^i\,\Phi(x,\theta,\bar\theta,z)\ , \tag{1.10}$$

where $\mathcal{D}_\alpha^i$ acts only on the components and anticommutes with the θ-variables. The algebra of $\mathcal{D}_\alpha^i$ and $\bar{\mathcal{D}}_{\dot\alpha i}$ is the same as for the supercovariant derivatives. The components of a superfield Φ may be regarded as the lowest components of superfields obtained from applying an appropriate polynomial $P(D,\bar D)$ of supercovariant derivatives to Φ. Eq. (1.10) implies that the generators $\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot\alpha i}$ act on components $P(D,\bar D)\,\Phi\,|_{\theta=\bar\theta=0}$ according to

$$\mathcal{D}_\alpha^i\left[P(D,\bar D)\,\Phi\right]_{\theta=\bar\theta=0} = \left[D_\alpha^i P(D,\bar D)\,\Phi\right]_{\theta=\bar\theta=0}\ . \tag{1.11}$$

If $\varphi(x,z) = \Phi(x,0,0,z)$ denotes the lowest component of Φ, it follows that

$$\big[P(D,\bar{D})\,\Phi\big]_{\theta=\bar{\theta}=0} = P(\mathcal{D},\bar{\mathcal{D}})\,\varphi(x,z)\ . \tag{1.12}$$

While the θ-expansion of a superfield terminates after a finite number of steps, the z-dependence in general is nonpolynomial, giving rise to an infinite tower of component fields. The supercovariant derivatives may be employed to impose constraints on superfields that eliminate all but a finite number of components without restricting their x-dependence. It is convenient, and we shall make use of it from now on throughout this thesis, to consider superfields living only on the subspace parametrized by $x, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}$ and to regard central charge transformations not as translations in some additional bosonic directions, but simply as transformations that map one superfield to a new superfield. Instead of ∂_z we denote the generator by δ_z, which then maps from one coefficient in the z-expansion of a general superfield to the next. Also, we confine ourselves to only a single real central charge, as the presence of a second one usually inhibits the formulation of finite multiplets. In symbols, one has the equivalence

$$\begin{aligned} &\Phi(x,\theta,\bar{\theta},z) = \Phi(x,\theta,\bar{\theta},0) + z\,\partial_z\Phi(x,\theta,\bar{\theta},z)\,|_{z=0} + \ldots \\ \Longleftrightarrow\quad &\delta_z:\ \Phi(x,\theta,\bar{\theta}) \mapsto \Phi^{(z)}(x,\theta,\bar{\theta}) \mapsto \ldots\ , \end{aligned}$$

and similar for the components. The purpose of superfield constraints then is to express all but at most a finite number of the images $\Phi^{(z)}, \Phi^{(zz)}$, etc. in terms of the primary superfield Φ and its spacetime derivatives.

The main reason why we altered our point of view concerning the central charge transformations is the similarity to (abelian) super Yang-Mills theories that arises when gauging the central charge. Let us briefly review the basics of supersymmetric gauge theories, mainly to introduce our conventions and notations. We shall denote an infinitesimal gauge transformation with spacetime dependent parameters $C^I(x)$ by $\Delta^{\mathrm{g}}(C)$. Tensor fields T are characterized by their homogeneous transformation law

$$\Delta^{\mathrm{g}}(C)\,T = C^I\delta_I T\ , \tag{1.13}$$

whereas the transformation of the gauge fields $\mathcal{A}_\mu^I$ involves the derivative of the parameters C^I,

$$\Delta^{\mathrm{g}}(C)\,\mathcal{A}_\mu^I = -\partial_\mu C^I - C^J\mathcal{A}_\mu^K f_{JK}{}^I\ . \tag{1.14}$$

Here the generators δ_I form a basis of a Lie algebra $\mathcal{G}$ with corresponding structure constants $f_{IJ}{}^K$,

$$[\,\delta_I\,,\,\delta_J\,] = f_{IJ}{}^K\delta_K\ ,\quad f_{[IJ}{}^L f_{K]L}{}^M = 0\ . \tag{1.15}$$

The transformation of the gauge fields is such that the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu^I\delta_I \tag{1.16}$$

of a tensor transforms again as a tensor. This implies that the generators δ_I commute with the $\mathcal{D}_\mu$, which in turn requires the gauge fields to transform in the adjoint representation of the Lie algebra $\mathcal{G}$ under the δ_I, i.e. $\delta_I \mathcal{A}_\mu^J = f_{KI}{}^J \mathcal{A}_\mu^K$. Note that the δ_I do not generate the full transformations of the gauge fields.
The commutator of two covariant derivatives involves the field strength $\mathcal{F}_{\mu\nu}^I$,

$$[\, \mathcal{D}_\mu \,,\, \mathcal{D}_\nu \,] = \mathcal{F}_{\mu\nu}^I \, \delta_I \ , \tag{1.17}$$

which is given by

$$\mathcal{F}_{\mu\nu}^I = \partial_\mu \mathcal{A}_\nu^I - \partial_\nu \mathcal{A}_\mu^I - \mathcal{A}_\mu^J \mathcal{A}_\nu^K f_{JK}{}^I \ , \tag{1.18}$$

and which satisfies the Bianchi identity

$$\varepsilon^{\mu\nu\rho\sigma} \, \mathcal{D}_\nu \mathcal{F}_{\rho\sigma}^I = 0 \ . \tag{1.19}$$

The field strength is a tensor that also transforms in the adjoint representation of the Lie algebra $\mathcal{G}$,

$$\Delta^{\mathrm{g}}(C) \, \mathcal{F}_{\mu\nu}^I = -C^J f_{JK}{}^I \mathcal{F}_{\mu\nu}^K \ . \tag{1.20}$$

Hence, in the abelian case the field strength is gauge invariant.
In [5] Grimm et al. have shown how to embed 1-form gauge fields into $N = 2$ supersymmetry multiplets. Let us first discuss the case without an explicit central charge, $\delta_z = 0$. In analogy to eq. (1.16) one extends the flat supercovariant derivatives D_α^i to super- and gaugecovariant derivatives D_α^i and imposes constraints on the field strengths such that only a minimal number of components survives. The Bianchi identities then fix the algebra to read

$$\begin{gathered}
\{ \mathrm{D}_\alpha^i \,,\, \bar{\mathrm{D}}_{\dot\alpha j} \} = -\mathrm{i} \delta_j^i \sigma_{\alpha\dot\alpha}^\mu \mathrm{D}_\mu \\
\{ \mathrm{D}_\alpha^i \,,\, \mathrm{D}_\beta^j \} = \varepsilon_{\alpha\beta} \varepsilon^{ij} \bar\phi^I \delta_I \qquad [\, \mathrm{D}_\alpha^i \,,\, \mathrm{D}_\mu \,] = \tfrac{\mathrm{i}}{2} (\sigma_\mu \bar{\mathrm{D}}^i \bar\phi^I)_\alpha \, \delta_I \\
\{ \bar{\mathrm{D}}_{\dot\alpha i} \,,\, \bar{\mathrm{D}}_{\dot\beta j} \} = \varepsilon_{\alpha\beta} \varepsilon^{ij} \phi^I \delta_I \qquad [\, \bar{\mathrm{D}}_{\dot\alpha i} \,,\, \mathrm{D}_\mu \,] = \tfrac{\mathrm{i}}{2} (\mathrm{D}_i \phi^I \sigma_\mu)_{\dot\alpha} \, \delta_I \ .
\end{gathered} \tag{1.21}$$

The generators δ_I act trivially on the supercovariant derivatives, i.e. they commute. We do not give the explicit realization of the D_α^i as in the following we need only their commutation relations. The calligraphic $\mathcal{D}_\alpha^i$ shall always generate supersymmetry transformations of component fields, and we remark that the relations (1.11) and (1.12) apply also to the present case when the D_α^i are replaced by D_α^i.
The so-called vector superfields ϕ^I transform as tensors in the adjoint representation under gauge transformations. The Bianchi identities imply that they are subject to constraints

$$\bar{\mathrm{D}}_{\dot\alpha i} \phi^I = 0 = \mathrm{D}_\alpha^i \bar\phi^I \ , \quad \mathrm{D}^{(i} \mathrm{D}^{j)} \phi^I = \bar{\mathrm{D}}^{(i} \bar{\mathrm{D}}^{j)} \bar\phi^I \ , \tag{1.22}$$

which lead to the field content of a complex scalar, a doublet of Weyl spinors, a real SU(2) triplet of auxiliary scalars and a real antisymmetric tensor[1],

$$\begin{gathered}\phi^I| \ , \quad \chi^{iI}_\alpha = \mathcal{D}^i_\alpha \phi^I| \ , \quad D^{ijI} = \tfrac{1}{2}\mathcal{D}^{(i}\mathcal{D}^{j)}\phi^I| \\ \mathcal{F}^I_{\mu\nu} = \tfrac{1}{4}(\mathcal{D}^i\sigma_{\mu\nu}\mathcal{D}_i\phi^I - \bar{\mathcal{D}}_i\bar\sigma_{\mu\nu}\bar{\mathcal{D}}^i\bar\phi^I)| \ ,\end{gathered} \tag{1.23}$$

the latter providing the field strength for the gauge potentials $\mathcal{A}^I_\mu$. Here and henceforth we shall employ the convention of labeling a superfield and its lowest component by the same symbol. As we are going to deal with up to three multiplets simultaneously and introduce a fair amount of abbreviations, a large number of symbols is needed, which calls for an economical notation. It should be clear in each equation which is which; when ambiguities might occur, we explicitly state whether the full superfield or merely a component field is meant.
From eq. (1.11) and the algebra (1.21) one derives the supersymmetry transformations of the tensor components of ϕ^I. The action of $\mathcal{D}^i_\alpha$ is found to be

$$\begin{gathered}\mathcal{D}^i_\alpha\phi^I = \chi^{iI}_\alpha \ , \quad \mathcal{D}^i_\alpha\bar\phi^I = 0 \\ \mathcal{D}^i_\alpha\chi^{jI}_\beta = \varepsilon_{\alpha\beta}\, D^{ijI} + \varepsilon^{ij}\mathcal{F}^I_{\mu\nu}\,\sigma^{\mu\nu}{}_{\alpha\beta} + \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{ij}\,\phi^J\bar\phi^K f_{JK}{}^I \ , \quad \mathcal{D}^i_\alpha\bar\chi^{jI}_{\dot\alpha} = \mathrm{i}\varepsilon^{ij}\,\mathcal{D}_{\alpha\dot\alpha}\bar\phi^I \\ \mathcal{D}^i_\alpha D^{jkI} = \mathrm{i}\varepsilon^{i(j}\,(\mathcal{D}_{\alpha\dot\alpha}\bar\chi^{k)\dot\alpha I} + \mathrm{i}\chi^{k)J}_\alpha\bar\phi^K f_{JK}{}^I) \\ \mathcal{D}^i_\alpha\mathcal{F}^I_{\mu\nu} = \mathrm{i}\,\mathcal{D}_{[\mu}(\sigma_{\nu]}\bar\chi^{iI})_\alpha\end{gathered} \tag{1.24}$$

while the action of $\bar{\mathcal{D}}_{\dot\alpha i}$ is readily obtained by complex conjugation. Since the gauge fields $\mathcal{A}^I_\mu$ do not occur linearly and undifferentiated in a θ-expansion of the ϕ^I, their supersymmetry transformations cannot be derived from eq. (1.11). We define the action of $\mathcal{D}^i_\alpha$ on $\mathcal{A}^I_\mu$ by

$$\mathcal{D}^i_\alpha\mathcal{A}^I_\mu = \tfrac{\mathrm{i}}{2}(\sigma_\mu\bar\chi^{iI})_\alpha \ , \tag{1.25}$$

which is compatible with the transformation of $\mathcal{F}^I_{\mu\nu}$. Then the supersymmetry algebra reads on all component fields

$$\begin{gathered}\{\mathcal{D}^i_\alpha\, ,\, \bar{\mathcal{D}}_{\dot\alpha j}\} = -\mathrm{i}\delta^i_j\,\sigma^\mu_{\alpha\dot\alpha}\big(\partial_\mu + \Delta^{\mathrm{g}}(\mathcal{A}_\mu)\big) \\ \{\mathcal{D}^i_\alpha\, ,\, \mathcal{D}^j_\beta\} = \varepsilon_{\alpha\beta}\,\varepsilon^{ij}\Delta^{\mathrm{g}}(\bar\phi) \qquad \{\bar{\mathcal{D}}_{\dot\alpha i}\, ,\, \bar{\mathcal{D}}_{\dot\beta j}\} = \varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{ij}\Delta^{\mathrm{g}}(\phi) \ .\end{gathered} \tag{1.26}$$

On tensors the combination $\partial_\mu + \Delta^{\mathrm{g}}(\mathcal{A}_\mu)$ is just the covariant derivative. We conclude that the commutator of two supersymmetry transformations yields a translation and a field dependent gauge transformation,

$$[\,\Delta(\xi)\, ,\, \Delta(\zeta)\,] = \epsilon^\mu\partial_\mu + \Delta^{\mathrm{g}}(C) \ , \tag{1.27}$$

with parameters

$$\epsilon^\mu = \mathrm{i}(\zeta_i\sigma^\mu\bar\xi^i - \xi_i\sigma^\mu\bar\zeta^i) \ , \quad C^I = \epsilon^\mu\mathcal{A}^I_\mu - \xi_i\zeta^i\,\bar\phi^I + \bar\xi^i\bar\zeta_i\,\phi^I \ . \tag{1.28}$$

[1]A bar denotes projection to the lowest component of a superfield by setting $\theta = \bar\theta = 0$.

Now let us compare the anticommutator of two spinor derivatives in eqs. (1.9) (substituting δ_z for $\partial_z = \partial_{\bar{z}}$) and (1.21). Evidently, an operator δ_z generating a rigid central charge transformation may formally be incorporated into the latter algebra by first extending the gauge group by an extra U(1) factor, the generator of which one identifies with δ_z, and then replacing the corresponding superfield with the constant background value i. Accordingly, the central charge is promoted to a local transformation by reintroducing the full vector superfield, denoted in the following by Z. This differs from the other ϕ^I in that it has a nonvanishing vacuum expectation value (vev),

$$\langle Z \rangle = \mathrm{i} \ . \tag{1.29}$$

It seems the dimensions have gone awry. If the generators δ_I are taken to be dimensionless, the corresponding vector superfields must have mass dimension unity. As is clear from its representation as a space derivative, however, δ_z has the dimension of an inverse length, which results in a shift of the dimension of Z. To compensate for this, the central charge coupling constant g_z that will be introduced with the Lagrangian (see next section) carries mass dimension -1.
To distinguish the components of the central charge vector multiplet from those of ordinary gauge multiplets, we denote them by

$$\big(Z \, , \ A_\mu \, , \ \lambda^i_\alpha \mid Y^{ij} \big) \ ,$$

and the abelian field strength of A_μ we write as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{1.30}$$

The tensor components are invariant under an infinitesimal central charge transformation $\Delta^{\mathrm{z}}(C)$, while A_μ transforms into the gradient of the parameter $C(x)$,

$$\Delta^{\mathrm{z}}(C) \big(Z \, , \ \lambda^i_\alpha \, , \ F_{\mu\nu} \, , \ Y^{ij} \big) = 0 \ , \quad \Delta^{\mathrm{z}}(C) \, A_\mu = -\partial_\mu C \ . \tag{1.31}$$

Note that the above discussion implies that the central charge multiplet is invariant under gauge transformations Δ^{g}, while the ordinary gauge multiplets are invariant under central charge transformations Δ^{z}.
The supersymmetry transformations can be copied from above. They are linear due to the abelian nature of the central charge,

$$\begin{aligned}
\mathcal{D}^i_\alpha Z = \lambda^i_\alpha \ , &\quad \mathcal{D}^i_\alpha \bar{Z} = 0 \\
\mathcal{D}^i_\alpha \lambda^j_\beta = \varepsilon_{\alpha\beta} Y^{ij} + \varepsilon^{ij} F_{\mu\nu} \, \sigma^{\mu\nu}{}_{\alpha\beta} \ , &\quad \mathcal{D}^i_\alpha \bar{\lambda}^j_{\dot{\alpha}} = \mathrm{i}\varepsilon^{ij} \, \partial_{\alpha\dot{\alpha}} \bar{Z} \\
\mathcal{D}^i_\alpha Y^{jk} &= \mathrm{i}\varepsilon^{i(j} \, \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{k)\dot{\alpha}} \\
\mathcal{D}^i_\alpha A_\mu = \tfrac{\mathrm{i}}{2} (\sigma_\mu \bar{\lambda}^i)_\alpha \ , &\quad \mathcal{D}^i_\alpha F_{\mu\nu} = \mathrm{i} \, \partial_{[\mu} (\sigma_{\nu]} \bar{\lambda}^i)_\alpha \ .
\end{aligned} \tag{1.32}$$

Note that the nonvanishing vev of Z does not break supersymmetry spontaneously.
Since the vector-tensor multiplet transforms trivially under gauge transformations Δ^{g} (see next chapter), the algebra we shall be dealing with mostly in this thesis includes

only the central charge generator δ_z, and for reference we list the commutation relations as they are to hold on tensor components,

$$\begin{aligned}
\{\mathcal{D}^i_\alpha\,,\,\bar{\mathcal{D}}_{\dot\alpha j}\} &= -\mathrm{i}\delta^i_j\,\sigma^\mu_{\alpha\dot\alpha}\mathcal{D}_\mu & [\,\mathcal{D}_\mu\,,\,\mathcal{D}_\nu\,] &= F_{\mu\nu}\,\delta_z \\
\{\mathcal{D}^i_\alpha\,,\,\mathcal{D}^j_\beta\} &= \varepsilon_{\alpha\beta}\,\varepsilon^{ij}\bar{Z}\delta_z & [\,\mathcal{D}^i_\alpha\,,\,\mathcal{D}_\mu\,] &= \tfrac{\mathrm{i}}{2}(\sigma_\mu\bar\lambda^i)_\alpha\delta_z \\
\{\bar{\mathcal{D}}_{\dot\alpha i}\,,\,\bar{\mathcal{D}}_{\dot\beta j}\} &= \varepsilon_{\dot\alpha\dot\beta}\,\varepsilon_{ij}Z\delta_z & [\,\bar{\mathcal{D}}_{\dot\alpha i}\,,\,\mathcal{D}_\mu\,] &= \tfrac{\mathrm{i}}{2}(\lambda_i\sigma_\mu)_{\dot\alpha}\delta_z\ .
\end{aligned} \tag{1.33}$$

The commutators involving δ_z vanish.

1.2 The Linear Multiplet

Once we have found a multiplet that realizes the $N=2$ supersymmetry algebra, be it with gauged central charge or not, the task is to construct an invariant action. In this section we discuss a procedure first developed by de Wit et al. [6] to derive possible Lagrangians from the so-called *linear multiplet*. By definition a linear superfield is a real Lorentz-scalar SU(2) triplet φ^{ij} which satisfies the constraints

$$\varphi^{ij} = \varphi^{ji}\ , \quad (\varphi^{ij})^* = \varphi_{ij}\ , \quad \mathcal{D}^{(i}_\alpha\varphi^{jk)} = 0 = \bar{\mathcal{D}}^{(i}_{\dot\alpha}\varphi^{jk)}\ . \tag{1.34}$$

Let us first neglect a possible central charge and suppose that the linear superfield transforms in some representation of the Lie group generated by the δ_I. As seen in the previous section, a central charge can easily be introduced by assigning to one of the δ_I the role of a central charge generator. The constraints then lead to a field content of two Weyl spinors, a complex scalar and a real vector in addition to the three real scalars which comprise the lowest components of the superfield,

$$\varphi^{ij}|\ , \quad \varrho^i_\alpha = \mathcal{D}_{\alpha j}\varphi^{ij}|\ , \quad S = \tfrac{1}{2}\mathcal{D}_i\mathcal{D}_j\varphi^{ij}|\ , \quad K^\mu = \tfrac{\mathrm{i}}{2}\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j\varphi^{ij}|\ . \tag{1.35}$$

Note that if φ^{ij} has (mass) dimension one, S and K^μ have dimension two and so must assume the role of auxiliary fields or, in the case of K^μ, field strengths. In the presence of a central charge the multiplet is larger as the action of δ_z on the components listed above remains undetermined and so leads to further fields $\delta_z\varphi^{ij}$, etc.
We obtain the supersymmetry transformations of the multiplet (1.35) by evaluating the algebra (1.21) on each component subject to the constraints on φ^{ij}. This gives

$$\begin{gathered}
\mathcal{D}^i_\alpha\varphi^{jk} = \tfrac{2}{3}\varepsilon^{i(j}\varrho^{k)}_\alpha \\
\mathcal{D}^i_\alpha\varrho^j_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}\left(\varepsilon^{ij}S - 3\bar\phi^I\delta_I\varphi^{ij}\right)\ , \quad \mathcal{D}^i_\alpha\bar\varrho^j_{\dot\alpha} = -\tfrac{\mathrm{i}}{2}(\varepsilon^{ij}K_{\alpha\dot\alpha} + 3\mathcal{D}_{\alpha\dot\alpha}\varphi^{ij}) \\
\mathcal{D}^i_\alpha S = \bar\phi^I\delta_I\varrho^i_\alpha\ , \quad \mathcal{D}^i_\alpha\bar S = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha}\bar\varrho^{\dot\alpha i} - 3\chi^I_{\alpha j}\delta_I\varphi^{ij} - \phi^I\delta_I\varrho^i_\alpha \\
\mathcal{D}^i_\alpha K^\mu = \left(2\sigma^{\mu\nu}\mathcal{D}_\nu\varrho^i + \tfrac{3}{2}\mathrm{i}\,\sigma^\mu\bar\chi^I_j\delta_I\varphi^{ij} + \mathrm{i}\bar\phi^I\,\sigma^\mu\delta_I\bar\varrho^i\right)_\alpha\ .
\end{gathered} \tag{1.36}$$

However, these transformations realize the supersymmetry algebra only if the vector K^μ satisfies a differential constraint, namely

$$\mathcal{D}_\mu K^\mu = -\frac{1}{2}\left(\phi^I\delta_I S + \bar\phi^I\delta_I\bar S + 2\chi^I_i\delta_I\varrho^i - 2\bar\chi^{iI}\delta_I\bar\varrho_i + 3D^I_{ij}\delta_I\varphi^{ij}\right)\ . \tag{1.37}$$

Evidently, this equation can only be solved when the linear multiplet is gauge invariant. Then the constraint reduces to a Bianchi identity which identifies K^μ as the dual field strength of a 2-form gauge field,

$$\delta_I \varphi^{ij} = 0 \quad \Rightarrow \quad \partial_\mu K^\mu = 0 \quad \Rightarrow \quad K^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} \ . \tag{1.38}$$

In this form the multiplet is known as the *tensor multiplet*, cf. [7].
What happens when we consider local central charge transformations instead of ordinary gauge transformations? We can simply replace the generators δ_I with δ_z and the multiplets ϕ^I with Z in the above equations. Then the constraint on K^μ reads

$$\mathcal{D}_\mu K^\mu = -\frac{1}{2}\,\delta_z\big(ZS + \bar{Z}\bar{S} + 2\lambda_i\varrho^i - 2\bar{\lambda}^i\bar{\varrho}_i + 3Y_{ij}\varphi^{ij}\big) \equiv -\frac{1}{2}\,\delta_z\hat{\mathcal{L}} \ , \tag{1.39}$$

for now the gauge multiplet transforms trivially under the generator δ_z. While it cannot be solved unless $\delta_z\varphi^{ij} = 0$, the constraint implies the existence of a gauge invariant action. Let us consider the expression

$$\mathcal{L} = \hat{\mathcal{L}} + 2A_\mu K^\mu \ .$$

Applying a local central charge transformation, we can replace $\delta_z\hat{\mathcal{L}}$ with the covariant derivative of K^μ using the constraint and then combine this with the transformed of the second term into a total derivative,

$$\begin{aligned}\Delta^z(C)\,\mathcal{L} &= C\delta_z\hat{\mathcal{L}} - 2\partial_\mu C\,K^\mu + 2CA_\mu\delta_z K^\mu \\ &= -2C\,\mathcal{D}_\mu K^\mu - 2\partial_\mu(CK^\mu) + 2C(\partial_\mu K^\mu + A_\mu\delta_z K^\mu) \\ &= -2\partial_\mu(CK^\mu) \ .\end{aligned}$$

Thus upon integration over spacetime $\int d^4x\,\mathcal{L}$ is invariant under gauged central charge transformations. Amazingly it is even supersymmetric, for we find after a little calculation

$$\mathcal{D}^i_\alpha\mathcal{L} = -\mathrm{i}\,\partial_\mu\big(2\bar{Z}\sigma^\mu\bar{\varrho}^i + 3\varphi^{ij}\sigma^\mu\bar{\lambda}_j - 4\mathrm{i}A_\nu\sigma^{\mu\nu}\varrho^i\big)_\alpha \ . \tag{1.40}$$

Altogether we have found a general prescription to construct invariant actions: If the components of the multiplets under consideration can be combined into a superfield $\mathcal{L}^{ij} = \mathcal{L}^{ji} = (\mathcal{L}_{ij})^*$ such that it satisfies the constraints

$$\mathcal{D}^{(i}_\alpha\mathcal{L}^{jk)} = 0 = \bar{\mathcal{D}}^{(i}_{\dot\alpha}\mathcal{L}^{jk)} \ , \tag{1.41}$$

then the Lagrangian

$$\mathcal{L} = \frac{1}{12}\left(Z\mathcal{D}_i\mathcal{D}_j + \bar{Z}\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j + 4\lambda_i\mathcal{D}_j - 4\bar{\lambda}_i\bar{\mathcal{D}}_j + 6Y_{ij} + 2\mathrm{i}A_\mu\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j\right)\mathcal{L}^{ij}| \tag{1.42}$$

provides us with a supersymmetric action that is invariant under local central charge transformations. Note that if the linear superfield $\mathcal{L}^{ij}$ is also invariant under gauge transformations Δ^{g}, this rule extends to ordinary gauge theories as well. When we do not consider fields that are subject to local central charge transformations, we replace the superfield Z by its background value $\langle Z\rangle = \mathrm{i}$, and the Lagrangian reduces to

$$\mathcal{L} = \frac{\mathrm{i}}{12}\left(\mathcal{D}_i\mathcal{D}_j - \bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\right)\mathcal{L}^{ij}| \ . \tag{1.43}$$

Occasionally we shall call $\mathcal{L}^{ij}$ the "pre-Lagrangian". There is no guarantee, however, that one can always find an $\mathcal{L}^{ij}$ which gives rise to a nontrivial Lagrangian, i.e. one which is not merely a total derivative.
It will not have escaped the reader's attention that, although we started from a superfield, we did not write the action formula as a superspace integral. Indeed we cannot with the formalism introduced so far. Only recently Dragon et al. [2] have found a manifestly supersymmetric version of eq. (1.42) using the harmonic superspace approach.

As a first application we use this recipe to determine the invariant action for $N=2$ vector multiplets. The most general linear superfield one can construct from superfields ϕ^I is given by

$$\mathcal{L}^{ij}_{\mathrm{sYM}} = -\mathrm{i}\,\mathcal{D}^i\mathcal{D}^j\mathcal{F}(\phi) + \mathrm{i}\,\bar{\mathcal{D}}^i\bar{\mathcal{D}}^j\bar{\mathcal{F}}(\bar\phi)\ , \quad \delta_I\mathcal{F}(\phi) = 0\ , \tag{1.44}$$

where $\mathcal{F}$ is a holomorphic function of the ϕ^I and $\bar{\mathcal{F}}$ its complex conjugate. That $\mathcal{L}^{ij}$ is symmetric in its SU(2) indices follows from the gauge invariance of $\mathcal{F}$, and the constraints (1.41) are satisfied by virtue of the chirality of the ϕ^I,

$$\mathcal{D}^{(i}_\alpha\mathcal{L}^{jk)}_{\mathrm{sYM}} = -\mathrm{i}\,\mathcal{D}^{(i}_\alpha\mathcal{D}^j\mathcal{D}^{k)}\mathcal{F}(\phi) + \mathrm{i}\,\mathcal{D}^{(i}_\alpha\bar{\mathcal{D}}^j\bar{\mathcal{D}}^{k)}\bar{\mathcal{F}}(\bar\phi) = \mathrm{i}\,\bar{\mathcal{D}}^{(i}\bar{\mathcal{D}}^j\mathcal{D}^{k)}_\alpha\bar{\mathcal{F}}(\bar\phi) = 0\ .$$

Using the algebra and the properties of $\mathcal{F}$, it is easy to show that the mixed generators $\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\mathcal{D}^i\mathcal{D}^j$ in eq. (1.43) give rise only to a total derivative,

$$\mathcal{L}_{\mathrm{sYM}} = \frac{1}{12}\,\mathcal{D}_i\mathcal{D}_j\,\mathcal{D}^i\mathcal{D}^j\mathcal{F}(\phi) + \partial^\mu\partial_\mu\,\mathcal{F}(\phi) + \text{c.c.}\ . \tag{1.45}$$

To obtain the usual super Yang-Mills Lagrangian, we choose

$$\mathcal{F}(\phi) = \frac{1}{8g^2}\,\delta_{IJ}\phi^I\phi^J\ , \tag{1.46}$$

where δ_{IJ} is an invariant tensor in the case of a compact gauge group and g a dimensionless coupling constant (see also section 3.4). Dropping all surface terms, we arrive after some algebra at

$$\begin{aligned} g^2\,\mathcal{L}_{\mathrm{sYM}} = &-\frac{1}{4}\,\mathcal{F}^{\mu\nu I}\mathcal{F}^I_{\mu\nu} + \frac{1}{2}\,\mathcal{D}^\mu\bar\phi^I\,\mathcal{D}_\mu\phi^I - \frac{\mathrm{i}}{4}\,\chi^{iI}\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\chi^I_i + \frac{1}{4}\,D^{ijI}D^I_{ij} \\ &+ \frac{1}{4}\left(\chi^{iI}\chi^J_i\bar\phi^K - \bar\chi^I_i\bar\chi^{iJ}\phi^K\right)f_{JK}{}^I + \frac{1}{8}\,(\phi^J\bar\phi^K f_{JK}{}^I)^2\ . \end{aligned} \tag{1.47}$$

For the central charge multiplet, which is abelian, interaction terms do not occur, and the Lagrangian is given simply by the sum of kinetic energies and the square of the auxiliary scalars,

$$g_z^2\,\mathcal{L}_{\mathrm{cc}} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\,\partial^\mu\bar Z\,\partial_\mu Z - \frac{\mathrm{i}}{4}\,\lambda^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\lambda_i + \frac{1}{4}\,Y^{ij}Y_{ij}\ . \tag{1.48}$$

Here the coupling constant g_z carries mass dimension -1 in order to render the action dimensionless.

1.3 The Hypermultiplet

A simple yet instructive example for a multiplet with a nontrivial central charge is the massive Fayet-Sohnius *hypermultiplet* [8, 4]. Although it contains no gauge fields by itself, we shall nevertheless demonstrate, as a warm-up for more complicated things to come, the gauging of the rigid transformation associated with the central charge. The hypermultiplet is described by two complex scalar superfields φ^i, $\bar{\varphi}_i = (\varphi^i)^*$ that form a doublet of the automorphism group SU(2) and, for rigid central charge, satisfy the constraints (for simplicity we take φ^i to be gauge invariant, $\delta_I \varphi^i = 0$)

$$D_\alpha^{(i}\varphi^{j)} = 0 = \bar{D}_{\dot\alpha}^{(i}\varphi^{j)} \ . \tag{1.49}$$

These imply that only φ^i itself contains independent components, while those of the central charge images $\varphi^{i(z)}$, etc. can be expressed in terms of the ones of φ^i and derivatives thereof. It is now a fundamental question whether, upon gauging the central charge, it suffices to simply replace the flat spinor derivatives with gaugecovariant ones in the constraints on a superfield, or whether there are obstructions that require modifications of the constraints. As we shall see in the next chapter, in general a naive "covariantization" leads to inconsistencies, and finding the proper constraints for the vector-tensor multiplet is quite an effort. However, in the case of the hypermultiplet it turns out that the first attempt is successful, i.e. the hypermultiplet with gauged central charge is described by

$$\mathcal{D}_\alpha^{(i}\varphi^{j)} = 0 = \bar{\mathcal{D}}_{\dot\alpha}^{(i}\varphi^{j)} \ . \tag{1.50}$$

Let us define the component fields as

$$\varphi^i| \ , \quad \chi_\alpha = \tfrac{1}{2}\mathcal{D}_{\alpha i}\varphi^i| \ , \quad \bar{\psi}_{\dot\alpha} = \tfrac{1}{2}\bar{\mathcal{D}}_{\dot\alpha i}\varphi^i| \ , \quad F^i = \delta_z \varphi^i| \ , \tag{1.51}$$

where the auxiliary scalars F^i do occur also in a θ-expansion of φ^i. One may easily verify that the supersymmetry transformations

$$\begin{aligned}
\mathcal{D}_\alpha^i \varphi^j &= \varepsilon^{ij}\chi_\alpha \ , \quad \mathcal{D}_\alpha^i \bar{\varphi}^j = -\varepsilon^{ij}\psi_\alpha \\
\mathcal{D}_\alpha^i \chi_\beta &= -\varepsilon_{\alpha\beta}\bar{Z}F^i \ , \quad \mathcal{D}_\alpha^i \bar{\chi}_{\dot\alpha} = -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\bar{\varphi}^i \\
\mathcal{D}_\alpha^i \bar{\psi}_{\dot\alpha} &= -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\varphi^i \ , \quad \mathcal{D}_\alpha^i \psi_\beta = \varepsilon_{\alpha\beta}\bar{Z}\bar{F}^i \\
\mathcal{D}_\alpha^i F^j &= \varepsilon^{ij}\delta_z\chi_\alpha \ , \quad \mathcal{D}_\alpha^i \bar{F}^j = -\varepsilon^{ij}\delta_z\psi_\alpha
\end{aligned} \tag{1.52}$$

represent the algebra (1.33) when δ_z acts as follows,

$$\begin{aligned}
\delta_z \chi_\alpha = -\frac{1}{Z}\left(\mathrm{i}\sigma^\mu \mathcal{D}_\mu \bar{\psi} + \lambda_i F^i\right)_\alpha \ , \quad \delta_z \bar{\psi}_{\dot\alpha} = -\frac{1}{\bar{Z}}\left(\mathrm{i}\mathcal{D}_\mu \chi\sigma^\mu + \bar{\lambda}_i F^i\right)_{\dot\alpha} \\
\delta_z F^i = \frac{1}{|Z|^2}\left(\mathcal{D}^\mu \mathcal{D}_\mu \varphi^i + \lambda^i \delta_z \chi + \bar{\lambda}^i \delta_z \bar{\psi} - Y^{ij}F_j\right) \ .
\end{aligned} \tag{1.53}$$

These equations have a peculiar structure. The covariant derivative acting on $\bar{\psi}$ in the expression for $\delta_z \chi$ contains the central charge generator δ_z, whose action is given by the second equation, which in turn involves a covariant derivative of χ. Hence, the central charge transformation of χ is given only implicitly as the equations are coupled. Let us insert the second one into the first,

$$\begin{aligned}|Z|^2\delta_z\chi_\alpha &= -\mathrm{i}A_{\alpha\dot\alpha}\bar Z\delta_z\bar\psi^{\dot\alpha} - \bar Z(\mathrm{i}\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} + \lambda_{\alpha i}F^i)\\ &= -\mathrm{i}A_{\alpha\dot\alpha}(\mathrm{i}\mathcal{D}^{\dot\alpha\beta}\chi_\beta - \bar\lambda_i^{\dot\alpha}F^i) - \bar Z(\mathrm{i}\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} + \lambda_{\alpha i}F^i)\\ &= A_{\alpha\dot\alpha}A^{\dot\alpha\beta}\delta_z\chi_\beta + A_{\alpha\dot\alpha}(\partial^{\dot\alpha\beta}\chi_\beta + \mathrm{i}\bar\lambda_i^{\dot\alpha}F^i) - \bar Z(\mathrm{i}\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} + \lambda_{\alpha i}F^i)\ .\end{aligned}$$

According to eq. (A.23) $A_{\alpha\dot\alpha}A^{\dot\alpha\beta} = A^\mu A_\mu\delta_\alpha^\beta$, so we have isolated $\delta_z\chi_\alpha$. Doing a similar calculation for $\bar\psi$, we conclude that

$$\begin{aligned}\delta_z\chi_\alpha &= -\frac{1}{\mathcal{E}}\left[\mathrm{i}\bar Z(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} - \mathrm{i}\lambda_{\alpha i}F^i) - A_{\alpha\dot\alpha}(\partial^{\dot\alpha\beta}\chi_\beta + \mathrm{i}\bar\lambda_i^{\dot\alpha}F^i)\right]\\ \delta_z\bar\psi_{\dot\alpha} &= -\frac{1}{\mathcal{E}}\left[\mathrm{i}Z(\partial_{\alpha\dot\alpha}\chi^\alpha - \mathrm{i}\bar\lambda_{\dot\alpha i}F^i) - A_{\alpha\dot\alpha}(\partial^{\dot\beta\alpha}\bar\psi_{\dot\beta} + \mathrm{i}\lambda_i^\alpha F^i)\right]\ ,\end{aligned} \tag{1.54}$$

where the abbreviation

$$\mathcal{E} \equiv |Z|^2 - A^\mu A_\mu \tag{1.55}$$

has been introduced. Since Z has a nonvanishing vev, $\mathcal{E}$ may be inverted at least formally. We can restructure the central charge transformation of F^i in like manner, for the covariant d'Alembertian acting on φ^i may be expanded as

$$\mathcal{D}^\mu\mathcal{D}_\mu\varphi^i = \Box\varphi^i + F^i\partial_\mu A^\mu + 2A^\mu\partial_\mu F^i + A^\mu A_\mu\delta_z F^i\ .$$

Thus one finds

$$\delta_z F^i = \frac{1}{\mathcal{E}}\left(\Box\varphi^i + F^i\partial_\mu A^\mu + 2A^\mu\partial_\mu F^i + \lambda^i\delta_z\chi + \bar\lambda^i\delta_z\bar\psi - Y^{ij}F_j\right). \tag{1.56}$$

Note that in the limit $Z = \mathrm{i}$, which corresponds to a rigid central charge, the transformations reduce to

$$\delta_z\chi = -\sigma^\mu\partial_\mu\bar\psi\ ,\quad \delta_z\bar\psi = -\bar\sigma^\mu\partial_\mu\chi\ ,\quad \delta_z F^i = \Box\varphi^i\ , \tag{1.57}$$

hence in the massless case they are trivial on-shell (cf. the Lagrangian given below).

In order to determine an invariant action, we apply the prescription (1.42) derived in the previous section. The constraints (1.50) imply that the combinations

$$\mathcal{L}_0^{ij} = -\bar\varphi^{(i}\overleftrightarrow{\delta_z}\varphi^{j)}\ ,\quad \mathcal{L}_{\mathrm{m}}^{ij} = -2\mathrm{i}\,m\bar\varphi^{(i}\varphi^{j)}\ ,\quad m\in\mathbb{R} \tag{1.58}$$

are both linear superfields, thus giving rise to two independent invariants. Let us consider the first; a straightforward computation leads to

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{2}\,\bar\varphi_i\mathcal{D}^\mu\mathcal{D}_\mu\varphi^i - \frac{1}{2}\,\varphi^i\mathcal{D}^\mu\mathcal{D}_\mu\bar\varphi_i - \frac{\mathrm{i}}{2}\left(\chi\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\chi + \psi\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\psi\right) + |Z|^2F^i\bar F_i\\ &+ \frac{1}{2}\,A_\mu\left(\varphi^i\overleftrightarrow{\mathcal{D}^\mu}\bar F_i + \bar\varphi_i\overleftrightarrow{\mathcal{D}^\mu}F^i + \mathrm{i}\chi\sigma^\mu\overleftrightarrow{\delta_z}\bar\chi + \mathrm{i}\psi\sigma^\mu\overleftrightarrow{\delta_z}\bar\psi\right)\ .\end{aligned} \tag{1.59}$$

We observe that the terms in the second line exactly cancel those in the first which involve a gauge potential, thereby reducing the covariant derivatives to partial ones. All that remains is a Lagrangian of (at least classically) free fields,

$$\mathcal{L}_0 = \partial^\mu \bar{\varphi}_i \, \partial_\mu \varphi^i - \frac{\mathrm{i}}{2} \left(\chi \sigma^\mu \overleftrightarrow{\partial}_\mu \bar{\chi} + \psi \sigma^\mu \overleftrightarrow{\partial}_\mu \bar{\psi} \right) + \mathcal{E} F^i \bar{F}_i \ , \tag{1.60}$$

where a total derivative has been dropped. Now consider the second linear superfield $\mathcal{L}_{\mathrm{m}}^{ij}$. It yields the Lagrangian

$$\begin{aligned} \frac{1}{m} \mathcal{L}_{\mathrm{m}} = \mathrm{i} A^\mu \big(\bar{\varphi}_i \overleftrightarrow{\partial}_\mu \varphi^i \big) + A_\mu (\chi \sigma^\mu \bar{\chi} - \psi \sigma^\mu \bar{\psi}) - \mathrm{i} \mathcal{E} (F^i \bar{\varphi}_i - \varphi^i \bar{F}_i) - \mathrm{i} \, Y_{ij} \bar{\varphi}^i \varphi^j \\ - \mathrm{i} (\bar{Z} \bar{\chi} \bar{\psi} - Z \chi \psi) - \mathrm{i} \, \varphi^i (\bar{\lambda}_i \bar{\chi} - \lambda_i \psi) + \mathrm{i} \, \bar{\varphi}_i (\lambda^i + \bar{\lambda}^i \bar{\psi}) \ . \end{aligned} \tag{1.61}$$

This one involves couplings of the gauge potential to combinations of the scalars and spinors which are reminiscent of U(1) currents, and indeed we find that the complete Lagrangian, i.e. the sum $\mathcal{L}_0 + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_{\mathrm{cc}}$,

$$\begin{aligned} \mathcal{L} = \nabla^\mu \bar{\varphi}_i \, \nabla_\mu \varphi^i - \frac{\mathrm{i}}{2} \left(\chi \sigma^\mu \overleftrightarrow{\nabla}_\mu \bar{\chi} + \psi \sigma^\mu \overleftrightarrow{\nabla}_\mu \bar{\psi} \right) + \mathcal{E} \, |F^i + \mathrm{i} m \varphi^i|^2 \\ - m^2 |Z|^2 \bar{\varphi}_i \varphi^i - \mathrm{i} m (\bar{Z} \bar{\chi} \bar{\psi} - Z \chi \psi) - \mathrm{i} m \, Y_{ij} \bar{\varphi}^i \varphi^j \\ - \mathrm{i} m \, \varphi^i (\bar{\lambda}_i \bar{\chi} - \lambda_i \psi) + \mathrm{i} m \, \bar{\varphi}_i (\lambda^i \chi + \bar{\lambda}^i \bar{\psi}) + \mathcal{L}_{\mathrm{cc}} \ , \end{aligned} \tag{1.62}$$

describes nothing but (a special kind of) $N = 2$ supersymmetric electrodynamics. Here the operator ∇_μ is defined by

$$\nabla_\mu \begin{pmatrix} \varphi^i \\ \chi \\ \bar{\psi} \end{pmatrix} = (\partial_\mu - \mathrm{i} m A_\mu) \begin{pmatrix} \varphi^i \\ \chi \\ \bar{\psi} \end{pmatrix} \ . \tag{1.63}$$

Hence, on-shell the gauged central charge generates just local U(1) transformations with an electric charge that is given by m (or rather $m g_z$ after a rescaling $A_\mu \rightarrow g_z A_\mu$). This may also be seen from the equation of motion for the auxiliary scalars $\bar{F}_i$ (the relation $\approx$ denotes on-shell equality),

$$\frac{\delta \mathcal{L}}{\delta \bar{F}_i} = \mathcal{E} (F^i + \mathrm{i} m \varphi^i) \approx 0 \ . \tag{1.64}$$

Since two superfields are equal if the lowest components coincide, we thus have

$$\delta_z \varphi^i \approx -\mathrm{i} m \varphi^i \tag{1.65}$$

for the full superfield. One may verify that this is in agreement with the eqs. (1.53). Note that since $\langle Z \rangle = \mathrm{i}$, the masses of the "electron" $(\chi, \bar{\psi})$ and its superpartners φ^i are given by the parameter m (where we assume $m \geq 0$),

$$M_{(\chi, \bar{\psi})} = M_\varphi = m \ , \tag{1.66}$$

so in the massless case the central charge is trivial on-shell as observed above. In this regard the model is different from conventional supersymmetric electrodynamics, where charge and mass are not related. What is more, whereas usually interactions of the matter fields with the gauge potential are tied to the kinetic terms through the covariant derivatives (the so-called minimal coupling), here these two derive from actions that are (off-shell) gauge invariant separately.
At last we remark that the prefactor $1/\mathcal{E}$, which accompanies the local central charge transformations, will prove to be a universal feature that we shall encounter again in the discussion of the vector-tensor multiplet. Note however, that here only the transformations are nonpolynomial, while the action is perfectly regular even off-shell.

Chapter 2
The Vector-Tensor Multiplet

The discovery of the vector-tensor multiplet by Sohnius, Stelle and West [9] dates back to the year 1980, yet our current knowledge about its various incarnations and possible interactions has been gathered only in the last three years. Its renaissance was triggered by the work of de Wit et al. [10] on $N=2$ supersymmetric vacua of heterotic string theory compactified on $K3 \times T^2$. The massless states in this theory comprise a vector and an antisymmetric tensor along with the dilaton, which organizes the perturbative expansion of a string theory. These three fields could be shown to fit into a vector-tensor multiplet. In string theory, an antisymmetric tensor is usually dualized into a pseudo-scalar, the axion, which in the case at hand results in an abelian $N=2$ vector multiplet, whose couplings have been studied extensively. However, not every vector multiplet can be converted into a vector-tensor multiplet (see [11] for details), which experiences much more stringent restrictions on its couplings. In any case, the duality transformation can be performed only on-shell, for the off-shell structure of the two multiplets is considerably different: the supersymmetry algebra of the vector-tensor multiplet contains a central charge in addition to the gauge transformations that are always present when gauge fields are involved.

The rediscovery of the vector-tensor multiplet spawned a lot of activity in this field. In [12] the superfield for the free multiplet was constructed for the first time, which subsequently could be generalized to include Chern-Simons couplings to nonabelian vector multiplets [13, 14]. In [15] an alternative formulation utilizing the harmonic superspace approach was presented. Already somewhat earlier, the central charge of the multiplet was gauged in [16, 17] as a preparatory step towards a coupling to supergravity (later achieved in [18], see also [19]), where the corresponding transformations would necessarily have to be realized locally. In the course of this, a second variant of the multiplet was discovered with nonlinear transformation laws, which give rise to self-interactions. These results were obtained by means of the so-called superconformal multiplet calculus, yet their complexity called for a formulation in terms of superfields. While in [20, 21] the nonlinear vector-tensor multiplet with rigid central charge could be derived from a set of superfield constraints in harmonic superspace, the problem of finding appropriate constraints describing the linear vector-tensor multiplet with gauged central charge was first tackled by Dragon and the author in conventional superspace [22, 23]. Finally, a general formalism for theories with gauged central charge was developed in [2], again employing the virtues of harmonic superspace, and a natural interpretation of the central charge of the linear vector-tensor multiplet as a remnant

of translations in six-dimensional spacetime was presented.
In the following chapters we give a derivation of the superfield constraints that underlie both the linear and nonlinear versions of the vector-tensor multiplet with gauged central charge. Furthermore, the origin of the nonpolynomial transformations and couplings is discussed in detail. Since we aim for an off-shell formulation, we shall not pass to the dual picture, however, but keep the antisymmetric tensor instead of replacing it with a scalar field.

2.1 Introducing the Multiplet

The multiplet consists of a real scalar, a vector and an antisymmetric tensor gauge field and a doublet of Weyl spinors, which accounts for $4+4$ (on-shell) degrees of freedom. An off-shell formulation requires in addition a real auxiliary scalar field, and we shall use the following notation for the components

$$\big(L\,,\ V_\mu\,,\ B_{\mu\nu}\,,\ \psi^i_\alpha\ |\ U \big)\ .$$

The field strength of V_μ and the dual field strength of $B_{\mu\nu}$ we will denote by $V_{\mu\nu}$ and H^μ, respectively,

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu\ ,\quad H^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}\ . \tag{2.1}$$

These are invariant under abelian gauge transformations

$$\Delta^V(\Theta)\, V_\mu = -\partial_\mu\, \Theta(x)\ ,\quad \Delta^B(\Omega)\, B_{\mu\nu} = -2\, \partial_{[\mu}\, \Omega_{\nu]}(x)\ , \tag{2.2}$$

the latter being reducible, i.e. they are inert to a change of the parameter Ω_μ by the gradient of some scalar. From our experience with super Yang-Mills theories we should expect that the supersymmetry algebra can be realized on the vector-tensor multiplet only modulo such gauge transformations, with field dependent parameters Θ and Ω_μ. The multiplet, the supersymmetry transformations of its components and an invariant action can be derived from a real scalar superfield, which we shall again label by its lowest component, subject to the constraints

$$D^{(i}D^{j)}L = 0\ ,\quad D^{(i}_\alpha \bar{D}^{j)}_{\dot\alpha} L = 0\ . \tag{2.3}$$

These give rise to the independent components

$$\begin{gathered} L|\ ,\quad \psi^i_\alpha = \mathrm{i}D^i_\alpha L|\ ,\quad U = \delta_z L| \\ G_{\alpha\beta} = \tfrac{1}{2}[\, D^i_\alpha\,,\, D_{\beta i}\,]\, L|\ ,\quad W_{\alpha\dot\alpha} = -\tfrac{1}{2}[\, D^i_\alpha\,,\, \bar{D}_{\dot\alpha i}\,]\, L|\ . \end{gathered} \tag{2.4}$$

The bispinor $G_{\alpha\beta} = G_{\beta\alpha}$ and its complex conjugate can be combined into a real antisymmetric tensor $G_{\mu\nu}$ according to eq. (A.21), while $W_{\alpha\dot\alpha}$ is equivalent to a real vector field W^μ.
Similarly to the case of the linear multiplet, the algebra (1.9) is realized provided that $G_{\mu\nu}$ and W^μ satisfy Bianchi identities,

$$\partial_\mu W^\mu = 0\ ,\quad \varepsilon^{\mu\nu\rho\sigma}\partial_\nu G_{\rho\sigma} = 0\ . \tag{2.5}$$

This allows to identify these components with the field strengths[1] (2.1) of the gauge potentials V_μ and $B_{\mu\nu}$. The reason for using different labels in the definition of the component fields will become clear when we investigate deformations of the superfield constraints (2.3) in the next chapters. There the relation between $G_{\mu\nu}$ and W^μ and the field strengths $V_{\mu\nu}$ and H^μ will be more complicated as the differential constraints (2.5) on the former also get modified. As we shall see, the transformations and actions can be formulated most easily in terms of components defined as in eq. (2.4) when interactions are introduced. In this section, however, in which only the free case is presented, there is no distinction between $G_{\mu\nu}$, W^μ and $V_{\mu\nu}$, H^μ.

The supersymmetry transformations of the components (2.4) read

$$\begin{aligned} \mathcal{D}^i_\alpha L &= -\mathrm{i}\psi^i_\alpha \ , \quad \mathcal{D}^i_\alpha U = -\mathrm{i}(\sigma^\mu\partial_\mu\bar\psi^i)_\alpha \\ \mathcal{D}^i_\alpha V_{\mu\nu} &= -2\,\partial_{[\mu}(\sigma_{\nu]}\bar\psi^i)_\alpha \ , \quad \mathcal{D}^i_\alpha H^\mu = 2(\sigma^{\mu\nu}\partial_\nu\psi^i)_\alpha \\ \mathcal{D}^i_\alpha\psi^j_\beta &= \tfrac{1}{2}\varepsilon^{ij}(\varepsilon_{\alpha\beta}U + \mathrm{i}V_{\mu\nu}\,\sigma^{\mu\nu}{}_{\alpha\beta}) \ , \quad \mathcal{D}^i_\alpha\bar\psi^j_{\dot\alpha} = \tfrac{1}{2}\varepsilon^{ij}\sigma^\mu_{\alpha\dot\alpha}(\partial_\mu L - \mathrm{i}H_\mu) \ , \end{aligned} \tag{2.6}$$

while those of the potentials are given by

$$\mathcal{D}^i_\alpha V_\mu = -(\sigma_\mu\bar\psi^i)_\alpha \ , \quad \mathcal{D}^i_\alpha B_{\mu\nu} = -2\mathrm{i}\,(\sigma_{\mu\nu}\psi^i)_\alpha \ . \tag{2.7}$$

The commutation relations of these involve a global central charge. The action of the generator δ_z reads

$$\begin{aligned} \delta_z L = U \ , \quad \delta_z U = \Box L \ , \quad \delta_z\psi^i = \sigma^\mu\partial_\mu\bar\psi^i \\ \delta_z V_{\mu\nu} = -2\,\partial_{[\mu}H_{\nu]} \ , \quad \delta_z H^\mu = \partial_\nu V^{\mu\nu} \end{aligned} \tag{2.8}$$

on the tensors, and on the gauge fields one has

$$\delta_z V_\mu = -H_\mu \ , \quad \delta_z B_{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}V^{\rho\sigma} \ . \tag{2.9}$$

We refrain from giving a detailed derivation of these results, as this will be done later on in the more general case of an algebra with a gauged central charge.
On the potentials V_μ and $B_{\mu\nu}$ the algebra (1.9) holds modulo gauge transformations (2.2). The commutator of two rigid supersymmetry transformations is given on all the fields[2] by

$$[\,\Delta(\xi)\,,\,\Delta(\zeta)\,] = \epsilon^\mu\partial_\mu + \Delta^z(C) + \Delta^V(\Theta) + \Delta^B(\Omega) \ , \tag{2.10}$$

with ϵ^μ as in eq. (1.28), $C = \mathrm{i}(\xi_i\zeta^i + \bar\xi^i\bar\zeta_i)$, and field dependent parameters

$$\begin{aligned} \Theta &= \epsilon^\mu V_\mu - L(\xi_i\zeta^i - \bar\xi^i\bar\zeta_i) \\ \Omega_\mu &= \epsilon_\mu L - B_{\mu\nu}\epsilon^\nu - V_\mu(\xi_i\zeta^i - \bar\xi^i\bar\zeta_i) \end{aligned} \tag{2.11}$$

[1]Occasionally we call H^μ the field strength of $B_{\mu\nu}$ for short, hoping not to confuse the reader by this abuse of denotation.

[2]It is understood that Δ^V and Δ^B act nontrivially only on V_μ and $B_{\mu\nu}$, respectively.

in the gauge transformations of V_μ and $B_{\mu\nu}$. Furthermore, Δ^z commutes with a supersymmetry transformation only modulo gauge transformations,

$$[\,\Delta^z(C)\,,\,\Delta(\xi)\,] = \Delta^V(\Theta) + \Delta^B(\Omega)\ , \tag{2.12}$$

where the parameters now read

$$\Theta = C(\xi_i\psi^i + \bar\xi^i\bar\psi_i)\ , \quad \Omega_\mu = \mathrm{i}\,C(\xi^i\sigma_\mu\bar\psi_i + \psi^i\sigma_\mu\bar\xi_i)\ . \tag{2.13}$$

To construct an invariant action for the vector-tensor multiplet it suffices to combine its components into a linear superfield, as shown in the previous chapter. From the constraints (2.3) on L it follows that the field

$$\mathcal{L}^{ij} = \kappa\, D^i L\, D^j L + \bar\kappa\, \bar D^i L\, \bar D^j L \tag{2.14}$$

with $\kappa \in \mathbb{C}$ constant has the desired properties, i.e. it is real, symmetric and satisfies $D^{(i}_\alpha \mathcal{L}^{jk)} = 0$. When calculating the Lagrangian using eq. (1.43) we find that the real part of κ gives rise to a total derivative, while the imaginary part provides the kinetic terms for the multiplet components. For $\kappa = \mathrm{i}$ one obtains

$$\mathcal{L}_{\mathrm{freeVT}} = \frac{1}{2}\,\partial^\mu L\,\partial_\mu L - \frac{1}{2}\,H^\mu H_\mu - \frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \mathrm{i}\psi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\psi_i + \frac{1}{2}\,U^2\ . \tag{2.15}$$

We observe that the central charge transformations (2.8) of the tensor fields vanish by virtue of the equations of motion. The gauge fields, however, transform nontrivially even on-shell. The conserved current that corresponds to this global symmetry of the action is given by

$$J^\mu_z = V^{\mu\nu}H_\nu\ . \tag{2.16}$$

Upon gauging the central charge transformations we therefore anticipate a coupling of this current to the gauge field A_μ to first order in the deformation of the free theory.

At last we would like to clarify the above statement about the conversion of the vector-tensor multiplet into an abelian vector multiplet. The equation of motion for $B_{\mu\nu}$ may be solved in terms of a real scalar field $a(x)$, which is then constrained by virtue of the Bianchi identity of the dual field strength H^μ,

$$\begin{aligned} \partial_{[\mu}H_{\nu]} \approx 0 \quad &\Rightarrow \quad H_\mu \approx \partial_\mu a \\ \partial_\mu H^\mu = 0 \quad &\Rightarrow \quad \Box a \approx 0\ . \end{aligned} \tag{2.17}$$

Hence, the antisymmetric tensor $B_{\mu\nu}$ describes one spin- and massless degree of freedom. Alternatively, one may consider H^μ to be a fundamental field and incorporate the Bianchi identity by means of a Lagrange multiplier,

$$-\tfrac{1}{2}H^\mu H_\mu - a\,\partial_\mu H^\mu = \tfrac{1}{2}\partial^\mu a\,\partial_\mu a - \tfrac{1}{2}(H_\mu - \partial_\mu a)^2 - \partial_\mu(aH^\mu)\ .$$

The supersymmetry transformation of H^μ,

$$\mathcal{D}^i_\alpha H^\mu = 2(\sigma^{\mu\nu}\partial_\nu\psi^i)_\alpha = (\sigma^\mu\bar\sigma^\nu\partial_\nu\psi^i - \partial^\mu\psi^i)_\alpha \approx -\partial^\mu\psi^i_\alpha\ ,$$

implies

$$\mathcal{D}^i_\alpha a = -\psi^i_\alpha \ , \tag{2.18}$$

which suggests to combine a and L into a complex scalar field that is then chiral,

$$\phi \equiv \tfrac{1}{2}(\mathrm{i}L - a) \quad \Rightarrow \quad \mathcal{D}^i_\alpha \phi = \psi^i_\alpha \ , \quad \bar{\mathcal{D}}_{\dot{\alpha}i}\phi = 0 \ . \tag{2.19}$$

Using $U \approx 0$, the Lagrangian (2.15) turns into

$$\mathcal{L}_{\mathrm{dual}} = 2\,\partial^\mu\bar{\phi}\,\partial_\mu\phi - \frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \mathrm{i}\psi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar{\psi}_i \ , \tag{2.20}$$

while the transformations of ψ^i and $\bar{\psi}_i$ read

$$\mathcal{D}^i_\alpha\psi^j_\beta \approx \tfrac{\mathrm{i}}{2}\varepsilon^{ij}\,V_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha\beta} \ , \quad \mathcal{D}^i_\alpha\bar{\psi}^j_{\dot{\alpha}} \approx \mathrm{i}\varepsilon^{ij}\partial_{\alpha\dot{\alpha}}\bar{\phi} \ . \tag{2.21}$$

Thus an on-shell equivalence has been established between the vector-tensor multiplet and an abelian vector multiplet.

2.2 Consistent Deformations

To couple the vector-tensor multiplet to an abelian vector multiplet such that the central charge transformations are realized locally, it will be necessary to modify the superfield constraints (2.3) which determine the multiplet. This is different from the hypermultiplet where the constraints could be retained when gauging the central charge. But also self-interactions and couplings to nonabelian vector multiplets are obtained from suitable deformations of the constraints. Instead of starting from a distinct Ansatz for each single case and then working out anew all the transformations and Bianchi identities, we treat all models simultaneously as far as possible by considering the most general deformation that does not alter the field content of the vector-tensor multiplet. The supersymmetry algebra imposes conditions on the constraints that restrict the possible deformations. These consistency conditions come in two kinds: First there are conditions that involve spacetime derivatives like the Bianchi identities (we shall call the differential constraints on W^μ and $G_{\mu\nu}$ so generically even if they cannot be solved, in which case the constraints are inconsistent), and second there are algebraic conditions without derivatives. We shall use the latter to single out possible constraints before trying to solve the conditions of the first kind. In the course of this we will encounter constraints that pass all hurdles save said Bianchi identities, so the consistency conditions of the second kind are necessary but not sufficient. Furthermore, seemingly different superfield constraints may be connected by a field redefinition. We do not distinguish such constraints as they do not lead to different theories. This will be of great help, for it allows to simplify the calculations by choosing certain "gauges". Let us consider the constraints

$$\mathcal{D}^{(i}\mathcal{D}^{j)}L = M^{ij} \ , \quad \mathcal{D}^{(i}_\alpha\bar{\mathcal{D}}^{j)}_{\dot{\alpha}}L = \tfrac{\mathrm{i}}{2}N^{ij}_{\alpha\dot{\alpha}} \ , \tag{2.22}$$

M^{ij} and $N^{ij}_{\alpha\dot\alpha}$ being arbitrary superfields with appropriate Lorentz and SU(2) transformation properties. The M^{ij} are (possibly complex) Lorentz scalars while the $N^{ij}_{\alpha\dot\alpha}$ may be converted into real Lorentz vectors N^{ij}_μ by means of the σ-matrices,

$$M^{ij} = M^{ji} = \left(\bar M_{ij}\right)^* \ , \quad N^{ij}_\mu \equiv \tfrac{1}{2}\bar\sigma^{\dot\alpha\alpha}_\mu N^{ij}_{\alpha\dot\alpha} = N^{ji}_\mu = \left(N_{\mu\, ij}\right)^* \ . \tag{2.23}$$

Although at this stage there is no apparent reason for taking L to be gauge invariant, we shall nevertheless require

$$\delta_I L = 0 \ , \quad \delta_I M^{ij} = 0 \ , \quad \delta_I N^{ij}_{\alpha\dot\alpha} = 0 \tag{2.24}$$

from the outset. We will justify this restriction in due course. Note, however, that M^{ij} and $N^{ij}_{\alpha\dot\alpha}$ may well depend also on superfields ϕ^I, etc. as long as these combine in a gauge invariant way. Moreover, $\delta_I L = 0$ does not exclude the possibility of $B_{\mu\nu}$ or V_μ transforming nontrivially under $\Delta^{\rm g}$, cf. section 3.4.
The independent tensor components of the multiplet can be defined similarly to the free case in the previous section,

$$\begin{gathered} L| \ , \quad \psi^i_\alpha = \mathrm{i}\mathcal{D}^i_\alpha L| \ , \quad U = \delta_z L| \\ G_{\alpha\beta} = \tfrac{1}{2}[\,\mathcal{D}^i_\alpha \, , \, \mathcal{D}_{\beta i}\,]\, L| \ , \quad W_{\alpha\dot\alpha} = -\tfrac{1}{2}[\,\mathcal{D}^i_\alpha \, , \, \bar{\mathcal{D}}_{\dot\alpha i}\,]\, L| \ . \end{gathered} \tag{2.25}$$

We will now evaluate the supersymmetry algebra (1.33) on each component, starting with the component of lowest dimension and using the results in the evalutation on the next component and so on, until the commutation relations have been verified on the whole multiplet. Note that, although we are going to work at the component level, every equation which involves only tensor fields may equally well be read as a relation for full-blown superfields.
From evalutating the anticommutators of $\mathcal{D}^i_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha i}$ on L we obtain the supersymmetry transformations of ψ^i and $\bar\psi_i$,

$$\mathcal{D}^i_\alpha \psi^j_\beta = \tfrac{\mathrm{i}}{2}\varepsilon^{ij}\left(\varepsilon_{\alpha\beta}\bar Z U - G_{\alpha\beta}\right) + \tfrac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\, M^{ij} \ , \tag{2.26}$$

$$\mathcal{D}^i_\alpha \bar\psi^j_{\dot\alpha} = \tfrac{1}{2}\varepsilon^{ij}\left(\mathcal{D}_{\alpha\dot\alpha} L - \mathrm{i}W_{\alpha\dot\alpha}\right) + \tfrac{1}{2}N^{ij}_{\alpha\dot\alpha} \ . \tag{2.27}$$

Whereas in the free case the parts symmetric in the SU(2) indices vanished, these are given in general by the deformations M^{ij} and $N^{ij}_{\alpha\dot\alpha}$. The requirement that δ_z commutes with the supersymmetry generators relates the transformation of U to the yet unknown central charge transform of ψ^i,

$$\mathcal{D}^i_\alpha U = -\mathrm{i}\delta_z \psi^i_\alpha \ . \tag{2.28}$$

This already completes the evaluation on L. Next we consider the spinors with dimension 3/2. Here we must go into greater detail, for the equations contain many irreducible components and actually provide all the missing transformations as well as all the consistency conditions! Let us start with

$$\{\mathcal{D}^i_\alpha \, , \, \mathcal{D}^j_\beta\}\, \psi^k_\gamma \stackrel{!}{=} \varepsilon_{\alpha\beta}\varepsilon^{ij}\bar Z\, \delta_z \psi^k_\gamma \ .$$

Using eqs. (2.26) and (2.28), this gives

$$0 = \mathrm{i}\varepsilon^{kj}\,(\mathrm{i}\varepsilon_{\beta\gamma}\bar{Z}\,\delta_z\psi^i_\alpha + \mathcal{D}^i_\alpha G_{\beta\gamma}) + \mathrm{i}\varepsilon_{\beta\gamma}\,\mathcal{D}^i_\alpha M^{jk} - \varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{Z}\,\delta_z\psi^k_\gamma + \left({}^i_\alpha \leftrightarrow {}^j_\beta\right) .$$

Symmetrizing in ijk, we obtain our first consistency condition (the second equation being the complex conjugate of the first),

$$\mathcal{D}^{(i}_\alpha M^{jk)} = 0\ , \quad \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{M}^{jk)} = 0\ . \tag{C.1}$$

Only such deformations M^{ij} that obey this condition can be taken into account. In the following we will assume M^{ij} to satisfy eq. (C.1).
If we symmetrize in the spinor indices $\alpha\beta\gamma$, we find that the spin-3/2 part of $\mathcal{D}^i_\alpha G_{\beta\gamma}$ vanishes. This must be so as the maximum helicity in nongravitational theories is ± 1. The remaining components all involve $\mathcal{D}^{\alpha i}G_{\alpha\beta}$. Thus we can express the action of $\mathcal{D}^i_\alpha$ on the self-dual part of $G_{\mu\nu}$ through $\delta_z\psi^i$ and M^{ij}. We find

$$\mathcal{D}^i_\alpha G_{\beta\gamma} = -2\,\varepsilon_{\alpha(\beta}\left(\mathrm{i}\bar{Z}\,\delta_z\psi^i - \tfrac{1}{3}\mathcal{D}_j M^{ij}\right)_{\gamma)}\ . \tag{2.29}$$

Now we consider

$$\{\mathcal{D}^i_\alpha\,,\,\bar{\mathcal{D}}^j_{\dot\alpha}\}\,\psi^k_\beta \overset{!}{=} \mathrm{i}\varepsilon^{ij}\,\mathcal{D}_{\alpha\dot\alpha}\psi^k_\beta\ .$$

With the supersymmetry transformations of ψ^i as above and using $[\,\mathcal{D}^i_\alpha\,,\,\mathcal{D}_{\beta\dot\alpha}]L = -\mathrm{i}\varepsilon_{\alpha\beta}\,\bar\lambda^i_{\dot\alpha}U$, this can be written as

$$\begin{aligned}0 = \mathrm{i}\varepsilon^{kj}\,(\mathcal{D}_{\beta\dot\alpha}\psi^i_\alpha + \varepsilon_{\alpha\beta}\bar\lambda^i_{\dot\alpha}U - \mathcal{D}^i_\alpha W_{\beta\dot\alpha}) - \mathrm{i}\varepsilon^{ik}\,\bar{\mathcal{D}}^j_{\dot\alpha}G_{\alpha\beta} - \mathrm{i}\varepsilon^{ij}\,\mathcal{D}_{\alpha\dot\alpha}\psi^k_\beta \\ + \mathrm{i}\varepsilon_{\alpha\beta}\,\varepsilon^{ik}\,(\bar\lambda^j_{\dot\alpha}U + \mathrm{i}\bar{Z}\,\delta_z\bar\psi^j_{\dot\alpha}) + \mathcal{D}^i_\alpha N^{jk}_{\beta\dot\alpha} + \mathrm{i}\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}^j_{\dot\alpha}M^{ik}\ .\end{aligned}$$

We decompose the equation into parts which are symmetric and antisymmetric in the indices $\alpha\beta$, respectively. Let us consider the former: symmetrized in ijk it provides us with a second consistency condition,

$$\mathcal{D}^{(i}_{(\beta}N^{jk)}_{\alpha)\dot\alpha} = 0\ , \quad \bar{\mathcal{D}}^{(i}_{(\dot\beta}N^{jk)}_{\dot\alpha)\alpha} = 0\ . \tag{C.2}$$

The remaining components determine the action of $\bar{\mathcal{D}}^i_{\dot\alpha}$ on $G_{\alpha\beta}$, of which we give the complex conjugate expression,

$$\mathcal{D}^i_\alpha\bar{G}_{\dot\alpha\dot\beta} = 2\,\mathcal{D}_{\alpha(\dot\alpha}\,\bar\psi^i_{\dot\beta)} - \tfrac{2}{3}\mathrm{i}\,\bar{\mathcal{D}}_{j(\dot\beta}N^{ij}_{\dot\alpha)\alpha}\ , \tag{2.30}$$

and supply the relation $\mathcal{D}^i_{(\alpha}W_{\beta)\dot\alpha} = \tfrac{1}{2}\bar{\mathcal{D}}^i_{\dot\alpha}G_{\alpha\beta}$. From the part antisymmetric in $\alpha\beta$ follows first of all a relation between M^{ij} and $N^{ij}_{\alpha\dot\alpha}$, which is a third consistency condition,

$$\bar{\mathcal{D}}^{(i}_{\dot\alpha}M^{jk)} = \tfrac{\mathrm{i}}{2}\,\mathcal{D}^{\alpha(i}N^{jk)}_{\alpha\dot\alpha}\ , \quad \mathcal{D}^{(i}_\alpha\bar{M}^{jk)} = \tfrac{\mathrm{i}}{2}\,\bar{\mathcal{D}}^{\dot\alpha(i}N^{jk)}_{\alpha\dot\alpha}\ . \tag{C.3}$$

Moreover, we obtain the central charge transformation of $\bar\psi_i$ and thus of ψ^i,

$$\begin{aligned}Z\delta_z\psi^i_\alpha &= \mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\bar\psi^{\dot\alpha i} - \mathrm{i}\lambda^i_\alpha U + \tfrac{\mathrm{i}}{3}\,\mathcal{D}_{\alpha j}\bar{M}^{ij} - \tfrac{1}{3}\,\bar{\mathcal{D}}^{\dot\alpha}_j N^{ij}_{\alpha\dot\alpha} \\ \bar{Z}\delta_z\bar\psi^i_{\dot\alpha} &= \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}\psi^{\alpha i} + \mathrm{i}\bar\lambda^i_{\dot\alpha}U - \tfrac{\mathrm{i}}{3}\,\bar{\mathcal{D}}_{\dot\alpha j}M^{ij} + \tfrac{1}{3}\,\mathcal{D}^\alpha_j N^{ij}_{\alpha\dot\alpha}\ ,\end{aligned} \tag{2.31}$$

and finally the part $\mathcal{D}^{\alpha i}W_{\alpha\dot\alpha}$, which together with eq. (2.30) gives the complete supersymmetry transformation of W^μ,

$$\mathcal{D}^i_\alpha W_{\beta\dot\beta} = -\mathcal{D}_{\beta\dot\beta}\psi^i_\alpha - \varepsilon_{\alpha\beta}\big(2\mathrm{i}\bar Z\,\delta_z\bar\psi^i + \bar\lambda^i U\big)_{\dot\beta} + \tfrac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\,\mathcal{D}^\gamma_j N^{ij}_{\gamma\dot\beta} - \tfrac{\mathrm{i}}{3}\,\mathcal{D}_{j(\alpha}N^{ij}_{\beta)\dot\beta}\ . \tag{2.32}$$

Before going any further, let us examine eqs. (2.31). We observe a structure similar to that of the central charge transformations of the spinors in the hypermultiplet, eqs. (1.53), namely the equations for $\delta_z\psi^i$ and $\delta_z\bar\psi^i$ are coupled by virtue of the covariant derivative. Let us try to solve for $\delta_z\psi^i$. For convenience we introduce the abbreviation

$$\eta^i_\alpha \equiv \lambda^i_\alpha U - \tfrac{1}{3}\,\mathcal{D}_{\alpha j}\bar M^{ij} - \tfrac{\mathrm{i}}{3}\,\bar{\mathcal{D}}^{\dot\alpha}_j N^{ij}_{\alpha\dot\alpha}$$

and calculate

$$\begin{aligned}|Z|^2\delta_z\psi^i_\alpha &= \mathrm{i}A_{\alpha\dot\alpha}\bar Z\delta_z\bar\psi^{\dot\alpha i} + \mathrm{i}\bar Z(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha i} - \eta^i_\alpha)\\ &= A_{\alpha\dot\alpha}(\mathcal{D}^{\dot\alpha\beta}\psi^i_\beta - \bar\eta^{\dot\alpha i}) + \mathrm{i}\bar Z(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha i} - \eta^i_\alpha)\\ &= A_{\alpha\dot\alpha}A^{\dot\alpha\beta}\delta_z\psi^i_\beta + A_{\alpha\dot\alpha}(\partial^{\dot\alpha\beta}\psi^i_\beta - \bar\eta^{\dot\alpha i}) + \mathrm{i}\bar Z(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha i} - \eta^i_\alpha)\ .\end{aligned}$$

Again the prefactor $\mathcal{E}$, defined in eq. (1.55), emerges. We have now eliminated $\delta_z\bar\psi^i$, provided that η^i or its complex conjugate does not contain such a term. We assume this to be the case[3]. Then the action of the central charge generator on ψ^i reads

$$\delta_z\psi^i_\alpha = \frac{1}{\mathcal{E}}\left[\,\mathrm{i}\bar Z(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha i} - \eta^i_\alpha) + A_{\alpha\dot\alpha}(\partial^{\dot\alpha\beta}\psi^i_\beta - \bar\eta^{\dot\alpha i})\right]\ . \tag{2.33}$$

This expression does not appear to be covariant with respect to local central charge transformations, as the gauge potential occurs explicitly. However, from eq. (2.31) it should be clear that $\delta_z\psi^i$ is indeed a tensor, and one may verify that all the differentiated gauge parameters cancel when calculating the central charge transformation of $\delta_z\psi^i$ proceeding from eq. (2.33). In what follows it is advantageous to use the manifestly covariant expression (2.31) rather than the complicated equation (2.33).
We resume the evaluation of the supersymmetry algebra with the anticommutator

$$\{\mathcal{D}^i_\alpha\,,\,\mathcal{D}^j_\beta\}\,\bar\psi^k_{\dot\alpha} \overset{!}{=} \varepsilon_{\alpha\beta}\varepsilon^{ij}\bar Z\,\delta_z\bar\psi^k_{\dot\alpha}\ .$$

From eq. (2.27) we see that this involves $\mathcal{D}^j_\beta W_{\alpha\dot\alpha}$, which is given in eq. (2.32). We find that the equation is fulfilled identically provided the consistency condition (C.2) holds, so we obtain no new information.
Next we investigate the consequences of the requirement that the central charge generator δ_z commute with the supersymmetry generator $\mathcal{D}^i_\alpha$,

$$[\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,\psi^j_\beta \overset{!}{=} 0\ .$$

In this equation we encounter two generators $\mathcal{D}^i_\alpha$, $\bar{\mathcal{D}}^i_{\dot\alpha}$ acting on the deformations M^{ij} and $N^{ij}_{\alpha\dot\alpha}$. From now on we assume the supersymmetry transformations of these fields

[3] All consistent constraints that we present in the following chapters have this property.

to satisfy the commutation relations (1.33). This we can take for granted if M^{ij} and $N^{ij}_{\alpha\dot\alpha}$ are composed only of the covariant components of the vector multiplets to which we wish to couple the vector-tensor multiplet. Of all the components of the latter only L and the spinors ψ^i can also enter the deformations, as we have already verified (by construction) the algebra to hold on those.
Let us first symmetrize in ij. The result can be rendered antisymmetric in $\alpha\beta$ using eq. (C.3), and we obtain

$$\begin{aligned}\delta_z\big(Y^{ij}L - \mathrm{i}\lambda^{(i}\psi^{j)} + \mathrm{i}\bar\lambda^{(i}\bar\psi^{j)} + \tfrac12 ZM^{ij} + \tfrac12\bar Z\bar M^{ij}\big) &= \\ = \mathcal{D}^\mu N^{ij}_\mu + \frac38\,\mathcal{D}_k\mathcal{D}^{(i}\bar M^{jk)} + \frac38\,\bar{\mathcal{D}}_k\bar{\mathcal{D}}^{(i}M^{jk)}\ .&\end{aligned} \tag{C.4}$$

This consistency condition differs from the ones found so far in that it is inhomogeneous. Whereas eqs. (C.1–3) admit vanishing M^{ij} and $N^{ij}_{\alpha\dot\alpha}$, we infer from eq. (C.4) that it is actually necessary to modify the constraints on L when gauging the central charge! According to eq. (C.3) we can express the term $\bar{\mathcal{D}}_k\bar{\mathcal{D}}^{(i}M^{jk)}$ and its complex conjugate through $N^{ij}_{\alpha\dot\alpha}$, so in the special case $N^{ij}_{\alpha\dot\alpha}=0$ condition (C.4) may be solved for the real part of ZM^{ij},

$$\tfrac12(ZM^{ij}+\bar Z\bar M^{ij}) = \mathrm{i}\lambda^{(i}\psi^{j)} - \mathrm{i}\bar\lambda^{(i}\bar\psi^{j)} - Y^{ij}L + \hat M^{ij}\ ,\quad \delta_z\hat M^{ij}=0\ . \tag{2.34}$$

This fact we shall exploit extensively in the next two chapters.
From (C.4) also follows why the vector-tensor superfield cannot transform nontrivially under gauge transformations Δ^{g}, i.e. $\delta_I L = 0$. If we relax this condition, then eq. (C.4) would read for $Z=\mathrm{i}$

$$\begin{aligned}D^{ijI}\delta_I L - \mathrm{i}\chi^{I(i}\delta_I\psi^{j)} + \mathrm{i}\bar\chi^{I(i}\delta_I\bar\psi^{j)} &= \frac{\mathrm{i}}{2}\,\delta_z(\bar M^{ij}-M^{ij}) - \frac12\,\phi^I\delta_I M^{ij} - \frac12\,\bar\phi^I\delta_I\bar M^{ij}\\ &\quad + \mathcal{D}^\mu N^{ij}_\mu + \frac38\,\mathcal{D}_k\mathcal{D}^{(i}\bar M^{jk)} + \frac38\,\bar{\mathcal{D}}_k\bar{\mathcal{D}}^{(i}M^{jk)}\ .\end{aligned}$$

and it is easily verified that no choice of M^{ij} and $N^{ij}_{\alpha\dot\alpha}$ can yield the expression on the left-hand side of the equation. Thus, there is no minimal coupling of the vector-tensor multiplet to a Yang-Mills potential.
Now we contract the commutator with ε_{ij}. The result can be further reduced to parts symmetric and antisymmetric in $\alpha\beta$, respectively. Taking the imaginary part of the latter we derive the first Bianchi identity,

$$\mathcal{D}_\mu W^\mu = \frac12\,\delta_z(\bar\lambda^i\bar\psi_i - \lambda_i\psi^i) - \frac{\mathrm{i}}{12}\,\mathcal{D}_i\mathcal{D}_j\bar M^{ij} + \frac{\mathrm{i}}{12}\,\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j M^{ij}\ . \tag{BI.1}$$

Note that the covariant derivative contains the central charge transformation of W^μ which we have not yet determined. The real part gives rise to

$$\begin{aligned}\delta_z\big[\,|Z|^2U - \tfrac{\mathrm{i}}{2}(\lambda_i\psi^i + \bar\lambda^i\bar\psi_i)\,\big] &= \mathcal{D}^\mu\mathcal{D}_\mu L + \frac{\mathrm{i}}{6}\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j N^{ij}_\mu\\ &\quad + \frac{1}{12}\,\mathcal{D}_i\mathcal{D}_j\bar M^{ij} + \frac{1}{12}\,\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j M^{ij}\ .\end{aligned} \tag{2.35}$$

Here we discover again the factor $\mathcal{E}$ accompanying the central charge generator, for the covariant d'Alembertian acting on L may be written as

$$\mathcal{D}^\mu \mathcal{D}_\mu L = \Box L + U\partial^\mu A_\mu + 2A^\mu \partial_\mu U + A^\mu A_\mu \delta_z U \ .$$

The last term then combines with $|Z|^2\delta_z U$ on the left-hand side of eq. (2.35) into $\mathcal{E}\delta_z U$. It remains to consider the part symmetric in $\alpha\beta$. Using eq. (A.21) we readily obtain

$$\delta_z\big(IG_{\mu\nu} - R\tilde{G}_{\mu\nu} - \tilde{\Sigma}_{\mu\nu}\big) = -2\,\mathcal{D}_{[\mu}W_{\nu]} + \tfrac{\mathrm{i}}{6}\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{D}^i\sigma^\rho\bar{\mathcal{D}}^j N_{ij}^\sigma \ , \tag{2.36}$$

with the abbreviations

$$I \equiv \mathrm{Im}\,Z \ , \quad R \equiv \mathrm{Re}\,Z \tag{2.37}$$

$$\Sigma_{\mu\nu} \equiv LF_{\mu\nu} + \mathrm{i}(\lambda_i\sigma_{\mu\nu}\psi^i - \bar{\psi}_i\bar{\sigma}_{\mu\nu}\bar{\lambda}^i) \ . \tag{2.38}$$

At last we require

$$[\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,\bar{\psi}^j_{\dot\alpha} \stackrel{!}{=} 0 \ .$$

We again decompose the commutator into SU(2) irreducible parts. Symmetrized in ij the equation is fulfilled identically when the conditions (C.1–3) hold. Antisymmetrized the real part provides the second Bianchi identity,

$$\begin{aligned} I\,\mathcal{D}_\nu\tilde{G}^{\mu\nu} + R\,\mathcal{D}_\nu G^{\mu\nu} = &-\frac{1}{2}\,U\partial^\mu|Z|^2 - \frac{1}{2}\,\delta_z(Z\psi^i\sigma^\mu\bar{\lambda}_i + \bar{Z}\lambda^i\sigma^a\bar{\psi}_i) \\ &- \frac{\mathrm{i}}{12}\,Z\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} - \frac{\mathrm{i}}{12}\,\bar{Z}\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j\bar{M}^{ij} \\ &- \frac{1}{12}\big(Z\mathcal{D}_i\mathcal{D}_j + \bar{Z}\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\big)N^{a\,ij} \ , \end{aligned} \tag{BI.2}$$

while the imaginary part gives rise to the central charge transformation of W^μ,

$$\begin{aligned} &\delta_z\big[\,|Z|^2W^\mu + \tfrac{\mathrm{i}}{2}L(Z\partial^\mu\bar{Z} - \bar{Z}\partial^\mu Z) + \tfrac{\mathrm{i}}{2}(Z\psi^i\sigma^\mu\bar{\lambda}_i - \bar{Z}\lambda^i\sigma^\mu\bar{\psi}_i)\,\big] = \\ &\quad = I\,\mathcal{D}_\nu G^{\mu\nu} - R\,\mathcal{D}_\nu\tilde{G}^{\mu\nu} + \frac{1}{12}\,Z\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} - \frac{1}{12}\,\bar{Z}\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j\bar{M}^{ij} \\ &\quad - \frac{\mathrm{i}}{12}\big(Z\mathcal{D}_i\mathcal{D}_j - \bar{Z}\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\big)N^{a\,ij} \ . \end{aligned} \tag{2.39}$$

With this the evaluation of the supersymmetry algebra on ψ^i is completed. We could already determine all the supersymmetry and central charge transformations of the covariant components of the vector-tensor multiplet. Evidently we cannot obtain any information on the gauge fields V_μ and $B_{\mu\nu}$ as long as the deformations M^{ij} and $N^{ij}_{\alpha\dot\alpha}$ have not been specified and the Bianchi identities solved. It is now a tedious exercise to check that the algebra holds also on W^μ, $G_{\mu\nu}$ and U and that we obtain no further consistency conditions.

2.3 The Ansatz

In this section we confine our investigation to couplings of the vector-tensor multiplet to just one abelian vector multiplet that gauges the central charge, which is our main objective. To this end we make an Ansatz for the constraints on L and apply the consistency conditions (C.1–3), i.e. those that do not contain spacetime derivatives. Since the only fields in the multiplets under consideration that transform nontrivially under the automorphism group SU(2) are given by $\mathcal{D}^i_\alpha L$, $\mathcal{D}^i_\alpha Z$, $\mathcal{D}^i\mathcal{D}^j Z$ and their complex conjugates, the most general Ansatz compatible with the properties (2.23) reads

$$\mathcal{D}^{(i}_\alpha \bar{\mathcal{D}}^{j)}_{\dot\alpha} L = a\,\mathcal{D}^{(i}_\alpha Z\,\bar{\mathcal{D}}^{j)}_{\dot\alpha} L - \bar a\,\bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar Z\,\mathcal{D}^{j)}_\alpha L + b\,\mathcal{D}^{(i}_\alpha Z\,\bar{\mathcal{D}}^{j)}_{\dot\alpha}\bar Z + c\,\mathcal{D}^{(i}_\alpha L\,\bar{\mathcal{D}}^{j)}_{\dot\alpha} L \tag{2.40}$$

$$\begin{aligned}\mathcal{D}^{(i}\mathcal{D}^{j)} L = {} & A\,\mathcal{D}^{(i} Z\,\mathcal{D}^{j)} L + B\,\bar{\mathcal{D}}^{(i}\bar Z\,\bar{\mathcal{D}}^{j)} L + C\,\mathcal{D}^i\mathcal{D}^j Z + D\,\mathcal{D}^i Z\,\mathcal{D}^j Z \\ & + E\,\bar{\mathcal{D}}^i\bar Z\,\bar{\mathcal{D}}^j\bar Z + F\,\mathcal{D}^i L\,\mathcal{D}^j L + G\,\bar{\mathcal{D}}^i L\,\bar{\mathcal{D}}^j L\ .\end{aligned} \tag{2.41}$$

Here the coefficients are arbitrary local functions of the superfields L, Z and $\bar Z$. $\bar a$ is the complex conjugate of a, and b and c must be real. Recall that since Z is an abelian vector superfield, it satisfies

$$\mathcal{D}^i\mathcal{D}^j Z = \mathcal{D}^{(i}\mathcal{D}^{j)} Z = \bar{\mathcal{D}}^i\bar{\mathcal{D}}^j\bar Z\ , \quad \mathcal{D}^i_\alpha\bar Z = 0 = \bar{\mathcal{D}}^i_{\dot\alpha} Z\ . \tag{2.42}$$

The first consistency condition (C.1), now written as a proper superfield equation, requires

$$\mathcal{D}^{(i}_\alpha\mathcal{D}^j\mathcal{D}^{k)} L = 0\ , \tag{2.43}$$

which simply expresses the fact that the spinor derivatives $\mathcal{D}^i_\alpha$ anticommute when symmetrized in the SU(2) indices,

$$\mathcal{D}^{(i}_\alpha\mathcal{D}^j\mathcal{D}^{k)} = -\varepsilon^{\gamma\beta}\mathcal{D}^{(j}_\beta\mathcal{D}^i_\alpha\mathcal{D}^{k)}_\gamma = -\tfrac12\varepsilon^{\gamma\beta}\varepsilon_{\alpha\gamma}\mathcal{D}^{(j}_\beta\mathcal{D}^i\mathcal{D}^{k)} = -\tfrac12\mathcal{D}^{(i}_\alpha\mathcal{D}^j\mathcal{D}^{k)} = 0\ . \tag{2.44}$$

When the Ansatz (2.41) is inserted, condition (2.43) translates into a set of nonlinear partial differential equations for the coefficient functions. Differentiations with respect to L and Z arise from the action of $\mathcal{D}^i_\alpha$ on the coefficients[4], while quadratic terms appear because we have to use the constraints (2.40), (2.41) when the spinor derivative acts on $\mathcal{D}^i_\alpha L$ or $\bar{\mathcal{D}}^i_{\dot\alpha} L$. Introducing the abbreviations

$$\partial \equiv \frac{\partial}{\partial Z}\ , \quad \bar\partial \equiv \frac{\partial}{\partial \bar Z}\ , \quad \partial_L \equiv \frac{\partial}{\partial L}\ , \tag{2.45}$$

we have for instance

$$\begin{aligned}\mathcal{D}^{(i}_\alpha\big(F\,\mathcal{D}^j L\,\mathcal{D}^{k)} L\big) &= \big(\partial F\,\mathcal{D}^{(i}_\alpha Z + \partial_L F\,\mathcal{D}^{(i}_\alpha L\big)\,\mathcal{D}^j L\,\mathcal{D}^{k)} L - 2F\,\mathcal{D}^{\beta(i} L\,\mathcal{D}^j_\alpha\mathcal{D}^{k)}_\beta L \\ &= \partial F\,\mathcal{D}^{(i}_\alpha Z\,\mathcal{D}^j L\,\mathcal{D}^{k)} L - F\,\mathcal{D}^{(i}_\alpha L\,\mathcal{D}^j\mathcal{D}^{k)} L\ ,\end{aligned}$$

[4]There is no differentiation with respect to $\bar Z$ since it is antichiral.

where the expression proportional to $\partial_L F$ vanishes by the same reasoning as for eq. (2.44). In this way condition (2.43) decomposes into ten linearly independent terms whose coefficients must vanish separately,

$$
\begin{aligned}
1)&\quad 0 = \partial F - \tfrac{1}{2}\partial_L A \\
2)&\quad 0 = \partial_L C - \tfrac{1}{2}A - CF \\
3)&\quad 0 = \partial C - D - \tfrac{1}{2}AC \\
4)&\quad 0 = \partial_L G + G(2c - F) \\
5)&\quad 0 = \partial G - \tfrac{1}{2}G(A - 4a) \\
6)&\quad 0 = \partial_L E - EF + \bar{a}B \\
7)&\quad 0 = \partial E - \tfrac{1}{2}AE + bB \\
8)&\quad 0 = \partial_L B + B(c - F) + 2\bar{a}G \\
9)&\quad 0 = \partial B - \tfrac{1}{2}B(A - 2a) + 2bG \\
10)&\quad 0 = \partial_L D - DF - \tfrac{1}{2}\partial A + \tfrac{1}{4}A^2\ .
\end{aligned}
\tag{2.46}
$$

Condition (C.2) may be recast into the form

$$
\mathcal{D}^{(i}_{(\alpha}\mathcal{D}^{j}_{\beta)}\bar{\mathcal{D}}^{k)}_{\dot\alpha}L = 0\ . \tag{2.47}
$$

Using the Ansatz (2.40) and proceeding as above, we obtain further differential equations,

$$
\begin{aligned}
11)&\quad 0 = \partial_L a - \partial c \\
12)&\quad 0 = \partial_L b - \partial\bar{a} + a\bar{a} + bc\ .
\end{aligned}
\tag{2.48}
$$

Condition (C.3) we write as

$$
\bar{\mathcal{D}}^{(i}_{\dot\alpha}\mathcal{D}^{j}\mathcal{D}^{k)}L = \mathcal{D}^{(j}\mathcal{D}^{k}\bar{\mathcal{D}}^{i)}_{\dot\alpha}L\ . \tag{2.49}
$$

This must hold as the anticommutator of $\mathcal{D}^i_\alpha$ and $\bar{\mathcal{D}}^j_{\dot\alpha}$ involves an ε^{ij} and thus vanishes when symmetrized in the SU(2) indices. The condition gives rise to ten more differential equations,

$$
\begin{aligned}
13)&\quad 0 = \partial_L(F - c) - G\bar{G} + c(F - c) \\
14)&\quad 0 = \bar\partial F - \partial_L\bar{a} - \tfrac{1}{2}B\bar{G} + \bar{a}(F - c) \\
15)&\quad 0 = \bar\partial C - E - \tfrac{1}{2}B\bar{C} - b - \bar{a}C \\
16)&\quad 0 = \bar\partial D - \partial b - \tfrac{1}{2}B\bar{E} - \bar{a}D + b(A - a) \\
17)&\quad 0 = \partial_L C - \tfrac{1}{2}B - \bar{C}G - a - cC \\
18)&\quad 0 = \partial_L D - \partial a - \bar{E}G - cD + a(A - a) \\
19)&\quad 0 = \partial_L(A - a) - \partial c - \bar{B}G + 2a(F - c) \\
20)&\quad 0 = \bar\partial A - \partial\bar{a} - \partial_L b - \tfrac{1}{2}B\bar{B} - a\bar{a} + b(2F - c) \\
21)&\quad 0 = \bar\partial G - \tfrac{1}{2}\partial_L B + \tfrac{1}{2}G(\bar{A} - 2\bar{a}) - \tfrac{1}{2}B(\bar{F} - c)
\end{aligned}
\tag{2.50}
$$

$$22) \quad 0 = \partial_L E - \tfrac{1}{2}\bar{\partial}B - \bar{D}G - cE + \tfrac{1}{2}B(\bar{a} + \tfrac{1}{2}\bar{A}) \ .$$

We may eliminate several derivatives in the above equations by virtue of the conditions (2.46) and (2.48),

$$\begin{aligned}
17)' \quad & 0 = C(F-c) + \tfrac{1}{2}(A-2a) - \tfrac{1}{2}B - \bar{C}G \\
18)' \quad & 0 = \partial(A-2a) - \tfrac{1}{2}(A-2a)^2 + 2D(F-c) - 2\bar{E}G \\
19)' \quad & 0 = \partial_L(A-2a) - \bar{B}G + 2a(F-c) \\
20)' \quad & 0 = \bar{\partial}(A-2a) - \tfrac{1}{2}B\bar{B} + 2bF \\
21)' \quad & 0 = \bar{\partial}G + \tfrac{1}{2}\bar{A}G + \tfrac{1}{2}B(2c - F - \bar{F}) \\
22)' \quad & 0 = \bar{\partial}B - 2E(F-c) - \tfrac{1}{2}B(\bar{A} - 2\bar{a}) + 2\bar{D}G \ .
\end{aligned} \tag{2.51}$$

Before we are trying to solve these equations, it is important to note that the solutions to the consistency conditions decompose into mutually disjoint equivalence classes, where two sets of constraints are deemed equivalent if they are related by a local superfield redefinition

$$L \;\rightarrow\; L = L(\hat{L}, Z, \bar{Z}) \ . \tag{2.52}$$

Each representative of such a class effectively describes the same physics. We may employ this to choose representatives which simplify the subsequent calculations as much as possible. Using

$$\mathcal{D}^i_\alpha L = L' \, \mathcal{D}^i_\alpha \hat{L} + \partial L \, \mathcal{D}^i_\alpha Z \tag{2.53}$$

and similar for $\bar{\mathcal{D}}^i_{\dot{\alpha}} L$, where $L' \equiv \partial_{\hat{L}} L \neq 0$, we rewrite the Ansatz (2.40), (2.41) in terms of $\hat{L}$. The coefficients as functions of $\hat{L}$ are then given by

$$\begin{aligned}
\hat{a} &= a + c\,\partial L - \partial L'/L' \\
\hat{b} &= (b + a\,\bar{\partial}L + \bar{a}\,\partial L + c\,\partial L\,\bar{\partial}L - \partial\bar{\partial}L)/L' \\
\hat{c} &= c\,L' - L''/L' \\
\hat{A} &= A + 2F\,\partial L - 2\partial L'/L' \\
\hat{B} &= B + 2G\,\bar{\partial}L \\
\hat{C} &= (C - \partial L)/L' \\
\hat{D} &= (D + A\,\partial L + F\,\partial L\,\partial L - \partial^2 L)/L' \\
\hat{E} &= (E + B\,\bar{\partial}L + G\,\bar{\partial}L\,\bar{\partial}L)/L' \\
\hat{F} &= F L' - L''/L' \\
\hat{G} &= G L' \ .
\end{aligned} \tag{2.54}$$

The differential equations (2.46), (2.48) and (2.50) are invariant under the above transformations in the sense that if A, B, etc. are solutions to the consistency conditions, then $\hat{A}$, $\hat{B}$, etc. satisfy the same equations with ∂_L replaced by $\partial_{\hat{L}}$. Consider for instance eq. 1) in (2.46),

$$\begin{aligned}\partial\hat{F} - \tfrac{1}{2}\partial_{\hat{L}}\hat{A} &= \partial\,(FL' - L''/L') - \tfrac{1}{2}\partial_{\hat{L}}(A + 2F\,\partial L - 2\partial L'/L') \\ &= \partial F L' + \partial L\,\partial_L F\,L' + F\,\partial L' - \partial L''/L' + L''\partial L'/L'^2 \\ &\quad - \tfrac{1}{2}(L'\partial_L A + 2L'\,\partial_L F\,\partial L + 2F\,\partial L' - 2\partial L''/L' + 2L''\partial L'/L'^2) \\ &= L'(\partial F - \tfrac{1}{2}\partial_L A) = 0\ .\end{aligned}$$

A superfield redefinition induces changes of the component fields. If we define the components of $\hat{L}$ similar to those of L in eq. (2.25), then one has

$$\psi^i = L'\hat{\psi}^i + \mathrm{i}\partial L\,\lambda^i \tag{2.55}$$

according to eq. (2.53). From this we readily obtain also the relations for W^μ and $G_{\mu\nu}$,

$$W^\mu = L'\hat{W}^\mu - \tfrac{1}{2}L''\,\hat{\psi}^i\sigma^\mu\hat{\bar{\psi}}_i - \tfrac{1}{2}\bar{\partial}\partial L\,\lambda^i\sigma^\mu\bar{\lambda}_i - \tfrac{\mathrm{i}}{2}(\partial L'\,\lambda^i\sigma^\mu\hat{\bar{\psi}}_i - \bar{\partial}L'\,\hat{\psi}^i\sigma^\mu\bar{\lambda}_i) \tag{2.56}$$

$$\begin{aligned}G_{\mu\nu} = {}& L'\hat{G}_{\mu\nu} + (\partial L + \bar{\partial}L)F_{\mu\nu} - \mathrm{i}(\partial L - \bar{\partial}L)\tilde{F}_{\mu\nu} - \tfrac{1}{2}L''(\hat{\psi}^i\sigma_{\mu\nu}\hat{\psi}_i + \hat{\bar{\psi}}^i\bar{\sigma}_{\mu\nu}\hat{\bar{\psi}}_i) \\ &+ \tfrac{1}{2}(\partial^2 L\,\lambda^i\sigma_{\mu\nu}\lambda_i + \bar{\partial}^2 L\,\bar{\lambda}^i\bar{\sigma}_{\mu\nu}\bar{\lambda}_i) - \mathrm{i}(\partial L'\,\lambda^i\sigma_{\mu\nu}\hat{\psi}_i - \bar{\partial}L'\,\hat{\bar{\psi}}^i\bar{\sigma}_{\mu\nu}\bar{\lambda}_i)\ ,\end{aligned} \tag{2.57}$$

while the auxiliary field simply transforms as $U = L'\hat{U}$.

2.3.1 Invariant Actions

Once a set of consistent constraints has been found, the construction of a linear superfield is the crucial step towards an invariant action. Similar to the derivation of the constraints themselves we start in full generality from an Ansatz for the pre-Lagrangian,

$$\begin{aligned}\mathcal{L}^{ij} = {}& \alpha\,\mathcal{D}^i L\,\mathcal{D}^j L + \bar{\alpha}\,\bar{\mathcal{D}}^i L\,\bar{\mathcal{D}}^j L + \beta\,\mathcal{D}^{(i}Z\,\mathcal{D}^{j)}L + \bar{\beta}\,\bar{\mathcal{D}}^{(i}\bar{Z}\,\bar{\mathcal{D}}^{j)}L \\ &+ \gamma\,\mathcal{D}^i\mathcal{D}^j Z + \delta\,\mathcal{D}^i Z\,\mathcal{D}^j Z + \bar{\delta}\,\bar{\mathcal{D}}^i\bar{Z}\,\bar{\mathcal{D}}^j\bar{Z}\ ,\end{aligned} \tag{2.58}$$

with γ real. The coefficients are again functions of L, Z and $\bar{Z}$. Whereas reality and symmetry in ij have already been taken into account, the coefficients are further constrained by the requirement

$$\mathcal{D}^{(i}_\alpha\mathcal{L}^{jk)} = 0\ .$$

Again this yields a set of differential equations analogous to the evaluation of the consistency condition (2.43). They read

$$\begin{aligned}0 &= \partial_L\gamma - \tfrac{1}{2}\beta - \alpha C \\ 0 &= \partial\gamma - \delta - \tfrac{1}{2}\beta C \\ 0 &= \partial_L\bar{\alpha} - \alpha G + 2\bar{\alpha}c \\ 0 &= \partial\bar{\alpha} - \tfrac{1}{2}\beta G + 2\bar{\alpha}a\end{aligned}$$

$$
\begin{aligned}
0 &= \partial\alpha - \tfrac{1}{2}\partial_L\beta - \tfrac{1}{2}\beta F + \tfrac{1}{2}\alpha A \\
0 &= \partial_L\bar\beta - \alpha B + 2\bar\alpha\bar a + \bar\beta c \\
0 &= \partial\bar\beta - \tfrac{1}{2}\beta B + 2\bar\alpha b + \bar\beta a \\
0 &= \partial\beta - 2\partial_L\delta + 2\alpha D - \tfrac{1}{2}\beta A \\
0 &= \partial_L\bar\delta - \alpha E + \bar\beta\bar a \\
0 &= \partial\bar\delta - \tfrac{1}{2}\beta E + \bar\beta b \ ,
\end{aligned}
\tag{2.59}
$$

and for given functions A, B, etc. determine the unknown coefficients α, β, etc.
Note that when $\bar{\mathcal{D}}^{\dot\alpha(i}N^{jk)}_{\alpha\dot\alpha} = 0$, for instance in the special case $N^{ij}_{\alpha\dot\alpha} = 0$, the combination

$$\kappa M^{ij} + \bar\kappa\bar M^{ij} \ , \quad \kappa \in \mathbb{C}$$

is real and hence a linear superfield by itself according to eqs. (C.1) and (C.3),

$$\bar{\mathcal{D}}^{\dot\alpha(i}N^{jk)}_{\alpha\dot\alpha} = 0 \quad \Rightarrow \quad \mathcal{D}^{(i}_{\alpha}(\kappa M^{jk)} + \bar\kappa\bar M^{jk)}) = 0 \ . \tag{2.60}$$

Thus it will turn up as a particular solution to the conditions (2.59).

2.3.2 Solutions for $Z = \mathrm{i}$

The general Ansatz (2.40), (2.41) does not reduce to the free constraints (2.3) in the limit $Z = \mathrm{i}$ but there remain terms quadratic in $D^i_\alpha L$. This suggests that the constraints (2.3) may not be the only possible description of the vector-tensor multiplet, and indeed, as mentioned in the introduction, Claus et al. have shown in [16] that there exists a nontrivial deformation[5] which gives rise to self-interactions. With the set of consistency conditions given above we can reproduce this result:
The case $Z = \mathrm{i}$ corresponds to

$$a = b = A = B = C = D = E = 0 \ , \quad \partial = \bar\partial \equiv 0 \ .$$

The equations (2.48) are satisfied identically, whereas (2.46) and (2.50) each provide a single condition on the remaining functions $c(L)$, $F(L)$ and $G(L)$, namely

$$
\begin{aligned}
0 &= (\partial_L + c)G - (F - c)G \\
0 &= (\partial_L + c)(F - c) - G\bar G \ .
\end{aligned}
\tag{2.61}
$$

These are invariant under field redefinitions $L = L(\hat L)$ and transformations

$$\hat c = c\,L' - L''/L' \ , \quad \hat F = FL' - L''/L' \ , \quad \hat G = GL'$$

as in (2.54). Since c transforms inhomogeneously, we may choose a gauge in which $c = 0$. Note that this does not fix the gauge completely, we are still free to shift and rescale L by real constant parameters,

$$L = \kappa\hat L + \varrho \ , \quad \kappa \in \mathbb{R}^* \ , \ \varrho \in \mathbb{R} \quad \Rightarrow \quad L''/L' = 0 \ . \tag{2.62}$$

[5] Nontrivial in the sense that it may not be removed by a field redefinition.

For $c = 0$, i.e. $N^{ij}_{\alpha\dot\alpha} = 0$, the consistency condition (C.4) can easily be evaluated. It reduces to

$$\begin{aligned} 0 &= \delta_z\big[D^{(i}D^{j)}L - \bar D^{(i}\bar D^{j)}L\big] \\ &= \delta_z\big[(F-\bar G)\,D^iL\,D^jL + (G-\bar F)\,\bar D^iL\,\bar D^jL\big] \ . \end{aligned} \tag{2.63}$$

Hence $G = \bar F$, and the equations (2.61) both imply

$$\partial_L F = F\bar F \ . \tag{2.64}$$

From this we infer first of all that the imaginary part of F is L-independent, thus

$$F = f(L) + \mathrm{i}\kappa \ , \ \kappa \in \mathbb{R} \quad \Rightarrow \quad \partial_L f = f^2 + \kappa^2 \ . \tag{2.65}$$

When $\kappa = 0$, we have two solutions. On the one hand $F_1 = 0$, which corresponds to the free constraints (2.3) as then all coefficients vanish. The second solution is

$$F_2 = -\frac{1}{L+\mu} \ , \quad \mu \in \mathbb{R} \ , \tag{2.66}$$

where μ may be removed by a field redefinition (2.62). In the case $\kappa \neq 0$ the general solution reads

$$F_3 = \kappa\big(\tan(\kappa L + \varrho) + \mathrm{i}\,\big) \ , \quad \varrho \in \mathbb{R} \ . \tag{2.67}$$

We may choose $\kappa = 1$ and $\varrho = 0$. Since we have fixed the gauge, the three solutions evidently yield distinct constraints that are not connected by a field redefinition. This may also be seen from the transformation law of the coefficient G: If $G = 0$ for one representative of a class of constraints, then this holds in the whole class. Moreover, there is no transition from the second to the third solution since G_2 is real while G_3 is not.

The constraints that correspond to F_3 were first discovered by Ivanov and Sokatchev in [21]. However, these are inconsistent, for the Bianchi identities (BI.1) and (BI.2) admit no local solution. We shall not demonstrate this fact, but remark that it shows that the conditions (C.1–4) are by no means sufficient.

The solution F_2 implies constraints

$$D^{(i}_\alpha \bar D^{j)}_{\dot\alpha} L = 0 \ , \quad D^{(i}D^{j)}L = -\frac{1}{L}\left(D^iL\,D^jL + \bar D^iL\,\bar D^jL\right) \ , \tag{2.68}$$

which are indeed consistent and describe what is known as the *nonlinear* vector-tensor multiplet. We shall first generalize these constraints to admit a gauged central charge before investigating them in any more detail. This will be done in chapter 4. At this point we just emphasize that the constraints may be rendered regular for $\hat L = 0$ by a field redefintion

$$L = \exp(-\kappa \hat L) \ , \quad \kappa \in \mathbb{R}^* \ , \tag{2.69}$$

which gives (omitting the hat)

$$D^{(i}_\alpha \bar D^{j)}_{\dot\alpha} L = \kappa\, D^{(i}_\alpha L\,\bar D^{j)}_{\dot\alpha} L \ , \quad D^{(i}D^{j)}L = 2\kappa\, D^iL\,D^jL + \kappa\,\bar D^iL\,\bar D^jL \ . \tag{2.70}$$

In this form they were first derived in [20] and are evidently a deformation of the free theory. While here the coefficients are constant, we shall nevertheless generalize the constraints (2.68), for these have the useful property of vanishing $N^{ij}_{\alpha\dot\alpha}$.

2.3.3 Generalization to $Z(x)$

In the general case of an x-dependent field Z the consistency conditions (C.1–3), which we have translated into a set of differential equations, do not determine completely the unknown coefficients in the Ansatz. This is quite obvious as the number of equations is not sufficient to fix the dependence of the coefficients on all three variables $L, Z, \bar{Z}$. Our goal is to generalize the solutions found in the previous section. To this end we simplify the Ansatz by setting

$$a = b = c = 0 \quad \Rightarrow \quad N^{ij}_{\alpha\dot{\alpha}} = 0 \ , \tag{2.71}$$

which can be justified a posteriori by showing that the resulting constraints do indeed yield the linear and nonlinear vector-tensor multiplet with gauged central charge. Note that we are still allowed to redefine

$$L = \kappa\hat{L} + f(Z) + \bar{f}(\bar{Z}) \ , \quad \kappa \in \mathbb{R}^* \ . \tag{2.72}$$

This is the general solution to the differential equations

$$L'' = \partial L' = \bar{\partial}\partial L = 0 \ ,$$

which according to eq. (2.54) guarantee the preservation of Ansatz (2.71).
The simplification is motivated by the fact that now condition (C.4) may easily be evaluated. With the reduced Ansatz put in, it reads

$$\begin{aligned} 0 = \delta_z \Big[&(\tfrac{1}{2}L + ZC)\, \mathcal{D}^i\mathcal{D}^j Z + (1 + ZA)\, \mathcal{D}^{(i} Z\, \mathcal{D}^{j)} L + (1 + ZB)\, \bar{\mathcal{D}}^{(i} \bar{Z}\, \bar{\mathcal{D}}^{j)} L \\ &+ ZD\, \mathcal{D}^i Z\, \mathcal{D}^j Z + ZE\, \bar{\mathcal{D}}^i \bar{Z}\, \bar{\mathcal{D}}^j \bar{Z} + ZF\, \mathcal{D}^i L\, \mathcal{D}^j L + ZG\, \bar{\mathcal{D}}^i L\, \bar{\mathcal{D}}^j L \Big] \\ &+ \text{c.c.} \end{aligned} \tag{2.73}$$

from which we infer

$$23) \quad 0 = 2 + ZA + \bar{Z}\bar{B} \qquad\qquad 24) \quad 0 = ZF + \bar{Z}\bar{G} \ , \tag{2.74}$$

as well as

$$25) \quad u(Z,\bar{Z}) = L + ZC + \bar{Z}\bar{C} \qquad\qquad 26) \quad v(Z,\bar{Z}) = ZD + \bar{Z}\bar{E} \ , \tag{2.75}$$

u and v being arbitrary L-independent functions that contribute only to δ_z-invariant terms inside the square brackets in eq. (2.73). Equations $23-26$) allow to eliminate B, D and G in the eqs. (2.46), (2.50) and (2.51), leaving the four unknown functions A, C, E and F, whereas u and v may be removed using the gauge freedom (2.72). To show this, let us first consider eq. 16) in (2.50). Replacing B and D according to the relations above, we obtain

$$0 = \bar{\partial}v - \bar{Z}(\bar{\partial}\bar{E} - \tfrac{1}{2}\bar{A}\bar{E}) \ .$$

Eq. 7) then implies $\bar{\partial} v = 0$. Next we consider eq. 15). Using eqs. 25), 3) and 26), we find

$$0 = \partial u - Z(\partial C - \tfrac{1}{2}AC) - \bar{Z}\bar{E} = \partial u - v \ .$$

Since v does not depend on $\bar{Z}$ and u is real, we conclude that there is a function $w(Z)$ such that

$$u(Z,\bar{Z}) = w(Z) + \bar{w}(\bar{Z}) \ , \quad v(Z) = \partial w(Z) \ . \tag{2.76}$$

Now let us perform a field redefinition (2.72) in eq. 25). With the transformation of C as in eq. (2.54), we calculate

$$\begin{aligned}\hat{L} + Z\hat{C} + \bar{Z}\hat{\bar{C}} &= \frac{1}{L'}\big(L - f - \bar{f}\big) + \frac{Z}{L'}\big(C - \partial f\big) + \frac{\bar{Z}}{L'}\big(\bar{C} - \bar{\partial}\bar{f}\big) \\ &= \frac{1}{L'}\big(w + \bar{w} - f - \bar{f} - Z\partial f - \bar{Z}\bar{\partial}\bar{f}\big) \ .\end{aligned}$$

The same redefinition applied to eq. 26) gives

$$\begin{aligned}Z\hat{D} + \bar{Z}\hat{\bar{E}} &= \frac{1}{L'}\big(ZD + \bar{Z}\bar{E} + \partial f\,(ZA + \bar{Z}\bar{B}) + \partial f\,\partial f\,(ZF + \bar{Z}\bar{G}) - Z\partial^2 f\big) \\ &= \frac{1}{L'}\,\partial\big(w - f - Z\partial f\big) \ ,\end{aligned}$$

where eqs. 23) and 24) have been used. So provided there is a function $g(Z)$ such that $\partial g = w$, a field redefinition with $f = g/Z$ yields (omitting the hats)

$$25) \quad 0 = L + ZC + \bar{Z}\bar{C} \qquad\qquad 26) \quad 0 = ZD + \bar{Z}\bar{E} \ . \tag{2.77}$$

Note that there remains a residual gauge freedom

$$L = \kappa\hat{L} + \frac{\varrho}{Z} + \frac{\bar{\varrho}}{\bar{Z}} \ , \quad \kappa \in \mathbb{R}^* \ , \ \varrho \in \mathbb{C} \ , \tag{2.78}$$

for g may be shifted by a complex constant ϱ.
Working in a gauge where $u = v = 0$, the 26 consistency conditions can be reduced to a set of 11 independent equations,

$$\begin{aligned}
&1) \quad 0 = 2\partial F - \partial_L A \\
&2) \quad 0 = \partial_L C - CF - \tfrac{1}{2}A \\
&3) \quad 0 = \partial C - \tfrac{1}{2}AC + \frac{\bar{Z}}{Z}\,\bar{E} \\
&4) \quad 0 = \partial_L \bar{F} - F\bar{F} \\
&5) \quad 0 = 2\partial\bar{F} - \Big(A + \frac{2}{Z}\Big)\bar{F} \\
&6) \quad 0 = \partial_L E - EF \\
&7) \quad 0 = \partial E - \tfrac{1}{2}AE
\end{aligned} \tag{2.79}$$

$$
\begin{aligned}
9)\quad & 0 = \partial \bar{A} - \tfrac{1}{2}\Big(A + \frac{2}{Z}\Big)\Big(\bar{A} + \frac{2}{\bar{Z}}\Big) \\
10)\quad & 0 = \partial A - \tfrac{1}{2}A^2 - 2\bar{E}\big(F - \bar{F}\big)\frac{\bar{Z}}{Z} \\
19)\quad & 0 = \partial_L A - \Big(A + \frac{2}{Z}\Big)\bar{F} \\
25)\quad & 0 = L + ZC + \bar{Z}\bar{C}\ .
\end{aligned}
$$

The trivial solution $M^{ij} = 0 = N^{ij}_{\alpha\dot{\alpha}}$ is now excluded as some of the equations are inhomogeneous. Eq. 4) in (2.79) we have already solved in the previous section, only now the integration constants may depend on Z and $\bar{Z}$. The solution that corresponds to F_3 we discard again since, as mentioned, it leads to inconsistent constraints, and the situation certainly does not improve when gauging the central charge. This leaves the two possibilities

$$
F_1 = 0\ , \qquad F_2 = -\frac{1}{L + h(Z,\bar{Z})}\ , \quad h \text{ real}\ . \tag{2.80}
$$

In the following two chapters we shall explore the consequences of each in detail.

Chapter 3
The Linear Case

In this chapter we present the linear vector-tensor multiplet with gauged central charge. Starting from the consistency conditions derived in the previous chapter, we determine the constraints that underlie the model and work out the supersymmetry and central charge transformations of the component fields. The Bianchi identities will be computed and solved in terms of gauge potentials. Then we follow the procedure outlined in sections 1.2 and 2.3.1 in order to derive an invariant action. After generalizing the model to include couplings to additional nonabelian vector multiplets, we conclude with a brief review of Henneaux-Knaepen models and their relation to the vector-tensor multiplet.

3.1 Consistent Constraints

Having singled out two possible coefficient functions F in the previous chapter, we shall now attempt to solve the consistency conditions (2.79) subject to the first solution $F_1 = 0$.
Eqs. 4) and 5) are satisfied identically, while from eqs. 1) and 19) we infer that A does not depend on L. The same holds for E according to eq. 6). Now consider eq. 10),

$$\partial A = \tfrac{1}{2} A^2 \ . \tag{3.1}$$

The general solution is given by $A_1 = 0$ and

$$A_2 = \frac{2}{\bar{h}(\bar{Z}) - Z} \ , \tag{3.2}$$

where $\bar{h}$ is an arbitrary function of $\bar{Z}$. Next let us differentiate eq. 25) with respect to L; using eq. 2) it follows that

$$0 = 1 + \tfrac{1}{2} ZA + \tfrac{1}{2} \bar{Z}\bar{A} + ZCF + \bar{Z}\bar{C}\bar{F} \ . \tag{3.3}$$

With $F = 0$ we find

$$A + \frac{2}{Z} = -\frac{\bar{Z}}{Z}\,\bar{A} \ , \tag{3.4}$$

which excludes first of all the solution $A_1 = 0$. When inserted into eq. 9) we obtain

$$\bar{\partial} A = \tfrac{1}{2} A\bar{A} \ , \tag{3.5}$$

which gives a condition on $\bar{h}$,

$$\bar{\partial}\bar{h} = \frac{\bar{h} - Z}{\bar{Z} - h} \ , \tag{3.6}$$

$h(Z)$ being the complex conjugate of $\bar{h}(\bar{Z})$. Differentiating once more with respect to $\bar{Z}$ yields

$$\bar{\partial}^2\bar{h} = \frac{\bar{\partial}\bar{h}}{\bar{Z} - h} - \frac{\bar{h} - Z}{(\bar{Z} - h)^2} = 0 \ ,$$

thus $\bar{\partial}\bar{h}$ is constant. Furthermore, the absolute value of the right-hand side of eq. (3.6) equals one, hence

$$\bar{\partial}\bar{h} = \mathrm{e}^{2\mathrm{i}\varphi} \quad \Rightarrow \quad \bar{h} = \mathrm{e}^{2\mathrm{i}\varphi}\bar{Z} + c \ , \tag{3.7}$$

where $\varphi \in \mathbb{R}$ and $c \in \mathbb{C}$ are constant. Inserting this expression back into eq. (3.6),

$$\mathrm{e}^{2\mathrm{i}\varphi} = \frac{\mathrm{e}^{2\mathrm{i}\varphi}\bar{Z} + c - Z}{\bar{Z} - \mathrm{e}^{-2\mathrm{i}\varphi}Z - \bar{c}} = \mathrm{e}^{2\mathrm{i}\varphi}\,\frac{\bar{Z} - \mathrm{e}^{-2\mathrm{i}\varphi}Z + \mathrm{e}^{-2\mathrm{i}\varphi}c}{\bar{Z} - \mathrm{e}^{-2\mathrm{i}\varphi}Z - \bar{c}} \ ,$$

we conclude

$$\mathrm{e}^{-2\mathrm{i}\varphi}c = -\bar{c} \quad \Rightarrow \quad c = \mathrm{i}r\,\mathrm{e}^{\mathrm{i}\varphi} \ , \quad r \in \mathbb{R} \ , \tag{3.8}$$

which eventually leads to the solution

$$A = \frac{2\,\mathrm{e}^{-\mathrm{i}\varphi}}{\mathrm{e}^{\mathrm{i}\varphi}\bar{Z} - \mathrm{e}^{-\mathrm{i}\varphi}Z + \mathrm{i}r} \ . \tag{3.9}$$

Eq. (3.4) requires $r = 0$, while the parameter φ may be removed by a U(1) rotation

$$Z \mapsto \mathrm{e}^{\mathrm{i}\varphi}Z \ , \quad \mathcal{D}^i_\alpha \mapsto \mathrm{e}^{-\mathrm{i}\varphi/2}\mathcal{D}^i_\alpha \ . \tag{3.10}$$

Having determined A, we continue with eq. 7),

$$\partial E = \tfrac{1}{2}AE \quad \Rightarrow \quad E = \tfrac{1}{2}A\bar{Z}\bar{\partial}\bar{g} \ , \tag{3.11}$$

where $\bar{g}(\bar{Z})$ is independent of Z, and the peculiar form chosen for E will soon proove beneficial. Eq. 2) fixes the L-dependence of C,

$$\partial_L C = \tfrac{1}{2}A \quad \Rightarrow \quad C = \tfrac{1}{2}LA + v(Z,\bar{Z}) \ , \tag{3.12}$$

while eq. 25) implies

$$0 = Zv + \bar{Z}\bar{v} \ . \tag{3.13}$$

It remains to solve eq. 3). When C and E are inserted, the L-dependent terms cancel and we arrive at

$$0 = \partial v - \tfrac{1}{2}A(v + \bar{Z}\partial g) \ , \tag{3.14}$$

which is readily solved by making an Ansatz $v = \frac{1}{2}A\bar{Z}\,u(Z,\bar{Z})$ leading to

$$0 = \partial u - \partial g \quad \Rightarrow \quad u = g(Z) + \bar{k}(\bar{Z})$$

for some function $\bar{k}$. Eq. (3.13) then requires $\bar{k} = \bar{g}$, so the general solution is given by

$$v = \frac{\bar{Z}}{\bar{Z} - Z}\left(g(Z) + \bar{g}(\bar{Z})\right) . \tag{3.15}$$

This finishes the solution to the consistency conditions (C.1–4) subject to the restriction (2.71) and $F = 0$. The complete set of coefficient functions reads

$$\begin{aligned} A = B = \frac{2}{\bar{Z} - Z} \quad , \quad C = \frac{1}{\bar{Z} - Z}\left(L + \bar{Z}g + \bar{Z}\bar{g}\right) \\ D = \frac{\bar{Z}\partial g}{\bar{Z} - Z} \quad , \quad E = \frac{\bar{Z}\bar{\partial}\bar{g}}{\bar{Z} - Z} \quad , \quad a = b = c = F = G = 0 \ , \end{aligned} \tag{3.16}$$

with some arbitrary function $g(Z)$. When inserted into the Ansatz (2.41), the g-dependent terms can be written as

$$\frac{\bar{Z}}{\bar{Z} - Z}\left[\mathcal{D}^i(g\,\mathcal{D}^j Z) + \bar{\mathcal{D}}^i(\bar{g}\,\bar{\mathcal{D}}^j \bar{Z})\right] , \tag{3.17}$$

and if there is a function $f(Z)$ with $\partial f = g$, they simplify to

$$\frac{\bar{Z}}{\bar{Z} - Z}\left[\mathcal{D}^i\mathcal{D}^j f(Z) + \bar{\mathcal{D}}^i\bar{\mathcal{D}}^j \bar{f}(\bar{Z})\right] . \tag{3.18}$$

We shall first consider the simplest case $g = 0$, which corresponds to the constraints

$$\begin{aligned} &\mathcal{D}_\alpha^{(i}\bar{\mathcal{D}}_{\dot\alpha}^{j)} L = 0 \\ &\mathcal{D}^{(i}\mathcal{D}^{j)} L = \frac{2}{\bar{Z} - Z}\left(\mathcal{D}^{(i}Z\,\mathcal{D}^{j)}L + \bar{\mathcal{D}}^{(i}\bar{Z}\,\bar{\mathcal{D}}^{j)}L + \tfrac{1}{2}L\,\mathcal{D}^i\mathcal{D}^j Z\right) . \end{aligned} \tag{3.19}$$

In the limit $Z = \mathrm{i}$ they reduce to the free constraints (2.3). We return to $g \neq 0$ in section 3.4.

3.2 Transformations and Bianchi Identities

By construction, the constraints (3.19) satisfy the necessary consistency conditions (C.1–4). The task now is to solve, if possible, the Bianchi identities (BI.1) and (BI.2). Then we would have shown the constraints to be consistent and could proceed to determine the invariant action. To do this, we need to compute $\mathcal{D}_{\alpha j}M^{ij}$, $\mathcal{D}_i\mathcal{D}_j M^{ij}$ and $\bar{\mathcal{D}}_{\dot\alpha i}\bar{\mathcal{D}}_{\alpha j}M^{ij}$, which suffices as in the case at hand M^{ij} is imaginary and $N^{ij}_{\alpha\dot\alpha} = 0$, cf. section 2.2.
In terms of component fields the deformation M^{ij} reads

$$M^{ij} = -\bar{M}^{ij} = \frac{1}{I}\left(\lambda^{(i}\psi^{j)} - \bar{\lambda}^{(i}\bar{\psi}^{j)} + \mathrm{i}LY^{ij}\right) . \tag{3.20}$$

Applying a supersymmetry generator $\mathcal{D}_{\alpha j}$ and summing over j, we obtain

$$\begin{aligned}\mathcal{D}_{\alpha j}M^{ij} = \frac{3}{2I}\big(&F_{\mu\nu}\sigma^{\mu\nu}\psi^i + \tfrac{\mathrm{i}}{2}G_{\mu\nu}\sigma^{\mu\nu}\lambda^i + \tfrac{\mathrm{i}}{2}W_\mu\sigma^\mu\bar\lambda^i - \mathrm{i}\partial_\mu\bar Z\sigma^\mu\bar\psi^i + Y^{ij}\psi_j \\ &- L\sigma^\mu\partial_\mu\bar\lambda^i - \tfrac{1}{2}\mathcal{D}_\mu L\,\sigma^\mu\bar\lambda^i - \tfrac{\mathrm{i}}{2}\bar Z U\lambda^i + \tfrac{\mathrm{i}}{2}\lambda_j M^{ij}\big)_\alpha\ ,\end{aligned} \tag{3.21}$$

where eqs. (2.26) and (2.27) have been employed. This expression enters the central charge transformation of ψ^i as well as the supersymmetry transformations of W^μ and $G_{\mu\nu}$, which may be cast into the form

$$\begin{aligned}\mathcal{D}^i_\alpha W^\mu &= \big(\mathrm{i}\bar Z\sigma^\mu\delta_z\bar\psi^i + \tfrac{1}{2}U\sigma^\mu\bar\lambda^i - \mathcal{D}^\mu\psi^i\big)_\alpha \\ \mathcal{D}^i_\alpha G_{\mu\nu} &= \big(2I\sigma_{\mu\nu}\delta_z\psi^i + U\sigma_{\mu\nu}\lambda^i + \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\,\sigma^\rho\mathcal{D}^\sigma\bar\psi^i\big)_\alpha\ .\end{aligned} \tag{3.22}$$

Next we calculate

$$\begin{aligned}\mathcal{D}_i\mathcal{D}_j M^{ij} = -\frac{3}{I}\big[&G^{\mu\nu}(\tilde F_{\mu\nu} + \mathrm{i}F_{\mu\nu}) + 2W^\mu\partial_\mu\bar Z + 2\mathrm{i}\,\mathcal{D}_\mu(L\partial^\mu\bar Z) \\ &+ \mathrm{i}\lambda^i\sigma^\mu\mathcal{D}_\mu\bar\psi_i + \mathrm{i}\mathcal{D}_\mu\psi^i\sigma^\mu\bar\lambda_i + 2\mathrm{i}\,\psi^i\sigma^\mu\partial_\mu\bar\lambda_i \\ &+ 2\mathrm{i}I\lambda_i\delta_z\psi^i - \tfrac{\mathrm{i}}{3}(\lambda_i\mathcal{D}_j + \bar\lambda_i\bar{\mathcal{D}}_j + 3Y_{ij})M^{ij}\big]\ ,\end{aligned} \tag{3.23}$$

and insert the result into eq. (BI.1),

$$I\mathcal{D}_\mu W^\mu = \frac{\mathrm{i}}{12}\,I\,\mathcal{D}_i\mathcal{D}_j M^{ij} - \tfrac{1}{2}I\,\lambda_i\delta_z\psi^i + \text{c.c.} = -W^\mu\partial_\mu I + \mathcal{D}_\mu\Lambda^\mu + \tfrac{1}{2}F_{\mu\nu}G^{\mu\nu}\ ,$$

where we introduced the abbreviation

$$\Lambda_\mu \equiv L\partial_\mu R + \tfrac{1}{2}(\psi^i\sigma_\mu\bar\lambda_i + \lambda^i\sigma_\mu\bar\psi_i)\ . \tag{3.24}$$

The first Bianchi identity thus reads

$$\mathcal{D}_\mu\big(IW^\mu - \Lambda^\mu\big) = \tfrac{1}{2}F_{\mu\nu}G^{\mu\nu}\ . \tag{3.25}$$

To determine the second one, we first apply $\bar{\mathcal{D}}_{\dot\alpha i}$ to eq. (3.21),

$$\begin{aligned}\bar{\mathcal{D}}_{\dot\alpha i}\mathcal{D}_{\alpha j}M^{ij} = \frac{3}{I}\,\sigma^\mu_{\alpha\dot\alpha}\big[&G_{\mu\nu}\partial^\nu R + \tilde G_{\mu\nu}\partial^\nu I + \mathcal{D}^\nu\Sigma_{\mu\nu} + \tilde F_{\mu\nu}W^\nu \\ &- \tfrac{1}{2}U\partial_\mu|Z|^2 - \tfrac{1}{2}\delta_z(Z\psi^i\sigma_\mu\bar\lambda_i + \bar Z\lambda^i\sigma_\mu\bar\psi_i)\big]\ .\end{aligned} \tag{3.26}$$

$\Sigma_{\mu\nu}$ has been defined in eq. (2.38). When put into eq. (BI.2), it follows that

$$\begin{aligned}I\,\mathcal{D}_\nu\tilde G^{\mu\nu} + R\,\mathcal{D}_\nu G^{\mu\nu} &= \tfrac{1}{6}I\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} - \tfrac{1}{2}U\partial^\mu|Z|^2 - \tfrac{1}{2}\delta_z(Z\psi^i\sigma_\mu\bar\lambda_i + \bar Z\lambda^i\sigma_\mu\bar\psi_i) \\ &= -\tilde G^{\mu\nu}\partial_\nu I - G^{\mu\nu}\partial_\nu R - \mathcal{D}_\nu\Sigma^{\mu\nu} - \tilde F^{\mu\nu}W_\nu\ ,\end{aligned}$$

and combining the derivatives, we eventually obtain

$$\mathcal{D}_\nu\big(I\tilde G^{\mu\nu} + RG^{\mu\nu} + \Sigma^{\mu\nu}\big) = -\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho}W_\sigma\ . \tag{3.27}$$

We observe that the Bianchi identities of W^μ and $G_{\mu\nu}$ are not independent of each other but constitute a coupled system of differential equations. We cannot solve them yet as the covariant derivatives contain the central charge generator δ_z, whose action on W^μ and $G_{\mu\nu}$ needs to be determined first. Since $N^{ij}_{\alpha\dot\alpha} = 0$, eq. (2.36) immediately gives

$$\delta_z\big(I\tilde G^{\mu\nu} + RG^{\mu\nu} + \Sigma^{\mu\nu}\big) = -\varepsilon^{\mu\nu\rho\sigma}\mathcal{D}_\rho W_\sigma \ , \tag{3.28}$$

whereas the determination of $\delta_z W^\mu$ is somewhat involved. According to eq. (2.39),

$$\begin{aligned} I\delta_z\big[\, & |Z|^2 W^\mu + \tfrac{\mathrm{i}}{2}L(Z\partial^\mu\bar Z - \bar Z\partial^\mu Z) + \tfrac{\mathrm{i}}{2}(Z\psi^i\sigma^\mu\bar\lambda_i - \bar Z\lambda^i\sigma^\mu\bar\psi_i)\,\big] = \\ &= I^2\,\mathcal{D}_\nu G^{\mu\nu} - IR\,\mathcal{D}_\nu\tilde G^{\mu\nu} + \tfrac{1}{6}IR\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} \\ &= I^2\,\mathcal{D}_\nu G^{\mu\nu} - G^{\mu\nu}R\partial_\nu R - R\big[\mathcal{D}_\nu(I\tilde G^{\mu\nu} + \Sigma^{\mu\nu}) + \tilde F^{\mu\nu}W_\nu\big] + \tfrac{1}{2}UR\,\partial^\mu|Z|^2 \\ &\quad + \tfrac{1}{2}R\,\delta_z(Z\psi^i\sigma_\mu\bar\lambda_i + \bar Z\lambda^i\sigma_\mu\bar\psi_i) \ . \end{aligned}$$

The expression in square brackets can be rewritten by means of the Bianchi identity (3.27),

$$\begin{aligned} I\delta_z\big[\, & |Z|^2 W^\mu + \tfrac{\mathrm{i}}{2}L(Z\partial^\mu\bar Z - \bar Z\partial^\mu Z) + \tfrac{\mathrm{i}}{2}(Z\psi^i\sigma^\mu\bar\lambda_i - \bar Z\lambda^i\sigma^\mu\bar\psi_i)\,\big] = \\ &= |Z|^2\,\mathcal{D}_\nu G^{\mu\nu} + \tfrac{1}{2}UR\,\partial^\mu|Z|^2 + \tfrac{1}{2}R\,\delta_z(Z\psi^i\sigma_\mu\bar\lambda_i + \bar Z\lambda^i\sigma_\mu\bar\psi_i) \ , \end{aligned}$$

from which follows

$$\delta_z\big(IW^\mu - \Lambda^\mu\big) = \mathcal{D}_\nu G^{\mu\nu} \ . \tag{3.29}$$

One can now easily check that the Bianchi identities, together with the central charge transformations just obtained, satisfy the integrability condition

$$\mathcal{D}_{[\mu}\mathcal{D}_{\nu]} = \tfrac{1}{2}F_{\mu\nu}\delta_z \ ,$$

which is the covariant analogue of $d^2 = 0$.
We first solve the constraint on W^μ. Let us split the covariant derivative in eq. (3.25) into the partial derivative and the central charge transformation, which we then replace using eq. (3.29),

$$\begin{aligned} \partial_\mu\big(IW^\mu - \Lambda^\mu\big) &= \tfrac{1}{2}F_{\mu\nu}G^{\mu\nu} - A_\mu\delta_z\big(IW^\mu - \Lambda^\mu\big) \\ &= \big(\tfrac{1}{2}F_{\mu\nu} - A_\mu\mathcal{D}_\nu\big)G^{\mu\nu} \\ &= \partial_\mu(G^{\mu\nu}A_\nu) \ . \end{aligned}$$

In the last step we employed the identity $A_{[\mu}\mathcal{D}_{\nu]} = A_{[\mu}\partial_{\nu]}$. We conclude that the terms in parentheses equal the dual field strength of an antisymmetric tensor gauge field $B_{\mu\nu}$,

$$IW^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu B_{\rho\sigma} - A_\nu\tilde G_{\rho\sigma}) + \Lambda^\mu \ . \tag{3.30}$$

Note how the first two terms on the right-hand side resemble a covariant derivative, and indeed we shall find that $B_{\mu\nu}$ transforms into $-\tilde G_{\mu\nu}$ under δ_z. In the limit $Z = \mathrm{i}$ we recover the relation $W^\mu = H^\mu$ just as in the free case.

To solve the Bianchi identity (3.27) we proceed along the same lines,

$$\begin{aligned}\partial_\nu\big(I\tilde{G}^{\mu\nu} + RG^{\mu\nu} + \Sigma^{\mu\nu}\big) &= -\varepsilon^{\mu\nu\rho\sigma}(\tfrac{1}{2}F_{\nu\rho}W_\sigma - A_\nu\mathcal{D}_\rho W_\sigma)\\ &= -\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(A_\rho W_\sigma)\ .\end{aligned}$$

Hence there is a vector field V_μ such that

$$I\tilde{G}^{\mu\nu} + RG^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}(\partial_\rho V_\sigma - A_\rho W_\sigma) - \Sigma^{\mu\nu}\ . \tag{3.31}$$

Again the terms in parentheses will turn out to be the covariant derivative of the potential V_μ. For $Z = \mathrm{i}$ the equation reduces to $G_{\mu\nu} = V_{\mu\nu}$.
We are not done yet, as the equations for W^μ and $G_{\mu\nu}$ are still coupled. To simplify the following calculations let us introduce the abbreviations

$$\mathcal{H}^\mu \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} + \Lambda^\mu \tag{3.32}$$

$$\mathcal{V}_{\mu\nu} \equiv \frac{I}{|Z|^2}\left(V_{\mu\nu} + \tilde{\Sigma}_{\mu\nu}\right) + \frac{R}{|Z|^2}\left(\tilde{V}_{\mu\nu} - \Sigma_{\mu\nu}\right)\ . \tag{3.33}$$

Then we first solve eq. (3.31) for $G_{\mu\nu}$,

$$G_{\mu\nu} = \mathcal{V}_{\mu\nu} - \frac{2I}{|Z|^2}\,A_{[\mu}W_{\nu]} - \frac{R}{|Z|^2}\,\varepsilon_{\mu\nu\rho\sigma}A^\rho W^\sigma\ , \tag{3.34}$$

and insert this into eq. (3.30),

$$IW^\mu = \mathcal{H}^\mu + \mathcal{V}^{\mu\nu}A_\nu - \frac{2I}{|Z|^2}\,A^{[\mu}W^{\nu]}A_\nu\ . \tag{3.35}$$

Collecting the terms with W^μ, this can be written as

$$IK^{\mu\nu}W_\nu = |Z|^2(\mathcal{H}^\mu + \mathcal{V}^{\mu\nu}A_\nu)\ , \tag{3.36}$$

where the field dependent matrix $K^{\mu\nu}$ is given by

$$K^{\mu\nu} = \eta^{\mu\nu}\mathcal{E} + A^\mu A^\nu\ , \tag{3.37}$$

$\mathcal{E}$ being the expression (1.55) that we have already encountered in the central charge transformation of the spinors ψ^i. To solve for W^μ, we need to invert $K^{\mu\nu}$. It can be easily checked that

$$(K^{-1})_{\mu\nu} = \frac{1}{\mathcal{E}}\left(\eta_{\mu\nu} - |Z|^{-2}A_\mu A_\nu\right)\ . \tag{3.38}$$

Due to the appearance of $\mathcal{E}$ in the denominator, this expression is nonpolynomial in the gauge field A_μ.
Having determined W^μ, we obtain $G_{\mu\nu}$ from eq. (3.34). The final solution to the Bianchi identities then reads

$$W^\mu = \frac{|Z|^2}{I\mathcal{E}}\left(\mathcal{H}^\mu + \mathcal{V}^{\mu\nu}A_\nu - |Z|^{-2}A^\mu A_\nu\mathcal{H}^\nu\right) \tag{3.39}$$

$$G_{\mu\nu} = \mathcal{V}_{\mu\nu} - \frac{2}{\mathcal{E}}\, A_{[\mu}\big(\mathcal{H}_{\nu]} + \mathcal{V}_{\nu]\rho}A^\rho\big) - \frac{R}{I\mathcal{E}}\, \varepsilon_{\mu\nu\rho\sigma}A^\rho\big(\mathcal{H}^\sigma + \mathcal{V}^{\sigma\lambda}A_\lambda\big)\ . \tag{3.40}$$

The fundamental fields of course are the gauge potentials V_μ and $B_{\mu\nu}$ rather than W^μ and $G_{\mu\nu}$, so we now have to determine their supersymmetry and central charge transformations as well. These can be obtained most easily from eqs. (3.30) and (3.31). Applying Δ^z to the former we have

$$\Delta^z(C)\,(IW^\mu - \Lambda^\mu) = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\big(\Delta^z(C)\,B_{\rho\sigma}\big) - G^{\mu\nu}\partial_\nu C + CA_\nu\,\delta_z G^{\mu\nu}\ .$$

On the other hand it follows from eq. (3.29) that

$$\Delta^z(C)\,(IW^\mu - \Lambda^\mu) = C\mathcal{D}_\nu G^{\mu\nu} = C\partial_\nu G^{\mu\nu} + CA_\nu\,\delta_z G^{\mu\nu}\ ,$$

and comparing the two expressions we find

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\big(\Delta^z(C)\,B_{\rho\sigma} + C\tilde{G}_{\rho\sigma}\big) = 0\ .$$

It comes as no surprise that the action of Δ^z on $B_{\mu\nu}$ is determined only modulo a gauge transformation Δ^B. We are free to choose a homogeneous transformation law, however, which is then generated by

$$\delta_z B_{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,G^{\rho\sigma}\ , \tag{3.41}$$

and the terms in parantheses in eq. (3.30) constitute a proper covariant derivative of $B_{\mu\nu}$. The central charge transformation of V_μ is derived in like manner. From the eqs. (3.31) and (3.28) we obtain

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\rho\big(\Delta^z(C)\,V_\sigma + CW_\sigma\big) = 0\ ,$$

so that we set

$$\delta_z V_\mu = -W_\mu\ . \tag{3.42}$$

Thus, formally the action of the central charge generator δ_z on V_μ and $B_{\mu\nu}$ has not changed upon gauging the symmetry, cf. equation (2.9). The difference is that now W^μ and $G_{\mu\nu}$ are not merely the field strengths but composite expressions that are moreover nonpolynomial in the gauge field A_μ. Therefore we expect that also the action will be nonpolynomial. Since E contains no derivatives this should not spoil locality.
The supersymmetry transformations of the gauge potentials can be determined in the same way as demonstrated for the central charge transformations. As this is a lengthy calculation we just give the result, again choosing the simplest form possible by neglecting any contribution that is a gauge transformation,

$$\mathcal{D}^i_\alpha V_\mu = -\big(\mathrm{i}\bar{Z}\sigma_\mu\bar{\psi}^i + \tfrac{1}{2}L\sigma_\mu\bar{\lambda}^i - A_\mu\psi^i\big)_\alpha \tag{3.43}$$

$$\mathcal{D}^i_\alpha B_{\mu\nu} = -2\mathrm{i}\,\big(I\sigma_{\mu\nu}\psi^i + \tfrac{1}{2}L\sigma_{\mu\nu}\lambda^i + A_{[\mu}\sigma_{\nu]}\bar{\psi}^i\big)_\alpha\ . \tag{3.44}$$

There is a short cut, however, that immediately yields the supersymmetry transformations modulo possible δ_z-invariant terms: Using eqs. (3.22), we calculate

$$\begin{aligned}\delta_z\,\mathcal{D}^i_\alpha V_\mu &= -\mathcal{D}^i_\alpha W_\mu + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,V_\mu \\ &= -\delta_z\big(\mathrm{i}\bar{Z}\sigma_\mu\bar{\psi}^i + \tfrac{1}{2}L\sigma_\mu\bar{\lambda}^i - A_\mu\psi^i\big)_\alpha + \partial_\mu\psi^i_\alpha + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,V_\mu\ ,\end{aligned}$$

and similarly for $B_{\mu\nu}$,

$$\begin{aligned}\delta_z\,\mathcal{D}^i_\alpha B_{\mu\nu} &= -\mathcal{D}^i_\alpha\tilde{G}_{\mu\nu} + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,B_{\mu\nu} \\ &= -2\mathrm{i}\,\delta_z\big(I\sigma_{\mu\nu}\psi^i + \tfrac{1}{2}L\sigma_{\mu\nu}\lambda^i + A_{[\mu}\sigma_{\nu]}\bar{\psi}^i\big)_\alpha - 2\mathrm{i}\,\partial_{[\mu}(\sigma_{\nu]}\bar{\psi}^i)_\alpha + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,B_{\mu\nu}\ .\end{aligned}$$

Comparing the δ_z-exact terms on the left and on the right, one obtains the previously found relations. In addition, the equations show that central charge transformations commute with supersymmetry transformations only modulo gauge transformations, a result we have stated already in the presentation of the free multiplet, section 2.1. Indeed, a more careful analysis reveals that eqs. (2.12) and (2.13) hold even in the present case, but with an x-dependent parameter C.

At last, we determine the commutator of two supersymmetry transformations. To this end, and to check the compatibility of the results just stated, we need to compute the anticommutators of two supersymmetry generators on the gauge potentials. This is again a tedious exercise of which we give no details. On V_μ one finds

$$\begin{aligned}\{\mathcal{D}^i_\alpha\,,\,\mathcal{D}^j_\beta\}\,V_\mu &= \varepsilon_{\alpha\beta}\,\varepsilon^{ij}\big(\bar{Z}\delta_z V_\mu - \mathrm{i}\partial_\mu(\bar{Z}L)\big) \\ \{\mathcal{D}^i_\alpha\,,\,\bar{\mathcal{D}}_{\dot\alpha j}\}\,V_\mu &= -\mathrm{i}\delta^i_j\big(\mathcal{D}_{\alpha\dot\alpha}V_\mu - \partial_\mu V_{\alpha\dot\alpha}\big)\ ,\end{aligned} \tag{3.45}$$

while on $B_{\mu\nu}$ the relations read

$$\begin{aligned}\{\mathcal{D}^i_\alpha\,,\,\mathcal{D}^j_\beta\}\,B_{\mu\nu} &= \varepsilon_{\alpha\beta}\,\varepsilon^{ij}\big(\bar{Z}\delta_z B_{\mu\nu} + 2\mathrm{i}\,\partial_{[\mu}(A_{\nu]}L + \mathrm{i}V_{\nu]})\big) \\ \{\mathcal{D}^i_\alpha\,,\,\bar{\mathcal{D}}_{\dot\alpha j}\}\,B_{\mu\nu} &= -\mathrm{i}\delta^i_j\big(\mathcal{D}_{\alpha\dot\alpha}B_{\mu\nu} + 2\,\partial_{[\mu}(B_{\nu]\rho} - \eta_{\nu]\rho}IL)\sigma^\rho_{\alpha\dot\alpha}\big)\ .\end{aligned} \tag{3.46}$$

On the other components of the vector-tensor multiplet the algebra (1.33) holds exactly by construction. We conclude that the commutator of two global supersymmetry transformations yields in addition to a translation and a local central charge transformation also gauge transformations of the potentials V_μ and $B_{\mu\nu}$,

$$[\,\Delta(\xi)\,,\,\Delta(\zeta)\,] = \epsilon^\mu\partial_\mu + \Delta^z(C) + \Delta^V(\Theta) + \Delta^B(\Omega)\ ,$$

where the parameters are given by

$$\begin{aligned}\epsilon^\mu &= \mathrm{i}(\zeta_i\sigma^\mu\bar{\xi}^i - \xi_i\sigma^\mu\bar{\zeta}^i) \\ C &= \epsilon^\mu A_\mu + \bar{\xi}^i\bar{\zeta}_i Z - \xi_i\zeta^i\bar{Z} \\ \Theta &= \epsilon^\mu V_\mu - \mathrm{i}L\,(\xi_i\zeta^i\bar{Z} + \bar{\xi}^i\bar{\zeta}_i Z) \\ \Omega_\mu &= \epsilon_\mu IL - B_{\mu\nu}\epsilon^\nu - V_\mu(\xi_i\zeta^i - \bar{\xi}^i\bar{\zeta}_i) + \mathrm{i}A_\mu L\,(\xi_i\zeta^i + \bar{\xi}^i\bar{\zeta}_i)\ .\end{aligned} \tag{3.47}$$

3.3 The Lagrangian

Now that we have found a consistent supersymmetry multiplet, the task is to construct an invariant action. With the general method outlined in section 1.2 and the Ansatz (2.58) this is pretty straightforward though tedious. At first we have to solve the differential equations (2.59) subject to the constraints (3.19) on L in order to determine the linear multiplet. With the coefficient functions as in eq. (3.16) (for $g = 0$), the equations (2.59) read

$$\begin{array}{llll} 1) & 0 = \partial_L\gamma - \tfrac{1}{2}\beta - \dfrac{\alpha L}{\bar{Z} - Z} & 2) & 0 = \partial\gamma - \delta - \dfrac{\beta L/2}{\bar{Z} - Z} \\ 3) & 0 = \partial_L\bar{\alpha} & 4) & 0 = \partial\bar{\alpha} \\ 5) & 0 = \partial\alpha - \tfrac{1}{2}\partial_L\beta + \dfrac{\alpha}{\bar{Z} - Z} & 6) & 0 = \partial_L\bar{\beta} - \dfrac{2\alpha}{\bar{Z} - Z} \\ 7) & 0 = \partial\bar{\beta} - \dfrac{\beta}{\bar{Z} - Z} & 8) & 0 = \partial\beta - 2\partial_L\delta - \dfrac{\beta}{\bar{Z} - Z} \\ 9) & 0 = \partial_L\bar{\delta} & 10) & 0 = \partial\bar{\delta}\ . \end{array}$$

From eqs. 3), 4) and 9), 10) it follows that $\alpha = \alpha(Z)$ and $\delta = \delta(Z)$, respectively. Since α does not depend on L, we can integrate eq. 6),

$$\beta = \frac{2\bar{\alpha}L}{Z - \bar{Z}} + \hat{\beta}(Z, \bar{Z})\ .$$

Next we insert this into eq. 8); the L-dependent terms cancel and the Z-dependence of $\hat{\beta}$ is fixed,

$$\partial\hat{\beta} = -\frac{\hat{\beta}}{Z - \bar{Z}} \qquad \Rightarrow \qquad \hat{\beta} = \frac{\bar{h}(\bar{Z})}{Z - \bar{Z}}\ .$$

Now we can determine α from eq. 5),

$$\partial\alpha = \frac{\alpha + \bar{\alpha}}{Z - \bar{Z}} \qquad \Rightarrow \qquad \alpha = \mathrm{i}(\kappa Z + \varrho)\ , \quad \kappa, \varrho \in \mathbb{R}\ . \tag{3.48}$$

We use this in eq. 7) to derive an analogous condition on $\bar{h}(\bar{Z})$,

$$\bar{\partial}\bar{h} = -\frac{h + \bar{h}}{Z - \bar{Z}} \qquad \Rightarrow \qquad \bar{h} = -2\mathrm{i}\,(\nu\bar{Z} + \mu)\ , \quad \nu, \mu \in \mathbb{R}\ .$$

With α and β known, eq. 1) may be integrated. The reality of γ requires $\nu = 0$, and we find

$$\gamma = -\frac{\mathrm{i}}{2}\,\frac{2\varrho + \kappa(Z + \bar{Z})}{Z - \bar{Z}}\,L^2 - \frac{\mathrm{i}\mu}{Z - \bar{Z}}\,L + \sigma(Z, \bar{Z})\ , \tag{3.49}$$

with σ real. It remains to solve eq. 2). When β and γ are inserted, the L-dependent terms drop out and we are left with $\delta = \partial\sigma$. Eq. 10) then implies

$$\sigma(Z, \bar{Z}) = f(Z) + \bar{f}(\bar{Z})\ , \tag{3.50}$$

where f is an arbitrary function of Z. We have thus found the most general linear multiplet one can build from the linear vector-tensor multiplet with gauged central charge.
When the coefficients are inserted into the Ansatz (2.58), it comes as no surprise that several terms group together to form the expression $\mathrm{i}\mathcal{D}^{(i}\mathcal{D}^{j)}L$, as this is evidently a linear superfield by itself (cf. the remark in section 2.3.1). The complete pre-Lagrangian finally reads

$$\mathcal{L}^{ij} = \mathcal{L}^{ij}_{\mathrm{linVT}} + \mathcal{L}^{ij}_{\mathrm{cc}} \;, \tag{3.51}$$

where

$$\begin{aligned}\mathcal{L}^{ij}_{\mathrm{linVT}} = {}& \mathrm{i}\varrho\left(\mathcal{D}^i L\,\mathcal{D}^j L - \bar{\mathcal{D}}^i L\,\bar{\mathcal{D}}^j L + L\,\mathcal{D}^{(i}\mathcal{D}^{j)}L\right) + \mathrm{i}\mu\,\mathcal{D}^{(i}\mathcal{D}^{j)}L \\ & + \mathrm{i}\kappa\Big[\frac{2L}{\bar{Z}-Z}\left(\bar{Z}\,\mathcal{D}^{(i}Z\,\mathcal{D}^{j)}L + Z\,\bar{\mathcal{D}}^{(i}\bar{Z}\,\bar{\mathcal{D}}^{j)}L + \tfrac{1}{4}L\,(Z+\bar{Z})\,\mathcal{D}^i\mathcal{D}^j Z\right) \\ & + Z\,\mathcal{D}^i L\,\mathcal{D}^j L - \bar{Z}\,\bar{\mathcal{D}}^i L\bar{\mathcal{D}}^j L\Big]\,, \quad \varrho,\mu,\kappa\in\mathbb{R}\;,\end{aligned} \tag{3.52}$$

and $\mathcal{L}^{ij}_{\mathrm{cc}}$ is the super Yang-Mills pre-Lagrangian as in eq. (1.44) with $\partial\mathcal{F}(Z) = f(Z)$. Without going into detail, the terms proportional to κ, which in the limit $Z = \mathrm{i}$ reduce to the real part of $D^i L\, D^j L$, can be shown to yield a Lagrangian that is a total derivative. Therefore we confine ourselves to $\kappa = 0$ in the following. The constant μ on the other hand can be removed by a shift of L, hence we also take $\mu = 0$.
This leaves the terms proportional to ϱ, where without loss of generality we can take $\varrho = 1$. Actually, they constitute a linear superfield irrespective of the precise form of M^{ij} as long as $N^{ij}_{\alpha\dot{\alpha}} = 0$ and $\bar{M}^{ij} = -M^{ij}$, for we have

$$\begin{aligned}&\mathcal{D}^{(i}_\alpha\left(\mathcal{D}^j L\,\mathcal{D}^{k)}L - \bar{\mathcal{D}}^j L\,\bar{\mathcal{D}}^{k)}L + L\,\mathcal{D}^j\mathcal{D}^{k)}L\right) = \\ &= -2\,\mathcal{D}^{\beta(i}L\,\mathcal{D}^j_\alpha\mathcal{D}^{k)}_\beta L + \mathcal{D}^{(i}_\alpha L\,\mathcal{D}^j\mathcal{D}^{k)}L = 0\;.\end{aligned}$$

For this reason we shall first compute the Lagrangian without specifying M^{ij}, the result of which then may also be used in the following section, where we extend the model by additional vector multiplets such that the properties of the deformations just mentioned are preserved.
Let us consider therefore

$$\mathcal{L}^{ij} = \mathrm{i}\left(\psi^i\psi^j - \bar{\psi}^i\bar{\psi}^j - LM^{ij}\right)\;, \quad \bar{M}^{ij} = -M^{ij}\;, \quad N^{ij}_{\alpha\dot{\alpha}} = 0\;, \tag{3.53}$$

and work out the Lagrangian according to eq. (1.42), with the supersymmetry transformations given in section 2.2. The first step is to apply a supersymmetry generator $\mathcal{D}_{\alpha j}$ to $\mathcal{L}^{ij}$,

$$\mathcal{D}_{\alpha j}\mathcal{L}^{ij} = \frac{3}{2}\left[\bar{Z}U\psi^i - G_{\mu\nu}\sigma^{\mu\nu}\psi^i - (W_\mu + \mathrm{i}\mathcal{D}_\mu L)\sigma^\mu\bar{\psi}^i - M^{ij}\psi_j - \tfrac{2}{3}\mathrm{i}L\,\mathcal{D}_j M^{ij}\right]_\alpha\;. \tag{3.54}$$

Next we apply a second generator $\mathcal{D}^\alpha_i$, sum over α and i, multiply with $Z/6$ and take the real part of the result. We find after some algebra

$$\frac{1}{12}\,Z\,\mathcal{D}_i\mathcal{D}_j\mathcal{L}^{ij} + \text{c.c.} = \frac{1}{2}\,I\left[\mathcal{D}^\mu L\,\mathcal{D}_\mu L - W^\mu W_\mu - 2\mathrm{i}\,\psi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar{\psi}_i + |Z|^2U^2\right]$$

$$
\begin{aligned}
&- \frac{1}{4}\, G^{\mu\nu}(I G_{\mu\nu} - R\tilde{G}_{\mu\nu}) - R W^\mu \mathcal{D}_\mu L + R\,\partial_\mu(\psi^i\sigma^\mu\bar\psi_i) \\
&- U(\bar Z \lambda_i\psi^i - Z\bar\lambda^i\bar\psi_i) + \mathrm{i} A_\mu U(\psi^i\sigma^\mu\bar\lambda_i - \lambda^i\sigma^\mu\bar\psi_i) \\
&+ \mathrm{i} A_\mu(\psi_i\sigma^\mu\bar\sigma^\nu\mathcal{D}_\nu\psi^i + \bar\psi^i\bar\sigma^\mu\sigma^\nu\mathcal{D}_\nu\bar\psi_i) + \frac{1}{4}\, I M^{ij} M_{ij} \\
&- \frac{\mathrm{i}}{12}\, L\big(Z\,\mathcal{D}_i\mathcal{D}_j + \bar Z\,\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\big) M^{ij} - \frac{2\mathrm{i}}{3}\, I(\psi_i\mathcal{D}_j + \bar\psi_i\bar{\mathcal{D}}_j) M^{ij} \\
&+ \frac{\mathrm{i}}{3}\, A_\mu(\psi_i\sigma^\mu\bar{\mathcal{D}}_j + \bar\psi_i\bar\sigma^\mu\mathcal{D}_j) M^{ij} \ .
\end{aligned}
\tag{3.55}
$$

We already recognize the properly normalized kinetic terms for L and ψ^i, keeping in mind that $\langle I\rangle = 1$. The naked gauge field A_μ appears due to the splitting of the covariant derivative of ψ^i as in the calculation leading to eq. (2.33). It remains to compute the mixed derivative of $\mathcal{L}^{ij}$ that couples to A_μ in eq. (1.42). It reads

$$
\begin{aligned}
\frac{\mathrm{i}}{6}\, A_\mu\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j\mathcal{L}^{ij} = {}& R U A_\mu W^\mu - I U A_\mu \mathcal{D}^\mu L - G^{\mu\nu} A_\mu W_\nu + \tilde G^{\mu\nu} A_\mu \partial_\nu L \\
&- \mathrm{i} A_\mu(\psi_i\mathcal{D}^\mu\psi^i + \bar\psi^i\mathcal{D}^\mu\bar\psi_i) - \frac{\mathrm{i}}{3}\, A_\mu(\psi_i\sigma^\mu\bar{\mathcal{D}}_j + \bar\psi_i\bar\sigma^\mu\mathcal{D}_j)\, M^{ij} \\
&+ \frac{1}{6}\, L A_\mu\, \mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} \ ,
\end{aligned}
\tag{3.56}
$$

and we observe that the terms in the second line cancel the corresponding ones in the previous equation. While in general M^{ij} has to be specified before the action of the supersymmetry generators can be computed, the last term in eq. (3.56) may be simplified by virtue of the Bianchi identity (BI.2). Since by assumption M^{ij} is imaginary and $N^{ij}_{\alpha\dot\alpha} = 0$, one has

$$
\begin{aligned}
\frac{1}{6}\, L A_\mu\, \mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} = \frac{1}{I}\, A_\mu L\big[& I\,\partial_\nu\tilde G^{\mu\nu} + R\,\partial_\nu G^{\mu\nu} + \tfrac{1}{2} U\partial^\mu|Z|^2 \\
&+ \tfrac{1}{2}\delta_z(Z\psi^i\sigma^\mu\bar\lambda_i + \bar Z\lambda^i\sigma^\mu\bar\psi_i)\big] \ .
\end{aligned}
\tag{3.57}
$$

The effect of the terms (3.56) is twofold; first they introduce the anticipated coupling of the current (2.16) to the gauge field A_μ (contained in the term $G^{\mu\nu}A_\mu W_\nu$, see below), and second they reduce the covariant derivatives to partial ones, similar to the case of the hypermultiplet (cf. $\mathcal{L}_0$ in eq. (1.59)). One finds for instance

$$
\mathcal{D}^\mu L\,\mathcal{D}_\mu L - 2 U A_\mu \mathcal{D}^\mu L = \partial^\mu L\,\partial_\mu L - A^\mu A_\mu U^2 \ .
$$

The resulting Lagrangian eventually reads

$$
\begin{aligned}
\mathcal{L} = {}& \frac{1}{2}\, I\big(\partial^\mu L\,\partial_\mu L - W^\mu W_\mu - 2\mathrm{i}\,\psi^i\sigma^\mu\overset{\leftrightarrow}{\partial}_\mu\bar\psi_i + E U^2\big) + \frac{R}{I}\, L A_\mu\partial_\nu G^{\mu\nu} + \mathcal{L}_M \\
&- \frac{1}{4}\, G^{\mu\nu}\big(I G_{\mu\nu} - R\tilde G_{\mu\nu} + 4 A_\mu W_\nu\big) + W^\mu\Lambda_\mu + \frac{1}{2}\,\tilde G^{\mu\nu}\Sigma_{\mu\nu} - W^\mu\partial_\mu(LR) \\
&+ \frac{1}{2I}\, L U A^\mu\partial_\mu|Z|^2 - \frac{\mathrm{i}}{2}\,\partial_\mu L\,(\psi^i\sigma^\mu\bar\lambda_i - \lambda^i\sigma^\mu\bar\psi_i) + \frac{\mathrm{i}}{2}\, Y_{ij}(\psi^i\psi^j - \bar\psi^i\bar\psi^j) \\
&- \frac{1}{2}\, U(\bar Z\lambda_i\psi^i - Z\bar\lambda^i\bar\psi_i) + \frac{\mathrm{i}}{2}\, A_\mu U(\psi^i\sigma^\mu\bar\lambda_i - \lambda^i\sigma^\mu\bar\psi_i) + R\,\partial_\mu(\psi^i\sigma^\mu\bar\psi_i)
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
&+\frac{\mathrm{i}}{2}\,F_{\mu\nu}(\psi_i\sigma^{\mu\nu}\psi^i+\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i)+\frac{1}{2I}\,LA_\mu\,\delta_z(Z\psi^i\sigma^\mu\bar\lambda_i+\bar Z\lambda^i\sigma^\mu\bar\psi_i)\\
&-\partial_\mu\big[L\tilde G^{\mu\nu}A_\nu+\mathrm{i}A_\nu(\psi_i\sigma^{\mu\nu}\psi^i+\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i)\big]\ ,
\end{aligned}
$$

where the part

$$
\begin{aligned}
\mathcal{L}_M=&-\frac{\mathrm{i}}{12}\,LZ\,\mathcal{D}_i\mathcal{D}_jM^{ij}-\frac{\mathrm{i}}{3}\,(L\lambda_i+2I\psi_i)\mathcal{D}_jM^{ij}+\frac{1}{8}\,IM_{ij}M^{ij}\\
&-\frac{1}{4}\big(\lambda_i\psi_j-\bar\lambda_i\bar\psi_j+\mathrm{i}LY_{ij}\big)M^{ij}+\text{c.c.}
\end{aligned}
\tag{3.59}
$$

has to be determined separately for each model, which however is easy as all the terms have been computed already for the Bianchi identities.
Now let us consider M^{ij} as in eq. (3.20). Using eqs. (3.21) and (3.23), we find

$$
\begin{aligned}
\mathcal{L}_M=&-W^\mu\Lambda_\mu-\frac{1}{2}\,\tilde G^{\mu\nu}\Sigma_{\mu\nu}-LR\,\partial_\mu W^\mu-\frac{R}{I}\,LA_\mu\partial_\nu G^{\mu\nu}-\frac{1}{2I}\,LUA^\mu\partial_\mu|Z|^2\\
&+\frac{1}{2}\,U(\bar Z\lambda_i\psi^i-Z\bar\lambda^i\bar\psi_i)-\frac{\mathrm{i}}{2}\,A_\mu U(\psi^i\sigma^\mu\bar\lambda_i-\lambda^i\sigma^\mu\bar\psi_i)+2\,\psi^i\sigma^\mu\bar\psi_i\,\partial_\mu R\\
&-\frac{1}{2}\,L^2\Box I-\mathrm{i}L(\psi^i\sigma^\mu\partial_\mu\bar\lambda_i-\partial_\mu\lambda^i\sigma^\mu\bar\psi_i)-\mathrm{i}Y_{ij}(\psi^i\psi^j-\bar\psi^i\bar\psi^j)\\
&-\mathrm{i}F_{\mu\nu}(\psi_i\sigma^{\mu\nu}\psi^i+\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i)-\frac{1}{2I}\,LA_\mu\,\delta_z(Z\psi^i\sigma^\mu\bar\lambda_i+\bar Z\lambda^i\sigma^\mu\bar\psi_i)\\
&+\frac{1}{4}\,IM_{ij}M^{ij}-\frac{1}{2}\,\partial_\mu\big[L\partial^\mu I+\mathrm{i}L(\psi^i\sigma^\mu\bar\lambda_i-\lambda^i\sigma^\mu\bar\psi_i)\big]\ .
\end{aligned}
\tag{3.60}
$$

When put into eq. (3.58), several terms cancel or combine into total derivatives, and we arrive at

$$
\begin{aligned}
\mathcal{L}_{\text{linVT}}=&\ \frac{1}{2}\,I\big(\partial^\mu L\,\partial_\mu L-W^\mu W_\mu-2\mathrm{i}\,\psi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\psi_i+EU^2\big)-\frac{1}{2}\,L^2\Box I\\
&-\frac{1}{4}\,G^{\mu\nu}\big(IG_{\mu\nu}-R\tilde G_{\mu\nu}+4A_\mu W_\nu\big)-\frac{\mathrm{i}}{2}\,Y_{ij}\,(\psi^i\psi^j-\bar\psi^i\bar\psi^j)\\
&-\frac{\mathrm{i}}{2}\,F_{\mu\nu}\,(\psi_i\sigma^{\mu\nu}\psi^i+\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i)-\frac{\mathrm{i}}{2}\,L\,(\psi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\lambda_i+\lambda^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\psi^i)\\
&+\psi^i\sigma^\mu\bar\psi_i\,\partial_\mu R+\frac{1}{4}\,IM^{ij}M_{ij}\\
&-\partial_\mu\big[LRW^\mu+L\tilde G^{\mu\nu}A_\nu+\mathrm{i}A_\nu(\psi_i\sigma^{\mu\nu}\psi^i+\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i)\\
&\qquad+\tfrac{1}{2}L^2\partial^\mu I-\psi^i\sigma^\mu\bar\psi_iR+\mathrm{i}L\,(\psi^i\sigma^\mu\bar\lambda_i-\lambda^i\sigma^\mu\bar\psi_i)\big]\ .
\end{aligned}
\tag{3.61}
$$

At last, we have to replace W^μ and $G_{\mu\nu}$ by the solutions to the Bianchi identities found in the previous section. We first insert $G_{\mu\nu}$ from eq. (3.34), which allows to combine the terms containing W^μ in a nice way,

$$
\begin{aligned}
&-\frac{1}{4}\,G^{\mu\nu}\big(IG_{\mu\nu}-R\tilde G_{\mu\nu}+4A_\mu W_\nu\big)-\frac{1}{2}\,I\,W^\mu W_\mu=\\
&=-\frac{1}{4}\,\mathcal{V}^{\mu\nu}\big(I\mathcal{V}_{\mu\nu}-R\tilde{\mathcal{V}}_{\mu\nu}\big)-\frac{I}{2|Z|^2}\,W^\mu K_{\mu\nu}W^\nu\ ,
\end{aligned}
$$

with $K_{\mu\nu}$ as in eq. (3.37). Next we replace W^μ using eq. (3.36). As each W^μ contributes an inverse of $K_{\mu\nu}$, the result

$$-\frac{I}{2|Z|^2}\,W^\mu K_{\mu\nu}W^\nu = -\frac{|Z|^2}{2I}\,\big(\mathcal{H}^\mu + \mathcal{V}^{\mu\rho}A_\rho\big)(K^{-1})_{\mu\nu}\big(\mathcal{H}^\nu + \mathcal{V}^{\nu\sigma}A_\sigma\big) \tag{3.62}$$

also involves the inverse matrix, giving rise to nonpolynomial but local couplings to the gauge field A_μ. When K^{-1} is inserted, we obtain

$$\begin{aligned} &-\frac{1}{4}\,G^{\mu\nu}\big(IG_{\mu\nu} - R\tilde{G}_{\mu\nu} + 4A_\mu W_\nu\big) - \frac{1}{2}\,I\,W^\mu W_\mu = \\ &\quad = -\frac{1}{4}\,\mathcal{V}^{\mu\nu}(I\mathcal{V}_{\mu\nu} - R\tilde{\mathcal{V}}_{\mu\nu}) - \frac{|Z|^2}{2I\mathcal{E}}\,(\mathcal{H}^\mu + \mathcal{V}^{\mu\nu}A_\nu)^2 + \frac{1}{2I\mathcal{E}}\,(A_\mu\mathcal{H}^\mu)^2\ . \end{aligned} \tag{3.63}$$

To conclude this section, let us concentrate on the gauge field part of the model by freezing the scalars to constants (in particular $Z = \mathrm{i}$ and $L = 0$) and neglecting the fermions. Dropping the total derivative, the complete Lagrangian (3.61) reduces to

$$\mathcal{L} = -\frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \frac{1}{2\mathcal{E}}\,(H^\mu + V^{\mu\nu}A_\nu)^2 + \frac{1}{2\mathcal{E}}\,(A_\mu H^\mu)^2 - \frac{1}{4g_z^2}\,F^{\mu\nu}F_{\mu\nu}\ , \tag{3.64}$$

where a kinetic term for A_μ originating from $\mathcal{L}_{\rm cc}$, eq. (1.48), has been added. Supersymmetry has now been broken explicitly of course, but the gauge invariances remain intact. After a rescaling $A_\mu \to g_z A_\mu$, such that all fields have canonical dimension one[1], we can expand the Lagrangian in powers of the coupling constant g_z, which gives up to first order

$$\mathcal{L} = -\frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \frac{1}{2}\,H^\mu H_\mu - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + g_z A_\mu V^{\mu\nu}H_\nu + O(g_z^2)\ , \tag{3.65}$$

and we recognize the coupling of A_μ to the current J_z^μ from eq. (2.16).
The Lagrangian (3.64) had previously been found outside the framework of supersymmetry in [24], and is actually but one example of a whole class of gauge theories known as Henneaux-Knaepen models, which we review in the last section of this chapter.

3.4 Chern-Simons Couplings

Let us now consider the more general solution (3.16) to the consistency conditions (C.1–4), containing an arbitrary holomorphic function $g(Z)$. We note that in every coefficient it is accompanied by a factor $\bar{Z}$. This may be removed by a field redefinition

$$L = \hat{L} + f(Z) + \bar{f}(\bar{Z})$$

with $\partial f = g$, for the transformation rules (2.54) give (dropping the hats)

$$C = \frac{1}{\bar{Z} - Z}\,\big(L + h + \bar{h}\big)\ , \quad D = \frac{\partial h}{\bar{Z} - Z}\ , \quad E = \frac{\bar{\partial}\bar{h}}{\bar{Z} - Z}\ , \tag{3.66}$$

[1]Recall that the coupling constant g_z has mass dimension -1.

while the other coefficients are unchanged. Here $h = Zg + f$. The functions u and v in eqs. 25) and 26) do not vanish anymore, but one has

$$L + ZC + \bar{Z}\bar{C} = -(h + \bar{h}) \ , \quad ZD + \bar{Z}\bar{E} = -\partial h \ ,$$

which however is compatible with condition (C.4). We now write the constraints as

$$\begin{aligned} &\mathcal{D}_\alpha^{(i}\bar{\mathcal{D}}_{\dot\alpha}^{j)} L = 0 \\ &\mathcal{D}^{(i}\mathcal{D}^{j)} L = \frac{2}{\bar{Z} - Z}\big(\mathcal{D}^{(i} Z\, \mathcal{D}^{j)} L + \bar{\mathcal{D}}^{(i} \bar{Z}\, \bar{\mathcal{D}}^{j)} L + \tfrac{1}{2} L\, \mathcal{D}^i\mathcal{D}^j Z \\ &\qquad\qquad - \mathrm{i}\, \mathcal{D}^i\mathcal{D}^j\mathcal{F} + \mathrm{i}\, \bar{\mathcal{D}}^i\bar{\mathcal{D}}^j\bar{\mathcal{F}}\big) \ , \end{aligned} \tag{3.67}$$

and it is easily verified that these satisfy the consistency conditions (C.1–4) for *any* function $\mathcal{F}$ that is chiral, $\bar{\mathcal{D}}_{\dot\alpha i}\mathcal{F} = 0$, and invariant under central charge and gauge transformations. In particular, $\mathcal{F}$ may be a gauge invariant combination of vector superfields ϕ^I, with appropriately extended spinor and covariant derivatives, i.e.

$$\mathcal{D}_\mu = \partial_\mu + A_\mu \delta_z + \mathcal{A}_\mu^I \delta_I \ , \quad \text{etc.} \ ,$$

where L and Z transform trivially under the δ_I. This provides a means of coupling the vector-tensor multiplet to additional (even nonabelian) vector multiplets, as long as the Bianchi identities admit such a coupling. We shall see that this is the case only for a very specific function $\mathcal{F}(Z, \phi)$.
To determine the Bianchi identities, we follow the steps in section 3.2. With the benefit of hindsight we take $\mathcal{F}$ to depend only on the ϕ^I, which simplifies the calculations considerably. The deformations then read

$$N_{\alpha\dot\alpha}^{ij} = 0 \ , \quad M^{ij} = -\bar{M}^{ij} = M_1^{ij} + M_2^{ij} \ , \tag{3.68}$$

with M_1^{ij} as in eq. (3.20) and

$$M_2^{ij} = \frac{1}{I}\left[\chi^{iI}\chi^{jJ}\mathcal{F}_{IJ} - \bar{\chi}^{iI}\bar{\chi}^{jJ}\bar{\mathcal{F}}_{IJ} + 2D^{ijI}(\mathcal{F}_I - \bar{\mathcal{F}}_I)\right] \ , \tag{3.69}$$

where a subscript on $\mathcal{F}$ denotes a differentiation with respect to ϕ and similar for $\bar{\mathcal{F}}$,

$$\mathcal{F}_{I_1\ldots I_n} \equiv \frac{\partial}{\partial\phi^{I_1}} \cdots \frac{\partial}{\partial\phi^{I_n}}\,\mathcal{F} \ , \quad \bar{\mathcal{F}}_{I_1\ldots I_n} \equiv \frac{\partial}{\partial\bar{\phi}^{I_1}} \cdots \frac{\partial}{\partial\bar{\phi}^{I_n}}\,\bar{\mathcal{F}} \ . \tag{3.70}$$

The derivatives of $\mathcal{F}$ are not independent of each other, for gauge invariance implies

$$0 = \delta_I\mathcal{F} = \delta_I\phi^K\,\mathcal{F}_K = -f_{IJ}{}^K\phi^J\mathcal{F}_K \ , \tag{3.71}$$

and differentiating once more with respect to ϕ we obtain another identity,

$$0 = f_{IJ}{}^K\mathcal{F}_K + f_{IL}{}^K\phi^L\mathcal{F}_{JK} \ . \tag{3.72}$$

We observe that, modulo the prefactor $1/I$, the expression M_2^{ij} is precisely the linear superfield from which one constructs the super Yang-Mills Lagrangian, cf. section 1.2. Applying a supersymmetry generator to M^{ij} yields

$$\begin{aligned}\mathcal{D}_{\alpha j}M^{ij} = \frac{3}{I}\big[&D^{ijI}\chi_j^J\mathfrak{F}_{IJ} + \mathcal{F}^I_{\mu\nu}\sigma^{\mu\nu}\chi^{iJ}\mathfrak{F}_{IJ} - \mathrm{i}\mathcal{D}_\mu(\bar{\mathfrak{F}}_I\sigma^\mu\bar\chi^{iI}) + \mathrm{i}\mathfrak{F}_I\sigma^\mu\mathcal{D}_\mu\bar\chi^{iI} \\ &- \tfrac{1}{2}\chi^{iI}\bar\phi^J f_{IJ}{}^K\mathfrak{F}_K + \tfrac{1}{3}(\chi^{iI}\chi^{jJ})\,\chi_j^K\,\mathfrak{F}_{IJK} + \tfrac{\mathrm{i}}{4}\lambda_j M_2^{ij}\big]_\alpha + \dots\ ,\end{aligned} \tag{3.73}$$

where only contributions from M_2^{ij} have been written explicitly, while the dots denote the terms already given in eq. (3.21) (where now $M^{ij} = M_1^{ij}$). Next we apply $\bar{\mathcal{D}}_{\dot\alpha i}$. Making frequent use of the above identities for the derivatives of $\mathfrak{F}$, we arrive at

$$\begin{aligned}\bar{\mathcal{D}}_{\dot\alpha i}\mathcal{D}_{\alpha j}M^{ij} = -\frac{3}{I}\,\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\mathcal{D}^\nu\big[&\,2(\mathfrak{F}_I - \bar{\mathfrak{F}}_I)\mathcal{F}^I_{\mu\nu} - 2\mathrm{i}\,(\mathfrak{F}_I + \bar{\mathfrak{F}}_I)\tilde{\mathcal{F}}^I_{\mu\nu} + \mathfrak{F}_{IJ}\,\chi^{iI}\sigma_{\mu\nu}\chi_i^J \\ &- \bar{\mathfrak{F}}_{IJ}\,\bar\chi_i^I\bar\sigma_{\mu\nu}\bar\chi^{iJ}\,\big] + \dots\ .\end{aligned} \tag{3.74}$$

Here the covariant derivative actually reduces to the partial derivative since the terms in square brackets are gauge invariant, and a similar remark as above applies to the dots. With the result from section 3.2, the second Bianchi identity (BI.2) takes the form

$$\mathcal{D}_\nu\big(I\tilde G^{\mu\nu} + RG^{\mu\nu} + \hat\Sigma^{\mu\nu}\big) = -\tilde F^{\mu\nu}W_\nu\ , \tag{3.75}$$

where $\hat\Sigma_{\mu\nu}$ is given by

$$\begin{aligned}\hat\Sigma_{\mu\nu} = \Sigma_{\mu\nu} - \mathrm{i}\,\big[\,&2(\mathfrak{F}_I - \bar{\mathfrak{F}}_I)\mathcal{F}^I_{\mu\nu} - 2\mathrm{i}\,(\mathfrak{F}_I + \bar{\mathfrak{F}}_I)\tilde{\mathcal{F}}^I_{\mu\nu} \\ &+ \mathfrak{F}_{IJ}\,\chi^{iI}\sigma_{\mu\nu}\chi_i^J - \bar{\mathfrak{F}}_{IJ}\,\bar\chi_i^I\bar\sigma_{\mu\nu}\bar\chi^{iJ}\,\big]\ .\end{aligned} \tag{3.76}$$

Since $(\hat\Sigma_{\mu\nu} - \Sigma_{\mu\nu})$ is δ_z-invariant, we can replace $\Sigma_{\mu\nu}$ with the extended expression in eq. (3.28) and thus in the solution (3.31). Hence, the second Bianchi identity does not restrict the ϕ-dependence of the function $\mathfrak{F}$. It is the first Bianchi identity for W^μ, however, that imposes a constraint on $\mathfrak{F}$. It now reads

$$\mathcal{D}_\mu\big(IW^\mu - \Lambda^\mu\big) = \frac{1}{4}\,F_{\mu\nu}G^{\mu\nu} + \frac{\mathrm{i}}{12}\,I\mathcal{D}_i\mathcal{D}_jM_2^{ij} - \frac{1}{12}\,(2Y_{ij} + \bar\lambda_i\bar{\mathcal{D}}_j)M_2^{ij} + \text{c.c.}\ ,$$

where the last term originates from $\mathcal{D}_i\mathcal{D}_jM_1^{ij}$. So let us apply a generator $\mathcal{D}_i^\alpha$ to eq. (3.73); the contribution from M_2^{ij} is

$$\begin{aligned}\mathcal{D}_i\mathcal{D}_jM_2^{ij} = \frac{12}{I}\,\big[\,&\mathfrak{F}_{IJ}\mathcal{D}^\mu\phi^I\,\mathcal{D}_\mu\bar\phi^J - \tfrac{1}{2}\mathfrak{F}_{IJ}(\mathcal{F}_{\mu\nu} - \mathrm{i}\tilde{\mathcal{F}}_{\mu\nu})^I\mathcal{F}^{\mu\nu J} - \mathrm{i}\mathfrak{F}_{IJ}\,\chi^{iI}\sigma^\mu\mathcal{D}_\mu\bar\chi_i^J \\ &+ \tfrac{1}{2}\mathfrak{F}_{IJ}D^I_{ij}D^{ijJ} + \tfrac{1}{2}\mathfrak{F}_{IJ}\,\chi^{iI}\chi_i^K\bar\phi^L f_{KL}{}^J - \tfrac{1}{2}\mathfrak{F}_{IJ}\,\bar\chi_i^I\bar\chi^{iK}\phi^L f_{KL}{}^J \\ &- \tfrac{1}{4}\mathfrak{F}_{IJ}(\phi^K\bar\phi^J f_{KL}{}^I)(\phi^M\bar\phi^N f_{MN}{}^L) + \partial_\mu\big((\bar{\mathfrak{F}}_I - \mathfrak{F}_I)\mathcal{D}^\mu\bar\phi^I\big) \\ &+ \tfrac{1}{2}\mathfrak{F}_{IJK}D^{ijI}\chi_i^J\chi_j^K + \tfrac{1}{2}\mathfrak{F}_{IJK}\,\mathcal{F}^I_{\mu\nu}\,\chi_i^J\sigma^{\mu\nu}\chi^{iK} \\ &+ \tfrac{1}{12}\mathfrak{F}_{IJKL}\,\chi^{iI}\chi^{jJ}\,\chi_i^K\chi_j^L\,\big] + \frac{\mathrm{i}}{I}\,(Y_{ij} + \lambda_i\mathcal{D}_j)M_2^{ij}\ .\end{aligned} \tag{3.77}$$

Clearly the imaginary part[2] of the expression inside the square brackets can combine into a total derivative only if $\mathcal{F}_{IJ}$ is constant and real. For a compact gauge group this fixes $\mathcal{F}$ modulo a normalization,

$$\mathcal{F}(\phi) = \frac{e}{2}\,\delta_{IJ}\phi^I\phi^J \;, \tag{3.78}$$

e being a coupling constant of mass dimension -1. Then the first Bianchi identity reduces to

$$\mathcal{D}_\mu\big(IW^\mu - \hat{\Lambda}^\mu\big) = \tfrac{1}{2}F_{\mu\nu}G^{\mu\nu} \;, \tag{3.79}$$

where

$$\begin{aligned}\hat{\Lambda}^\mu = \Lambda^\mu - e\,\big[\,&2\varepsilon^{\mu\nu\rho\sigma}(\mathcal{A}^I_\nu\partial_\rho\mathcal{A}^I_\sigma - \tfrac{1}{3}\mathcal{A}^I_\nu\mathcal{A}^J_\rho\mathcal{A}^K_\sigma f_{JK}{}^I)\\ &+ \mathrm{i}(\phi-\bar{\phi})^I\mathcal{D}^\mu(\phi+\bar{\phi})^I - \chi^{iI}\sigma^\mu\bar{\chi}^I_i\,\big]\end{aligned} \tag{3.80}$$

contains the nonabelian Chern-Simons form that lends its name to this section. It originates from the term $\mathcal{F}^I_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu I}$, which can be written as a total derivative using the Jacobi identity and the antisymmetry of the structure constants $f_{IJ}{}^K\delta_{KL}$,

$$\begin{aligned}\tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^I_{\mu\nu}\mathcal{F}^I_{\rho\sigma} &= 2\varepsilon^{\mu\nu\rho\sigma}\big(\partial_\mu\mathcal{A}^I_\nu\,\partial_\rho\mathcal{A}^I_\sigma - f_{JK}{}^I\mathcal{A}^J_\mu\mathcal{A}^K_\nu\,\partial_\rho\mathcal{A}^I_\sigma\\ &\qquad + \tfrac{1}{4}\mathcal{A}^J_\mu\mathcal{A}^K_\nu\mathcal{A}^L_\rho\mathcal{A}^M_\sigma\,f_{JK}{}^I f_{LM}{}^I\big)\\ &= 2\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\big(\mathcal{A}^I_\nu\partial_\rho\mathcal{A}^I_\sigma - \tfrac{1}{3}\mathcal{A}^I_\nu\mathcal{A}^J_\rho\mathcal{A}^K_\sigma f_{JK}{}^I\big)\;.\end{aligned}$$

Again, one can replace Λ_μ with $\hat{\Lambda}_\mu$ in eq. (3.29), and we conclude that for the function (3.78) the constraints (3.67) are consistent, since the Bianchi identities can be solved exactly as in section 3.2. Note however that, although $\hat{\Lambda}^\mu$ is δ_I-invariant, a full gauge transformation Δ^{g} yields

$$\Delta^{\mathrm{g}}(C)\,\hat{\Lambda}^\mu = 2e\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(C^I\partial_\rho\mathcal{A}^I_\sigma)\;. \tag{3.81}$$

Hence, in order to render the solution (3.30) to the first Bianchi identity Δ^{g}-invariant, we have to cancel the contribution from $\hat{\Lambda}^\mu$ by assigning to $B_{\mu\nu}$ the nontrivial transformation law

$$\Delta^{\mathrm{g}}(C)\,B_{\mu\nu} = -4e\,C^I\partial_{[\mu}\mathcal{A}^I_{\nu]}\;. \tag{3.82}$$

V_μ on the other hand remains gauge invariant.
What happens when we choose $e\phi = Z$, i.e. $\mathcal{F} = Z^2/2e$? This corresponds to the function $h = -2\mathrm{i}Z/e$ in the coefficients (3.66), and by a field redefinition

$$L = \hat{L} + \frac{\mathrm{i}}{e}\,(Z-\bar{Z}) \tag{3.83}$$

we can achieve $\hat{h} = 0$. Thus the Bianchi identities have singled out a function $\mathcal{F}(Z)$ that may be gauged away, which shows that, modulo field redefintions, the constraints (3.19) uniquely describe the linear vector-tensor multiplet with gauged central charge.

[2]The real part is exactly the super Yang-Mills Lagrangian (1.45) for an arbitrary prepotential $\mathcal{F}$.

An invariant action can now easily be written down, as in the previous section we have used only the two properties $N^{ij}_{\alpha\dot\alpha} = 0$ and $\bar M^{ij} = -M^{ij}$ in the derivation of the Lagrangian (3.58), and these are also valid in the case at hand. Therefore, we just need to determine $\mathcal{L}_M$ according to eq. (3.59). As all the ingredients have been given above, this is merely a matter of inserting and combining terms, and we proceed immediately to the final Lagrangian, which reads (modulo a total derivative)

$$\begin{aligned}\mathcal{L} = {}& \mathcal{L}_{\rm linVT}\left(\Lambda^\mu \to \hat\Lambda^\mu,\, \Sigma_{\mu\nu} \to \hat\Sigma_{\mu\nu}\right) + \mathcal{L}_{\rm cc} \\ & - 2e\left[\, \mathrm{i}\mathcal{F}^I_{\mu\nu}\psi_i\sigma^{\mu\nu}\chi^{iI} - \psi^i\sigma^\mu\bar\chi^I_i\mathcal{D}_\mu\bar\phi^I + (\phi-\bar\phi)^I\psi^i\sigma^\mu\mathcal{D}_\mu\bar\chi^I_i + \mathrm{i}D^{ijI}\psi_i\chi^I_j \right. \\ & \qquad \left. + \tfrac{\mathrm{i}}{2}\psi^i\chi^J_i\bar\phi^K f_{JK}{}^I\phi^I + \text{c.c.}\,\right] - e\,\partial_\mu L\,(\phi+\bar\phi)^I\partial^\mu(\phi+\bar\phi)^I \\ & + (eL + 1/4g^2)\left[\, 2\mathcal{D}^\mu\phi^I\,\mathcal{D}_\mu\bar\phi^I - \mathcal{F}^{\mu\nu I}\mathcal{F}^I_{\mu\nu} - \mathrm{i}\chi^{iI}\sigma^\mu\overleftrightarrow{\mathcal{D}}_\mu\bar\chi^I_i + D^I_{ij}D^{ijI} \right. \\ & \qquad \left. + \left(\chi^{iI}\chi^J_i\bar\phi^K - \bar\chi^I_i\bar\chi^{iJ}\phi^K\right)f_{JK}{}^I + \tfrac{1}{2}\left(\phi^J\bar\phi^K f_{JK}{}^I\right)^2\,\right]\,.\end{aligned} \tag{3.84}$$

$\mathcal{L}_{\rm linVT}$ has been given in eq. (3.61) and depends on Λ^μ and $\Sigma_{\mu\nu}$ through the composite fields W^μ and $G_{\mu\nu}$. Upon replacing Λ^μ with $\hat\Lambda^\mu$ in the generalized field strength $\mathcal{H}^\mu$, eq. (3.32), a coupling of the Chern-Simons form to the tensor gauge field $B_{\mu\nu}$ emerges to zeroth order in g_z, cf. (3.63). For the pure gauge field part we find

$$\begin{aligned}\mathcal{L} = {}& -\frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \frac{1}{2\mathcal{E}}\,(\hat H^\mu + V^{\mu\nu}A_\nu)^2 + \frac{1}{2\mathcal{E}}\,(A_\mu\hat H^\mu)^2 - \frac{1}{4g_z^2}\,F^{\mu\nu}F_{\mu\nu} \\ & - \frac{1}{4g^2}\,\mathcal{F}^{\mu\nu I}\mathcal{F}^I_{\mu\nu}\,,\end{aligned} \tag{3.85}$$

where

$$\hat H^\mu = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\left[\,\partial_\nu B_{\rho\sigma} - 4e\left(\mathcal{A}^I_\nu\partial_\rho\mathcal{A}^I_\sigma - \tfrac{1}{3}\mathcal{A}^I_\nu\mathcal{A}^J_\rho\mathcal{A}^K_\sigma f_{JK}{}^I\right)\,\right]\,. \tag{3.86}$$

3.5 Henneaux-Knaepen Models

We conclude the chapter by showing how the special gauge couplings we have found as a result of supersymmetry fit into a more general scheme devised by Henneaux and Knaepen in [25]. When formulated in D spacetime dimensions, these models involve interactions of $(D-2)$-form gauge fields with gauge potentials of lesser form degree and include as a subset the so-called Freedman-Townsend models [26, 27], which describe nonpolynomial self-couplings of $(D-2)$-forms. In four dimensions the field content consists of 2-form and ordinary 1-form gauge potentials. While it has been shown by Brandt and the author in [28] that every four-dimensional Henneaux-Knaepen model admits an $N = 1$ supersymmetric generalization, the only known example of such a model possessing two supersymmetries is the (linear) vector-tensor multiplet with gauged central charge.

Let us now collectively denote the antisymmetric tensors as $B_{\mu\nu A}$ and the vector fields as A^a_μ, with field strengths

$$H^\mu_A = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma A}\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu\,. \tag{3.87}$$

In the case of the vector-tensor multiplet with gauged central charge we would have two 1-forms A^1_μ, A^2_μ, one of which being identical to what we used to call V_μ, and just one 2-form $B_{\mu\nu}$.
A Lagrangian that is invariant under abelian gauge transformations

$$\Delta^z(C)\, A^a_\mu = -\partial_\mu C^a\ , \quad \Delta^B(\Omega)\, B_{\mu\nu A} = -2\, \partial_{[\mu}\Omega_{\nu]A} \tag{3.88}$$

is given simply in terms of the field strengths,

$$\mathcal{L} = -\frac{1}{2}\, H^\mu_A H^A_\mu - \frac{1}{4}\, F^{\mu\nu}_a F^a_{\mu\nu}\ . \tag{3.89}$$

The key observation is that the action has in addition global symmetries generated by

$$\delta_a A^b_\mu = -H^A_\mu T^b_{A\,a}\ , \quad \delta_a B_{\mu\nu A} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}_b T^b_{A\,a}\ , \tag{3.90}$$

where the $T^a_{A\,b}$ are, at this stage, arbitrary real constants. The corresponding Noether currents read

$$J^\mu_a = T^b_{A\,a} F^{\mu\nu}_b H^A_\nu\ . \tag{3.91}$$

Comparing with section 2.1, we observe that the δ_a generalize the rigid central charge transformations of the free vector-tensor multiplet. When more than one antisymmetric tensor is considered, there are also nontrivial second-order currents

$$J^{\mu\nu A} = \tfrac{1}{2} f_{BC}{}^A \varepsilon^{\mu\nu\rho\sigma} H^B_\rho H^C_\sigma\ , \tag{3.92}$$

which are conserved on-shell for any constants $f_{AB}{}^C = f_{[AB]}{}^C$, but do not correspond to any global symmetry of $\mathcal{L}$.
Henneaux and Knaepen have been able to simultaneously deform the free action (3.89) and the gauge transformations (3.88) such that the symmetries generated by the δ_a are realized locally. At first order in the deformation parameter g the gauge fields couple to the respective currents, giving rise to so-called Freedman-Townsend vertices. At second order in g a condition arises on the as yet arbitrary constants $T^a_{A\,b}$ and $f_{AB}{}^C$, namely the $f_{AB}{}^C$ have to satisfy a Jacobi identity, which identifies them as structure constants of a Lie algebra, while the matrices T_A are required to define a real representation of the same,

$$f_{[AB}{}^D f_{C]D}{}^E = 0\ , \quad [\, T_A\, ,\, T_B\,] = f_{AB}{}^C T_C\ . \tag{3.93}$$

It is possible, and convenient, to present the resulting model in a first order formulation, where the Lagrangian and the transformations are polynomial. To this end, we introduce auxiliary vector fields W^A_μ, which may be eliminated later on to obtain the nonpolynomial form. The complete Lagrangian then reads

$$\mathcal{L}_{\rm HK} = -\frac{1}{4}\, \varepsilon^{\mu\nu\rho\sigma} W^A_{\mu\nu} B_{\rho\sigma A} - \frac{1}{4}\, \mathcal{F}^{\mu\nu a}\big(\delta_{ab}\mathcal{F}^b_{\mu\nu} + c_{ab}\tilde{\mathcal{F}}^b_{\mu\nu}\big) + \frac{1}{2}\, \delta_{AB} W^{\mu A} W^B_\mu\ , \tag{3.94}$$

where $c_{ab} \in \mathbb{R}$ and

$$W_{\mu\nu}^A = \partial_\mu W_\nu^A - \partial_\nu W_\mu^A + g f_{BC}{}^A W_\mu^B W_\nu^C \tag{3.95}$$

$$\mathfrak{F}_{\mu\nu}^a = \nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a \ , \quad \nabla_\mu A_\nu^a = \partial_\mu A_\nu^a + g W_\mu^A T_{A\,b}^a A_\nu^b \ . \tag{3.96}$$

$W_{\mu\nu}^A$ resembles a nonabelian Yang-Mills field strength, in particular it satisfies the Bianchi identity

$$\varepsilon^{\mu\nu\rho\sigma} \nabla_\nu W_{\rho\sigma}^A = \varepsilon^{\mu\nu\rho\sigma} (\partial_\nu W_{\mu\nu}^A + g W_\nu^B f_{BC}{}^A W_{\rho\sigma}^C) = 0 \ , \tag{3.97}$$

but due to the presence of the last term in $\mathcal{L}_{\rm HK}$ there is no gauge transformation associated with W_μ^A that leaves $\mathcal{L}_{\rm HK}$ invariant. The term proportional to the constants c_{ab}, which may be chosen arbitrarily, slightly extends the original model of Henneaux and Knaepen. As we shall see, it gives rise to Chern-Simons couplings, which however neither include the nonabelian ones of the previous section, nor do they describe the nonlinearities of the self-interacting vector-tensor multiplet we are going to construct in the following chapter. This suggests that the Henneaux-Knaepen models presented here admit a further generalization.
The conditions (3.93) are sufficient[3] to render $\int d^4x \, \mathcal{L}_{\rm HK}$ invariant under the following two sets of gauge transformations: First one may vary just the $B_{\mu\nu A}$,

$$\Delta^B(\Omega) \, B_{\mu\nu A} = -2 \, \nabla_{[\mu} \Omega_{\nu]A} \ , \quad \Delta^B(\Omega) \, A_\mu^a = 0 \ , \quad \Delta^B(\Omega) \, W_\mu^A = 0 \ , \tag{3.98}$$

where the action of the "covariant derivative" ∇_μ on the parameters $\Omega_{\mu A}$ is given by

$$\nabla_\mu \Omega_{\nu A} = \partial_\mu \Omega_{\nu A} - g W_\mu^B f_{BA}{}^C \Omega_{\nu C} \ . \tag{3.99}$$

By means of the Bianchi identity (3.97) one easily verifies that $\mathcal{L}_{\rm HK}$ changes by a total derivative only. The second set subsumes what we have previously encountered as local central charge transformations (hence the denomination Δ^z),

$$\begin{aligned} \Delta^z(C) \, A_\mu^a &= -\nabla_\mu C^a \ , \quad \Delta^z(C) \, W_\mu^A = 0 \\ \Delta^z(C) \, B_{\mu\nu A} &= g \, (c_{ab} \mathfrak{F}_{\mu\nu}^a - \delta_{ab} \tilde{\mathfrak{F}}_{\mu\nu}^a) \, T_{A\,c}^b C^c \ . \end{aligned} \tag{3.100}$$

Here ∇_μ acts on the C^a as it does on the A_μ^a, eq. (3.96). In view of the relation

$$\Delta^z(C) \, \mathfrak{F}_{\mu\nu}^a = -[\, \nabla_\mu \, , \, \nabla_\nu \,] \, C^a = -g W_{\mu\nu}^A T_{A\,b}^a C^b \ , \tag{3.101}$$

the variation of the $B_{\mu\nu A}$ evidently cancels the one of the A_μ^a, thus $\mathcal{L}_{\rm HK}$ is Δ^z-invariant. Since $W_{\mu\nu}^A \approx 0$ by virtue of the equations of motion for the $B_{\mu\nu A}$, the transformations commute on-shell, so the algebra of gauge transformations is in fact abelian to all orders in the coupling constant.

[3]We emphasize that the Lie algebra need not be compact and that the metrics δ_{ab}, δ_{AB} as well as the constants c_{ab} need not be invariant tensors.

Let us now eliminate the auxiliary fields in order to make contact with the vector-tensor multiplet. We rearrange $\mathcal{L}_{\rm HK}$ such that the W_μ^A effectively decouple from the dynamical fields. Dropping a total derivative, this gives

$$\begin{aligned}\mathcal{L}_{\rm HK} &= \frac{1}{2}\, W_\mu^A K_{AB}^{\mu\nu} W_\nu^B - W_\mu^A \mathcal{H}_A^\mu - \frac{1}{4}\,\delta_{ab}\, F^{\mu\nu a} F_{\mu\nu}^b \\ &= -\frac{1}{2}\,\mathcal{H}_A^\mu\,(K^{-1})_{\mu\nu}^{AB}\,\mathcal{H}_B^\nu - \frac{1}{4}\,\delta_{ab}\,F^{\mu\nu a}F_{\mu\nu}^b \\ &\quad + \frac{1}{2}\left[W_\mu^A - (K^{-1})_{\mu\rho}^{AC}\mathcal{H}_C^\rho\right] K_{AB}^{\mu\nu}\left[W_\nu^B - (K^{-1})_{\nu\sigma}^{BD}\mathcal{H}_D^\sigma\right] ,\end{aligned} \tag{3.102}$$

with the abbreviations

$$\mathcal{H}_A^\mu = H_A^\mu + g\, T_{A\,c}^a A_\nu^c (\delta_{ab} F^{\mu\nu b} + c_{ab}\tilde{F}^{\mu\nu b}) \tag{3.103}$$

$$\begin{aligned}K_{AB}^{\mu\nu} &= \delta_{AB}\eta^{\mu\nu} - \tfrac{1}{2} g\, f_{AB}{}^C \varepsilon^{\mu\nu\rho\sigma} B_{\rho\sigma C} \\ &\quad - g^2\, T_{A\,c}^a T_{B\,d}^b\,(\delta_{ab}\eta^{\mu\nu}A^{\rho c}A_\rho^d - \delta_{ab}A^{\mu d}A^{\nu c} - c_{ab}\varepsilon^{\mu\nu\rho\sigma}A_\rho^c A_\sigma^d) ,\end{aligned} \tag{3.104}$$

and $(K^{-1})_{\mu\rho}^{AC} K_{CB}^{\rho\nu} = \delta_\mu^\nu\,\delta_B^A$. Hence, on-shell we can replace W_μ^A with $(K^{-1})_{\mu\nu}^{AB}\mathcal{H}_B^\nu$ in the above transformations and omit the second line in eq. (3.102), leaving a Lagrangian that is nonpolynomial in the fields and the coupling constant g. Since $K_{AB}^{\mu\nu}$ and its inverse do not involve derivatives, the action remains local, however. Note that the $\mathcal{H}_A^\mu$ include Chern-Simons terms $\tilde{F}^{\mu\nu b}A_\nu^c$, which in the Lagrangian couple to the field strengths of the 2-forms with coefficients $c_{ab}T_{A\,c}^a$.
The pure gauge field part of the linear vector-tensor multiplet, given in eq. (3.64), is now recovered by making the identification

$$A_\mu^1 = A_\mu\ , \quad A_\mu^2 = V_\mu\ , \quad B_{\mu\nu 1} = B_{\mu\nu}\ , \tag{3.105}$$

together with the choice

$$T_{1\,b}^a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \quad c_{ab} = 0\ , \tag{3.106}$$

which conforms to $f_{11}{}^1 = 0$ for a single antisymmetric tensor. When substituted in eqs. (3.103), (3.104), these coefficients yield the expressions

$$\mathcal{H}^\mu = H^\mu + gF^{\mu\nu 2}A_\nu^1\ , \quad K^{\mu\nu} = \eta^{\mu\nu}(1 - g^2 A^{\rho 1}A_\rho^1) + g^2 A^{\mu 1}A^{\nu 1}$$
$$K_{\mu\nu}^{-1} = \frac{\eta_{\mu\nu} - g^2 A_\mu^1 A_\nu^1}{1 - g^2 A^{1\rho}A_\rho^1}\ ,$$

which coincide exactly with their counterparts in the supersymmetric model after replacing the scalars by their background values.
At last we point out that what made the construction of the bosonic Henneaux-Knaepen models possible in the first place was the introduction of auxiliary fields, resulting in polynomial actions and transformations (a feature shared by the $N = 1$ supersymmetric versions in [28]). As yet, we do not know how to do this in the $N = 2$ supersymmetric case.

Chapter 4
The Nonlinear Case

In section 2.3.2 we have argued that there exist two inequivalent sets of constraints describing the vector-tensor multiplet. So far, we have been dealing only with the first one and its generalizations to admit couplings to vector multiplets. We shall now show how the second set gives rise to a new feature, namely self-interactions of the vector-tensor multiplet. As announced, this will be done from the beginning in the presence of a gauged central charge, but in somewhat less detail than in the previous chapter, for the steps from the constraints to the Bianchi identities and ultimately to the Lagrangian are essentially the same. From the latter in particular we give only the purely bosonic part, as we are mostly interested in the gauge field interactions.

4.1 Consistent Constraints

Let us resume the evaluation of the consistency conditions in section 2.3.3. We recall the second solution to eq. 4) of the system of differential equations (2.79),

$$F = -\frac{1}{L + h(Z, \bar{Z})} \ , \quad h \text{ real.}$$

We now continue by solving eq. 5) for A,

$$A + \frac{2}{Z} = \frac{2}{\bar{F}}\,\partial\bar{F} = -\frac{2\partial h}{L + h} \ . \tag{4.1}$$

Then it can easily be checked that also eqs. 1) and 19) hold identically. When put into eq. 9), we obtain a condition on h, namely

$$\bar{\partial}\partial h = 0 \ . \tag{4.2}$$

From eq. 10) follows another condition,

$$0 = \partial A - \tfrac{1}{2}A^2 = -\frac{2}{L + h}\left(\partial^2 h + \frac{2}{Z}\,\partial h\right) , \tag{4.3}$$

which allows to determine h completely,

$$h = \frac{\varrho}{Z} + \frac{\bar{\varrho}}{\bar{Z}} + \mu \ , \quad \varrho \in \mathbb{C} \ , \ \mu \in \mathbb{R} \ . \tag{4.4}$$

We observe that the combination $L + h$ is, modulo the constant μ, precisely of the form (2.78), thus we can achieve $\varrho = 0$ by a field redefinition, thereby fixing the gauge modulo rescalings of L by a constant parameter. Next we insert A and F into eq. 2),

$$(L+\mu)\partial_L C + C = -\frac{1}{Z}\left(L+\mu\right) \ , \tag{4.5}$$

the general solution to which is given by

$$C = \frac{v(Z,\bar{Z})}{L+\mu} - \frac{L+\mu}{2Z} \ . \tag{4.6}$$

Eq. 25) then implies

$$\mu(L+\mu) = Zv + \bar{Z}\bar{v} \quad \Rightarrow \quad \mu = 0 \ . \tag{4.7}$$

From eqs. 6) and 7) we obtain the dependence of E on L and Z, respectively,

$$L\partial_L E + E = 0 = Z\partial E + E \quad \Rightarrow \quad E = \frac{\bar{\partial}\bar{g}}{ZL} \ , \tag{4.8}$$

where $\bar{g}(\bar{Z})$ is independent of Z. At last, we consider eq. 3), which requires

$$Z\partial v + v = -\partial g \ . \tag{4.9}$$

By differentiating eq. (4.7) with respect to Z, we find

$$0 = Z\partial v + v + \bar{Z}\partial\bar{v} = \partial(\bar{Z}\bar{v} - g) \quad \Rightarrow \quad \bar{v} = \frac{g}{\bar{Z}} + \bar{u}(\bar{Z}) \ , \tag{4.10}$$

and finally from eq. (4.7) the relation $u = -g/Z$. The coefficient functions thus read

$$\begin{gathered} A = -\frac{2}{Z} \ , \quad C = -\frac{L}{2Z} - \frac{1}{ZL}\left(g - \bar{g}\right) \ , \quad E = \frac{\bar{\partial}\bar{g}}{ZL} \ , \quad D = -\frac{\partial g}{ZL} \\ F = -\frac{1}{L} \ , \quad G = \frac{\bar{Z}}{ZL} \ , \quad a = b = c = B = 0 \ , \end{gathered} \tag{4.11}$$

where $g(Z)$ is some arbitrary holomorphic function. Similar to the case of the linear vector-tensor multiplet, the g-dependent terms combine to

$$-\frac{1}{ZL}\left[\mathcal{D}^i(g\,\mathcal{D}^j Z) - \bar{\mathcal{D}}^i(\bar{g}\,\bar{\mathcal{D}}^j\bar{Z})\right] = -\frac{1}{ZL}\left[\mathcal{D}^i\mathcal{D}^j f(Z) - \bar{\mathcal{D}}^i\bar{\mathcal{D}}^j\bar{f}(\bar{Z})\right] \ , \tag{4.12}$$

provided g can be integrated, $\partial f = g$. In the following we consider the case $g = 0$ only, for it can be shown [2] that the Bianchi identities again single out a function $g(Z)$ which may be removed by a superfield redefinition. We shall not generalize the model to include Chern-Simons couplings to nonabelian vector multiplets (this can be found in the reference just mentioned), but Chern-Simons-like terms for V_μ and A_μ arise automatically, as we will see. The constraints we are now going to investigate read

$$\begin{aligned} &\mathcal{D}^{(i}_\alpha\bar{\mathcal{D}}^{j)}_{\dot{\alpha}} L = 0 \ , \\ &\mathcal{D}^{(i}\mathcal{D}^{j)} L = -\frac{1}{ZL}\left(2L\,\mathcal{D}^{(i}Z\,\mathcal{D}^{j)}L + \tfrac{1}{2}L^2\,\mathcal{D}^i\mathcal{D}^j Z + Z\mathcal{D}^i L\,\mathcal{D}^j L - \bar{Z}\bar{\mathcal{D}}^i L\,\bar{\mathcal{D}}^j L\right) \ . \end{aligned} \tag{4.13}$$

4.2 Transformations and Bianchi Identities

To determine the Bianchi identities, we need to calculate the action of the supersymmetry generators on the deformation

$$M^{ij} = \frac{1}{ZL}\left(2\mathrm{i}\,L\lambda^{(i}\psi^{j)} - L^2 Y^{ij} + Z\psi^i\psi^j - \bar{Z}\bar{\psi}^i\bar{\psi}^j\right) . \tag{4.14}$$

Note that contrary to the case of the linear vector-tensor multiplet, M^{ij} is neither real nor imaginary, which makes things a little more complicated. Applying $\mathcal{D}_{\alpha j}$, we obtain

$$\begin{aligned}\mathcal{D}_{\alpha j}M^{ij} = \frac{3}{2ZL}\big[\,&2\mathrm{i}L\,Y^{ij}\psi_j - \mathrm{i}L^2\sigma^\mu\partial_\mu\bar{\lambda}^i + \bar{Z}LU\lambda^i + G_{\mu\nu}\sigma^{\mu\nu}(\mathrm{i}Z\psi^i - L\lambda^i)\\ &+ 2\mathrm{i}L\,F_{\mu\nu}\sigma^{\mu\nu}\psi^i - \bar{Z}(\mathcal{D}_\mu L - \mathrm{i}W_\mu)\,\sigma^\mu\bar{\psi}^i + M^{ij}(\mathrm{i}Z\psi_j - L\lambda_j)\\ &+ (\psi^i\psi^j)\lambda_j + (\lambda^i\psi^j)\psi_j\,\big]_\alpha ,\end{aligned} \tag{4.15}$$

while the action of $\bar{\mathcal{D}}_{\dot{\alpha}j}$ now cannot be derived by complex conjugation of the above expression but needs to be computed separately. One finds

$$\begin{aligned}\bar{\mathcal{D}}_{\dot{\alpha}j}M^{ij} = -\frac{3}{2ZL}\big[\,&2L\psi^i\sigma^\mu\partial_\mu Z + \mathrm{i}L\,\mathcal{D}_\mu(L\lambda^i\sigma^\mu) - LW_\mu\lambda^i\sigma^\mu + \mathrm{i}|Z|^2U\bar{\psi}^i\\ &+ Z(\mathcal{D}_\mu L + \mathrm{i}W_\mu)\,\psi^i\sigma^\mu + \mathrm{i}\bar{Z}\,G_{\mu\nu}\,\bar{\psi}^i\bar{\sigma}^{\mu\nu} - \mathrm{i}\bar{Z}\,\bar{M}^{ij}\bar{\psi}_j\\ &+ (\bar{\psi}^i\bar{\psi}^j)\bar{\lambda}_j + (\bar{\lambda}^i\bar{\psi}^j)\bar{\psi}_j\,\big]_{\dot{\alpha}} .\end{aligned} \tag{4.16}$$

The central charge transformation of W^μ and the Bianchi identity for $G_{\mu\nu}$ involve the real and imaginary part of $Z\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij}$, respectively. After a lengthy calculation, we arrive at

$$\begin{aligned}\frac{1}{6}\,Z\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} = {}&U(\mathrm{i}\bar{Z}\partial^\mu Z + \lambda^i\sigma^\mu\bar{\lambda}_i) - (\tilde{G}^{\mu\nu} + \mathrm{i}G^{\mu\nu})\partial_\nu Z - (\mathrm{i}\tilde{F}^{\mu\nu} - F^{\mu\nu})W_\nu\\ &- \mathcal{D}_\nu(L\tilde{F}^{\mu\nu} + \mathrm{i}LF^{\mu\nu} + 2\psi^i\sigma^{\mu\nu}\lambda_i) + \mathrm{i}\bar{Z}\,\lambda^i\sigma^\mu\delta_z\bar{\psi}_i\\ &+ \frac{1}{2L}\big[\,(IG^{\mu\nu} - R\tilde{G}^{\mu\nu})\mathcal{D}_\nu L + (I\tilde{G}^{\mu\nu} + RG^{\mu\nu})W_\nu\\ &\quad - |Z|^2UW^\mu + 2\mathrm{i}\,\psi^i\sigma^{\mu\nu}\psi_i\,\partial_\nu Z - 2\mathrm{i}Z\,\psi^i\mathcal{D}^\mu\psi_i\\ &\quad - 2\mathrm{i}\,\lambda^i\sigma^{\mu\nu}\psi_i(W_\nu - \mathrm{i}\mathcal{D}_\nu L) + \tfrac{2}{3}\mathrm{i}Z\,\psi_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} + \text{c.c.}\,\big] .\end{aligned} \tag{4.17}$$

When inserted into eq. (BI.2), only a few terms survive,

$$\begin{aligned}I\,\mathcal{D}_\nu\tilde{G}^{\mu\nu} + R\,\mathcal{D}_\nu G^{\mu\nu} &= -\frac{1}{2}\,U\bar{Z}\partial^\mu Z - \frac{1}{2}\,\bar{Z}\,\lambda^i\sigma^\mu\delta_z\bar{\psi}_i - \frac{\mathrm{i}}{12}\,Z\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_j M^{ij} + \text{c.c.}\\ &= -\tilde{G}^{\mu\nu}\partial_\nu I - G^{\mu\nu}\partial_\nu R - \mathcal{D}_\nu(LF^{\mu\nu} + \mathrm{i}\lambda_i\sigma^{\mu\nu}\psi^i - \mathrm{i}\bar{\psi}_i\bar{\sigma}_{\mu\nu}\bar{\lambda}^i)\\ &\quad - \tilde{F}^{\mu\nu}W_\nu ,\end{aligned}$$

and the second Bianchi identity is found to read

$$\mathcal{D}_\nu\tilde{\mathcal{G}}^{\mu\nu} = -\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho}W_\sigma , \tag{4.18}$$

where we have introduced the abbreviation[1]

$$\mathcal{G}_{\mu\nu} \equiv IG_{\mu\nu} - R\tilde{G}_{\mu\nu} - \tilde{\Sigma}_{\mu\nu} \ . \tag{4.19}$$

A comparison with eq. (3.27) shows that this is exactly the same constraint as for the linear vector-tensor multiplet! This was to be expected, however, for according to eq. (2.36) the action of the central charge generator δ_z on $\mathcal{G}_{\mu\nu}$ depends only on $N^{ij}_{\alpha\dot{\alpha}}$, which we chose to be zero in both cases,

$$\delta_z \tilde{\mathcal{G}}^{\mu\nu} = -\varepsilon^{\mu\nu\rho\sigma}\mathcal{D}_\rho W_\sigma \ . \tag{4.20}$$

Therefore, the second Bianchi identity in the case at hand could have deviated from eq. (3.27) at most by δ_z-invariant terms under the covariant derivative. Due to this correspondence, we can simply copy the solution from section 3.2,

$$\mathcal{G}_{\mu\nu} = V_{\mu\nu} - 2A_{[\mu}W_{\nu]} \ , \tag{4.21}$$

and it is obvious that also the central charge and supersymmetry transformations of the gauge potential V_μ are the same,

$$\delta_z V_\mu = -W_\mu \ , \quad \mathcal{D}^i_\alpha V_\mu = -\big(\mathrm{i}\bar{Z}\sigma_\mu\bar{\psi}^i + \tfrac{1}{2}L\sigma_\mu\bar{\lambda}^i - A_\mu\psi^i\big)_\alpha \ , \tag{4.22}$$

for the second relation follows from the first, which in turn is a consequence of eqs. (4.20) and (4.21).
We observe that the expressions just derived are linear in the components of the vector-tensor multiplet. Nonlinearities enter through the constraint on W^μ, the central charge transformation of which we obtain by multiplying eq. (2.39) with L and inserting the real part of expression (4.17),

$$\begin{aligned}
&L\,\delta_z\big[\,|Z|^2W^\mu + \tfrac{\mathrm{i}}{2}L(Z\partial^\mu\bar{Z} - \bar{Z}\partial^\mu Z) + \tfrac{\mathrm{i}}{2}(Z\psi^i\sigma^\mu\bar{\lambda}_i - \bar{Z}\lambda^i\sigma^\mu\bar{\psi}_i)\,\big] = \\
&\quad = IL\,\mathcal{D}_\nu G^{\mu\nu} - RL\,\mathcal{D}_\nu\tilde{G}^{\mu\nu} + \frac{L}{12}\big[\,Z\,\mathcal{D}_i\sigma^\mu\bar{\mathcal{D}}_jM^{ij} + \text{c.c.}\,\big] \\
&\quad = -|Z|^2UW^\mu + \tilde{\mathcal{G}}^{\mu\nu}W_\nu + \mathcal{D}_\nu(LIG^{\mu\nu} - LR\tilde{G}^{\mu\nu}) + |Z|^2\delta_z(\psi^i\sigma^\mu\bar{\psi}_i) \\
&\qquad - L\,\mathcal{D}_\nu\tilde{\Sigma}^{\mu\nu} - \tfrac{\mathrm{i}}{2}L\,\delta_z(Z\psi^i\sigma^\mu\bar{\lambda}_i - \bar{Z}\lambda^i\sigma^\mu\bar{\psi}_i) - \mathrm{i}U(Z\psi^i\sigma^\mu\bar{\lambda}_i - \bar{Z}\lambda^i\sigma^\mu\bar{\psi}_i) \\
&\qquad - \tfrac{\mathrm{i}}{2}LU(Z\partial^\mu\bar{Z} - \bar{Z}\partial^\mu Z + 2\mathrm{i}\,\lambda^i\sigma^\mu\bar{\lambda}_i) + (\lambda_i\sigma^{\mu\nu}\psi^i + \bar{\psi}_i\bar{\sigma}^{\mu\nu}\bar{\lambda}^i)\,\mathcal{D}_\nu L \\
&\qquad + \mathrm{i}\,\mathcal{D}_\nu(Z\psi^i\sigma^{\mu\nu}\psi_i - \bar{Z}\bar{\psi}^i\bar{\sigma}^{\mu\nu}\bar{\psi}_i) \ .
\end{aligned}$$

Here we have expressed $\bar{\mathcal{D}}_jM^{ij}$ in terms of $\delta_z\bar{\psi}^i$ rather than using eq. (4.16),

$$\tfrac{\mathrm{i}}{3}Z\psi_i\sigma^\mu\bar{\mathcal{D}}_jM^{ij} = Z\psi_i\sigma^\mu(\mathrm{i}\bar{\lambda}^iU - \mathrm{i}\bar{\sigma}^\nu\mathcal{D}_\nu\psi^i - \bar{Z}\delta_z\bar{\psi}^i) \ .$$

The above equation can now be written as

$$\delta_z\mathcal{W}^\mu = \mathcal{D}_\nu(L\mathcal{G}^{\mu\nu} + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu} + \Pi^{\mu\nu}) + \tilde{\mathcal{G}}^{\mu\nu}W_\nu \ , \tag{4.23}$$

[1] This we could have done already in section 2.2, where the combination occured for the first time. However, it is only now that equations simplify considerably when formulated in terms of $\mathcal{G}_{\mu\nu}$.

where the real composite fields

$$\begin{aligned}\mathcal{W}^\mu \equiv\ & |Z|^2(LW^\mu - \psi^i\sigma^\mu\bar\psi_i) + \tfrac{\mathrm{i}}{2}L^2(Z\partial^\mu\bar Z - \bar Z\partial^\mu Z + \mathrm{i}\lambda^i\sigma^\mu\bar\lambda_i)\\ & + \mathrm{i}L\,(Z\psi^i\sigma^\mu\bar\lambda_i - \bar Z\lambda^i\sigma^\mu\bar\psi_i)\end{aligned} \tag{4.24}$$

$$\Pi^{\mu\nu} \equiv \mathrm{i}(Z\psi^i\sigma^{\mu\nu}\psi_i - \bar Z\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i) \tag{4.25}$$

are both bilinear in the components of the vector-tensor multiplet. In view of eq. (2.56) one might search for a superfield redefintion which simplifies $\mathcal{W}^\mu$. In the limit $Z = \mathrm{i}$ we would have

$$LW^\mu - \psi^i\sigma^\mu\bar\psi_i = LL'\,\hat W^\mu - (L'^2 + \tfrac12 LL'')\,\hat\psi^i\sigma^\mu\bar{\hat\psi}_i\ ,$$

so the spinors could be removed indeed (with $L \sim \hat L^{1/3}$). However, this would merely shift complications from one place to another, as both $\mathcal{W}^\mu$ and W^μ occur in the Bianchi identities and their solutions. Hence, we stick to our original choice (4.13) for the constraints.

It is quite an effort to derive the first Bianchi identity from eq. (BI.1). Applying $\bar{\mathcal{D}}_i^{\dot\alpha}$ to eq. (4.16) gives

$$\begin{aligned}\frac{\mathrm{i}}{6}\,Z\,\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j M^{ij} =\ & \mathrm{i}L\Box Z - 2(W^\mu - \mathrm{i}\mathcal{D}^\mu L)\,\partial_\mu Z - Z\lambda_i\delta_z\psi^i + 2\mathrm{i}\,\partial_\mu\lambda^i\sigma^\mu\bar\psi_i\\ & - \frac{1}{L}\Big[\,\tfrac{\mathrm{i}}{2}Z(W_\mu - \mathrm{i}\mathcal{D}_\mu L)^2 + \tfrac{\mathrm{i}}{4}\bar Z G^{\mu\nu}(G_{\mu\nu} + \mathrm{i}\tilde G_{\mu\nu}) + \tfrac{\mathrm{i}}{2}Z|Z|^2U^2\\ & \quad + \lambda^i\sigma^\mu\bar\psi_i\,(W_\mu - \mathrm{i}\mathcal{D}_\mu L) + \mathrm{i}F_{\mu\nu}\,\bar\psi^i\bar\sigma^{\mu\nu}\bar\psi_i - Z\,\mathcal{D}_\mu(\psi^i\sigma^a\bar\psi_i)\\ & \quad - 2\,\psi^i\sigma^\mu\bar\psi_i\,\partial_\mu Z + ZU\psi^i\lambda_i - \mathrm{i}Y^{ij}\bar\psi_i\bar\psi_j - \tfrac{\mathrm{i}}{4}\bar Z\bar M_{ij}\bar M^{ij}\\ & \quad + \tfrac13 Z\,(\psi_i\mathcal{D}_j\bar M^{ij} - \bar\psi_i\bar{\mathcal{D}}_j M^{ij})\,\Big]\ .\end{aligned} \tag{4.26}$$

This we insert into eq. (BI.1) and multiply the equation with $|Z|^2L$, upon which the covariant divergence of $\mathcal{W}^\mu$ emerges,

$$\begin{aligned}\mathcal{D}_\mu\mathcal{W}^\mu =\ & -\tfrac14(IG_{\mu\nu} - R\tilde G_{\mu\nu})(I\tilde G^{\mu\nu} + RG^{\mu\nu}) - \tfrac12 G_{\mu\nu}(Z\,\lambda_i\sigma^{\mu\nu}\psi^i + \bar Z\,\bar\psi_i\bar\sigma^{\mu\nu}\bar\lambda^i)\\ & + \tfrac12 F_{\mu\nu}\Pi^{\mu\nu} - \tfrac{\mathrm{i}}{2}\big[\,\tfrac14 Z(ZM_{ij} - 4\mathrm{i}\,\lambda_i\psi_j)M^{ij} + (ZY_{ij} - \lambda_i\lambda_j)\psi^i\psi^j\\ & \qquad - \lambda_i\psi_j\,\lambda^i\psi^j - \text{c.c.}\,\big]\ .\end{aligned}$$

By virtue of eqs. (A.35) and (A.25) the four-fermion terms that remain when M^{ij} is inserted can be written as the product of an antisymmetric tensor with its dual,

$$\begin{aligned}&\tfrac14 Z(ZM_{ij} - 4\mathrm{i}\,\lambda_i\psi_j)M^{ij} + (ZY_{ij} - \lambda_i\lambda_j)\psi^i\psi^j - \lambda_i\psi_j\,\lambda^i\psi^j - \text{c.c.} =\\ &= -\tfrac12(\lambda_i\lambda_j\,\psi^i\psi^j + \lambda_i\psi_j\,\lambda^j\psi^i) - \text{c.c.}\\ &= -\tfrac{\mathrm{i}}{4}\varepsilon^{\mu\nu\rho\sigma}\,(\lambda_i\sigma_{\mu\nu}\psi^i - \bar\psi_i\bar\sigma_{\mu\nu}\bar\lambda^i)\,(\lambda_j\sigma_{\rho\sigma}\psi^j - \bar\psi_j\bar\sigma_{\rho\sigma}\bar\lambda^j)\\ &= \tfrac{\mathrm{i}}{2}(\Sigma_{\mu\nu} - LF_{\mu\nu})\,(\tilde\Sigma^{\mu\nu} - L\tilde F^{\mu\nu})\ ,\end{aligned}$$

which together with the relation

$$-\tfrac12 G_{\mu\nu}(Z\,\lambda_i\sigma^{\mu\nu}\psi^i + \bar Z\,\bar\psi_i\bar\sigma^{\mu\nu}\bar\lambda^i) = -\tfrac12(IG_{\mu\nu} - R\tilde G_{\mu\nu})\,(\Sigma^{\mu\nu} - LF^{\mu\nu})$$

results in

$$\mathcal{D}_\mu \mathcal{W}^\mu = -\tfrac{1}{4}\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu} + \tfrac{1}{2}F_{\mu\nu}(L\mathcal{G}^{\mu\nu} + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu} + \Pi^{\mu\nu}) \ . \tag{4.27}$$

Let us now split the covariant derivative and express $\delta_z\mathcal{W}^\mu$ by means of eq. (4.23),

$$\begin{aligned}
\partial_\mu \mathcal{W}^\mu &= \mathcal{D}_\mu \mathcal{W}^\mu - A_\mu \delta_z \mathcal{W}^\mu \\
&= (\tfrac{1}{2}F_{\mu\nu} - A_\mu\partial_\nu)\,(L\mathcal{G}^{\mu\nu} + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu} + \Pi^{\mu\nu}) - \tfrac{1}{4}(\mathcal{G}_{\mu\nu} + 4A_\mu W_\nu)\tilde{\mathcal{G}}^{\mu\nu} \\
&= \partial_\mu(L\mathcal{G}^{\mu\nu}A_\nu + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu}A_\nu + \Pi^{\mu\nu}A_\nu) - \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu V_\nu + A_\mu W_\nu)\,(\partial_\rho V_\sigma - A_\rho W_\sigma) \\
&= \partial_\mu(L\mathcal{G}^{\mu\nu}A_\nu + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu}A_\nu + \Pi^{\mu\nu}A_\nu - \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}V_\nu\partial_\rho V_\sigma) \ .
\end{aligned}$$

Here we have inserted the solution for $\mathcal{G}_{\mu\nu}$ in the third step. The first Bianchi identity may thus be solved in terms of an antisymmetric tensor gauge field $B_{\mu\nu}$,

$$\mathcal{W}^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu B_{\rho\sigma} - V_\nu\partial_\rho V_\sigma) + (L\mathcal{G}^{\mu\nu} + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu} + \Pi^{\mu\nu})A_\nu \ , \tag{4.28}$$

which proves that the constraints (4.13), and their flat limit (2.68) in particular, are indeed consistent. We observe that they give rise to abelian Chern-Simons terms both for the gauge potential V_μ of the vector-tensor multiplet and for the vector field A_μ associated with the central charge.

We are interested in W^μ rather than in $\mathcal{W}^\mu$, of course, as the former determines the central charge transformation of V_μ (and also the one of $B_{\mu\nu}$, see below), while the latter had been introduced merely as an auxiliary means to simplify our calculations. When $\mathcal{G}_{\mu\nu}$ is replaced in eq. (4.28), we find that the prefactors of W^μ combine into the matrix $K^{\mu\nu}$ given in eq. (3.37) just as for the linear vector-tensor multiplet,

$$LK^{\mu\nu}W_\nu = \mathcal{H}^\mu - \tfrac{1}{2}\tilde{V}^{\mu\nu}V_\nu + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu}A_\nu + (LV^{\mu\nu} + \Pi^{\mu\nu})A_\nu \ , \tag{4.29}$$

where $\mathcal{H}^\mu$ is now defined by

$$\begin{aligned}
\mathcal{H}^\mu \equiv{}& \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} + |Z|^2\psi^i\sigma^\mu\bar{\psi}_i - \mathrm{i}L\,(Z\psi^i\sigma^\mu\bar{\lambda}_i - \bar{Z}\lambda^i\sigma^\mu\bar{\psi}_i) \\
&- \tfrac{\mathrm{i}}{2}L^2(Z\partial^\mu\bar{Z} - \bar{Z}\partial^\mu Z + \mathrm{i}\lambda^i\sigma^\mu\bar{\lambda}_i) \ .
\end{aligned} \tag{4.30}$$

Inverting $K^{\mu\nu}$ then yields

$$\begin{aligned}
W^\mu = \frac{1}{L\mathcal{E}}\big(&\mathcal{H}^\mu - \tfrac{1}{2}\tilde{V}^{\mu\nu}V_\nu + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu}A_\nu + (LV^{\mu\nu} + \Pi^{\mu\nu})A_\nu \\
&- |Z|^{-2}A^\mu A_\nu\mathcal{H}^\nu + \tfrac{1}{2}|Z|^{-2}A^\mu\tilde{V}^{\nu\rho}A_\nu V_\rho\big) \ ,
\end{aligned} \tag{4.31}$$

with $\mathcal{E}$ as in eq. (1.55). It remains to determine the transformations of the gauge field $B_{\mu\nu}$. The central charge transformation we obtain most easily by applying $\Delta^z(C)$ to eq. (4.28) and comparing the result with $C\delta_z\mathcal{W}^\mu$ as it follows from eq. (4.23). This gives

$$\begin{aligned}
0 ={}& \varepsilon^{\mu\nu\rho\sigma}\partial_\nu\big[\,\Delta^z(C)B_{\rho\sigma} + C(L\tilde{\mathcal{G}}_{\rho\sigma} - \tfrac{1}{2}L^2F_{\rho\sigma} + \tilde{\Pi}_{\rho\sigma} - V_\rho W_\sigma)\,\big] \\
&- \varepsilon^{\mu\nu\rho\sigma}C\,W_\nu(\mathcal{G}_{\rho\sigma} - V_{\rho\sigma}) \ ,
\end{aligned}$$

and as the terms proportional to C vanish by virtue of the antisymmetry of the ε-tensor, we conclude that

$$\delta_z B_{\mu\nu} = V_{[\mu}W_{\nu]} - L\,\varepsilon_{\mu\nu\rho\sigma}(\partial^\rho V^\sigma - A^\rho W^\sigma) + \tfrac{1}{2}L^2 F_{\mu\nu} - \tilde{\Pi}_{\mu\nu} \ . \tag{4.32}$$

As we had seen in section 3.4, the occurence of a Chern-Simons term in the generalized field strength of $B_{\mu\nu}$ requires the gauge transformation associated with the corresponding vector field to act nontrivially on $B_{\mu\nu}$ in order to render the field strength gauge invariant. According to eq. (4.28), the change of V_μ by the gradient of some scalar field Θ is to be accompanied by the transformation

$$\Delta^V(\Theta)\, B_{\mu\nu} = -\tfrac{1}{2}\Theta V_{\mu\nu} \ . \tag{4.33}$$

At last, the supersymmetry transformation of $B_{\mu\nu}$ follows from the one of $\delta_z B_{\mu\nu}$,

$$\begin{aligned}
\delta_z \mathcal{D}^i_\alpha B_{\mu\nu} &= \mathcal{D}^i_\alpha\big(V_{[\mu}W_{\nu]} - L\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{D}^\rho V^\sigma + \tfrac{1}{2}L^2F_{\mu\nu} - \tilde{\Pi}_{\mu\nu}\big) + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,B_{\mu\nu}\\
&= \delta_z\big[\,V_{[\mu}\sigma_{\nu]}\,(\mathrm{i}\bar{Z}\bar{\psi}^i + \tfrac{1}{2}L\bar{\lambda}^i) - V_{[\mu}A_{\nu]}\,\psi^i - \bar{Z}L\sigma_{\mu\nu}\,(2\mathrm{i}\,Z\psi^i - L\lambda^i)\\
&\qquad - \mathrm{i}LA_{[\mu}\sigma_{\nu]}\,(2\mathrm{i}\,\bar{Z}\bar{\psi}^i + \tfrac{1}{2}L\bar{\lambda}^i)\,\big]_\alpha + \partial_{[\mu}\big(V_{\nu]}\psi^i + 2\bar{Z}L\sigma_{\nu]}\bar{\psi}^i - \tfrac{\mathrm{i}}{2}L^2\sigma_{\nu]}\bar{\lambda}^i\big)_\alpha\\
&\quad + \tfrac{1}{2}\psi^i_\alpha V_{\mu\nu} + [\,\delta_z\,,\,\mathcal{D}^i_\alpha\,]\,B_{\mu\nu} \ .
\end{aligned}$$

From this we infer that (modulo δ_z-invariant terms, which can be neglected however)

$$\begin{aligned}
\mathcal{D}^i_\alpha B_{\mu\nu} = \big[\,&V_{[\mu}\sigma_{\nu]}\,(\mathrm{i}\bar{Z}\bar{\psi}^i + \tfrac{1}{2}L\bar{\lambda}^i) - V_{[\mu}A_{\nu]}\,\psi^i - \bar{Z}L\sigma_{\mu\nu}\,(2\mathrm{i}\,Z\psi^i - L\lambda^i)\\
&- \mathrm{i}LA_{[\mu}\sigma_{\nu]}\,(2\mathrm{i}\,\bar{Z}\bar{\psi}^i + \tfrac{1}{2}L\bar{\lambda}^i)\,\big]_\alpha \ ,
\end{aligned} \tag{4.34}$$

and it is easily verified that on all the component fields supersymmetry and central charge transformations commute modulo gauge transformations,

$$[\,\Delta^z(C)\,,\,\Delta(\xi)\,] = \Delta^V(\Theta) + \Delta^B(\Omega) \ ,$$

where Δ^V now acts on both V_μ and $B_{\mu\nu}$. Here the parameters read explicitly

$$\Theta = C(\xi_i\psi^i + \bar{\xi}^i\bar{\psi}_i) \ , \quad \Omega_\mu = C\,\mathrm{Re}(V_\mu\xi_i\psi^i + 2\bar{Z}L\,\xi_i\sigma_\mu\bar{\psi}^i - \tfrac{\mathrm{i}}{2}L^2\,\xi_i\sigma_\mu\bar{\lambda}^i) \ . \tag{4.35}$$

Finally, a straightforward though tedious computation of the supersymmetry commutation relations on $B_{\mu\nu}$ results in

$$\begin{aligned}
\{\mathcal{D}^i_\alpha\,,\,\mathcal{D}^j_\beta\}\,B_{\mu\nu} &= \varepsilon_{\alpha\beta}\,\varepsilon^{ij}\big(\bar{Z}\delta_z B_{\mu\nu} - \partial_{[\mu}(\mathrm{i}V_{\nu]}\bar{Z}L + A_{\nu]}\bar{Z}L^2) - \tfrac{\mathrm{i}}{2}\bar{Z}LV_{\mu\nu}\big)\\
\{\mathcal{D}^i_\alpha\,,\,\bar{\mathcal{D}}_{\dot{\alpha}j}\}\,B_{\mu\nu} &= -\mathrm{i}\delta^i_j\big(\mathcal{D}_{\alpha\dot{\alpha}}B_{\mu\nu} + 2\,\partial_{[\mu}(B_{\nu]\rho} - \tfrac{1}{2}\eta_{\nu]\rho}|Z|^2L^2)\sigma^\rho_{\alpha\dot{\alpha}} - \tfrac{1}{2}V_{\alpha\dot{\alpha}}V_{\mu\nu}\big) \ ,
\end{aligned} \tag{4.36}$$

which implies that the parameters ϵ^μ, C and Θ on the right-hand side of the equation

$$[\,\Delta(\xi)\,,\,\Delta(\zeta)\,] = \epsilon^\mu\partial_\mu + \Delta^z(C) + \Delta^V(\Theta) + \Delta^B(\Omega)$$

coincide with those for the linear vector-tensor multiplet, eqs. (3.47), while Ω_μ reads

$$\Omega_\mu = \tfrac{1}{2}\epsilon_\mu|Z|^2L^2 - B_{\mu\nu}\epsilon^\nu - \tfrac{\mathrm{i}}{2}V_\mu L(\xi_i\zeta^i\bar{Z} + \bar{\xi}^i\bar{\zeta}_i Z) - \tfrac{1}{2}A_\mu L^2(\xi_i\zeta^i\bar{Z} - \bar{\xi}^i\bar{\zeta}_i Z) \ . \tag{4.37}$$

4.3 The Lagrangian

Now everything is set to determine the invariant action, which we again derive from a linear superfield that is the solution to the differential equations (2.59). When the coefficient functions (4.11) (with $g = 0$) are inserted, they read

$$
\begin{array}{llll}
1) & 0 = \partial_L\gamma - \tfrac{1}{2}\beta + \dfrac{L}{2Z}\,\alpha & 2) & 0 = \partial\gamma - \delta + \dfrac{L}{4Z}\,\beta \\
3) & 0 = \partial_L\bar{\alpha} - \dfrac{\bar{Z}}{ZL}\,\alpha & 4) & 0 = \partial\bar{\alpha} - \dfrac{\bar{Z}}{2ZL}\,\beta \\
5) & 0 = \partial\alpha - \tfrac{1}{2}\partial_L\beta + \dfrac{\beta}{2L} - \dfrac{\alpha}{Z} & 6) & 0 = \partial_L\bar{\beta} \\
7) & 0 = \partial\bar{\beta} & 8) & 0 = \partial\beta - 2\partial_L\delta + \dfrac{\beta}{Z} \\
9) & 0 = \partial_L\bar{\delta} & 10) & 0 = \partial\bar{\delta}\ .
\end{array}
$$

From eqs. 6), 7) and 9), 10) we infer that $\beta = \beta(Z)$ and $\delta = \delta(Z)$, respectively. β is then fully determined through eq. 8),

$$\partial\beta = -\frac{\beta}{Z} \quad \Rightarrow \quad \beta = -\frac{2\kappa}{Z}\ , \quad \kappa \in \mathbb{C}\ . \tag{4.38}$$

Eq. 5) now fixes the Z-dependence of α,

$$\partial\alpha = \frac{1}{Z}\left(\alpha + \frac{\kappa}{L}\right) \quad \Rightarrow \quad \alpha = Z\,h(\bar{Z}, L) - \frac{\kappa}{L}\ ,$$

which we insert into eq. 4) to obtain a condition on the function h,

$$\bar{\partial}h = -\frac{\bar{\kappa}}{L\bar{Z}^2} \quad \Rightarrow \quad h = k(L) + \frac{\bar{\kappa}}{L\bar{Z}}\ .$$

Eq. 3) holds if

$$L\partial_L k = \bar{k} \quad \Rightarrow \quad k = \mathrm{i}\frac{\mu}{L} - \varrho L\ , \quad \mu, \varrho \in \mathbb{R}\ .$$

Eq. 1) requires $\mu = 0$ due to the reality of γ, so α finally reads

$$\alpha = \frac{\bar{\kappa}Z - \kappa\bar{Z}}{L\bar{Z}} - \varrho ZL\ . \tag{4.39}$$

We can easily integrate eq. 1) and obtain

$$\gamma = \frac{1}{6}\,\varrho L^3 - \frac{\bar{\kappa}Z + \kappa\bar{Z}}{2|Z|^2}\,L + \sigma(Z, \bar{Z})\ , \quad \sigma \text{ real}\ . \tag{4.40}$$

Eventually, eq. 2) yields the same relation between δ and σ as in the case of the linear vector-tensor multiplet,

$$\delta = \partial\sigma \quad \Rightarrow \quad \sigma = f(Z) + \bar{f}(\bar{Z})\ .$$

This of course is not surprising as these functions determine the Lagrangian for the central charge vector multiplet, which does not depend at all on the constraints on L. Again similar to the case of the linear vector-tensor multiplet, the terms proportional to κ combine into the real part of $\kappa\,\mathcal{D}^{(i}\mathcal{D}^{j)}L$,

$$\begin{aligned}&\frac{\bar\kappa Z-\kappa\bar Z}{L}\Big(\frac{1}{\bar Z}\,\mathcal{D}^iL\,\mathcal{D}^jL-\frac{1}{Z}\,\bar{\mathcal{D}}^iL\,\bar{\mathcal{D}}^jL\Big)-\frac{2\kappa}{Z}\,\mathcal{D}^{(i}Z\,\mathcal{D}^{j)}L-\frac{2\bar\kappa}{\bar Z}\,\bar{\mathcal{D}}^{(i}\bar Z\,\bar{\mathcal{D}}^{j)}L\\&-\frac{\bar\kappa Z+\kappa\bar Z}{2|Z|^2}\,L\,\mathcal{D}^i\mathcal{D}^jZ=\kappa\,\mathcal{D}^{(i}\mathcal{D}^{j)}L+\bar\kappa\,\bar{\mathcal{D}}^{(i}\bar{\mathcal{D}}^{j)}L\ ,\end{aligned}$$

which is a linear superfield by itself. It has been shown in [2], however, that it gives rise only to a total derivative Lagrangian for any value of κ. Therefore we choose $\kappa=0$. Altogether the pre-Lagrangian for the nonlinear vector-tensor multiplet with gauged central charge reads

$$\mathcal{L}^{ij}=\mathcal{L}^{ij}_{\text{nlinVT}}+\mathcal{L}^{ij}_{\text{cc}}\ ,\tag{4.41}$$

where the remaining parameter in

$$\mathcal{L}^{ij}_{\text{nlinVT}}=-\varrho L\left(Z\,\mathcal{D}^iL\,\mathcal{D}^jL+\bar Z\,\bar{\mathcal{D}}^iL\,\bar{\mathcal{D}}^jL-\tfrac16L^2\,\mathcal{D}^i\mathcal{D}^jZ\right)\tag{4.42}$$

may be chosen as $\varrho=1/\langle L\rangle$, which turns out to yield the proper normalization of the kinetic terms.
For the sake of simplicity we confine ourselves to the purely bosonic part of the action. According to the general prescription (1.42) for the Lagrangian we need to compute the second supersymmetry variations of $\mathcal{L}^{ij}_{\text{nlinVT}}$. When acting with $\mathcal{D}_{\alpha j}$ on this field,

$$\begin{aligned}\mathcal{D}_{\alpha j}\mathcal{L}^{ij}_{\text{nlinVT}}=\frac{3\varrho}{2}\big[\,&\bar ZL\,(\mathcal{D}_\mu L-\mathrm{i}W_\mu)\,\sigma^\mu\bar\psi^i+\mathrm{i}ZL\,G_{\mu\nu}\sigma^{\mu\nu}\psi^i-\mathrm{i}|Z|^2LU\psi^i\\&+\tfrac{\mathrm{i}}{3}L^3\sigma^\mu\partial_\mu\bar\lambda^i-\mathrm{i}L^2\,Y^{ij}\psi_j-\mathrm{i}\bar Z\bar\psi^i\bar\psi^j\,\psi_j-\tfrac{\mathrm{i}}{3}Z(\psi^i\psi^j)\,\psi_j\\&+\tfrac12L(\psi^i\psi^j)\,\lambda_j-\tfrac12L(\lambda^i\psi^j)\,\psi_j\,\big]_\alpha\ ,\end{aligned}\tag{4.43}$$

the fermion trilinears can be neglected. We now need to apply $\mathcal{D}^\alpha_i$ and $\bar{\mathcal{D}}_{\dot\alpha i}$ only to the remaining spinors, which results for the former in (the relation $\simeq$ denotes omission of fermions)

$$\begin{aligned}\mathcal{D}_i\mathcal{D}_j\mathcal{L}^{ij}_{\text{nlinVT}}\simeq\frac{3\varrho}{2Z}\big[\,&2|Z|^2L\left(\mathcal{D}^\mu L\,\mathcal{D}_\mu L-W^\mu W_\mu-2\mathrm{i}\,W^\mu\mathcal{D}_\mu L+|Z|^2U^2\right)\\&-Z^2L\,G^{\mu\nu}(\mathrm{i}\tilde G_{\mu\nu}-G_{\mu\nu})-\tfrac43L^3Z\Box\bar Z-L^3\,Y^{ij}Y_{ij}\,\big]\ ,\end{aligned}\tag{4.44}$$

while the latter yields the expression

$$\begin{aligned}\bar{\mathcal{D}}_{\dot\alpha i}\mathcal{D}_{\alpha j}\mathcal{L}^{ij}_{\text{nlinVT}}\simeq-3\mathrm{i}\varrho L\,\sigma^\mu_{\alpha\dot\alpha}\big[\,&(IG_{\mu\nu}-R\tilde G_{\mu\nu})W^\nu-(I\tilde G_{\mu\nu}+RG_{\mu\nu})\mathcal{D}^\nu L\\&+|Z|^2U\mathcal{D}_\mu L+\tfrac13L^2\partial^\nu F_{\mu\nu}\,\big]\ .\end{aligned}\tag{4.45}$$

Putting it all together, we find

$$\begin{aligned}\mathcal{L}_{\text{nlinVT}} \simeq \varrho L\Big[&\frac{1}{2}\,|Z|^2\big(\partial^\mu L\,\partial_\mu L - W^\mu W_\mu + EU^2\big) - \frac{1}{6}\,L^2(Z\Box\bar{Z} + \bar{Z}\Box Z)\\ &- \frac{1}{4}\,\mathcal{G}^{\mu\nu}\big(\mathcal{G}_{\mu\nu} + 4A_\mu W_\nu + 2\varepsilon_{\mu\nu\rho\sigma}\,\partial^\rho(LA^\sigma)\big) + LW_\mu\tilde{F}^{\mu\nu}A_\nu\\ &+ \frac{1}{12}\,L^2F^{\mu\nu}F_{\mu\nu} - \frac{1}{12}\,L^2\,Y^{ij}Y_{ij}\Big] + \frac{\varrho}{3}\,\partial_\mu\,(L^3F^{\mu\nu}A_\nu)\ ,\end{aligned} \tag{4.46}$$

where for consistency the fermions contained in the composite fields W^μ and $\mathcal{G}_{\mu\nu}$ have to be set to zero. The latter depends on the former according to eq. (4.21), so we first replace $\mathcal{G}_{\mu\nu}$,

$$\begin{aligned}&-\frac{1}{4}\,\mathcal{G}^{\mu\nu}\big(\mathcal{G}_{\mu\nu} + 4A_\mu W_\nu + 2\varepsilon_{\mu\nu\rho\sigma}\,\partial^\rho(LA^\sigma)\big) =\\ &= -\frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \tilde{V}^{\mu\nu}\partial_\mu(LA_\nu) - LW_\mu\tilde{F}^{\mu\nu}A_\nu + A^\mu W^\nu A_{[\mu}W_{\nu]}\ .\end{aligned}$$

Then the terms linear in W^μ cancel, while the bilinear ones combine into $W^\mu K_{\mu\nu}W^\nu$ just as for the linear vector-tensor multiplet,

$$\begin{aligned}\mathcal{L}_{\text{nlinVT}} \simeq \varrho L\Big[&\frac{1}{2}\,|Z|^2\big(\partial^\mu L\,\partial_\mu L + EU^2\big) - \frac{1}{2}\,W^\mu K_{\mu\nu}W^\nu - \frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \frac{1}{4}\,L\tilde{F}^{\mu\nu}V_{\mu\nu}\\ &+ \frac{1}{12}\,L^2F^{\mu\nu}F_{\mu\nu} - \frac{1}{6}\,L^2(Z\Box\bar{Z} + \bar{Z}\Box Z) - \frac{1}{12}\,L^2\,Y^{ij}Y_{ij}\Big]\\ &+ \frac{\varrho}{6}\,\partial_\mu\,\big(2L^3F^{\mu\nu}A_\nu - 3L^2\tilde{V}^{\mu\nu}A_\nu\big)\ .\end{aligned} \tag{4.47}$$

Again the nonpolynomial interactions arise from inverting $K_{\mu\nu}$. Using eqs. (3.38) and (4.29), the substitution of W^μ gives

$$\begin{aligned}-\frac{\varrho}{2}\,L\,W^\mu K_{\mu\nu}W^\nu &= -\frac{\varrho}{2L}\,(LK^{\mu\rho}W_\rho)\,(K^{-1})_{\mu\nu}\,(LK^{\nu\sigma}W_\sigma)\\ &= -\frac{\varrho}{2L\mathcal{E}}\,\big[\,\mathcal{H}^\mu - \tfrac{1}{2}\tilde{V}^{\mu\nu}V_\nu + \tfrac{1}{2}L^2\tilde{F}^{\mu\nu}A_\nu + (LV^{\mu\nu} + \Pi^{\mu\nu})A_\nu\,\big]^2\\ &\quad + \frac{\varrho}{2L\mathcal{E}|Z|^2}\,\big(A_\mu\mathcal{H}^\mu - \tfrac{1}{2}A_\mu\tilde{V}^{\mu\nu}V_\nu\big)^2\ .\end{aligned} \tag{4.48}$$

At last, let us neglect also fluctuations of the scalars around their background values $\langle Z\rangle = \mathrm{i}$ and $\langle L\rangle = 1/\varrho$. Then only the gauge potentials V_μ, $B_{\mu\nu}$ and A_μ remain, and after rescaling

$$A_\mu \to g_zA_\mu\ ,\quad B_{\mu\nu} \to B_{\mu\nu}/\varrho\ ,$$

such that both fields have canonical dimension 1, we find[2] (dropping a total derivative)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\,V^{\mu\nu}V_{\mu\nu} - \frac{1}{4}\,\big(1 - g_z^2/3\varrho^2\big)F^{\mu\nu}F_{\mu\nu}\\ &- \frac{1}{2\mathcal{E}}\,\big(H^\mu - \tfrac{1}{2}\varrho\tilde{V}^{\mu\nu}V_\nu + g_zV^{\mu\nu}A_\nu + \tfrac{1}{2}\varrho^{-1}g_z^2\tilde{F}^{\mu\nu}A_\nu\big)^2\\ &+ \frac{g_z^2}{2\mathcal{E}}\,\big(A_\mu H^\mu - \tfrac{1}{2}\varrho A_\mu\tilde{V}^{\mu\nu}V_\nu\big)^2\ .\end{aligned} \tag{4.49}$$

[2]Evidently, positivity of the kinetic energies requires $3\varrho^2 > g_z^2$.

Apart from the normalisation of A_μ, this Lagrangian follows from the one in eq. (3.64), when in the latter we make the substitution

$$H^\mu \to H^\mu - \tfrac{1}{2}\varrho\tilde{V}^{\mu\nu}V_\nu + \tfrac{1}{2}\varrho^{-1}g_z^2\tilde{F}^{\mu\nu}A_\nu \ , \tag{4.50}$$

which introduces couplings of H^a to Chern-Simons terms of both V_μ and A_μ. At first glance, these seem to fit into the structure of the Henneaux-Knaepen models, eq. (3.103). However, it is easily verified that no choice of the parameters c_{ab}, which govern the Chern-Simons couplings, can result in the specific combination

$$\mathcal{H}^\mu = H^\mu - \tfrac{1}{2}\varrho\tilde{V}^{\mu\nu}V_\nu + \tfrac{1}{2}\varrho^{-1}g_z^2\tilde{F}^{\mu\nu}A_\nu + g_z V^{\mu\nu}A_\nu \tag{4.51}$$

as displayed above: A comparison with eq. (3.103) shows that again the second column of the matrix $T_{1\,b}^{a}$ vanishes (otherwise at least one of the two terms $V^{\mu\nu}V_\nu$, $F^{\mu\nu}V_\nu$ would be present), hence the coefficients $c_{ab}T_{1\,c}^{a}$ can at most yield a Chern-Simons term for $A_\mu^1 = A_\mu$.
We conclude that the Henneaux-Knaepen models, in their current formulation, cannot account for the type of gauge field interactions described by the nonlinear vector-tensor multiplet with local central charge. Most likely, however, the former admit a generalization which then includes also the case presented here. Work in this direction is in progress, but as yet we cannot report on results.

Conclusions and Outlook

In the present thesis we have given a derivation of the superfield constraints which describe the two versions of the vector-tensor multiplet in presence of a gauged central charge. Key to this was the formulation of consistency conditions every deformation of the free model has to meet. We stress that these may be, and have been to a certain extent in [2], employed to determine superfield constraints that yield even more general models than the ones presented here, like for instance the linear vector-tensor multiplet with global scale and chiral invariance first obtained in [17] by means of the superconformal multiplet calculus. This involves a coupling to another abelian vector multiplet with a nonvanishing background value (the nonlinear one with gauged central charge possesses these invariances without further modifications).

Even in the case of a single vector multiplet, however, the consistency conditions turned out to be insufficient when starting from a completely general Ansatz for the constraints. While we were able to find solutions to the differential equations on the coefficients that provide the sought generalizations of the two different vector-tensor multiplets, we cannot exclude further solutions which may not be obtained from the known ones merely by a field redefintion. However, what has been shown is that each solution must reduce in the limit $Z = \mathrm{i}$ to either of two possible versions with global central charge. Since we have found two corresponding classes of deformations, we venture the assertion that no third one exists.

Unfortunately, as yet we do not know how to determine in a manifestly supersymmetric way whether a given set of constraints is really compatible with the supersymmetry algebra. While the consistency conditions (C.1–4) provide a preliminary selection of superfield constraints, it is still necessary in each case to solve the Bianchi identities at the component level in order to verify their validity.

Of course, the ultimate goal is to describe the vector-tensor multiplet in terms of an unconstrained superfield, as it is possible for the hypermultiplet in harmonic superspace at the expense of a finite number of off-shell components [1].

What we consider the most exciting feature of the vector-tensor multiplet (rather than its relevance to certain string theory compactifications, which is beyond the scope of this thesis), is the similarity of its local central charge transformations to the kind of gauge transformations that occur in the new class of theories by Henneaux and Knaepen. It is natural to ask for supersymmetric versions of these models. While this problem could be solved completely for $N = 1$, in the case of two supersymmetries the only known example we have presented here suffers from the explicit nonpolynomial dependence on the central charge gauge field. As is clear in view of the complexity of our component calculations, a first-order superfield formulation is indispensable. It is

likely to exist only in harmonic superspace, which admits unconstrained prepotentials for $N = 2$ vector multiplets that could serve as the necessary auxiliary superfields.
In fact, the superfield constraints we have found in this thesis can readily be converted into constraints on a corresponding harmonic superfield, cf. [2]. However, presumably the 2-forms have to be embedded in other multiplets than the ones of the vector-tensor variety, since the latter always introduce in addition as many vectors as there are tensors, which is likely to prevent a formulation of pure Freedman-Townsend models. The only other multiplet known to include a 2-form gauge field (apart from the yet to be constructed double-tensor multiplet which, as the name suggests, would hardly be an alternative) is the so-called tensor multiplet [7] we have encountered briefly in section 1.2.
Finally, we note that also the purely bosonic Henneaux-Knaepen models deserve further study concerning a possible extension of the Chern-Simons couplings. As has been shown in the last chapter, the gauge field part of the nonlinear vector-tensor multiplet with local central charge hints at a generalization that exceeds the one included already in [28] and in our exposition of the models in section 3.5.

Appendix A
Conventions

A.1 Vectors and Spinors

We denote Lorentz vector indices as usual by small letters from the middle of the greek alphabet, while those from the beginning are reserved for two-component Weyl spinors, which are used exclusively in this thesis. Small letters from the middle of the latin alphabet denote SU(2) spinors in the fundamental representation and run also from 1 to 2.
The signature of the Minkowski metric follows the convention in particle physics,

$$\eta_{\mu\nu} = \mathrm{diag}\,(1,\,-1,\,-1,\,-1)\;. \tag{A.1}$$

Parantheses and square brackets denote symmetrization and antisymmetrization of the enclosed indices respectively,

$$V_{(A_1\ldots A_n)} = \frac{1}{n!}\sum_{\pi\in S_n} V_{A_{\pi(1)}\ldots A_{\pi(n)}} \tag{A.2}$$

$$V_{[A_1\ldots A_n]} = \frac{1}{n!}\sum_{\pi\in S_n} \mathrm{sgn}(\pi)\, V_{A_{\pi(1)}\ldots A_{\pi(n)}}\;, \tag{A.3}$$

where $A \in \{\mu, \alpha, \dot\alpha, i\}$. The Levi-Civita tensor $\varepsilon^{A_1\ldots A_d}$ is antisymmetric upon interchange of any two indices, and the following relations hold,

$$\varepsilon_{A_1\ldots A_d} = \eta_{A_1B_1}\ldots\eta_{A_dB_d}\,\varepsilon^{B_1\ldots B_d}\,, \quad \eta_{AB} = \mathrm{diag}\,(1,\,-1,\ldots,\,-1) \tag{A.4}$$

$$\varepsilon^{0\ldots(d-1)} = 1\,, \quad \varepsilon_{0\ldots(d-1)} = (-)^{d-1} \tag{A.5}$$

$$\varepsilon_{A_1\ldots A_d}\,\varepsilon^{B_1\ldots B_d} = (-)^{d-1} d!\,\delta^{[B_1}_{A_1}\ldots\delta^{B_d]}_{A_d}\,. \tag{A.6}$$

The Hodge dual of an antisymmetric Lorentz tensor is denoted by a tilde,

$$\tilde F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\;. \tag{A.7}$$

Our conventions concerning Weyl spinors agree with those in [29]. Indices are raised and lowered by means of the ε-tensors according to

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\;, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \tag{A.8}$$

$$\bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \ , \quad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \ , \tag{A.9}$$

and the following summation rule is used,

$$\psi\chi = \psi^{\alpha}\chi_{\alpha} \ , \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \ . \tag{A.10}$$

Similarly, an SU(2) spinor may be converted into a spinor transforming in the contragredient representation by means of an ε-tensor, which is invariant under SU(2),

$$\varphi_i = \varepsilon_{ij}\varphi^j \ , \quad \varphi^i = \varepsilon^{ij}\varphi_j \ . \tag{A.11}$$

However, we always spell out SU(2) indices even when contracted. Complex conjugation raises and lowers SU(2) indices. Due to the reality properties of ε^{ij},

$$(\varepsilon^{ij})^* = \varepsilon^{ij} = -\varepsilon_{ij} \ , \tag{A.12}$$

a change of sign has to be taken into account whenever complex conjugation applies also to an implicit ε-tensor,

$$(\varphi^i)^* = \bar{\varphi}_i \qquad \Rightarrow \qquad (\varphi_i)^* = -\bar{\varphi}^i \ . \tag{A.13}$$

A.2 σ-Matrices

The σ-matrices

$$\sigma^{\mu} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \tag{A.14}$$

provide the link between the proper orthochronous Lorentz group and its universal covering $SL(2,\mathbb{C})$. The index structure of these hermitian matrices is

$$\sigma^{\mu}_{\alpha\dot{\beta}} \ , \quad (\sigma^{\mu}_{\alpha\dot{\beta}})^* = \sigma^{\mu}_{\beta\dot{\alpha}} \ , \tag{A.15}$$

and $\bar{\sigma}$-matrices with upper indices are defined by

$$\bar{\sigma}^{\mu\,\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\sigma^{\mu}_{\delta\dot{\gamma}} = (1,\, -\vec{\sigma})^{\dot{\alpha}\beta} \ . \tag{A.16}$$

Lorentz vector indices can be converted into spinor indices and vice versa,

$$V_{\alpha\dot{\beta}} = \sigma^{\mu}_{\alpha\dot{\beta}}V_{\mu} \ , \quad V^{\mu} = \tfrac{1}{2}\bar{\sigma}^{\mu\,\dot{\beta}\alpha}V_{\alpha\dot{\beta}} \ . \tag{A.17}$$

The generators of the Lorentz group in the two inequivalent spinor representations are given by

$$\begin{gathered} \sigma^{\mu\nu} = \tfrac{1}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}) \, , \quad \bar{\sigma}^{\mu\nu} = \tfrac{1}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}) \\ ({\sigma^{\mu\nu}}_{\beta}{}^{\alpha})^* = -{\bar{\sigma}^{\mu\nu\,\dot{\alpha}}}_{\dot{\beta}} \ . \end{gathered} \tag{A.18}$$

We use a shorthand notation for $\sigma^{\mu\nu}$-matrices whose indices have been lowered by means of the ε-tensor,

$$\sigma^{\mu\nu}{}_{\alpha\beta} = -(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\ , \quad \bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta} = (\varepsilon\bar{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta} \tag{A.19}$$

They are symmetric in the spinor indices,

$$\sigma^{\mu\nu}{}_{\alpha\beta} = \sigma^{\mu\nu}{}_{\beta\alpha}\ , \quad \bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta} = \bar{\sigma}^{\mu\nu}{}_{\dot\beta\dot\alpha}\ . \tag{A.20}$$

Using the $\sigma^{\mu\nu}$-matrices, an antisymmetric tensor $F_{\mu\nu}$ can be decomposed into its "self-dual" and "anti-selfdual" part,

$$\begin{aligned} F_{\alpha\dot\alpha\,\beta\dot\beta} = \varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} + \varepsilon_{\alpha\beta}\bar{F}_{\dot\alpha\dot\beta}\ , &\quad \tilde{F}_{\alpha\dot\alpha\,\beta\dot\beta} = \mathrm{i}\,\varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} - \mathrm{i}\,\varepsilon_{\alpha\beta}\bar{F}_{\dot\alpha\dot\beta} \\ F_{\alpha\beta} = -F_{\mu\nu}\,\sigma^{\mu\nu}{}_{\alpha\beta}\ , &\quad \bar{F}_{\dot\alpha\dot\beta} = F_{\mu\nu}\,\bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta}\,. \end{aligned} \tag{A.21}$$

There are numerous relations between the quantities defined so far. The ones used frequently in this thesis shall be listed here.
Identities containing two σ-matrices:

$$\sigma^{\mu}_{\alpha\dot\alpha}\sigma_{\mu\,\beta\dot\beta} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\ , \quad \sigma^{\mu}_{\alpha\dot\alpha}\,\bar{\sigma}^{\dot\beta\beta}_{\mu} = 2\,\delta^{\beta}_{\alpha}\,\delta^{\dot\beta}_{\dot\alpha} \tag{A.22}$$

$$(\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}{}^{\beta} = \eta^{\mu\nu}\,\delta^{\beta}_{\alpha} + 2\,\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\ , \quad (\bar{\sigma}^{\mu}\sigma^{\nu})^{\dot\alpha}{}_{\dot\beta} = \eta^{\mu\nu}\,\delta^{\dot\alpha}_{\dot\beta} + 2\,\bar{\sigma}^{\mu\nu\,\dot\alpha}{}_{\dot\beta} \tag{A.23}$$

$$\sigma^{[\mu}_{\alpha\dot\alpha}\sigma^{\nu]}_{\beta\dot\beta} = \varepsilon_{\alpha\beta}\,\bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,\sigma^{\mu\nu}{}_{\alpha\beta} \tag{A.24}$$

$$\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\mathrm{i}\,\sigma^{\mu\nu}\,, \quad \varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = -2\mathrm{i}\,\bar{\sigma}^{\mu\nu} \tag{A.25}$$

$$\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\,\alpha\dot\alpha}\sigma_{\sigma\,\beta\dot\beta} = -2\mathrm{i}\,(\varepsilon_{\alpha\beta}\,\bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,\sigma^{\mu\nu}{}_{\alpha\beta})\ . \tag{A.26}$$

Identities containing three σ-matrices:

$$\sigma^{\mu\nu}\sigma^{\rho} = \tfrac{1}{2}(\eta^{\nu\rho}\sigma^{\mu} - \eta^{\mu\rho}\sigma^{\nu} + \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\sigma}) \tag{A.27}$$

$$\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho} = \tfrac{1}{2}(\eta^{\nu\rho}\bar{\sigma}^{\mu} - \eta^{\mu\rho}\bar{\sigma}^{\nu} - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\sigma}) \tag{A.28}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu\rho} = \tfrac{1}{2}(\eta^{\mu\nu}\bar{\sigma}^{\rho} - \eta^{\mu\rho}\bar{\sigma}^{\nu} - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\sigma}) \tag{A.29}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu\rho} = \tfrac{1}{2}(\eta^{\mu\nu}\sigma^{\rho} - \eta^{\mu\rho}\sigma^{\nu} + \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\sigma}) \tag{A.30}$$

$$\sigma^{\mu\nu}{}_{\alpha\beta}\,\sigma_{\nu\,\gamma\dot\alpha} = -\varepsilon_{\gamma(\beta}\,\sigma^{\mu}_{\alpha)\dot\alpha}\ , \quad \bar{\sigma}^{\mu\nu}{}_{\dot\alpha\dot\beta}\,\sigma_{\nu\,\alpha\dot\gamma} = -\sigma^{\mu}_{\alpha(\dot\alpha}\,\varepsilon_{\dot\beta)\dot\gamma}\ . \tag{A.31}$$

Identities containing four σ-matrices:

$$\begin{aligned} \sigma^{\mu\nu}\sigma^{\rho\sigma} &= \tfrac{1}{2}(\eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\mu\rho}\sigma^{\nu\sigma} + \eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho}) \\ &\quad + \tfrac{1}{4}(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma} + \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}) \end{aligned} \tag{A.32}$$

$$\begin{aligned} \bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} &= \tfrac{1}{2}(\eta^{\mu\sigma}\bar{\sigma}^{\nu\rho} - \eta^{\mu\rho}\bar{\sigma}^{\nu\sigma} + \eta^{\nu\rho}\bar{\sigma}^{\mu\sigma} - \eta^{\nu\sigma}\bar{\sigma}^{\mu\rho}) \\ &\quad + \tfrac{1}{4}(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma} - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}) \end{aligned} \tag{A.33}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\sigma_{\nu}{}^{\rho}{}_{\gamma}{}^{\delta} = \tfrac{1}{2}(\delta^{\beta}_{\gamma}\sigma^{\mu\rho}{}_{\alpha}{}^{\delta} - \delta^{\delta}_{\alpha}\sigma^{\mu\rho}{}_{\gamma}{}^{\beta}) + \tfrac{1}{4}\eta^{\mu\rho}\,(\varepsilon_{\alpha\gamma}\,\varepsilon^{\beta\delta} + \delta^{\delta}_{\alpha}\,\delta^{\beta}_{\gamma}) \tag{A.34}$$

$$\sigma^{\mu\nu}{}_{\alpha\beta}\,\sigma_{\mu\nu\,\gamma\delta} = -2\,\varepsilon_{\alpha(\gamma}\,\varepsilon_{\delta)\beta}\ , \quad \sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}_{\mu\nu}{}^{\dot{\gamma}}{}_{\dot{\delta}} = 0\ . \tag{A.35}$$

These identities imply among others the following two useful relations: Let $F_{\mu\nu}$, $G_{\mu\nu}$, $H_{\mu\nu}$ be antisymmetric tensors and V_μ, W_μ be vectors. Then one has

$$F_\alpha{}^\beta G_\beta{}^\alpha + \bar{F}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{H}^{\dot{\beta}}{}_{\dot{\alpha}} = \mathrm{i}\tilde{F}^{\mu\nu}(G_{\mu\nu} - H_{\mu\nu}) - F^{\mu\nu}(G_{\mu\nu} + H_{\mu\nu}) \tag{A.36}$$

$$F_\alpha{}^\beta V_{\beta\dot{\alpha}} + W_{\alpha\dot{\beta}}\bar{F}^{\dot{\beta}}{}_{\dot{\alpha}} = \mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\tilde{F}_{\mu\nu}(V^\nu - W^\nu) - \sigma^\mu_{\alpha\dot{\alpha}}F_{\mu\nu}(V^\nu + W^\nu)\ . \tag{A.37}$$

A.3 Multiplet Components

In the course of the present thesis we encounter numerous supersymmetry multiplets. For quick reference we now list their components. As explained in section 1.1 the corresponding superfields are labeled by the same letter as used for the lowest component (if there are several components of the same dimension, the first in the respective list provides the superfield label). Symbols separated by a semicolon denote field strengths. Components of the vector-tensor multiplet:

$$L\ ,\ V_\mu\ ,\ B_{\mu\nu}\ ,\ \psi^i_\alpha\ ,\ U\ ;\ G_{\mu\nu}\ ,\ W^\mu\ ;\ V_{\mu\nu}\ ,\ H^\mu\ .$$

Components of the central charge vector multiplet:

$$Z\ ,\ \bar{Z}\ ,\ A_\mu\ ,\ \lambda^i_\alpha\ ,\ Y^{ij}\ ;\ F_{\mu\nu}\ .$$

Components of additional vector multiplets:

$$\phi^I\ ,\ \bar{\phi}^I\ ,\ \mathcal{A}^I_\mu\ ,\ \chi^I_\alpha\ ,\ D^{ijI}\ ;\ \mathcal{F}^I_{\mu\nu}\ .$$

Components of the linear multiplet:

$$\varphi^{ij}\ (\mathcal{L}^{ij})\ ,\ \varrho^i_\alpha\ ,\ S\ ,\ \bar{S}\ ,\ K^\mu\ .$$

Components of the hypermultiplet:

$$\varphi^i\ ,\ \bar{\varphi}_i\ ,\ \chi_\alpha\ ,\ \bar{\psi}_{\dot{\alpha}}\ ,\ F^i\ ,\ \bar{F}_i\ .$$

References

[1] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469.

[2] N. Dragon, E. Ivanov, S. Kuzenko, E. Sokatchev and U. Theis, Nucl. Phys. B 538 (1999) 411.

[3] R. Haag, J. Łopuszanski and M.F. Sohnius, Nucl. Phys. B 88 (1975) 257.

[4] M.F. Sohnius, Nucl. Phys. B 138 (1978) 109.

[5] R. Grimm, M.F. Sohnius and J. Wess, Nucl. Phys. B 133 (1978) 275.

[6] B. de Wit, J.W. van Holten and A. van Proeyen, Phys. Lett. B 95 (1980) 51.

[7] B. de Wit and J.W. van Holten, Nucl. Phys. B 155 (1979) 530.

[8] P. Fayet, Nucl. Phys. B 113 (1976) 135.

[9] M.F. Sohnius, K.S. Stelle and P.C. West, Phys. Lett. B 92 (1980) 123.

[10] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B 451 (1995) 53.

[11] R. Siebelink, Nucl. Phys. B 524 (1998) 86.

[12] A. Hindawi, B.A. Ovrut and D. Waldram, Phys. Lett. B 392 (1997) 85.

[13] R. Grimm, M. Hasler and C. Herrmann, Int. J. Mod. Phys. A 13 (1998) 1805.

[14] I. Buchbinder, A. Hindawi and B.A. Ovrut, Phys. Lett. B 413 (1997) 79.

[15] N. Dragon, S. Kuzenko and U. Theis, Eur. Phys. J. C 4 (1998) 717.

[16] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, Phys. Lett. B 373 (1996) 81.

[17] P. Claus, B. de Wit, M. Faux and P. Termonia, Nucl. Phys. B 491 (1997) 201.

[18] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, Nucl. Phys. B 512 (1998) 148.

[19] G. Akemann, R. Grimm, M. Hasler and C. Herrmann, Class. Quant. Grav. 16 (1999) 1617.

[20] N. Dragon and S. Kuzenko, Phys. Lett. B 420 (1998) 64.

[21] E. Ivanov and E. Sokatchev, Phys. Lett. B 429 (1998) 35.

[22] N. Dragon and U. Theis, *Gauging the central charge.* hep-th/9711025.

[23] N. Dragon and U. Theis, Phys. Lett. B 446 (1999) 314.

[24] F. Brandt and N. Dragon, in: *Theory of Elementary Particles. Proceedings of the 31st International Symposium Ahrenshoop.* H. Dorn, D. Lüst and G. Weigt, eds. (Wiley-VCH, Weinheim, 1998) 149.

[25] M. Henneaux and B. Knaepen, Phys. Rev. D 56 (1997) 6076.

[26] V.I. Ogievetsky and I.V. Polubarinov, J. Nucl. Phys. (USSR) 4 (1966) 216, Soviet. J. Nucl. Phys. 4 (1967) 156.

[27] D. Freedman and P.K. Townsend, Nucl. Phys. B 177 (1981) 282.

[28] F. Brandt and U. Theis, Nucl. Phys. B 550 (1999) 495 (to appear).

[29] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd Ed. Princeton University Press, Princeton (1992).

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