Topological Invariants of Isolated Determinantal Singularities

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Abstract

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by Matthias Zach

Keywords: Determinantal singularity, Milnor fiber, homology groups.

The presented thesis contains an explicit treatment of the deformation theory of determinantal singularities based on $K_{\mathcal{V}}$-equivalence and a careful construction of versal determinantal deformations. The necessary theory involved is gathered from the scattered literature and distinct viewpoints on the subject from different research groups are discussed. Based on the existence of versal determinantal deformations, we construct the determinantal Milnor fiber of a determinantal singularity as the preimage of the corresponding generic determinantal variety under a stabilization of the defining matrix considered as a map germ. In general, the determinantal Milnor fiber is a Whitney stratified space, which is unique for a given determinantal singularity up to homeomorphism.

We then turn to the study of topological invariants of the determinantal Milnor fiber. We describe the work done by different groups on the vanishing Euler-characteristic and reprove a formula for its computation from polar multiplicities for smoothable isolated determinantal singularities.

In Chapter 3, we introduce the Tjurina modification in family to reduce topological questions about determinantal singularities to questions about local complete intersections. In case of isolated singularities in the Tjurina transform of a given determinantal singularity, this enables us to explicitly determine the distinct homology groups of the Milnor fiber and we deduce some formulas on their interplay with the space of infinitesimal deformations.

Finally, we pick up the theory for the topology of non-isolated singularities, to also treat the case when the Tjurina transform is singular along a whole projetive line. To this end, we generalize certain connectivity results for the Milnor fibers of non-isolated singularities with one-dimensional singular locus to complete intersections with line singularities. This enables us to prove that for certain matrix sizes for smoothable isolated determinantal singularities we always find "characteristic cycles" which are directly related to the determinantal structure. They are the only contributions to the homology of the Milnor fiber below the middle degree. This phenomenon can not be observed for isolated complete intersection singularities.

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Zusammenfassung

Fakultät für Mathematik und Physik
Institut für Algebraische Geometrie

Topologische Invarianten Isolierter Determinantieller Singularitäten

von Matthias Zach

Schlagworte: Determinantielle Singularität, Milnor Faser, Homologiegruppen.

Die vorgelegte Arbeit beinhaltet eine explizite Darstellung der Deformationstheorie determinantieller Singularitäten basierend auf $K_\mathcal{V}$-Äquivalenz und eine Konstruktion verseller determinantieller Deformationen. Die dazu notwendige Theorie wird aus der verstreuten Literatur zusammengestellt und es werden unterschiedliche Sichtweisen von verschiedenen Arbeitsgruppen diskutiert. Basierend auf der Existenz verseller determinantieller Deformationen konstruieren wir die determinantielle Milnor-Faser als das Urbild der assoziierten generischen determinantiellen Varietät unter einer Stabilisierung der definierenden Matrix aufgefasst als Abbildungskeim. Im Allgemeinen ist die determinantielle Milnor Faser ein Whitney-stratifizierter Raum, welcher eindeutig bis auf Homöomorphie ist.

Wir wenden uns dann der Frage nach topologischen Invarianten der determinantiellen Milnor-Faser zu. Wir geben eine Beschreibung der Beiträge verschiedener Arbeitsgruppen zur verschwindenden Euler-Characteristik und erarbeiten einen neuen Beweis für eine Formel zu ihrer Berechnung durch Polar-Multiplizitäten für glättbare isolierte determinantielle Singularitäten.

In Kapitel 3 führen wir Tjurina-Modifikationen in Familie ein und reduzieren so Fragen zur Topologie von determinantiellen Singularitäten auf Fragen über lokal vollständige Durchschnitte. Im Fall von isolierten Singularitäten in der Tjurina-Transformierten ermöglicht uns dies eine genaue Beschreibung der Homologiegruppen der Milnor-Faser und wir leiten einge Formeln über ihre Beziehung zum Raum der infinitesimalen Deformationen her.

Im letzten Kapitel benützen wir die Theorie zur Topologie von nicht-isolierten Singularitäten, um auch den Fall behandeln zu können, in dem der singuläre Ort der Tjurina-Transformierten eine projektive Gerade ist. Dazu verallgemeinern wir bestimmte Resultate über den Zusammenhang von Milnor-Fasern nicht-isolierter Singularitäten mit ein-dimensionalem singulären Ort für vollständige Durchschnitte, welche singulär entlang einer Gerade sind. Dies ermöglicht es uns zu beweisen, dass für gewisse Matrixgrößen für glättbare determinantielle Singularitäten immer "characteristische Zykel" existieren, welche direkt mit der determinantiellen Struktur in Verbindung stehen. Sie sind die einzigen Beiträge zur Homologie der Milnor-Faser unterhalb des mittleren Grades. Dieses Phänomen kann nicht für isolierte komplette Durchschnitte beobachtet werden.

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List of Symbols

$\mathcal{O}_N$	ring of convergent power series in N complex variables at the origin
$\mathcal{O}_{X,p}$	local ring of the structure sheaf of a complex space X at a point p
$A^{\wedge t}$	homomorphism induced by A in the t-th exterior power
$A_{I,J}$	submatrix of A determined by the ordered multiindices I and J
$A^{\wedge t}_{I,J}$	entry of $A^{\wedge t}$ corresponding to the (I,J)-th minor
$\langle B \rangle$	ideal in a given ring R generated by the elements of a subset $B \subset R$
$\dim R$	Krull-dimension of a ring R
$\operatorname{codim}_R I$	codimension of an ideal I in a ring R
$\operatorname{pd}(M)$	projective dimension of a module M
$\operatorname{Supp}(M)$	support of a module M
$\operatorname{projdim} M$	projective dimension of a module M
$\operatorname{Ass} M$	associated primes of a module M
$M \hat{\otimes}_{\mathbb{C}} N$	analytic tensor product of two modules M and N
$\mathcal{F}$	coherent sheaf
$\mathcal{F}_p$	stalk of the sheaf $\mathcal{F}$ at a point p
$p.\mathcal{F}$	fiber of the sheaf $\mathcal{F}$ at a point p
$\Gamma(X, \mathcal{F})$	sections in $\mathcal{F}$ over X
$\mathcal{F}(X)$	also sections in $\mathcal{F}$ over X
$H^i(X, \mathcal{F})$	cohomology group of the sheaf $\mathcal{F}$ on a space X
(X, p)	germ of a complex space X at a point p
$\operatorname{Mat}(m, n; R)$	space of $m \times n$-matrices with entries in a ring R
$\Re z$	real part of a complex number z
$\Im z$	imaginary part of a complex number z
$\operatorname{rank} A$	rank of a matrix A
$\operatorname{Crit}(f)$	critical locus of a function f
$\operatorname{Sing}(X)$	singular locus of a space X
$H_i(X)$	homology group of a space X with integer coefficients
$H_i(A, B)$	relative homology group of the pair (A, B)
(A, B)	pair of topological spaces A with $B \subset A$
$H_p(\mathbf{C}_\bullet)$	homology of a complex $\mathbf{C}_\bullet$

Dem Tintenfisch gewidmet.

Prerequisites and Notation

In this thesis, we give a precise description of the Milnor fiber for determinantal singularities and address questions about their topology. We assume the reader to be familiar with the basic theory of complex analytic spaces and singularities, commutative algebra, as well as algebraic topology. Some standard references for these topics are for example [5], [6], [57], [38], [22], and [42].

In the text we will be confronted with coherent sheaves and their holomorphic sections as well as with vector bundles and continuous or differentiable sections in them. We try to adapt standard notation from analytic geometry and sheaf cohomology and from differential geometry depending on the context. One disadvantage is of this is that in the one notation $T_{X,p}$ denotes the stalk of the tangent sheaf of a space X at p, while the similar expression T_pX is only the fiber of the corresponding vector bundle in the other notation. To avoid confusion, we therefore write $p.\mathcal{F}$ for the fiber of a sheaf $\mathcal{F}$ at a point p and $\mathcal{F}_p$ for its stalk.

Another conflict of notation arises for space germs (X, p) and pairs of spaces (A, B), especially because we will be considering germs (Y, E) not only at points, but along compact subspaces $E \subset Y$. However, we hope that it is clear from the context, which kind of object we mean.

Concerning the notation used in algebraic topology, it should be pointed out that by $H_i(X)$ we always denote the i-th homology group of a topological space X with *integer* coefficients. In any case we provide a list of symbols and an index for the objects used in this thesis.

Chapter 1

Deformations of Determinantal Singularities

We give the common definition of determinantal singularities and gather the results concerning their algebraic and geometric properties. While the material itself is not new, the specific exposition and motivation is carried out by the author. We try to reflect different viewpoints on the subject, which appear in the common literature [11], [10], [22], and contemporary research [18], [19], [61], [62], [49], [16], [15], [17], [20], [59], [8], [23], [28], [27].

This chapter aims at the development of *versal determinantal deformations*. To this end, we formulate a notion of equivalence for determinantal singularities and equivalence of map germs into the space of matrices. For the latter, we give an explicit formulation of $K_{\mathcal{V}}$-equivalence following J. Damon and then continue with a hybrid system showing that a semi-universal unfolding of the map germ leads to a versal determinantal deformation of the underlying determinantal singularity. At the core of this interplay lays the fact that a determinantal singularity inherits a free resolution of its defining ideal from the *generic determinantal variety*. This is used to establish flatness of determinantal deformations. In the end, we discuss the relation to semi-universal deformations of space germs in the sense of Grauert, Schlessinger and the work of M. Schaps.

1.1 Determinantal Singularities

Definition 1.1.1. A germ of a complex space $(X, 0) \subset (\mathbb{C}^N, 0)$ is called a *determinantal singularity of type* (m, n, t) if there is a matrix

$$A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$$

with holomorphic entries in the ring of convergent power series $\mathcal{O}_N = \mathbb{C}\{x_1, \ldots, x_N\}$ such that

$$\mathcal{O}_{X,0} = \mathcal{O}_N / \langle A^{\wedge t} \rangle$$

and

$$\mathrm{codim}_{\mathcal{O}_N} \langle A^{\wedge t} \rangle = (m - t + 1)(n - t + 1). \tag{1.1}$$

Here $\langle A^{\wedge t} \rangle$ denotes the ideal generated by the t-minors of A. We also say that the quotient ring $\mathcal{O}_{X,0}$ is a *determinantal ring*. An *isolated determinantal singularity* is a determinantal singularity $(X, 0)$ such that $0 \in X$ is the only singular point.

The notation for the ideal of t-minors is explained as follows: If we consider A as a homomorphism of free $\mathcal{O}_N$-modules, then $A^{\wedge t}$ denotes the matrix representing the induced map on the t-th exterior powers (cf. Appendix A.1). For any matrix $B \in \mathrm{Mat}(m,n;\mathcal{O}_N)$ we denote by $\langle B \rangle$ the ideal generated by its entries, which, in case $B = A^{\wedge t}$ for some A, are just the t-minors of A.

Remark 1.1.2. A priori for a given singularity $(X,0)$ there are many matrices describing a structure of $(X,0) \subset (\mathbb{C}^N,0)$ as a determinantal singularity. Suppose that the t-minors of some $A \in \mathrm{Mat}(m,n;\mathcal{O}_N)$ generate the vanishing ideal of $(X,0)$ and one of the entries $a_{i,j}$ of A is a unit in $\mathcal{O}_N$. Without loss of generality, we may assume $(i,j) = (1,1)$, and we can apply row and column operations on A to reduce to the form

$$A \sim \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}.$$

Since these operations do not alter the ideal generated by the t-minors, we see that $(X,0)$ is also a determinantal singularity of type $(m-1,n-1,t-1)$ by means of $\tilde{A}$.

Let $\mathfrak{m} = \langle \underline{x} \rangle \subset \mathcal{O}_N$ be the maximal ideal. If there was a linear dependence $A \cdot \lambda = 0$ of the columns of A with $\lambda = \left(\lambda_1,\dots,\lambda_n\right)^T \in \mathcal{O}_N^n$, but $\lambda \notin \mathfrak{m} \cdot \mathcal{O}_N^n$ then we can, without loss of generality, assume $\lambda_1 = 1$. This means that the first column can be expressed by the others and consequently, if we let $\tilde{A}$ be the matrix obtained from A by deleting the first column, then $\langle A^{\wedge t} \rangle = \langle \tilde{A}^{\wedge t-1} \rangle$ are the same ideals in $\mathcal{O}_N$. The same argument holds for the rows of A.

For these reasons, we always assume that all entries of the describing matrix are in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_N$, i.e. they are non-units, and that neither the columns nor the rows of A admit relations with coefficients in $\mathcal{O}_N \setminus \mathfrak{m}$.

But, as we can see in the following example 1.1.3, for a given singularity even these minimality conditions are in general not sufficient to uniquely determine the matrix A from $(X,0)$. Therefore, if we speak of a determinantal singularity of type (m,n,t), we do not only mean the germ $(X,0) \subset (\mathbb{C}^N,0)$, but also the matrix A.

Example 1.1.3. a) A complete intersection of codimension d is a determinantal singularity of type $(d,1,1)$.

b) Let $(X,0) \subset (\mathbb{C}^3,0)$ be the A_1-surface singularity given by the equation $f = x^2 - yz = 0$. It is a determinantal singularity in two different ways. On the one hand we can consider the equation f as a 1×1-matrix, which gives $(X,0)$ the structure of a determinantal singularity of type $(1,1,1)$. On the other hand we have

$$f = \det \begin{pmatrix} x & y \\ z & x \end{pmatrix},$$

which makes it a determinantal singularity of type $(2,2,2)$.

With these examples in mind we shall develop a notion of equivalence for determinantal singularities parallel to contact equivalence. Recall that two germs $(X,0), (Y,0) \subset (\mathbb{C}^N,0)$ are *contact equivalent* if there is a germ of a diffeomorphism $\Phi : (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$, or equivalently an automorphism Φ^* of $\mathcal{O}_N$ such that Φ takes $(X,0)$ to $(Y,0)$ and the other way round for Φ^{-1}.

In terms of algebra this translates to the following: Let J be the ideal in $\mathcal{O}_N$ defining $(Y,0)$ and I the ideal of $(X,0)$. Then we have

$$\begin{tikzcd} J \arrow[r, "\Phi^*"] \arrow[rr, bend right, "\mathrm{id}_J"'] & I \arrow[r, "(\Phi^{-1})^*"] \arrow[rr, bend right, "\mathrm{id}_I"'] & J \arrow[r, "\Phi^*"] & I\,. \end{tikzcd}$$

These maps naturally induce homomorphisms on the $\mathbb{C}$-vector spaces $I/\mathfrak{m}I$ and $J/\mathfrak{m}J$ which have to be isomorphisms. Let $a_1,\dots,a_n$ be a minimal set of generators of I and $b_1,\dots,b_n$ of J. Using Nakayama's Lemma, we see that there exist invertible matrices $F \in \mathrm{Mat}(1,1;\mathcal{O}_N), G \in \mathrm{Mat}(n,n;\mathcal{O}_N)$ such that

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} = F \cdot \begin{pmatrix} \Phi^*(b_1) & \cdots & \Phi^*(b_n) \end{pmatrix} \cdot G^{-1}$$

and thus, the submodules I and J of $\mathcal{O}_N = \mathcal{O}_N^1$ are identified by Φ^*.

For determinantal singularities we do not compare the ideals but the defining matrices.

Definition 1.1.4. Let $(X,0)$ and $(Y,0) \subset (\mathbb{C}^N,0)$ be two determinantal singularities of type (m,n,t) given by matrices A and $B \in \mathrm{Mat}(m,n;\mathcal{O}_N)$. We say $(X,0)$ and $(Y,0)$ are *equivalent as determinantal singularities* if there exists an automorphism Φ^* of $\mathcal{O}_N$ and invertible matrices $F \in \mathrm{Mat}(m,m;\mathcal{O}_N), G \in \mathrm{Mat}(n,n;\mathcal{O}_N)$ such that

$$A = F \cdot (\Phi^* B) \cdot G^{-1}$$

in $\mathrm{Mat}(m,n;\mathcal{O}_N)$.

The analogy should be obvious. Here we can see the minimality conditions from Remark 1.1.2 in action. Just like we describe an ideal as a submodule of $\mathcal{O}_N$ by a minimal set of generators, the columns of A give a minimal set of generators of the image of A in $\mathcal{O}_N^m$ when considered as an element of $\mathrm{Hom}_{\mathcal{O}_N}(\mathcal{O}_N^n, \mathcal{O}_N^m)$ and similarly for the rows. Requiring F and G to be invertible is no restriction once minimality of A and B is assumed.

Contact equivalence of $(X,0)$ and $(Y,0)$ now follows directly from Corollary A.1.2 since the matrices F and G in Definition 1.1.4 are invertible.

There is another point of view for describing determinantal singularities. We can regard the matrix A as a germ of a map

$$A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m,n;\mathbb{C}), 0).$$

Then $(X,0)$ appears as a degeneracy locus of A, namely as the preimage of the *generic determinantal variety*

$$M_{m,n}^t := \{M \in \mathrm{Mat}(m,n;\mathbb{C}) : \mathrm{rank}\, M < t\} \tag{1.2}$$

under the map A. We will see that a determinantal singularity of type (m,n,t) inherits many properties from its associated generic determinantal variety $M_{m,n}^t$. For the course of our studies, it is therefore beneficial to have a good understanding of these: In the following, we will collect some well known facts.

Lemma 1.1.5 ([11], Proposition 1.1)**.** *The space* $M^t_{m,n} \subset \mathrm{Mat}(m,n;\mathbb{C})$ *is irreducible of codimension* $(m-t+1)(n-t+1)$ *and its singular locus is precisely*

$$\operatorname{Sing} M^t_{m,n} = M^{t-1}_{m,n}.$$

This shows where the condition (1.1) on the codimension for a determinantal singularity comes from: It is the *expected codimension* of the preimage of the generic determinantal variety.

A more involved computation shows that the spaces

$$\{0\} = M^1_{m,n} \subsetneq M^2_{m,n} \subsetneq \cdots \subsetneq M^{\min\{m,n\}}_{m,n} \subset \mathrm{Mat}(m,n;\mathbb{C}) \tag{1.3}$$

form a complex analytic *Whitney stratification* of the space $\mathrm{Mat}(m,n;\mathbb{C})$ with strata

$$\Sigma^t_{m,n} = M^t_{m,n} \setminus M^{t-1}_{m,n},$$

see e.g. [3]. For the definition of Whitney stratifications see the Appendix. On one hand, by considering maps, one automatically leaves the realm of intrinsic properties of $(X_0,0)$ because one has to take into account the behaviour of the map outside the singularity as well. On the other hand, since the fundamental work done by Milnor ([53], cf. Theorem 2.1.13), the benefits of this viewpoint for topological questions about the singularity are evident. For determinantal singularities, we will see a lot of interplay between map germs and their unfoldings and space germs and their deformations.

While the definition of an unfolding of a map is rather trivial, the deformation theory of space germs is much more involved. One reason for the interest in determinantal singularities is that (parts of) their deformation theory is accessible via perturbations of the defining matrix as a map germ. The rest of this chapter is devoted to the exposition of this interplay.

1.2 Some Notions in Commutative Algebra

All the results of this section are well known and gathered from standard sources. Mostly we refer to [22] and [10].

1.2.1 Flatness

Recall that a module M over a commutative ring R is called *flat* if the functor $-\otimes_R M$ from the category of R-modules to itself is exact.

Definition 1.2.1. Let $(B,\mathfrak{n})$ be a local ring with residue field k and R_0 a Noetherian k-algebra. A *deformation* of R_0 over $(B,\mathfrak{n})$ is given by a flat B-algebra R such that the fiber over $\mathfrak{n}$

$$\begin{tikzcd} R\otimes_B B/\mathfrak{n} & R \arrow[l] \\ B/\mathfrak{n} \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

is isomorphic to R_0. A deformation of a germ $(X_0,0) \subset (\mathbb{C}^N,0)$ over a $\mathbb{C}$-algebra B is a deformation of $\mathcal{O}_{X_0,0}$.

Remark 1.2.2. In practice, for any deformation of a ring $\mathcal{O}_{X_0,0} = \mathbb{C}\{\underline{x}\}/I$ the ring B will either be an Artinian $\mathbb{C}$-algebra or another power series ring $\mathbb{C}\{u_1, \ldots, u_t\}$ or quotient $\mathbb{C}\{\underline{u}\}/T$ thereof. In the first case, the deformation is called a *formal deformation*. In the second case R will be of the form $\mathbb{C}\{\underline{x}, \underline{u}\}/\tilde{I}$, where $\tilde{I}$ is an ideal such that

$$\mathbb{C}\{\underline{x}, \underline{u}\}/\tilde{I} + \langle \underline{u} \rangle \cong \mathbb{C}\{\underline{x}\}/I \cong \mathcal{O}_{X_0,0}.$$

Here we get a geometric realization of the above diagram reversing the arrows

$$\begin{tikzcd}
(X_0,0) \arrow[r] \arrow[d] & (X,0) \arrow[d] \\
\{0\} \arrow[r] & (Y,p)
\end{tikzcd}$$

where $(X, 0)$ and (Y, p) are the germs of complex spaces associated to $\mathbb{C}\{\underline{x}, \underline{u}\}/\tilde{I}$ and $\mathbb{C}\{\underline{u}\}/T$, respectively.

In general, it is difficult to create flat families. The following theorem gives the probably most common criterion for flatness. We present it as it can be found in [22, Theorem 6.8]:

Theorem 1.2.3 (Local Criterion for Flatness)**.** *Let $\phi : (B, \mathfrak{n}) \to (S, \mathfrak{m})$ be a homomorphism of Noetherian local rings and M a finitely generated S-module. Then M is flat over B if and only if*

$$\operatorname{Tor}_1^B(B/\mathfrak{n}, M) = 0.$$

In view of the standard situations we will encounter as described above, we deduce another criterion for our purposes. This one can e.g. be found in [7, Proposition 3.1].

Suppose we are given $(X_0, 0) \subset (\mathbb{C}^N, 0)$ by means of an ideal $I \in \mathbb{C}\{\underline{x}\}$. There is up to isomorphism a unique minimal free resolution

$$0 \longleftarrow \mathbb{C}\{\underline{x}\}/I \longleftarrow \mathbb{C}\{\underline{x}\} \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots \tag{1.4}$$

of the quotient ring $\mathcal{O}_{X_0,0}$, where the F_i are free $\mathbb{C}\{\underline{x}\}$-modules.

Now, let B be some analytic algebra and $S = \mathbb{C}\{\underline{x}\}\hat{\otimes}_{\mathbb{C}}B$. Here $\hat{\otimes}_{\mathbb{C}}$ denotes the analytic tensor product, see e.g. [33] We will basically encounter two cases: If $B = \mathbb{C}\{\underline{u}\}/T$ for some ideal T, then $\mathbb{C}\{\underline{x}\}\hat{\otimes}_{\mathbb{C}}\mathbb{C}\{\underline{u}\}/T = \mathbb{C}\{\underline{x}, \underline{u}\}/T$, where T is considered as an ideal in $\mathbb{C}\{\underline{x}, \underline{u}\}$ in the obvious sense. In the other case B is Artinian and $\hat{\otimes}_{\mathbb{C}}$ reduces to the usual tensor product.

Let $\mathfrak{n}$ be the maximal ideal of B. Suppose $\tilde{I}$ is an ideal in S such that

$$\mathbb{C}\{\underline{x}\}/I \cong S/(\tilde{I} + \mathfrak{n}) = S \otimes_B B/\mathfrak{n}.$$

By the right exactness of the tensor product, this amounts to saying that any free presentation

$$0 \longleftarrow S/\tilde{I} \longleftarrow S \longleftarrow P_1$$

of the quotient $S/\tilde{I}$ spezializes to the presentation of $\mathcal{O}_{X_0,0}$ in (1.4).

Lemma 1.2.4 (Lifting of Relations)**.** *If in the above setup, there is a free resolution*

$$0 \longleftarrow S/\tilde{I} \longleftarrow S \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots \tag{1.5}$$

of the ring $S/\tilde{I}$ as an S-module which is taken isomorphically to a given free resolution (1.4) of $\mathbb{C}\{\underline{x}\}/I$ by the functor $B/\mathfrak{n} \otimes_B -$, then $\operatorname{Tor}_1^B(B/\mathfrak{n}, S/\tilde{I}) = 0$ and $S/\tilde{I}$ is flat over B.

Proof. All the P_i are free S-modules. But S itself is a free B-module and hence we can regard (1.5) as a free resolution of B-modules as well. By definition $\operatorname{Tor}_1^B(B/\mathfrak{n}, S/\tilde{I})$ is the homology group of the complex obtained from (1.5) by applying $B/\mathfrak{n} \otimes_B -$. If this is isomorphic to (1.4) then there is no homology, because (1.4) was a resolution. The claim now follows from the Local Criterion for Flatness, Theorem 1.2.3. □

Remark 1.2.5. Of course, to check for the vanishing of the first Tor only, it would be sufficient if the complex (1.5) was specializing to (1.4) in the first four terms only. But if this is the case, then one can see inductively that this must also hold for all other terms of the resolution. Hence the requirement in Lemma 1.2.4 is not unnecessarily strong.

1.2.2 Perfect Ideals and Modules

We start with an algebraic notion of height, dimension and codimension.

Definition 1.2.6. Let R be a commutative Noetherian ring. For a prime ideal $P \subset R$, the height is defined as the supremum of the lengths of strictly descending chains of prime ideals Q_i from P:

$$\operatorname{height} P = \sup\{r \in \mathbb{N} : P \supsetneq Q_1 \supsetneq Q_2 \supsetneq \cdots \supsetneq Q_r \supsetneq 0\}. \tag{1.6}$$

If $I \subset R$ is an arbitrary ideal, its height is

$$\operatorname{height} I = \inf\{\operatorname{height} P : P \supset I \text{ prime }\}. \tag{1.7}$$

The dimension of R is the supremum of the heights of all its maximal ideals. For an R-module M we let $\dim M$ be the dimension of $R/\operatorname{Ann} M$. We define the codimension of an ideal I by

$$\operatorname{codim} I = \dim R - \dim R/I.$$

It follows directly from the definition that we have an inequality

$$\operatorname{height} I \leq \operatorname{codim} I \tag{1.8}$$

for all ideals $I \subset R$.

Krull's Principal Ideal Theorem (see e.g. [22, Theorem 10.2] [1]) asserts that the height of an ideal $I \subset R$ is bounded from above by its number of generators.

[1] In [22], the height of an ideal is referred to as codimension.

The "Converse of the Principal Ideal Theorem" (see e.g. [22], Corollary 10.5) on the other hand assures that for any given ideal I of height c we can find a sequence of elements $x_1, \dots, x_c \in I$ such that for all $0 < i \le c$ one has $\operatorname{height}\langle x_1, \dots, x_i\rangle = i$.

To construct such a sequence, one has to choose the x_i successively such that x_{i+1} is not contained in the union of minimal primes over $\langle x_1, \dots, x_i\rangle$. Recall from primary decomposition that for a given finite module M over a Noetherian ring R, the set of zero divisors on M is given by the union of associated primes $\bigcup_{P \in \operatorname{Ass} M} P$, and that all primes minimal over $\operatorname{Ann} M$ are contained in $\operatorname{Ass} M$ (see e.g. [22, Theorem 3.1]).

Thus if we let $M = R/\langle x_1, \dots, x_i\rangle$ and choose x_{i+1} to be a nonzerodivisor on M, then this is in general a stronger condition on x_{i+1} than just not being contained in the primes minimal over $\langle x_1, \dots, x_i\rangle$. There is a name for sequences of nonzerodivisors:

Definition 1.2.7. Let R be a commutative Noetherian ring and M an R-module. An element $x \in R$ which is not a zero divisor on M is called a *regular element* on M. A sequence of elements $x_1, \dots, x_r \in R$ is a *regular sequence* on M if for all $0 \le i < r$ the element x_{i+1} is regular on $M/\langle x_1, \dots, x_i\rangle M$ and $M \neq \langle x_1, \dots, x_r\rangle M$.

Clearly, the maximal length of a regular sequence in I on R is bounded by the height of I. But, in general, this number can be strictly smaller and, hence, has a name of its own.

Definition 1.2.8. Let I be an ideal in a commutative Noetherian ring R and M a finitely generated R-module. The *grade* of I on M is the number

$$\operatorname{grade}(I, M) := \sup\{r \in \mathbb{N} : \exists\, (x_1, \dots, x_r) \in I \text{ regular sequence on } M\}.$$

By what has been said above for $M = R$, one has

$$\operatorname{grade}(I, R) \le \operatorname{height} I. \tag{1.9}$$

The great advantage of the notion of grade is the following: If one wants to compute the height of a given ideal I, one could start with a prime P minimal over I and successively choose $Q_1 \subset P$ minimal over $\langle 0\rangle$, then $Q_2 \subset P$ minimal over Q_1 and so on. One eventually ends up with a maximal chain of prime ideals in P. But the length of this chain need not be equal to $\operatorname{height} I$.

For the computation of $\operatorname{grade}(I, R)$, on the other hand, any maximal regular sequence in I on R coming from a successive choice of elements x_i already has length $\operatorname{grade}(I, R)$. The next theorem assures that this is even true for arbitrary modules M over R.

Theorem 1.2.9 ([10], Theorem 1.2.5)**.** *Let R be a commutative Noetherian ring, I an ideal in R, and M an R-module. If $\mathbf{x} = (x_1, \dots, x_r)$ is a regular sequence in I on M, then there is a natural isomorphism*

$$\operatorname{Hom}_R(R/I, M/\mathbf{x}M) \cong \operatorname{Ext}_R^r(R/I, M),$$

and for all $0 \le i < r$, the modules $\operatorname{Ext}_R^i(R/I, M)$ are zero. In particular,

$$\operatorname{grade}(I, M) = \min\{r \in \mathbb{N} : \operatorname{Ext}_R^r(R/I, M) \neq 0\},$$

and any regular sequence in I on M can be extended to a maximal regular sequence of length $\operatorname{grade}(I, M)$.

Using the description of grade in terms of Ext, one can extend this notion to arbitrary modules.

Definition 1.2.10. Let M be a module over a commutative Noetherian ring R. The grade of M is $\operatorname{grade} M = \min\{r \in \mathbb{N} : \operatorname{Ext}^r_R(M, R) \neq 0\}$.

Since the Ext-modules over Noetherian rings can be computed from a projective resolution of the first factor, the lengths of such resolutions come into play for the computation of grade.

Definition 1.2.11. Let R be a commutative Noetherian ring and M an R-module. The *projective dimension* of M is the minimal length of a projective resolution

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_r \longleftarrow 0$$

of M over R. If no finite projective resolution of M over R exists, we write $\operatorname{projdim} M = \infty$.

Clearly we have an inequality

$$\operatorname{grade} M \leq \operatorname{projdim} M. \tag{1.10}$$

For the case of equality there is a special name.

Definition 1.2.12. A module M over a commutative Noetherian ring R is *perfect* if

$$\operatorname{projdim} M = \operatorname{grade} M.$$

An ideal I is perfect if R/I is a perfect module over R.

1.2.3 Cohen-Macaulay rings

Definition 1.2.13. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and M a finitely generated R-module. The depth of M is defined as

$$\operatorname{depth} M := \operatorname{grade}(\mathfrak{m}, M).$$

In case $\operatorname{depth} M = \dim M$ we say that M is *Cohen-Macaulay*. The ring R is Cohen-Macaulay if it is Cohen-Macaulay as a module over itself. An arbitrary commutative Noetherian ring S is called Cohen-Macaulay if for all prime ideals $P \subset S$, the localized ring S_P is Cohen-Macaulay.

From (1.9), we obtain an inequality

$$\operatorname{depth} R \leq \dim R - \dim R/\mathfrak{m} = \dim R \tag{1.11}$$

because $\dim R/\mathfrak{m} = 0$.

The power series ring $\mathbb{C}\{x_1, \ldots, x_n\}$ and the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ are both Cohen-Macaulay. Even more: They are *regular*:

Definition 1.2.14. A commutative Noetherian local ring $(R, \mathfrak{m})$ is called *regular* if its maximal ideal $\mathfrak{m}$ can be generated by a regular sequence on R. Again, an arbitrary commutative Noetherian ring S is regular if all of its localizations at prime ideals are regular.

In particular, every regular local ring is Cohen-Macaulay. Gathering the inequalities (1.8) and (1.9), we obtain

$$\operatorname{grade}(I, R) \leq \operatorname{height} I \leq \operatorname{codim} I.$$

The next theorem tells us why we should care about working in Cohen-Macaulay rings: In this case the inequalities become equalities.

Theorem 1.2.15 ([10], Theorem 2.1.2). *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and M a Cohen-Macaulay R-module.*

i) One has an equality

$$\operatorname{grade}(I, M) = \dim M - \dim M/IM$$

for all ideals $I \subset M$.

ii) A sequence $\mathbf{x} = (x_1, \ldots, x_r)$ of elements in R is a regular sequence on M if and only if $\dim M/\mathbf{x}M = \dim M - r$.

We finally describe the connection to the preceeding section by exhibiting the interplay of perfect ideals and Cohen-Macaulay rings. The following formula is well known (see e.g. [22] or [10]) and will be useful later on.

Theorem 1.2.16 (Auslander Buchsbaum Formula). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with* $\operatorname{projdim} M < \infty$. *One has an equality*

$$\operatorname{depth} M + \operatorname{projdim} M = \operatorname{depth} R.$$

Now we have the following theorem.

Theorem 1.2.17 ([10],Theorem 2.1.5). *Let R be a Cohen-Macaulay ring and M a finitely generated R-module with* $\operatorname{projdim} M < \infty$

i) If M is perfect, then it is Cohen-Macaulay.

ii) The converse holds when R is local.

Proof. In case R is a local ring and $M = R/I$, i.e. if $I \subset R$ is a perfect ideal, the assertions follow from the Auslander Buchsbaum Formula 1.2.16. One has

$$\begin{aligned}
\operatorname{depth} R/I &= \operatorname{depth} R - \operatorname{projdim} R/I \\
&\leq \dim R - \operatorname{grade}(I, R) \\
&= \dim R - \operatorname{codim} I \\
&= \dim R/I
\end{aligned}$$

by (1.10) and Theorem 1.2.15 i). Now if I is perfect, then $\operatorname{projdim} R/I = \operatorname{grade}(I, R)$ and hence $\operatorname{depth} R/I = \dim R/I$. On the other hand, provided the latter equality, we may deduce perfectness of I.

For a full proof see [10, Theorem 2.1.5]. □

1.2.4 The Koszul Complex

We first define the Koszul complex. Let R be a commutative ring and $x_1, \dots, x_n \in R$ be arbitrary elements. We can consider $\mathbf{x} = (x_1, \dots, x_n)$ as an element of the free module R^n. For any $p \in \mathbb{N}_0$, we define [2]

$$\mathbf{x}\wedge : \bigwedge^p R^n \to \bigwedge^{p+1} R^n, \quad \omega \mapsto \mathbf{x} \wedge \omega.$$

From this, we build the *Koszul cocomplex* in $\mathbf{x}$ over R:

$$\mathbf{K}^\bullet(\mathbf{x}) : \ 0 \longrightarrow \bigwedge^0 R^n \xrightarrow{\mathbf{x}\wedge} \bigwedge^1 R^n \xrightarrow{\mathbf{x}\wedge} \cdots \xrightarrow{\mathbf{x}\wedge} \bigwedge^n R^n \longrightarrow 0 \tag{1.12}$$

The group of p-cochains is thus given by $\mathbf{K}_p(\mathbf{x}) = \bigwedge^p R^n$.

This is a complex of free modules. The *Koszul complex* in $\mathbf{x}$ on R is defined by dualizing $\mathbf{K}^\bullet(\mathbf{x})$:

$$\mathbf{K}_\bullet(\mathbf{x}) : \ 0 \longleftarrow (\bigwedge^0 R^n)^\vee \xleftarrow{\mathbf{x}^\vee} (\bigwedge^1 R^n)^\vee \xleftarrow{\mathbf{x}^\vee} \cdots \xleftarrow{\mathbf{x}^\vee} (\bigwedge^n R^n)^\vee \longleftarrow 0\,.$$

The notation $\mathbf{x}^\vee$ is motivated from the fact that $\mathbf{x} \in R^n$ can be considered as a homomorphism $R \to R^n$, $1 \mapsto \mathbf{x}$. We find this as the first nontrivial map in (1.12). In this sense $\mathbf{x}^\vee$ is the natural extension of the map dual to $\mathbf{x}$ to the exterior powers.

On the other hand, given a homomorphism

$$\Psi : R^n \to R, \quad e_i \mapsto x_i,$$

we will also speak of the Koszul complex associated to Ψ and just write Ψ for all the maps $\mathbf{x}^\vee$.

Using the duality $(\bigwedge^p R^n)^\vee \cong \bigwedge^p (R^\vee)^n$ in (A.5) from the appendix, and identifying R with $R^\vee$ in the canonical way, we obtain the final definition of the Koszul complex:

$$\mathbf{K}_\bullet(\mathbf{x}) : \ 0 \longleftarrow \bigwedge^0 R^n \xleftarrow{\mathbf{x}^\vee} \bigwedge^1 R^n \xleftarrow{\mathbf{x}^\vee} \cdots \xleftarrow{\mathbf{x}^\vee} \bigwedge^n R^n \longleftarrow 0\,. \tag{1.13}$$

Definition 1.2.18. For a module M over R we define the Koszul complex and the Koszul cocomplex in $\mathbf{x}$ on M as

$$\mathbf{K}_\bullet(\mathbf{x}; M) := \mathbf{K}_\bullet(\mathbf{x}) \otimes_R M \quad \text{and} \quad \mathbf{K}^\bullet(\mathbf{x}; M) := \mathbf{K}^\bullet(\mathbf{x}) \otimes_R M. \tag{1.14}$$

Recall that for a splitting $F = P \oplus Q$ of a free module into two free parts P and Q we obtain a decomposition of the exterior algebra:

$$\bigwedge^n F = \bigoplus_{p+q=n} \bigwedge^p P \otimes_R \bigwedge^q Q. \tag{1.15}$$

We use this to show the following:

[2] See the appendix A.1 for a definition of the exterior powers $\bigwedge^p R^n$ and exterior multiplication $\wedge$

Lemma 1.2.19. *For any two sets* $\mathbf{x} = (x_1, \ldots, x_m)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *of elements in* R *there is a natural description*

$$\mathbf{K}_r(\mathbf{x}, \mathbf{y}) \cong \bigoplus_{p+q=r} \mathbf{K}_p(\mathbf{x}) \otimes \mathbf{K}_q(\mathbf{y})$$

which identifies $\mathbf{K}_\bullet(\mathbf{x}, \mathbf{y})$ *with the total complex of the double complex* $C_{p,q} = \bigwedge^p R^m \otimes \bigwedge^q R^n$ *with the boundary maps*

$$\begin{aligned} \mathbf{y}^\vee : C_{p,q} \to C_{p,q-1}, &\quad \omega \otimes \eta \mapsto (-1)^p \omega \otimes \mathbf{y}^\vee(\eta), \\ \mathbf{x}^\vee : C_{p,q} \to C_{p-1,q}, &\quad \omega \otimes \eta \mapsto \mathbf{x}^\vee(\omega) \otimes \eta. \end{aligned}$$

Proof. Direct computation. □

This can be used to build the Koszul complex inductively on the number of elements x_i. The basic building block of this iteration is the Koszul complex in one element $\mathbf{x} = (x_1)$, which is just

$$\mathbf{K}_\bullet(\mathbf{x}) : \ 0 \longleftarrow \bigwedge^0 R^1 \xleftarrow{\cdot x_1} \bigwedge^1 R^1 \longleftarrow 0 \ .$$

For any given R-module M, the homology of the Koszul complex $\mathbf{K}(x_1; M)$ in x_1 on M is

$$\begin{aligned} H_0(\mathbf{K}(x_1; M)) &= M/x_1 M \\ H_1(\mathbf{K}(x_1; M)) &= 0 :_M x_1 \\ H_j(\mathbf{K}(x_1; M)) &= 0 \quad \text{for all } j \neq 0, 1. \end{aligned}$$

We see that x_1 is a nonzerodivisor on M if and only if all homology groups of the associated Koszul complex vanish outside degree zero. This fact generalizes:

Lemma 1.2.20. *Let* R *be a commutative ring,* $x_1, \ldots, x_n \in R$*, and* M *an* R*-module. If* $\mathbf{x} = (x_1, \ldots, x_n)$ *is a regular sequence on* M*, then the Koszul complex* $\mathbf{K}_\bullet(\mathbf{x}; M)$ *in* $\mathbf{x}$ *on* M *has the homology groups*

$$\mathbf{H}_p(\mathbf{x}; M) \cong \begin{cases} M/\langle x_1, \ldots, x_n \rangle M & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} .$$

A special case deserves attention:

Corollary 1.2.21. *In case* $M = R$ *in Lemma 1.2.20 and a for a regular sequence* $\mathbf{x} = (x_1, \ldots, x_n)$ *on* R*, the Koszul complex gives a free resolution of the quotient* $R/\langle \mathbf{x} \rangle$*.*

Proof. (of Lemma 1.2.20). We do induction on the number of elements $n = \#\mathbf{x}$. If $n = 1$, this is just the definition of a nonzerodivisor on M. So suppose we did already prove the claim for the Koszul complex in n elements and we are given a further element $y \in R$. Consider the Koszul complex $\mathbf{K}_\bullet(x_1, \ldots, x_n, y)$ in $\mathbf{x}$ and y and its decomposition according to Lemma

1.2.19 as the total complex of the double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 & \bigwedge^n R^n \otimes \bigwedge^0 R^1 \arrow[l] \arrow[d, "\mathbf{x}^\vee"] & & \bigwedge^n R^n \otimes \bigwedge^1 R^1 \arrow[ll, "y^\vee"'] \arrow[d, "\mathbf{x}^\vee"] & 0 \arrow[l] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 & \bigwedge^2 R^n \otimes \bigwedge^0 R^1 \arrow[l] \arrow[d, "\mathbf{x}^\vee"] & & \bigwedge^2 R^n \otimes \bigwedge^1 R^1 \arrow[ll, "y^\vee"'] \arrow[d, "\mathbf{x}^\vee"] & 0 \arrow[l] \\
0 & \bigwedge^1 R^n \otimes \bigwedge^0 R^1 \arrow[l] \arrow[d, "\mathbf{x}^\vee"] & & \bigwedge^1 R^n \otimes \bigwedge^1 R^1 \arrow[ll, "y^\vee"'] \arrow[d, "\mathbf{x}^\vee"] & 0 \arrow[l] \\
0 & \bigwedge^0 R^n \otimes \bigwedge^0 R^1 \arrow[l] \arrow[d] & & \bigwedge^0 R^n \otimes \bigwedge^1 R^1 \arrow[ll, "y^\vee"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & & 0 &
\end{tikzcd} \quad . \tag{1.16}$$

We obtain $\mathbf{K}_\bullet(\mathbf{x}, y; M)$ from (1.16) by tensoring with M. Observe that both columns of (1.16) are canonically isomorphic to $\mathbf{K}_\bullet(\mathbf{x})$ since $\bigwedge^0 R^1 \cong \bigwedge^1 R^1 \cong R$.

By the induction hypothesis, $(x_1, \dots, x_n)$ is a regular M-sequence and, hence, the columns of (1.16) tensored with M are exact. The homology of the associated total complex in degree zero is obvious and anyway independent of whether or not $(x_1, \dots, x_n, y)$ is a regular sequence. In degree one the homology is zero if and only if y is a regular element on $M/\mathbf{x}M$. For all higher degrees, the assertion follows from a simple diagram chase using the exactness of the columns. □

There is a converse of Lemma 1.2.20. We cite it for sake of completeness, but we shall not need it.

Theorem 1.2.22 ([22], Theorem 17.4)**.** *Let M be an R-module and $I = \langle x_1, \dots, x_r \rangle$ a finitely generated ideal. One has*

$$\operatorname{grade}(I, M) = r - j,$$

where

$$j = \max\{i \in \mathbb{N} : H_i(\mathbf{K}_\bullet(\underline{x}; M)) \neq 0\}.$$

1.2.5 Dimension of Base and Fiber

We gather some theorems and lemmas concerning the interplay of the preceeding sections with deformation theory. The main source is [22].

Theorem 1.2.23 ([22], Theorem 10.10). *Let $(B,\mathfrak{n}) \to (R,\mathfrak{m})$ be a homomorphism of local rings. Then*

$$\dim R \leq \dim B + \dim R/\mathfrak{n}R.$$

If R is flat as a B-module, then equality holds.

The converse of the second statement is in general false. However, there are certain conditions, which allow its deduction (cf. [22, Theorem 18.16]):

Theorem 1.2.24. *Let $(B,\mathfrak{n}) \to (R,\mathfrak{m})$ be a homomorphism of local rings where $(B,\mathfrak{n})$ is regular and $(R,\mathfrak{m})$ is Cohen-Macaulay. Then R is flat over B if and only if*

$$\dim R = \dim B + \dim R/\mathfrak{n}R.$$

Proof. One direction directly follows from 1.2.23. For the other one let $\mathbf{u} = (u_1,\ldots,u_r)$ be a regular B-sequence generating $\mathfrak{n}$. By abuse of notation, we also write u_i for their images in R. According to the local criterion for flatness 1.2.3 for R to be flat over B, it is sufficient to show

$$\operatorname{Tor}_1^B(B/\mathfrak{n}, R) = 0.$$

This can be computed from a free resolution of $B/\mathfrak{n}$ over B. According to Lemma 1.2.20, such a resolution is given by the Koszul complex $\mathbf{K}_\bullet(\mathbf{u})$. But tensoring with R gives

$$\operatorname{Tor}_i^R(B/\mathfrak{n}, R) \cong H_i(\mathbf{K}_\bullet(\mathbf{u}; R)).$$

Now if $\dim R/\mathfrak{n}R = \dim R - \dim B = \dim R - r$, then according to Theorem 1.2.15, $\mathbf{u} = (u_1,\ldots,u_r)$ is a regular sequence on R and, hence, all homology groups $H_i(\mathbf{K}_\bullet(\mathbf{u}; R))$ vanish in degree $\neq 0$ due to Lemma 1.2.20. □

The next lemma assures that flatness behaves well under taking hyperplane sections.

Lemma 1.2.25. *Let $(B,\mathfrak{n}) \to (R,\mathfrak{m})$ be a homomorphism of Noetherian local rings, and suppose that M is an R-module which is flat over B. For any nonzerodivisor $x \in \mathfrak{m}$ on $M/\mathfrak{n}M$, also M/xM is a flat B-module.*

Proof. This follows directly from the long exact sequence of Tor. Consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0\ .$$

Applying $- \otimes_B B/\mathfrak{n}$, we obtain an exact sequence

$$\operatorname{Tor}_1^B(M, B/\mathfrak{n}) \longrightarrow \operatorname{Tor}_1^B(M/xM, B/\mathfrak{n}) \longrightarrow M/\mathfrak{n}M \xrightarrow{\cdot x} M/\mathfrak{n}M\ .$$

Now, by assumption on M, $\operatorname{Tor}_B^1(M, B/\mathfrak{n})$ vanishes. On the other hand the kernel of multiplication by x is trivial because x was a nonzerodivisor on $M/\mathfrak{n}M$. Thus the assertion follows from the Local Criterion for Flatness, Theorem 1.2.3. □

1.3 Determinantal Deformations

The condition on the codimension in the definition of determinantal singularities 1.1.1 is natural in the following sense. It is a fact known as the *generalized principal ideal theorem*, proved by Eagon and Northcott [18, Theorem 3], that for any ideal I generated by the t-minors of an $(m \times n)$-matrix A with entries in a commutative Noetherian ring R, the height of I in R is bounded from above by the number $(m-t+1)(n-t+1)$. Since this is also the codimension of the generic determinantal variety $M^t_{m,n}$, it is also the expected codimension of its preimage under the map given by A. The definition of a determinantal singularity makes sure that this bound is attained.

In this section we will develop the theory of determinantal deformations of a given determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$. As it turns out, the only requirement on $(X_0, 0)$ for the deformation theory to be well behaved is to have expected codimension. As a guiding example, we first recall the case of a complete intersection singularity.

1.3.1 Deformations of Complete Intersections

Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type $(d, 1, 1)$. This means the ideal I of $(X_0, 0)$ in $\mathcal{O}_N$ is generated by d elements $f_1, \dots, f_d$, and $\operatorname{codim} I = \operatorname{codim}(X_0, 0) = d$. In view of Lemma 1.2.4, a first step to understanding deformations of $(X_0, 0)$ is to know a free resolution of $\mathcal{O}_{X_0,0}$ over $\mathcal{O}_N$.

Since $\mathcal{O}_N$ is Cohen-Macaulay, Theorem 1.2.15 asserts that $\mathbf{f} = (f_1, \dots, f_d)$ is a regular sequence on $\mathcal{O}_N$. Therefore, a free resolution is given by the Koszul complex

$$0 \longleftarrow \mathcal{O}_{X_0,0} \longleftarrow \mathbf{K}_\bullet(\mathbf{f})$$

according to Lemma 1.2.20. Because the length of this free resolution is d, we may deduce that $\mathcal{O}_{X_0,0}$ is a Cohen-Macaulay $\mathcal{O}_N$-module by using the Auslander-Buchsbaum Formula.

Now suppose we are given another analytic algebra $(B, \mathfrak{n})$ and we want to deform $(X_0, 0)$ over $(B, \mathfrak{n})$. We need to give the ideal $\tilde{I}$ in the ring $S := \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B$ for the total space $(X, 0)$ of the deformation in $(\mathbb{C}^N, 0) \times \operatorname{Spec} B$. In order for $\mathcal{O}_{X,0} = S/\tilde{I}$ to specialize to $\mathcal{O}_{X_0,0} = \mathcal{O}_N / I$ there have to be lifts $F_i \in \tilde{I}$, which reduce to f_i modulo $\mathfrak{n}S$. The next theorem says that any ideal of the form $\tilde{I} = \langle F_1, \dots, F_d \rangle$ already defines a flat family.

Theorem 1.3.1. *Let $I \subset \mathcal{O}_N$ be a complete intersection ideal generated by the elements of a regular sequence $\mathbf{f} = (f_1, \dots, f_d)$. If $(B, \mathfrak{n})$ is any Noetherian local $\mathbb{C}$-algebra and F_i elements in $S := \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B$ reducing to $F_i \otimes_B B/\mathfrak{n} = f_i$ for $i = 1, \dots, d$, then $S/\langle F_1, \dots, F_d \rangle$ is flat over B.*

Proof. We first prove the theorem in the case $B = \mathbb{C}\{\underline{u}\} = \mathcal{O}_k$ for some $\underline{u} = u_1, \dots, u_k$. In this case, $S = \mathbb{C}\{\underline{x}, \underline{u}\} = \mathcal{O}_{N+k}$ is Cohen-Macaulay and flat as a B-algebra. One has

$$\dim S = \dim S/\mathfrak{n}S + \dim B = \dim \mathcal{O}_N + \dim B$$

according to Theorem 1.2.23. Let $J := \langle F_1, \ldots, F_d \rangle$. For the extension of rings S/I over B, we can say

$$\dim S/J \leq \dim \mathcal{O}_N/I + \dim B = \dim \mathcal{O}_N + \dim B - d = \dim S - d.$$

We see that J is an ideal of codimension d generated by d elements in the Cohen-Macaulay ring S and, hence, a complete intersection as well. It follows that the Koszul complex in $(F_1, \ldots, F_c)$ over S gives a free resolution of J as an S-module. It obviously specializes to the Koszul complex in the f_i and, hence, flatness follows from Lemma 1.2.4.

For a general analytic algebra $B = \mathcal{O}_k/T$ for some ideal T, let $\tilde{F}_i \in \mathbb{C}\{\underline{x}, \underline{u}\}$ be any lifts of the $F_i \in \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B$ under the canonical projection $\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} \mathcal{O}_k \to \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B$. From the above said, it follows that the family defined by these $\tilde{F}_i$ over $\mathcal{O}_k$ is flat. To show flatness of S/J over B, we may now use the fact that base-change preserves flatness, since

$$S/J \cong \mathcal{O}_{N+k}/\langle \tilde{F}_1, \ldots, \tilde{F}_d \rangle \otimes_{\mathcal{O}_k} B.$$

□

From the perspective of determinantal singularities, Theorem 1.3.1 can be rephrased as:

For a determinantal singularity of type $(d, 1, 1)$ any family coming from a perturbation of the defining matrix is flat.

The proof consisted of two essential steps: First, the reduction to deformations over analytic algebras $B = \mathcal{O}_k$, and secondly, the exploitation of expected codimension to give a free resolution of $\mathcal{O}_{X,0}$ spezializing to a free resolution of $\mathcal{O}_{X_0,0}$. We aim to show that this pattern works for determinantal singularities of arbitrary type.

Theorem 1.3.2. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$, and let $(B, \mathfrak{n})$ be any local $\mathbb{C}$-algebra. Set $S := \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B$, and let $\mathbf{A} \in \operatorname{Mat}(m, n; S)$ be any matrix such that $\mathbf{A} \otimes_B B/\mathfrak{n} = A$. Then $S/\langle \mathbf{A}^{\wedge t} \rangle$ is flat over B.*

We postpone the proof of Theorem 1.3.2.

Definition 1.3.3. Deformations of a determinantal singularity as in Theorem 1.3.2 are called *determinantal deformations*.

Remark 1.3.4. One can consider a determinantal deformation also as an unfolding of maps. Namely, if $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$ is as in Theorem 1.3.2, then

$$\mathbf{A} : (\mathbb{C}^N, 0) \times \operatorname{Spec} B \to (\operatorname{Mat}(m, n; \mathbb{C}), 0)$$

is a family of maps parametrized by $\operatorname{Spec} B$. Here, $\operatorname{Spec} B$ can be a formal scheme in case B was an Artinian algebra or if $B = \mathcal{O}_{Y,p}$ for some space (Y, p), then $\operatorname{Spec} B = (Y, p)$.

Families of maps are naturally well behaved. The hard part is to establish flatness of the induced family of preimages $\mathbf{A}^{-1}(M_{m,n}^t, 0) \to \operatorname{Spec} B$ for a given analytic set $(M_{m,n}^t, 0) \subset (\operatorname{Mat}(m, n; \mathbb{C}), 0)$.

The first step to understand a special determinantal singularity $(X_0, 0)$ of type (m, n, t) is to understand the generic singularity $(M_{m,n}^t, 0) \subset (\operatorname{Mat}(m, n; \mathbb{C}), 0)$.

In some sense, one could see this already in the case where $(X, 0)$ is a complete intersection singularity given by a matrix

$$A = \begin{pmatrix} a_1 & \dots & a_d \end{pmatrix}^T.$$

Here the generic "singularity" is a (smooth) point $V = \{0\} \in \mathbb{C}^c$. If $y_1, \dots, y_d$ are the coordinates on $\mathbb{C}^d$, then the ideal $J = \langle y_1, \dots, y_d \rangle$ defining $\{0\}$ is generated by the regular sequence $(y_1, \dots, y_d)$ and a free resolution of $\mathcal{O}_{V,0} = \mathcal{O}_d / J$ is given by the Koszul complex. The free resolution of $\mathcal{O}_{X_0,0}$, the quotient by the ideal of 1-minors of A, is then inherited from the resolution of $\mathcal{O}_{V,0}$ by substituting the entries a_i of A for the variables y_i.

This phenomenon that a special determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ given by a matrix $A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ inherits a free resolution of its local ring $\mathcal{O}_{X_0,0}$ as an $\mathcal{O}_N$-module from its generic singularity $(M^t_{m,n}, 0) \subset (\mathrm{Mat}(m, n; \mathbb{C}), 0)$ holds in general, as we shall see now. It is the key ingredient for the existence of determinantal deformations.

1.3.2 Inheritance of Projective Resolutions

We start by observing that the term "Koszul complex" itself is merely a name for an algorithm or pattern for how to construct a complex for a given set of c elements in a ring. In [19], Eagon and Northcott extract and describe this fact in their notion of a *universal complex* on c parameters. Apparently, being algebraists, they seem to be targeted on the greatest possible generality of their results and thus work over $\mathbb{Z}$ instead of any field. For us, however, it is more convenient to work over a field k ($\mathbb{Q}$ or $\mathbb{C}$ will do) because the description of projective resolutions of the generic determinantal ideals is easier in this case. For example, we will be able to use the results by Lascoux [49] on free resolutions of the ideals defining the generic determinantal varieties $M^t_{m,n}$.

We will reformulate the definitions and their theorem in this sense.

Definition 1.3.5 ([19])**.** Let $k[\underline{Y}] = k[Y_1, \dots, Y_p]$ be a polynomial ring over a field k. A complex

$$\mathbf{K}_\bullet(Y_1, \dots, Y_p) : \; 0 \longleftarrow K_0 \longleftarrow K_1 \longleftarrow K_2 \longleftarrow \cdots$$

of projective $k[\underline{Y}]$-modules is called a *universal projective complex on p parameters over k.*

Given any universal projective complex on p parameters $\mathbf{K}_\bullet(Y_1, \dots, Y_p)$ over a field k and p elements $a_1, \dots, a_p$ in a k-algebra R, we obtain a new complex of projective R-modules as follows. There is a unique homomorphism $\phi : k[\underline{Y}] \to R$ sending 1 to 1 and Y_i to a_i, which makes R into a $k[\underline{Y}]$-module. Set

$$\mathbf{K}_\bullet(a_1, \dots, a_p; R) := \mathbf{K}_\bullet(Y_1, \dots, Y_p) \otimes_{k[\underline{Y}]} R. \tag{1.17}$$

Each term of this complex is naturally a projective R-module. We write

$$\begin{aligned} M(\underline{a}; R) &:= \mathrm{coker}(K_1(\underline{a}; R) \to K_0(\underline{a}; R)), && (1.18)\\ I(\underline{a}; R) &:= \mathrm{Ann}_R\, M(\underline{a}; R), && (1.19) \end{aligned}$$

for the natural augmentation module and its annihilator.

Theorem 1.3.6 ([19],Proposition 4)**.** *Let k be a field, $I \subset k[\underline{Y}] = k[Y_1, \dots, Y_p]$ a proper ideal, and*

$$0 \longleftarrow k[\underline{Y}]/I \longleftarrow K_0 \longleftarrow K_1 \longleftarrow \cdots \longleftarrow K_\mu \longleftarrow 0 \tag{1.20}$$

a projective resolution of $k[\underline{Y}]/I$ over $k[\underline{Y}]$ of minimal length, i.e. $\mu = \mathrm{pd}(k[\underline{Y}]/I)$. If R is any Noetherian k-algebra and $a_1, \dots, a_p \in R$, then

$$\mathrm{grade}(I(\underline{a}; R), R) \leq \mu;$$

and if equality holds, then

$$0 \longleftarrow R/I(\underline{a}; R) \longleftarrow K_0(\underline{a}; R) \longleftarrow \cdots \longleftarrow K_\mu(\underline{a}; R) \longleftarrow 0$$

is a projective resolution of $R/I(\underline{a}; R)$. In particular $I(\underline{a}; R)$ is a perfect ideal.

Remark 1.3.7. In [19], the authors formulate this result for ideals I in the ring $\mathbb{Z}[\underline{Y}]$, for which $\mathbb{Z}[\underline{Y}]/I$ is a torsion-free $\mathbb{Z}$-module. They show that in this case, the resolution (1.20) is a *generically acyclic complex* K. But the only step in the proof of Theorem 1.3.6 where they really use this fact is when they deduce that for any ring R, the complex

$$K \otimes_{\mathbb{Z}} R$$

is again exact. If we work over a field k and take the tensor product with a k-algebra R, then this is a trivial fact, since R is naturally flat as a k-module.

Remark 1.3.8. Theorem 1.3.6 also holds if we work over $\mathbb{C}$ and replace $k[\underline{Y}]$ by the ring of convergent power series $\mathbb{C}\{\underline{Y}\}$. In this case, we have to restrict to R being another analytic algebra for the ring homomorphism $\mathbb{C}\{\underline{Y}\} \to R$ to make sense.

Theorem 1.3.9. *Let $f : (\mathbb{C}^q, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map and $(V, 0) \subset (\mathbb{C}^p, 0)$ a germ defined by an ideal $\langle h_1, \dots, h_r \rangle$ such that $\mathcal{O}_{V,0}$ is Cohen-Macaulay. If the preimage $(X_0, 0) := f^{-1}(V, 0) \subset (\mathbb{C}^q, 0)$ has expected codimension*

$$\mathrm{codim}(X_0, 0) = \mathrm{codim}(V, 0),$$

then for any local $\mathbb{C}$-algebra $(B, \mathfrak{n})$ and unfolding F of f over B, the ring

$$\mathcal{O}_q \hat{\otimes}_{\mathbb{C}} B / \langle F^* h_1, \dots, F^* h_r \rangle$$

is a flat B-module.

$$\begin{tikzcd}
(X_0,0) \arrow[r] \arrow[d] & (X,0) \arrow[r] \arrow[d] & (V,0) \arrow[d] \\
(\mathbb{C}^q,0) \arrow[r] \arrow[d] & (\mathbb{C}^q,0) \times \mathrm{Spec}\, B \arrow[r, "F"] \arrow[d] & (\mathbb{C}^p,0) \\
\{0\} \arrow[r] & \mathrm{Spec}\, B &
\end{tikzcd}$$

In other words, if we let $(X, 0) := F^{-1}(V, 0) \subset (\mathbb{C}^q, 0) \times \mathrm{Spec}\, B$, then the family given by the projection $(X, 0) \to \mathrm{Spec}\, B$ is a deformation of $(X_0, 0)$.

Proof. Again, we first prove the statement for $B = \mathcal{O}_k$ for some k. Since both $\mathcal{O}_p$ and $\mathcal{O}_{V,0}$ are Cohen-Macaulay, the Auslander Buchsbaum formula gives

$$\operatorname{pd}_{\mathcal{O}_p} \mathcal{O}_{V,0} = \operatorname{codim}(V,0) =: c,$$

and by assumption, this coincides with $\operatorname{codim}(X_0,0)$. We want to show that also the total space $(X,0)$ has expected codimension. Let $S = \mathcal{O}_{q+k}$ be the local ring of the ambient space $(\mathbb{C}^q,0) \times (\mathbb{C}^k,0)$ of $(X,0)$ and $J \subset S$ its defining ideal. Due to Theorem 1.2.23, we conclude

$$\dim \mathcal{O}_{X,0} \leq \dim \mathcal{O}_k + \dim \mathcal{O}_{X_0,0} = k + q - c = \dim S - c.$$

Since S is Cohen-Macaulay, we deduce from Theorem 1.2.15

$$\operatorname{codim} J = \operatorname{height} J = \operatorname{grade}(J, \mathcal{O}_{q+k}).$$

But J is nothing but $I(\underline{F}; \mathcal{O}_{q+k})$ for the complex $\mathbf{K}_\bullet$ coming from a minimal free resolution of $\mathcal{O}_{V,0}$. Because $(V,0)$ was assumed to be Cohen-Macaulay, the length μ of the free resolution $\mathbf{K}_\bullet$ is equal to $c = \operatorname{codim}(V,0)$. From Theorem 1.3.6, it follows that $\mathbf{K}_\bullet(\underline{F}; \mathcal{O}_{q+k})$ gives a free resolution of $\mathcal{O}_{X,0}$ over $\mathcal{O}_{q+k}$.

Now, flatness of $\mathcal{O}_{X,0}$ over $\mathcal{O}_k$ follows from Theorem 1.2.24. Alternatively, one could argue that $\mathbf{K}_\bullet(\underline{f}; \mathcal{O}_q)$ is a free resolution of $\mathcal{O}_{X_0,0}$ over $\mathcal{O}_q$ and that the resolution $\mathbf{K}_\bullet(\underline{F}; \mathcal{O}_q)$ specializes to it. In this case, flatness follows from Lemma 1.2.4.

For general $B = \mathcal{O}_k/T$, we conclude as in the proof of Theorem 1.3.1, using a lift and the flatness of base change. □

Remark 1.3.10. From the proof of Theorem 1.3.9 we see that for the preimage $(X_0,0) = f^{-1}(V,0)$ of a Cohen-Macaulay germ $(V,0)$, not only every perturbation of f induces a well behaved deformation of $(X_0,0)$, but $(X_0,0)$ inherits a free resolution of $\mathcal{O}_{X_0,0}$ from a free resolution of $\mathcal{O}_{V,0}$. Moreover, this free resolution is preserved under deformations. This is why we need to have a good understanding of the generic determinantal varieties.

1.3.3 The Generic Determinantal Singularities

We start by observing that all the ideals defining the varieties $M^t_{m,n}$ are homogeneous polynomial ideals. We will therefore in this section consider graded resolutions. They coincide with minimal resolutions when seen as analytic ideals in $\mathcal{O}_{m\cdot n}$. As usual for a graded ring

$$S = \bigoplus_{d\in\mathbb{Z}} S_d,$$

we will denote the degree d part by S_d. Any module M over S will be assumed to be graded as well, i.e. $M = \bigoplus_{d\in\mathbb{Z}} M_d$, and multiplication by elements of S has to be graded in the sense that degrees add up:

$$\cdot : S_d \times M_e \to M_{e+d} \subset M.$$

A homomorphism of graded modules $\varphi : M \to N$ preserves degrees. For any graded module M, we denote by $M(d)$ the same module with its grading shifted by d, i.e.

$$(M(d))_e = M_{e+d}. \tag{1.21}$$

For details on graded resolutions and their parallelism to minimal resolutions over local rings, see e.g. [22] or [10].

Consider $X = M^m_{m,n}$ for $m < n$, i.e. the variety cut out by the maximal minors of the matrix

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,n} \end{pmatrix} \in \mathrm{Mat}(m, n; \mathcal{O}_{m\cdot n}). \tag{1.22}$$

Let $R = \mathbb{C}[\underline{y}] := \mathbb{C}[y_{1,1}, \dots, y_{m,n}]$ and $I = \langle Y^{\wedge m} \rangle \subset R$. A minimal graded resolution of R/I is given by the Eagon-Northcott complex first described by Eagon and Northcott in [18]. We will reproduce the exposition of its construction from [22, Appendix A2.6]. The Eagon-Northcott complex is constructed for arbitrary rings R and homomorphisms $\varphi : R^n \to R^m$. But the reader may keep in mind the case $R = \mathbb{C}[\underline{y}]$ and φ the homomorphism represented by the matrix Y in the following.

Let $\varphi : R^n \to R^m$ be a homomorphism of free modules over a ring R. Let $\underline{x} = (x_1, \dots, x_m)$ be the canonical generators of the free module R^m and set $S = R[x_1, \dots, x_m]$. We consider S as a graded R-algebra with its natural grading by degree in $\underline{x}$.

There is a canonical map of S-modules induced by φ:

$$\Phi : R^n \otimes_R S(-1) \to S, \quad e_j \otimes 1 \mapsto \sum_i y_{i,j} \cdot x_j.$$

Here the e_j are the standard generators of R^n. We may identify $R^n \otimes_R S(-1)$ with $S(-1)^n$. Also, we have

$$\bigwedge^p S(d)^r \cong S(p \cdot d)^{\binom{r}{p}} \cong S(p \cdot d) \otimes_R \bigwedge^p R^r. \tag{1.23}$$

Consider the Koszul complex associated to Φ on S:

$$0 \longleftarrow \bigwedge^0 S(-1)^n \xleftarrow{\Phi} \bigwedge^1 S(-1)^n \longleftarrow \cdots \longleftarrow \bigwedge^n S(-1)^n \longleftarrow 0\,.$$

Using (1.23), we obtain

$$0 \longleftarrow S \xleftarrow{\Phi} S(-1) \otimes \bigwedge^1 R^n \longleftarrow \cdots \longleftarrow S(-n) \otimes \bigwedge^n R^n \longleftarrow 0\,. \tag{1.24}$$

This can be viewed as a complex of graded, free R-modules. We now dualize over R applying $\mathrm{Hom}_R(-, R)$. To fix notation we let

$$S^* := \mathrm{Hom}_R(S, R), \quad S^*_d = \mathrm{Hom}_R(S_d, R). \tag{1.25}$$

The S_d^* give a natural grading on S^*. Also, although a priori being only an R-module, S^* has a natural structure as an S-module via

$$(s, \varphi) \mapsto s \cdot \varphi := \varphi \circ (a \mapsto s \cdot a).$$

Note, however, that the grading is in a sense reversed, i.e. we have

$$\cdot : S_d \times S_e^* \to S_{e-d}^*.$$

Identifying $\mathrm{Hom}_R(\bigwedge^p R^n, R)$ with $\bigwedge^{n-p} R^n$ using the canonical orientation of $\bigwedge^n R^n$ in (A.9) as usual, the dualized Koszul complex takes the form

$$0 \longrightarrow S^* \otimes \bigwedge^n R^n \longrightarrow S^*(-1) \otimes \bigwedge^{n-1} R^n \longrightarrow \cdots$$

$$\cdots \longrightarrow S^*(-n+1) \otimes \bigwedge^2 R^n \longrightarrow S^*(-n) \otimes \bigwedge^1 R^n \longrightarrow 0 \tag{1.26}$$

Now for $d \in \mathbb{N}_0$, consider the degree d part, a so called *strand*, of the complex:

$$0 \longrightarrow S_d^* \otimes \bigwedge^n R^n \longrightarrow S_{d-1}^* \otimes \bigwedge^{n-1} R^n \longrightarrow S_{d-2}^* \otimes \bigwedge^{n-2} R^n \longrightarrow \cdots$$

$$\cdots \longrightarrow S_1^* \otimes \bigwedge^{n-d+1} R^n \longrightarrow S_0^* \otimes \bigwedge^{n-d} R^n \longrightarrow 0 \tag{1.27}$$

Note that, since $S_k^* = 0$ for all $k < 0$, this complex ends prematurely if $d < n$. Something similar happens for the degree e part of (1.24):

$$0 \longleftarrow S_e \xleftarrow{\Phi} S_{e-1} \otimes \bigwedge^1 R^n \longleftarrow \cdots \tag{1.28}$$

$$\cdots \longleftarrow S_1 \otimes \bigwedge^{e-1} R^n \longleftarrow S_0 \otimes \bigwedge^e R^n \longleftarrow 0$$

For $e = n-m-d$ we now define a "splice map" ε from the right end of (1.27) to the degree e part of the right end of (1.28) as follows. We can identify

$$S_0^* \otimes \bigwedge^{n-d} R^n \cong \mathrm{Hom}_R(R, R) \otimes \bigwedge^{n-d} R^n \cong \bigwedge^{n-d} R^n$$

and

$$S_0 \otimes \bigwedge^{n-m-d} R^n \cong R \otimes \bigwedge^{n-m-d} R^n \cong \bigwedge^m R^m \otimes \bigwedge^{n-m-d} R^n.$$

Now $\varepsilon : \bigwedge^{n-d} R^n \to \bigwedge^m R^m \otimes \bigwedge^{n-m-d} R^n$ is given by the contraction with $\varphi^{\wedge m}$ (see Appendix A.1, equation (A.13) for a definition).

One can check that the splice map ε is indeed compatible with the other differentials. Thus we obtain a family of complexes

$$\begin{array}{c} 0 \longrightarrow S_d^* \otimes \bigwedge^n R^n \longrightarrow S_{d-1}^* \otimes \bigwedge^{n-1} R^n \longrightarrow \cdots \\ \cdots \longrightarrow S_0^* \otimes \bigwedge^{n-d} R^n \xrightarrow{\varepsilon} S_0 \otimes \bigwedge^{n-m-d} R^n \longrightarrow \cdots \\ \cdots \longrightarrow S_{n-m-d-1} \otimes \bigwedge^1 R^n \longrightarrow S_{n-m-d} \longrightarrow 0 \end{array} \tag{1.29}$$

for all $d \in \mathbb{N}$.

Definition 1.3.11. The *Eagon-Northcott complex* is defined to be (1.29) for $d = n - m$.

The family of complexes for $d < n - m$ consists of the *Buchsbaum-Rim complexes*.

For $d = n - m$, i.e. if (1.29) is the Eagon-Northcott complex, the splice map takes the following form:

$$\varepsilon : \bigwedge^m R^n \to R, \quad e_I \mapsto \varphi_{I,K}^{\wedge m},$$

where $K = (1, \ldots, m)$ is the only possible ordered multiindex of degree m. Thus, the image of ε is the ideal generated by the maximal minors of the matrix representing φ. Also, the target of ε is the last nonzero term in the complex.

Theorem 1.3.12. *In case $R = \mathbb{C}[\underline{y}]$, and φ represented by Y as in 1.22, the Eagon-Northcott gives a free graded resolution of $R/\langle Y^{\wedge m}\rangle$. Also the Buchsbaum-Rim complexes are exact except at their right end and, hence, describe a free resolution of their final cokernel.*

Proof. See [18] for the Eagon-Northcott complex. Both cases including the Buchsbaum-Rim complex are treated in [22, Theorem A2.10][3]. Another construction of the Eagon-Northcott complex is described in [12]. □

Corollary 1.3.13. *For all $n > m > 0$, the variety $M_{m,n}^m \subset \operatorname{Mat}(m, n; \mathbb{C})$ is Cohen-Macaulay at the origin.*

Proof. This follows directly from the Auslander-Buchsbaum formula 1.2.16 and the fact that the length of the Eagon-Northcott complex is equal to the codimension: $n - m + 1$. □

Remark 1.3.14. We include the Buchsbaum-Rim complexes in Theorem (1.3.12), because of the following fact. Let $d = n - m - 1$. Then (1.29) ends with

$$\cdots \longrightarrow S_0^* \otimes \bigwedge^{m+1} R^n \xrightarrow{\varepsilon} S_0 \otimes \bigwedge^1 R^n \xrightarrow{\varphi} S_1.$$

Thus the first Buchsbaum-Rim complex gives a free resolution of the module presented by φ.

[3] The proof in [22] is formulated for more general situations. But historically, the exactness of the Eagon-Northcott complex was proved for this case first, while the general case was deduced using generically acyclic complexes as introduced above.

Giving resolutions of the rings of the $M^t_{m,n}$ for $t < \min\{m,n\}$, i.e. the case of non-maximal minors, has taken much longer. Lascoux was the first one to construct them over rings R which contain the rationals, $\mathbb{Q}$, in [49]. Working over $\mathbb{Q}$ was necessary for him, since he heavily used representation theory and Schur functors. His ideas were then picked up and people tried to carry over his results to the integers. For example, in [2] the case of $(m-1)$-minors of $m \times n$-matrices is treated.

It should be pointed out, however, that despite the fact that Lascoux was the first to give explicit resolutions, Hochster and Eagon proved the following theorem already seven years earlier in [46].

Theorem 1.3.15 ([46], Corollary 4). *Let Y be as in (1.22) and $R = K[\underline{y}] = K[(y_{i,j})_{i,j}]$ for a Noetherian domain K. The ideal $\langle Y^{\wedge t}\rangle \subset R$ is perfect of grade $(m-t+1)(n-t+1)$ for all $0 < t \leq \min\{m,n\}$.*

In particular, if K is a field, $(M^t_{m,n}, 0) \subset (\mathrm{Mat}(m,n;\mathbb{C}), 0)$ is Cohen-Macaulay.

In the proof the authors show that the length of a minimal free resolution must be equal to $(m+t-1)(n+t-1)$. However, they do not construct it.

We are now in the position to prove Theorem 1.3.2.

Proof. (of Theorem 1.3.2) From Theorem 1.3.15 we know that every generic determinantal singularity $(M^t_{m,n}, 0) \subset (\mathrm{Mat}(m,n;\mathbb{C}), 0)$ is Cohen-Macaulay. Hence, Theorem 1.3.9 is applicable and the result follows. □

Finally we may furnish an explicit corollary out of what has already been hinted in Remark 1.3.10.

Corollary 1.3.16. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m,n,t) given by a matrix $A \in \mathrm{Mat}(m,n;\mathbb{C})$. Then a minimal free resolution of $\mathcal{O}_{X_0,0}$ as an $\mathcal{O}_N$-module is given by substituting the entries $a_{i,j}$ of A for the $y_{i,j}$ in a minimal graded resolution of $\mathcal{O}_{M^t_{m,n},0}$ as an $\mathcal{O}_{m\cdot n}$-module.*

Proof. This directly follows from Theorem 1.3.6 and the fact that $(M^t_{m,n}, 0)$ is Cohen-Macaulay 1.3.15. Also, one uses that minimal graded resolutions over $\mathbb{C}[\underline{y}]$ give minimal free resolutions over $\mathbb{C}\{\underline{y}\}$. □

1.4 Versal Families

Any matrix $A \in \mathrm{Mat}(m,n;\mathcal{O}_N)$ defining a determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ can be regarded as a map germ

$$A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m,n;\mathbb{C}), 0)$$

and determinantal deformations of $(X_0, 0)$ are precisely those coming from a perturbation of A. Hence, the deformation theory of $(X_0, 0)$ is closely related to unfoldings of the map A.

In this section we will first develop a construction of semi-universal unfoldings of map germs into the space of matrices. Then we will discuss its implications for the determinantal deformations of a determinantal singularity and finally compare the determinantal deformations with the semi-universal deformation of the underlying space germ.

1.4.1 Versal Unfoldings of Map Germs into the Space of Matrices

The first step in the development of the theory of *versal unfoldings* is an adequate notion of equivalence of maps. What we will present now is basically $\mathcal{K}_V$-equivalence - an idea originally suggested by James Damon. In his book [16] he treats a more general case and we will reformulate many of his results there explicitly for our purposes. In this process we are, of course, able to drastically simplify the exposition.

Since we are dealing with maps to the space of matrices $\mathrm{Mat}(m, n; \mathbb{C})$ and we are doing geometry, we might prefer to work independently of any chosen basis or local coordinates. This suggests the following definition.

Definition 1.4.1. Two map germs $A_1, A_2 : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ are called equivalent if there is a germ of an analytic diffeomorphism $\Phi : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ at 0 and two germs

$$F : (\mathbb{C}^N, 0) \to \mathrm{GL}(m; \mathcal{O}_N), \quad G : (\mathbb{C}^N, 0) \to \mathrm{GL}(n; \mathcal{O}_N)$$

such that

$$A_2 = F \cdot (\Phi^* A_1) \cdot G^{-1}.$$

The reader may note the compatibility with the definition of equivalence of determinantal singularities 1.1.4. Note that we do not require that $0 \in \mathbb{C}^N$ is mapped to the zero matrix. For determinantal singularities the notion of minimality allowed us to reduce to this case. But for reasons that will become appearent later, it will be more convenient to consider this more general setup in the context of map germs. If we forget about the matrix structure of the target space, we can, however, assume that P is just the origin of some $\mathbb{C}^p$ again.

Definition 1.4.2. An unfolding of a map germ $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ over a local $\mathbb{C}$-algebra $(B, \mathfrak{n})$ is a commutative diagram

$$\begin{tikzcd}
(\mathbb{C}^N, 0) \arrow[d, "{(\mathrm{id},0)}"'] \arrow[dr, "f"] & \\
(\mathbb{C}^N, 0) \times \operatorname{Spec} B \arrow[r, "F"] \arrow[dr, "\pi_2"] & (\mathbb{C}^p, 0) \times \operatorname{Spec} B \arrow[d, "\pi_2"] \\
& \operatorname{Spec} B
\end{tikzcd} \tag{1.30}$$

where π_2 is the projection to the second factor.
If $B = \mathcal{O}_q$ for some q, then $\operatorname{Spec} B$ stands for $(\mathbb{C}^q, 0)$ and F is called an *unfolding* of f on q parameters.
If $(B, \mathfrak{n})$ is Artinian, then F is called an *infinitesimal unfolding* of f.
If furthermore $\mathfrak{n}^2 = 0$, then F is called a *first order unfolding* of f.

Remark 1.4.3. The commutativity of the lower triangle of (1.30) implies that F corresponds to a morphism of B-algebras. Geometrically this means that, if we interpret F as a function on two arguments $(x, u) \in \mathbb{C}^N \times \operatorname{Spec} B$, then $F(x, u) = (\tilde{F}(x, u), u) \in \mathbb{C}^p \times \operatorname{Spec} B$ for some $\tilde{F}(x, u)$, which takes values in $\mathbb{C}^p$. In other words $\tilde{F} \in (\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B)^p$. Since F is compeletely determined by commutativity over B and $\tilde{F}$, we will, by abuse of notation, also write $F \in (\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B)^p$.

For any map $\Psi : \operatorname{Spec} B' \to \operatorname{Spec} B$ of local $\mathbb{C}$-algebras and F an unfolding of f over B, we obtain an *induced unfolding* $F' = \Psi^* F$ via Ψ

$$\begin{tikzcd}
(\mathbb{C}^N, 0) \times_{\operatorname{Spec} B} \operatorname{Spec} B' \arrow[r] \arrow[d] \arrow[rr, bend left, "\tilde{F}'"] & (\mathbb{C}^N, 0) \times \operatorname{Spec} B \arrow[r, "\tilde{F}"] \arrow[d] & (\mathbb{C}^p, 0) \\
\operatorname{Spec} B' \arrow[r, "\Psi"] & \operatorname{Spec} B &
\end{tikzcd} \tag{1.31}$$

over $\operatorname{Spec} B'$. In terms of functions this means

$$\tilde{F}'(x, v) = (\Psi^* F)^{\sim}(x, v) = \tilde{F}(x, \Psi(v)).$$

Remark 1.4.4. If we want to classify unfoldings of maps, it is no restriction to consider only unfoldings with smooth base $(\mathbb{C}^k, 0)$, i.e. over power series rings $\mathcal{O}_k$. If B is any analytic algebra, then it is of the form $\mathcal{O}_k/J$ for some ideal $J \subset \mathcal{O}_k$. Now, if $\tilde{F} \in (\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B)^p$ is an unfolding of $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$, we can consider any lift

$$\tilde{F}' \in (\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} \mathcal{O}_k)^p$$

of $\tilde{F}$. This is an unfolding of F over $\mathcal{O}_k$ and $\tilde{F}$ is induced from $\tilde{F}'$ via the natural map $\mathcal{O}_k \to B$.

Definition 1.4.5. Let $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), P)$ be a given map germ. Two unfoldings

$$\mathbf{A}_1, \mathbf{A}_2 : (\mathbb{C}^N, 0) \times \operatorname{Spec} B \to (\operatorname{Mat}(m, n; \mathbb{C}), 0) \times \operatorname{Spec} B$$

of A over B are called equivalent if there is an unfolding of the identity

$$\Phi : (\mathbb{C}^N, 0) \times \operatorname{Spec} B \to (\mathbb{C}^N, 0) \times \operatorname{Spec} B, \quad \Phi|_{(\mathbb{C}^N,0)\times\{0\}} = \operatorname{id}_{(\mathbb{C}^N,0)},$$

and two unfoldings of map germs

$$\begin{aligned} F &\in \operatorname{GL}(m; \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B), \qquad & F \otimes_B \mathbb{C} &= \mathbf{1}_m \\ G &\in \operatorname{GL}(n; \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B), \qquad & G \otimes_B \mathbb{C} &= \mathbf{1}_n \end{aligned}$$

such that

$$\mathbf{A}_2 = F \cdot (\Phi^* \mathbf{A}_1) \cdot G^{-1}$$

as unfoldings over B.

A unfolding $\mathbf{A}$ of A over B is trivial, if $\mathbf{A}$ is equivalent to the unfolding given by $A \hat{\otimes}_{\mathbb{C}} 1_B$.

Definition 1.4.6. For a given $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), P)$ let $\operatorname{Inf}(A)$ be the space of equivalence classes of unfoldings of A over $\mathbb{C}[\varepsilon]/\varepsilon^2$.

Remark 1.4.7. $\operatorname{Inf}(A)$ is what Damon calls the *extended tangent space* to A in $\operatorname{Mat}(m, n; \mathcal{O}_N)$ in [16].

Lemma 1.4.8. *The space* $\mathrm{Inf}(A)$ *for a given* $A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ *is canonically isomorphic to the* $\mathcal{O}_N$*-module*

$$\mathrm{Mat}(m, n; \mathcal{O}_N) / \left\langle \frac{\partial A}{\partial x_i} : i = 1, \ldots, N \right\rangle + \langle \mathrm{im}(g) \rangle \tag{1.32}$$

where $\mathrm{im}(g)$ *is the image of the map*

$$g : \mathrm{Mat}(m, m; \mathcal{O}_N) \times \mathrm{Mat}(n, n; \mathcal{O}_N) \to \mathrm{Mat}(m, n; \mathcal{O}_N), \quad (F, G) \mapsto F \cdot A + A \cdot G.$$

Proof. The algebra $B := \mathbb{C}[\varepsilon]/\varepsilon^2$ has a natural splitting as $\mathbb{C} \oplus \varepsilon \cdot \mathbb{C}$, which induces an isomorphism

$$V \hat{\otimes}_{\mathbb{C}} \mathbb{C}[\varepsilon]/\varepsilon^2 \cong V \oplus \varepsilon \cdot V$$

for any $\mathbb{C}$-vector space V. An unfolding $\tilde{A}$ of A over B is, hence, given by

$$\tilde{A} = A_0 \oplus \varepsilon \cdot A_1 \in \mathrm{Mat}(m, n; \mathcal{O}_N) \hat{\otimes}_{\mathbb{C}} B$$

with $A_0 = A$. From this we see that $\tilde{A}$ is trivial if and only if $A_1 = 0$.

Suppose $\tilde{A}$ is equivalent to the trivial unfolding. Then there exist

$$\begin{array}{rcl} \Phi = \mathrm{id}_{\mathbb{C}^N, 0} + \varepsilon \cdot \Phi_1(\underline{x}) & \in & \mathcal{O}_N^N \hat{\otimes}_{\mathbb{C}} B \\ F = \mathbf{1}_m + \varepsilon \cdot F_1(\underline{x}) & \in & \mathrm{Mat}(m, m; \mathcal{O}_N) \hat{\otimes}_{\mathbb{C}} B \\ G = \mathbf{1}_n + \varepsilon \cdot G_1(\underline{x}) & \in & \mathrm{Mat}(n, n; \mathcal{O}_N) \hat{\otimes}_{\mathbb{C}} B \end{array}$$

such that

$$\tilde{A} = A_0 + \varepsilon A_1 = F \cdot (\Phi^*(A + \varepsilon \cdot 0)) \cdot G^{-1}.$$

Performing the Taylor expansion for the entries of $A = A_0$ and G^{-1}, we obtain for the right hand side

$$\begin{aligned} & (\mathbf{1}_m + \varepsilon \cdot F_1) \cdot \left(A_0 \circ \mathrm{id}_{(\mathbb{C}^N, 0)} + \varepsilon \cdot \sum_{i=1}^N \Phi_1^i \frac{\partial A_0}{\partial x_i} \circ \mathrm{id}_{(\mathbb{C}^N, 0)} \right) \cdot (\mathbf{1}_n - \varepsilon \cdot G_1) \\ = \quad & A_0 + \varepsilon \cdot \left(F_1 \cdot A_0 + \left(\sum_{i=1}^N \Phi_1^i \frac{\partial A_0}{\partial x_i} \right) - A_0 \cdot G_1 \right), \end{aligned}$$

where Φ_1^i is the i-th component of Φ_1. It follows that $A_1 \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ is a trivial unfolding if and only if it is zero in the quotient (1.32). □

Lemma 1.4.8 gives the set $\mathrm{Inf}(A)$ of a given map germ $A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ the nice structure of a finitely generated $\mathcal{O}_N$-module. By definition it covers all equivalence classes of unfoldings over $\mathbb{C}[\varepsilon]/\varepsilon^2$. If $\kappa := \dim_{\mathbb{C}} \mathrm{Inf}(A) < \infty$, we can choose elements $G_1, \ldots, G_\kappa \in \mathrm{Mat}(m, n; \mathcal{O}_N)$, which reduce to a $\mathbb{C}$-basis of $\mathrm{Inf}(A)$. From this we built the unfolding

$$\mathbf{A} = A + \sum_{i=1}^{\kappa} u_i \cdot G_i \tag{1.33}$$

over $B := \mathbb{C}[u_1, \ldots, u_\kappa]/\langle u_1, \ldots, u_\kappa \rangle^2$ with coordinates $u_1, \ldots, u_\kappa$.

Let $\mathfrak{n} = \langle u_1, \dots, u_\kappa \rangle$ be the maximal ideal. The tangent space to $(\operatorname{Spec} B, 0)$ at 0 is

$$0.T_{\mathbb{C}^\kappa} \cong \operatorname{Hom}_{\mathbb{C}}(\mathfrak{n}/\mathfrak{n}^2, \mathbb{C}) = (\mathfrak{n}/\mathfrak{n}^2)^\vee.$$

There is a canonical isomorphism $\psi : T_{\operatorname{Spec} B,0} \xrightarrow{\cong} \operatorname{Inf}(A)$ given by

$$\psi : \varphi \mapsto \sum_{i=1}^{\kappa} \varphi(u_i) \cdot G_i. \tag{1.34}$$

This generalizes to arbitrary local $\mathbb{C}$-algebras $(B', \mathfrak{n}')$ and unfoldings $\tilde{\mathbf{A}}$ of A over B'. Let $v_1, \dots, v_\tau$ be a minimal set of generators of $\mathfrak{n}'/\mathfrak{n}'^2$. If $\varphi \in T_{\operatorname{Spec} B',0}$, then we have

$$\Psi : \varphi \mapsto \sum_{i=1}^{\tau} \varphi(u_i) \cdot \frac{\partial \tilde{\mathbf{A}}}{\partial u_i} \otimes_B B/\mathfrak{n} \in \operatorname{Inf}(A). \tag{1.35}$$

This homomorphism of vector spaces Ψ is called the *determinantal Kodaira-Spencer map* of the unfolding.

In case $\mathfrak{n}'^2 = 0$ in B', i.e. if $\tilde{\mathbf{A}}$ is a first order unfolding, the dual map of $\psi^{-1} \circ \Psi : T_{\operatorname{Spec} B',0} \to T_{\operatorname{Spec} B,0}$ uniquely determines a homomorphism of algebras $\Phi : B \to B'$. It is now easy to see that we necessarily have

$$\tilde{\mathbf{A}} = \mathbf{A} \otimes_B B',$$

i.e. that $\tilde{\mathbf{A}}$ can be written as an unfolding of A induced from $\mathbf{A}$ via Φ.

In other words, the unfolding given by (1.33) is a universal object in the sense that any other first order unfolding can be obtained from it in an essentially unique way. The following definition is the generalization of this idea.

Definition 1.4.9. Let $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathcal{O}_N), P)$ be a map germ. An unfolding $\mathbf{A}$ of A over $(\mathbb{C}^k, 0)$ is called a *versal unfolding* of A if any other unfolding $\mathbf{A}'$ over some complex space germ $(Y, 0)$ is equivalent to an unfolding induced from $\mathbf{A}$ via some map $\Psi : (Y, 0) \to (\mathbb{C}^k, 0)$.

An unfolding $\mathbf{A}$ of A is called *infinitesimally versal* if any first order unfolding of A can be obtained from $\mathbf{A}$ via some homomorphism $\Psi^* : \mathcal{O}_k \to \mathbb{C}[\varepsilon]/\varepsilon^2$.

Moreover, an infinitesimally versal unfolding $\mathbf{A}$ is *semi-universal* if the dimension k of the base is equal to $\dim_{\mathbb{C}} \operatorname{Inf}(A)$.

Theorem 1.4.10. *Let $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), 0)$ be a holomorphic map germ with $\kappa := \dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$, and suppose $G_1, \dots, G_\kappa \in \operatorname{Mat}(m, n; \mathcal{O}_N)$ reduce to a $\mathbb{C}$-basis of $\operatorname{Inf}(A)$. Then the unfolding of A over $(\mathbb{C}^\kappa, 0)$ given by*

$$\mathbf{A} = A + \sum_{i=1}^{\kappa} u_i \cdot G_i \in \operatorname{Mat}(m, n; \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} \mathcal{O}_k),$$

where the u_i are the coordinates of $(\mathbb{C}^k, 0)$, is semi-universal.

The key idea for the proof is extracted in the following reduction lemma.

Lemma 1.4.11. *Let $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), P)$ be a holomorphic map germ and suppose $\mathbf{A} \in \operatorname{Mat}(m, n; \mathcal{O}_{N+1})$ is a 1-parameter unfolding of $\mathbf{A}$ with* *unfolding parameter* v. *If there exists a vector field* $\xi \in T_{\mathbb{C}^{N+1},0}$ *with* $\mathrm{d}v(\xi) = 1$ *and matrices*

$$F \in \mathrm{Mat}(m, m; \mathcal{O}_{N+1}), \quad G \in \mathrm{Mat}(n, n; \mathcal{O}_{N+1})$$

such that

$$\xi(\mathbf{A}) = F \cdot \mathbf{A} + \mathbf{A} \cdot G,$$

then $\mathbf{A}$ *is trivial.*

Proof. Associated to ξ there is a holomorphic flow

$$\Phi : U \times D \to \mathbb{C}^{N+1}$$

defined on some open neighborhood $U \subset \mathbb{C}^{N+1}$ of the origin and some open disk $D \subset \mathbb{C}$ (cf. Theorem A.4.1). As usual, let (x, v) be the coordinates of $\mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$. Denote the second argument of Φ by t. By the assumption $\mathrm{d}v(\xi) = 1$ we have $\Phi((x, v), t) = (x, v + t)$. In particular if we restrict the first argument of Φ to the x-plane $\mathbb{C}^N \times \{0\}$, then

$$\Phi : (U \cap \mathbb{C}^N \times \{0\}) \times D \to \mathbb{C}^{N+1}$$

is a holomorphic diffeomorphism onto its image $W \subset \mathbb{C}^{N+1}$ preserving $t = v$. We will use this as our new coordinate system (x, v) of $\mathbb{C}^{N+1}$ on W around the origin. In these coordinates $\xi = \frac{\partial}{\partial v}$ is the constant vector field.

Now consider the complex ordinary differential equation given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} M(x, t) &= F(x, t) \cdot M(x, t) + M(x, t) \cdot G(x, t) \\ M(x, 0) &= M_0(x) \end{cases} \tag{1.36}$$

for a function M on W taking values in $\mathrm{Mat}(m, n; \mathbb{C})$. According to A.4.5 there exist solution operators $L(x, t) \in \mathrm{GL}(m; \mathbb{C})$ and $R(x, t) \in \mathrm{GL}(n; \mathbb{C})$ for 1.36 describing the solution as

$$M(x, t) = L(x, t) \cdot M_0(x) \cdot R(x, t)$$

on some neighborhood of the origin.

But, since the solution of 1.36 is unique and $\mathbf{A}(x, t)$ is a solution with $\mathbf{A}(x, 0) = A(x)$, in the new coordinates we find

$$\mathbf{A}(x, t) = L(x, t) \cdot A(x) \cdot G(x, t).$$

By the definition of triviality this finishes the proof. $\square$

Proof. (of Theorem 1.4.10). Let $\tilde{\mathbf{A}} \in \mathrm{Mat}(m, n; \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B)$ be any other unfolding of A over some base $\mathrm{Spec}\, B$.

We will first prove the theorem for the case $B = \mathcal{O}_r$ for some $r \in \mathbb{N}$ and then the general case will follow as usual. Let $\underline{v} = (v_1, \ldots, v_r)$ be the coordinates of $(\mathbb{C}^r, 0)$. Then $\tilde{\mathbf{A}}$ is of the form

$$A + \sum_{j=1}^{r} v_r \cdot H_j$$

for some matrices $H_j \in \mathrm{Mat}(m,n;\mathcal{O}_N \hat{\otimes}_{\mathbb{C}} \mathcal{O}_r)$. We will write $\mathcal{O}_{N+\kappa}$ for $\mathbb{C}\{\underline{x},\underline{u}\}$, $\mathcal{O}_{N+r}$ for $\mathbb{C}\{\underline{x},\underline{v}\}$ and $\mathcal{O}_{N+\kappa+r}$ for $\mathbb{C}\{\underline{x},\underline{\kappa},\underline{v}\}$. Consider the "composite unfolding" given by

$$\mathcal{A} := A + \left(\sum_{i=1}^{\kappa} u_i \cdot G_i\right) + \left(\sum_{j=1}^{r} v_j \cdot H_i\right)$$

in $\mathrm{Mat}(m,n;\mathcal{O}_{N+\kappa+r})$ and its associated space of *relative infinitesimal unfolding*

$$\mathrm{Inf}^{\mathrm{rel}}(\mathcal{A}) := \mathrm{Mat}(m,n;\mathcal{O}_{N+\kappa+r}) / \left\langle \frac{\partial \mathcal{A}}{\partial \underline{x}} \right\rangle + \langle F \cdot \mathcal{A} + \mathcal{A} \cdot G \rangle. \tag{1.37}$$

Here $\langle \partial \mathcal{A}/\partial \underline{x} \rangle$ denotes the submodule generated by all partial derivatives with respect to the x_i, and $\langle F \cdot \mathcal{A} + \mathcal{A} \cdot G \rangle$ is the submodule generated by left- and right-multiplication with square matrices F and G in $\mathcal{O}_{N+\kappa+r}$ as usual.

Clearly $\mathrm{Inf}^{\mathrm{rel}}(\mathcal{A})/\langle \underline{u},\underline{v}\rangle \cong \mathrm{Inf}(A)$ and the G_i give a $\mathbb{C}$-basis of it. By the Weierstrass Finiteness Theorem it follows that $\mathrm{Inf}^{\mathrm{rel}}(\mathcal{A})$ is a finite $\mathcal{O}_{\kappa+r}$-module and the $(G_i)_{i=1}^{\kappa}$ generate $\mathrm{Inf}^{\mathrm{rel}}(\mathcal{A})$ over $\mathcal{O}_{\kappa+r}$. We deduce that there is an expression

$$\begin{aligned} \frac{\partial \mathcal{A}}{\partial v_r} &= \sum_{i=1}^{\kappa} a_i(\underline{u},\underline{v}) \cdot G_i + \sum_{j=1}^{N} b_j(\underline{x},\underline{u},\underline{v}) \cdot \frac{\partial \mathcal{A}}{\partial x_j} + \\ & \quad F(\underline{x},\underline{u},\underline{v}) \cdot \mathcal{A} + \mathcal{A} \cdot G(\underline{x},\underline{u},\underline{v}) \end{aligned}$$

in $\mathrm{Mat}(m,n;\mathcal{O}_{N+\kappa+r})$ for some $a = (a_1,\dots,a_\kappa)$, $b = (b_1,\dots,b_N)$, $F \in \mathrm{Mat}(m,m;\mathcal{O}_{N+\kappa+r})$, and $G \in \mathrm{Mat}(n,n;\mathcal{O}_{N+\kappa+r})$. If we let

$$\xi = \frac{\partial}{\partial v_r} - \sum_{i=1}^{\kappa} a_i \cdot \frac{\partial}{\partial u_i} - \sum_{j=1}^{N} b_j \cdot \frac{\partial}{\partial x_j},$$

then we can rewrite this expression as

$$\xi(\mathcal{A}) = F \cdot \mathcal{A} + \mathcal{A} \cdot G$$

with $\mathrm{d}v_r(\xi) = 1$. It follows from Lemma 1.4.11 that $\mathcal{A}$ regarded as an unfolding of $\mathcal{A}|_{\{v_r=0\}}$ by v_r over $(\mathbb{C},0)$ is trivial.

We may therefore change coordinates on $(\mathbb{C}^{N+\kappa+r-1},0)\times(\mathbb{C},0)$ in such a way that v_r is preserved and the new ones agree with the old ones on $(\mathbb{C}^{N+\kappa+r-1},0)\times\{0\} = \{v_r = 0\}$. If we let p_r be the projection to $(\mathbb{C}^{N+\kappa+r-1},0)$ in these new coordinates, then

$$\mathcal{A} = p^* \mathcal{A}|_{v_r=0}.$$

By induction on r we finally obtain a morphism

$$p : (\mathbb{C}^{N+\kappa+r},0) \to (\mathbb{C}^{N+\kappa},0)$$

such that

$$\mathcal{A} = p^* \mathcal{A}|_{\{\underline{v}=0\}} = p^* \mathbf{A}$$

and hence $\tilde{\mathbf{A}} = \mathcal{A}|_{\{\underline{u}=0\}}$ is induced from $\mathbf{A}$.

For arbitrary $B = \mathcal{O}_r/T$ the result follows by taking a lift $\tilde{\mathbf{A}}'$ of the unfolding to $\mathcal{O}_r$ and finding a morphism $\Psi : (\mathbb{C}^r, 0) \to (\mathbb{C}^\kappa, 0)$ such that $\tilde{\mathbf{A}}' = \Psi^*\mathbf{A}$. If we let $\iota : \operatorname{Spec} B \hookrightarrow (\mathbb{C}^r, 0)$ be the (scheme-theoretic) inclusion, then

$$\tilde{\mathbf{A}} = \Psi^*\mathbf{A} \otimes_{\mathcal{O}_r} B = (\Psi \circ \iota)^*\mathbf{A}.$$

□

1.4.2 Versal Determinantal Deformations

If $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$ describes a determinantal singularity $(X_0, 0)$ of type (m, n, t) and $\kappa := \dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$ so that a semi-universal unfolding $\mathbf{A}$ of A exists, then clearly $\mathbf{A}$ also covers all determinantal deformations of $(X_0, 0)$. More precisely, if we let

$$\begin{tikzcd} (X_0,0) \arrow[r, hook] \arrow[d] & (X,0) \arrow[r, hook] \arrow[d, "u"'] & (\mathbb{C}^N,0) \times (\mathbb{C}^\kappa,0) \arrow[dl, "u"] \\ \{0\} \arrow[r, hook] & (\mathbb{C}^\kappa,0) & \end{tikzcd} \tag{1.38}$$

be deformation of $(X_0, 0)$ defined by the ideal $\langle \mathbf{A}^{\wedge t} \rangle \subset \mathcal{O}_{N+\kappa}$, then for any other germ $(Y, 0)$ and determinantal deformation of $(X_0, 0)$ over $\mathcal{O}_{Y,0}$ there is a morphism $\Phi : \mathcal{O}_\kappa \to \mathcal{O}_{Y,0}$, through which the deformation of $(Y, 0)$ is obtained as a pullback of (1.38). In other words: (1.38) is a *versal determinantal deformation* of $(X_0, 0)$. But it is not clear that for all nontrivial unfoldings of A also the space germ $(X_0, 0)$ is deformed in a nontrivial way. What one would like to have is a *semi-universal determinantal deformation*. We give a precise meaning to these notions.

Definition 1.4.12. Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$. A determinantal deformation of $(X_0, 0)$ over an analytic algebra B given by a matrix $\mathbf{A} \in \operatorname{Mat}(m, n; \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} B)$ is called *versal determinantal deformation* of $(X_0, 0)$ if any other deformation of the *space germ* $(X_0, 0)$ coming from an unfolding of A is equivalent to a deformation obtained from the one given by $\mathbf{A}$ via pullback.

A versal determinantal deformation is called *semi-universal* if the dimension of the tangent space of the base B at 0 is minimal among all versal determinantal deformations.

As we already saw in the development of semi-universal unfoldings of map germs, the tangent space of the base of a versal unfolding encodes all first order unfoldings. Requiring the tangent space of the base of a semi-universal determinantal deformation to have minimal dimension, is therefore equivalent to saying that its elements uniquely represent all first order deformations up to equivalence. It follows that, if the deformation over B is semi-universal, then for any other determinantal deformation of $(X_0, 0)$ over some base $(S, 0)$ and the corresponding map $\Psi : (S, 0) \to \operatorname{Spec} B$ the differential $\mathrm{d}\Psi$ is uniquely determined.

But it seems to be difficult to get a control over this minimality. We do not have an analogue of the space $\operatorname{Inf}(A)$, with which we could begin to build a semi-universal deformation of $(X_0, 0)$ by first classifying the first order determinantal deformations.

What we can do, however, is to compare versal determinantal deformations of $(X_0, 0)$ with its semi-universal deformation as a space germ - if they exist. Recall the following fundamental Theorem by Grauert:

Theorem 1.4.13 ([32]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a singularity with $\dim_{\mathbb{C}} T^1_{X_0,0} < \infty$. Then there exists a semi-universal* [4] *analytic deformation*

$$\begin{tikzcd}
(X_0,0) \arrow[r, hook] \arrow[d] & (X,0) \arrow[r, hook] \arrow[d, "u"] & (\mathbb{C}^N,0) \times (S,0) \arrow[dl, "u"] \\
\{0\} \arrow[r, hook] & (S,0) &
\end{tikzcd}$$

of $(X_0, 0)$ over some analytic base $(S, 0)$.

Here, in the context of deformations of space germs, the term *versal* means what it should: Any other deformation $(X', 0) \longrightarrow (S', 0)$ of $(X_0, 0)$ as a space germ is equivalent to one obtained from the versal one via pullback $\Psi : (S', 0) \to (S, 0)$. The interesting part is the minimality condition in the definition of semi-universality. If $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is defined by the ideal $I \subset \mathcal{O}_N$, then the $T^1_{X_0,0}$ is defined as

$$T^1_{X_0,0} := \mathrm{Hom}_{\mathcal{O}_N}(I, \mathcal{O}_{X_0,0})/T_{\mathbb{C}^N,0}. \tag{1.39}$$

Here $\mathrm{Hom}_{\mathcal{O}_N}(I, \mathcal{O}_{X_0,0})$ is the stalk of the normal bundle of $(X_0, 0)$ in $(\mathbb{C}^N, 0)$ at the origin, and the action by elements $\xi \in T_{\mathbb{C}^N,0}$ is given by

$$\xi : I \to \mathcal{O}_{X_0,0}, \quad f \mapsto \xi(f) + I.$$

A direct calculation shows that if $(X_0, 0)$ is smooth at 0, then $T^1_{X_0,0} = 0$. Consequently, the coherent analytic sheaf associated to $T^1_{X_0,0}$ is supported in the singular locus of $(X_0, 0)$ and in particular the requirement $\dim_{\mathbb{C}} T^1_{X_0,0}$ in the statement of Theorem 1.4.13 is fulfilled for all isolated singularities. The dimension

$$\tau := \dim_{\mathbb{C}} T^1_{X_0,0}$$

is called the *Tjurina number* of the singularity $(X_0, 0)$.

The space $T^1_{X_0,0}$ is the equivalent object - in the realm of deformations of space germs - of the space $\mathrm{Inf}(A)$ in the following sense: It classifies all first order deformations of $(X_0, 0)$, i.e. deformations over $\mathbb{C}[\varepsilon]/\varepsilon^2$, up to equivalence. For a more detailed treatment of the $T^1_{X_0,0}$, the reader may consult e.g. [7], [64], [38] or also the original article by Schlessinger [63].

Definition 1.4.14. A versal deformation of an arbitrary singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ as a space germ over a base $(S, 0)$ is *semi-universal* if $\dim_{\mathbb{C}} T^1_{X_0,0}$ is equal to the dimension of the tangent space of $(S, 0)$ at 0.

Clearly, the tangent space of the base of a semi-universal deformation is minimal among all versal deformations of the given singularity.

[4] In [32] Grauert speaks of a "versal deformation". However, this translates to the more common notion of a semi-universal deformation, which we shall adapt here.

From the definitions of semi-universal unfoldings of map germs $A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0)$ to the space of matrices and semi-universal deformations of space germs it follows that if we have a determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, t) given by A with $\kappa := \dim_{\mathbb{C}} \mathrm{Inf}(A) < \infty$ and also $\tau := \dim_{\mathbb{C}} T^1_{X_0,0} < \infty$ so that the semi-universal unfolding of A and the semi-universal deformation of $(X_0, 0)$ both exist, then we have a comparison map

$$\Phi : (\mathbb{C}^\kappa, 0) \to (S, 0), \tag{1.40}$$

where $(S, 0)$ is the base of a semi-universal deformation of $(X_0, 0)$. In the following we will discuss this map with the view towards the question, whether or not a semi-universal determinantal deformation can be constructed.

The idea for the proof of Theorem 1.4.13 by Grauert is to use the techniques developed by Schlessinger in [63], from which one obtains the existence of a formal semi-universal deformation in the setup of Theorem 1.4.13. By the latter we mean a deformation over a formal scheme associated to an algebra of the form $\mathbb{C}[[u_1, \dots, u_k]]/T$, where $\mathbb{C}[[\underline{u}]]$ denotes the ring of formal power series. Grauert proved that if such a formal semi-universal deformation exists, then it exists already in the rings of convergent power series and therefore enjoys all the functorial properties provided by the theory developed by Schlessinger plus a concrete geometric realisation.

The application of Schlessingers approach to determinantal deformations has been pursued by M. Schaps in [62]. She gives a criterion for a determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ to have a semi-universal determinantal deformation. However, large parts of [62] are devoted to the exposition of examples, in which these criteria are not met.

Example 1.4.15. i) Let $(X_0, 0) \subset (\mathbb{C}^3, 0)$ of type $(2, 2, 2)$ be given by the matrix

$$A = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

The ideal I of $(X_0, 0)$ is thus generated by the equation $f = x^2 - yz$ and we recognize the well-known A_1 surface singularity. A basis of $\mathrm{Inf}(A)$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence if we let u be the deformation parameter in the semi-universal unfolding of A, then the induced deformation of the space germ $(X_0, 0)$ comes from a perturbation of f by $-u^2$.

The semi-universal deformation of $(X_0, 0)$ as a space germ on the other hand is given by the perturbation of f by a constant v. It follows that the comparison map (1.40) takes the form

$$\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad u \mapsto v = u^2.$$

In other words: The base of the semi-universal unfolding of A is a $2:1$ cover of the base of the semi-universal deformation of $(X_0, 0)$.

ii) (Pinkham, [58]) Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be the cone over the rational normal curve of degree 4. As a determinantal singularity it is given by the

2×2-minors of the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

One easily checks that the following matrices give a basis for $\operatorname{Inf}(A)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and hence $\dim_{\mathbb{C}} \operatorname{Inf}(A) = 3$. Let u_1, u_2 and u_3 be the deformation parameters corresponding to these matrices.

In [58] Pinkham shows explicitly that the semi-universal deformation of $(X_0, 0)$ as a space germ has a base $(S, 0) \subset (\mathbb{C}^4, 0)$ of the following form. Let $v_1, \dots, v_4$ be the coordinates of $\mathbb{C}^4$. Then $(S, 0)$ consists of two components: The plane $H = \{v_4 = 0\}$ and the line $L = \{v_1 = v_2 = v_3 = 0\}$. Furthermore, the comparison map $\Phi : (\mathbb{C}^3, 0) \to (S, 0)$ takes $(\mathbb{C}^3, 0)$ isomorphically to $(H, 0)$. In this case the determinantal deformations of $(X_0, 0)$ coming from A embed as a component of the base of the semi-universal deformation.

The deformation of $(X_0, 0)$ by v_4 along the line L can be described as follows. Its total space over $(L, 0) \cong (\mathbb{C}, 0)$ is given by the 2×2-minors of the matrix

$$B = \begin{pmatrix} x_1 & x_2 & x_3 - v_4 \\ x_2 & x_3 & x_4 \\ x_3 - v_4 & x_4 & x_5 \end{pmatrix}.$$

Note that despite being described by the minors of a matrix, $(X_0, 0)$ does not become a determinantal singularity via B: It does not have expected codimension. However, the induced deformation from the perturbation by v_4 is apparently flat. The reason behind this is that B is a symmetric matrix, and in the appropriate deformation theory for this setup $(X_0, 0)$ does have the correct codimension.

iii) Consider the A_1 threefold singularity in $(\mathbb{C}^4, 0)$ as a determinantal singularity of type $(2, 2, 2)$ via the matrix

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

There are no nontrivial determinantal deformations of this singularity. For the space germ on the other hand we have $T^1_{X_0,0} \cong \mathbb{C}$ so that the comparison map takes the form

$$\Phi : \{\mathrm{pt}\} \to (\mathbb{C}, 0).$$

iv) This example is taken from Schaps [62]. It also appears in [13] and was recently picked up by Frühbis-Krüger in [24], where the computations for the comparison map Φ are carried out explicitly.

Let $(X_0,0) \subset (\mathbb{C}^4,0)$ be the union of the four coordinate axis. This is a determinantal singularity via any matrix

$$\begin{pmatrix} x_1 & \alpha \cdot x_2 & \beta \cdot x_3 & \gamma \cdot x_4 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}$$

for general values $\alpha, \beta, \gamma \in \mathbb{C}$. Using row and column operations and local coordinate changes, one can always bring this matrix to the form

$$A := \begin{pmatrix} x_1 & 0 & x_3 & \gamma' \cdot x_4 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}$$

with $\gamma' \notin \{0,1\}$. One can show that the following matrices give a $\mathbb{C}$-basis of $\operatorname{Inf}(A)$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the base of the semi-universal unfolding of A is $(\mathbb{C}^5,0)$. Let $u_1,\dots,u_5$ be the standard coordinates of this space corresponding to the five matrices above.

Computations of Rim[5] and independently of Buchweitz [13] have shown that the base $(S,0)$ of the semi-universal deformation of $(X_0,0)$ is isomorphic to the cone of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ into $\mathbb{P}^7$ and thus also of dimension 5. Consider the comparison map

$$\Phi : (\mathbb{C}^5,0) \to (S,0).$$

It is easy to see that the perturbation by u_5 alone does not change the ideal generated by the 2-minors of A in $\mathcal{O}_4$: This is a non-trivial deformation of the map germ A which induces a trivial deformation of the underlying space germ! Accordingly, as the computations by Frühbis-Krüger show, Φ is a contraction of the u_5-axis, i.e. the set

$$C := \{(u_1,\dots,u_5) \in \mathbb{C}^5 : u_1 = \dots = u_4 = 0\}$$

is mapped to the point $0 \in S$, but outside C the map Φ is a local diffeomorphism.

It is evident from this list of examples that there is no general pattern of how the comparison map Φ behaves. It neither needs to be injective nor surjective. The last example even shows that Φ can have non-finite fibers.

One could hope that in a semi-universal unfolding of the map A the parameters, which lead to trivial deformations of $(X_0,0)$ can be singled out on an infinitesimal level: If $v \in T_{\mathbb{C}^\kappa}(0) \cong \operatorname{Inf}(A)$ is any element of the tangent space of the base of the semi-universal unfolding, then one could ask whether the induced infinitesimal deformation of $(X_0,0)$ given by $\mathrm{d}\Phi(v)$ is zero. The candidate for the tangent space of a semi-universal determinantal deformation of $(X_0,0)$ would be the quotient $\operatorname{Inf}(A)/\ker \mathrm{d}\Phi$. However, Example 1.4.15 i) also shows that it is pointless to try to construct

[5]The computations are attributed to Rim in [62] without further reference

a semi-universal determinantal deformation of $(X_0, 0)$ from this quotient, since $\mathrm{d}\Phi|_0$ is the zero map.

We conclude this section by sketching a possible step towards a construction of a semi-universal determinantal deformation. However, the author can not precisely estimate the benefits of it.

Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by $A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$. We define

$$\mathrm{Inf}_t(A) := \mathrm{Inf}(A) \otimes_{\mathcal{O}_N} \mathcal{O}_{X_0,0}. \tag{1.41}$$

The following theorem shows that if we start to build a determinantal deformation of $(X_0, 0)$ from the infinitesimal ones described by this space, then we indeed obtain a versal determinantal deformation.

Theorem 1.4.16. *Let $(X_0, 0)$ be as above with $\gamma := \dim_{\mathbb{C}} \mathrm{Inf}_t(A) < \infty$ and let $(G_i)_{i=1}^{\gamma} \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ be matrices, which reduce to a $\mathbb{C}$-basis of $\mathrm{Inf}_t(A)$. The determinantal deformation of $(X_0, 0)$ given by*

$$\mathbf{A} = A + \sum_{i=1}^{\gamma} u_i \cdot G_i \tag{1.42}$$

over $\mathcal{O}_\gamma = \mathbb{C}\{\underline{u}\}$ is a versal determinantal deformation of $(X_0, 0)$.

Remark 1.4.17. There is no guarantee for any minimality of this deformation. Note, however, that $\dim_{\mathbb{C}} \mathrm{Inf}_t(A)$ might be finite also in cases where $\dim_{\mathbb{C}} \mathrm{Inf}(A)$ is not. Thus $\dim_{\mathbb{C}} \mathrm{Inf}(A) < \infty$ need not be a necessary criterion for a versal determinantal deformation of $(X_0, 0)$ to exist.

The proof of Theorem 1.4.16 is straightforward given the proof of Theorem 1.4.10. It also uses a reduction lemma similar to Lemma 1.4.11.

Lemma 1.4.18. *Suppose $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is described by the ideal $I = \langle f_1, \ldots, f_n \rangle \subset \mathcal{O}_N$ and that $F_1, \ldots, F_n \in \mathcal{O}_N \hat{\otimes}_{\mathbb{C}} \mathcal{O}_1 = \mathcal{O}_{N+1}$ are power series such that $F_i(x, 0) = f_i(x)$. Let u be the additional coordinate, i.e. the variable of $\mathcal{O}_1$. Consider the family $(X, 0) \subset (\mathbb{C}^N, 0) \times (\mathbb{C}, 0)$ over $(\mathbb{C}, 0)$ defined by the F_i. If there exists a vector field $\xi \in T_{\mathbb{C}^{N+1},0}$ with $\mathrm{d}u(\xi) = 1$ such that*

$$\xi(F_i) \in \langle F_1, \ldots, F_n \rangle,$$

then there is a commutative diagram

$$\begin{tikzcd} (X,0) \arrow[rr, "\cong"] \arrow[dr, "u"'] & & (X_0,0) \times (\mathbb{C},0) \arrow[dl, "u"] \\ & (\mathbb{C},0) & \end{tikzcd}$$

and the family $(X, 0)$ is trivial.

Proof. Suppose such a vector field $\xi \in T_{\mathbb{C}^{N+1},0}$ exists. Let $\Phi : (\mathbb{C}^N, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}^{N+1}, 0)$ be the flow of ξ associating to any point (x, v) the point in $\mathbb{C}^{N+1}$ reached by traveling for time v starting from $(x, 0) \in \mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$. Because $\mathrm{d}u(\xi) = 1$ the map Φ commutes with u, i.e. $u(\Phi(x, v)) = v$, and since the differential of Φ at the origin is the identity, Φ is a germ of a diffeomorphism. We may, therefore, after a change of coordinates, which preserves $u = v$, assume that $\xi = \frac{\partial}{\partial u}$.

By assumption we find matrices $\Lambda \in \mathrm{Mat}(n, n; \mathcal{O}_{N+1})$ such that

$$\xi(F_i) = \frac{\partial F_i}{\partial u}(x, u) = \sum_{j=1}^{n} \Lambda_{i,j}(x, u) F_j(x, u).$$

Along a flow line through a fixed $x \in \mathbb{C}^{N+1}$ this is a linear differential equation on $F = \begin{pmatrix} F_1 & \cdots & F_n \end{pmatrix}^T$ which, by the general theory on those equations, has a unique solution operator $U_x(u)$. By this we mean a matrix $U_x \in \mathrm{GL}(n; \mathcal{O}_1)$ such that

$$F(x, u) = U_x(u) \cdot F(x, 0)$$

for u small enough. The claim of the lemma follows directly from the observation that U depends holomorphically on x. $\square$

Proof. (of Theorem 1.4.16). The proof is very similar to the proof of Theorem 1.4.10 and we will make use of the notation there.

Suppose $\mathcal{A} \in \mathrm{Mat}(m, n; \mathcal{O}_{N+\gamma+k})$ is a k-parameter unfolding of $\mathbf{A}$ with unfolding parameters $v = (v_1, \dots, v_k)$. We will show that on the level of analytic space germs the deformations by v are trivial.

We define the space of *relative infinitesimal deformations* as

$$\mathrm{Inf}_t^{\mathrm{rel}}(\mathcal{A}) := \mathrm{Inf}^{\mathrm{rel}}(\mathcal{A}) / \langle \mathcal{A}^{\wedge t} \rangle \, \mathrm{Mat}(m, n; \mathcal{O}_{N+\gamma+1}).$$

Just as in the proof of Theorem 1.4.10 we obtain a vector field

$$\xi = \frac{\partial}{\partial v_r} - \sum_i a_i \cdot \frac{\partial}{\partial u_i} - \sum_j b_j \cdot \frac{\partial}{\partial x_j} \in T_{\mathbb{C}^{N+\kappa+1},0}$$

such that

$$\xi(\mathcal{A}) = F(x, u, v) \cdot \mathcal{A} - \mathcal{A} \cdot G(x, u, v) + H$$

with the difference that now we also have a term H, which is a matrix with entries in $\langle \mathcal{A}^{\wedge t} \rangle$.

Consider now $\mathcal{A}^{\wedge t}$ as a vector with $\binom{m}{t} \cdot \binom{n}{t}$ components in $\mathcal{O}_{N+\kappa+1}$. For any two ordered multiindices I and J we write $(\cdot)^{\wedge t}_{I,J}$ for the function on $\mathrm{Mat}(m, n; \mathbb{C})$ associating the minor of rows in I and columns in J to a matrix A. Along ξ we find

$$\begin{aligned} \xi\left(\mathcal{A}^{\wedge t}_{I,J}\right) &= \sum_{i,j} \frac{\partial (\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A}) \cdot \xi(\mathcal{A}_{i,j}) \\ &= \sum_{i,j} \frac{\partial (\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A}) \cdot (F \cdot \mathcal{A} + \mathcal{A} \cdot G + H)_{i,j}\,. \end{aligned}$$

The terms $\frac{\partial(\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A})$ can be computed using row- or column expansion of determinants:

$$\frac{\partial(\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A}) = \begin{cases} 0 & \text{if } i \notin I \text{ or } j \notin J \\ (-1)^{p+q}\mathcal{A}^{\wedge t-1}_{I\setminus\{i\},J\setminus\{j\}} & \text{otherwise} \end{cases} \tag{1.43}$$

where in the second case i is the p-th entry of the ordered multiindex I and j is the q-th of J.

With this at hand we can exand the first two summands on the right hand side:

$$\begin{aligned} \sum_{i,j} \frac{\partial(\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A}) \cdot (F \cdot \mathcal{A})_{i,j} &= \sum_{i\in I, j\in J} (-1)^{p+q}\mathcal{A}^{\wedge t-1}_{I\setminus\{i\},J\setminus\{j\}} \cdot \left(\sum_k F_{i,k} \cdot \mathcal{A}_{k,j}\right) \\ &= \sum_{k,i\in I} F_{i,k}(-1)^p \sum_{j\in J}(-1)^q \mathcal{A}^{\wedge t-1}_{I\setminus\{i\},J\setminus\{j\}} \cdot \mathcal{A}_{k,j} \\ &= \sum_{k,i\in I} F_{i,k}(-1)^p \mathcal{A}^{\wedge t}_{(I\setminus\{i\})\cup\{k\},J} \end{aligned}$$

and similarly for the term involving G. The resulting expressions clearly are in $\langle \mathcal{A}^{\wedge t}\rangle$ and so is

$$\sum_{i,j} \frac{\partial(\cdot)^{\wedge t}_{I,J}}{\partial y_{i,j}}(\mathcal{A}) \cdot H_{i,j}.$$

Hence, Lemma 1.4.18 is applicable and the deformation by v_k is trivial. The proof is now concluded as for Theorem 1.4.10. □

1.4.3 Complete Intersections and Cohen-Macaulay Schemes of Codimension 2

After what has been said in the previous section about the (im-)possibility to construct semi-universal determinantal deformations, we would like to single out two important classes of determinantal singularities, for which the notions of a semi-universal unfolding of the defining matrix considered as a map germ and the semi-universal deformation of the underlying space germ coincide.

The first class consists of the complete intersection singularities. If $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is a complete intersection given by a matrix $A \in \mathrm{Mat}(1, d; \mathcal{O}_N)$, then clearly every deformation of $(X_0, 0)$ is determinantal. A direct computation shows that there is a canonical isomorphism

$$\mathrm{Inf}(A) \cong T^1_{X_0,0}. \tag{1.44}$$

Thus a semi-universal unfolding of A exists if and only if the semi-universal deformation of $(X_0, 0)$ exists; and this is the case if and only if $(X_0, 0)$ has an isolated singularity at the origin. It is well known (see e.g. [57]) that a semi-universal deformation of $(X_0, 0)$ can be constructed from a $\mathbb{C}$-basis of the space $T^1_{X_0,0}$ in the same way that we constructed the semi-universal unfolding of map germs in 1.4.10. Consequently for complete intersections the base of a semi-universal deformation is smooth of dimension $\tau = \dim_{\mathbb{C}} T^1_{X_0,0}$. It follows from (1.44) that the comparison map Φ

between the base $(\mathbb{C}^\kappa, 0)$ of the semi-universal unfolding of A and the base $(\mathbb{C}^\tau, 0)$ of the semi-universal deformation of $(X_0, 0)$ has a differential of full rank at the origin we deduce:

Theorem 1.4.19. *For an isolated determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of type $(1, d, 1)$ the comparison map Φ from the base of a semi-universal unfolding of the defining matrix A to the base of a semi-universal deformation of $(X_0, 0)$ is an isomorphism.*

The second class of singularities which we would like to consider are singularities $(X_0, 0) \subset (\mathbb{C}^N, 0)$, which are Cohen-Macaulay and of codimension 2. As a consequence of the Hilbert Burch Theorem below, they are determinantal singularities in a canonical way. For them the semi-universal unfolding of the defining matrix A and the semi-universal deformation of the space germ $(X_0, 0)$ also coincide. This was found by M. Schaps and published in an article [61] preceeding [62].

Theorem 1.4.20 (Hilbert-Burch)**.** *Let $I \subset \mathcal{O}_N$ be an ideal of codimension 2 such that $\mathcal{O}_N/I$ is Cohen-Macaulay. Then the minimal resolution of $\mathcal{O}_N/I$ as an $\mathcal{O}_N$-module takes the form*

$$0 \longleftarrow \mathcal{O}_N/I \longleftarrow \mathcal{O}_N \xleftarrow{f} \mathcal{O}_N^{t+1} \xleftarrow{A} \mathcal{O}_N^t \longleftarrow 0 \tag{1.45}$$

for some matrix $A \in \mathrm{Mat}(t+1, t; \mathcal{O}_N)$ and $I = \langle A^{\wedge t} \rangle$ as ideals in $\mathcal{O}_N$.

Conversely suppose $A \in \mathrm{Mat}(t+1, t; \mathcal{O}_N)$ is any matrix such that the ideal $I := \langle A^{\wedge t} \rangle$ has codimension 2. If we let

$$f = \begin{pmatrix} \delta_1 & \cdots & \delta_{t+1} \end{pmatrix},$$

where δ_i is $(-1)^i$ times the determinant of A after deleting the i-th row, then (1.45) gives a minimal free resolution of $\mathcal{O}_N/I$.

Proof. We show the second part. By assumption $(X_0, 0)$ is a determinantal singularity of type $(t, t+1, t)$. In this case the Eagon-Northcott complex (1.29) for the generic determinantal Singularity $(M^t_{t,t+1}, 0)$ takes the form

$$0 \longrightarrow S_1^* \otimes \bigwedge^{t+1} R^{t+1} \xrightarrow{\varphi} S_0^* \otimes \bigwedge^t R^{t+1} \xrightarrow{\varphi^{\wedge t}} S_0 \otimes \bigwedge^0 R^{t+1} \longrightarrow 0\,.$$

According to Corollary 1.3.16, we only need to substitute $\mathcal{O}_N$ for R and the matrix A for φ to obtain a resolution of $\mathcal{O}_{X_0,0}$ as an $\mathcal{O}_N$-module. The result is exactly (1.45).

For the other direction see e.g. [22] or also [7]. □

Schaps made use of this fact to prove the following theorem.

Theorem 1.4.21 (Schaps, [61])**.** *Any deformation of a Cohen-Macaulay codimension 2 singularity is determinantal.*

Along the same lines one can prove:

Lemma 1.4.22 (Frühbis-Krüger, [23])**.** *For any Cohen-Macaulay codimension 2 singularity $(X_0, 0)$ given by a matrix $A \in \mathrm{Mat}(t, t+1; \mathcal{O}_N)$ one has a canonical isomorphism*

$$\mathrm{Inf}(A) \cong T^1_{X_0,0}.$$

Just like for the isolated complete intersection singularities one can show:

Theorem 1.4.23. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a Cohen-Macaulay codimension 2 singularity of type $(t, t+1, t)$ with defining matrix A as in the Hilbert-Burch Theorem 1.4.20. If $\operatorname{Inf}(A)$ is of finite dimension so that the semi-universal unfolding of A exists, then the comparison map Φ from the base of the semi-universal unfolding of A to the base of the semi-universal deformation of $(X_0, 0)$ is an isomorphism.*

Note that in this theorem the singularity does not need to be isolated.

Proof. This follows from the fact that the semi-universal deformation of $(X_0, 0)$ is smooth of dimension $\tau = \dim_{\mathbb{C}} T^1_{X_0,0}$ in the same way as for complete intersection singularities. In [61] M. Schaps announces to give a proof for this fact in a subsequent paper. However it seems that this paper did not appear. For a construction of the semi-universal deformation see e.g. [57]. □

For the author the Cohen-Macaulay codimension 2 case was the starting point of his investigations in the field of determinantal singularities and a good part of the research exhibited in this thesis was done for isolated Cohen-Macaulay codimension 2 singularities. The fact that they are determinantal in a canonical way and that any deformation is determinantal makes a comparison of analytic and topological invariants, like the Tjurina number and the vanishing Euler-characteristic of the singularity (which we will define later), more reasonable. The definition of the Tjurina number of an isolated singularity does not require any determinantal structure whatsoever, while the vanishing Euler-characteristic - as we will define it - depends on the choice of a matrix describing a given singularity.

Chapter 2

Topological Invariants of Singularities

This chapter is devoted to the development of the notion of Milnor fibers. We first recall the classical definitions and theorems concerning topological invariants of isolated complete intersection singularities (ICIS). Essentially, the material can also be found in the standard sources such as [53], [5], [6], and [31]. We reproduce the existence and uniqueness of Milnor fibers for ICIS as a motivation for our considerations for determinantal singularities. Then we state the main results concerning Milnor and Tjurina number from [53], [70], [35], [34], and [35].

For determinantal singularities, which are not always smoothable, we develop the notion of a stabilization. After some preparations, we recover the notion of an essentially isolated determinantal singularity (EIDS) as defined in [20]; and prove the existence and uniqueness of a determinantal Milnor fiber for this class of singularities.

Finally, we reprove a formula for the computation of the vanishing Euler characteristic in terms of polar multiplicities for isolated determinantal singularities, which admit a determinantal smoothing. This was already done in [8] and [59], but during the writing of this thesis, the author pointed out a mistake in a result, which was used in [8]; and this made it necessary to find a new proof. The one presented here is due to the author. However in the same time the authors of [8] independently came up with an erratum, which turned out to use similar methods. For a third proof, the reader may consult [59].

2.1 Smoothings, Milnor Fibers and Topology

As stated above, the results of this section are not new. It is collected and reformulated in a concise way from the mentioned standard sources.

Suppose we are given a singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ and we want to study its change in topology under deformation. The first thing to do is to single out a concrete neighborhood of $0 \in X_0$, which we want to observe. Clearly, the result should only depend on the germ and not on the chosen representative. All this is contained in the notion of a *Milnor ball*, which we will introduce now.

There exists a complex analytic Whitney stratification of X_0 coming from a strictly ascending chain of analytic subspaces

$$\{0\} \subset X_0^{(0)} \subset \cdots \subset X_0^{(d)} = X_0$$

of X_0 at 0, see e.g. [40], [45]. In this setting the strata are, of course, given by the sets $S_i := X_0^{(i)} \setminus X_0^{(i-1)}$. For the definition of Whitney stratifications see the Appendix A.3. We will usually assume that the stratification is minimal in the sense that the singular locus $\operatorname{Sing} X_0^{(i)}$ is equal to $X_0^{(i-1)}$ and not only contained in it for all $i > 0$.

Lemma 2.1.1. *There is an $\varepsilon > 0$ such that the sphere $S_{\varepsilon'} \subset \mathbb{C}^N$ of radius ε'^2 around the origin intersects all strata $X_0^{(i)}$ of X_0 transversally for all $\varepsilon \geq \varepsilon' > 0$.*

The proof is standard and using the Curve Selection Lemma as it can be found in [53]. Although the statement there is formulated for algebraic sets, the proof is known to carry over to real analytic sets as well.

Proof. Let
$$\rho : \mathbb{C}^N \to \mathbb{R}, \quad x \mapsto |x|^2$$
be the squared distance function from the origin. We show that ρ does not have critical points on any of the strata of the given Whitney stratification.

Set
$$K_i = \overline{\left\{ x \in X_0^{(i+1)} \setminus X_0^{(i)} : \mathrm{d}\rho = 0 \text{ in } x.\Omega^1_{X_0^{(i+1)} \setminus X_0^{(i)}} \right\}}.$$
Clearly, K_i is a closed analytic subset of $X_0^{(i+1)}$ and $K_i \setminus X_0^{(i)}$ satisfies the conditions of the Curve Selection Lemma. Suppose $0 \in K$. Then according to the Curve Selection Lemma there is a real analytic curve
$$\gamma : [0, \delta) \to K_i, \quad \gamma(0) = 0, \quad \gamma(t) \in K_i \setminus X_0^{(i)} \quad \forall t > 0.$$
Along this curve we find
$$\rho(\gamma(t)) = \int_0^t \mathrm{d}\rho(\gamma(\tau)) \cdot \dot{\gamma}(\tau) \, \mathrm{d}\tau = 0$$
since by assumption $\mathrm{d}\rho(\gamma(\tau)) = 0$ for all $\tau > 0$. $K \subset X_0^{(0)}$ be the critical locus of ρ on $X_0^{(0)}$. This contradicts the choice of γ.

Since there are only finitely many K_i and ρ is bounded from below on each of them, there is a minimal $\varepsilon > 0$ for which the assertion follows. □

Definition 2.1.2. For any ε as in Lemma 2.1.1 the ball B_ε of radius ε^2 around the origin is called a *Milnor ball* for $(X_0, 0)$. The space $X_0 \cap \partial B_\varepsilon$ is called the *link of the singularity* $(X_0, 0)$

Corollary 2.1.3. *For ε as in Lemma 2.1.1 the space $\overline{X}_0 = B_\varepsilon \cap X_0$ is contractible, i.e. the map*
$$\overline{X}_0 \to \overline{X}_0, \quad x \mapsto 0$$
is homotopic to the identity on $\overline{X}_0$.

Proof. This is an easy consequence of Thom's First Isotopy Lemma A.3.2. Lemma 2.1.1 shows that the squared distance function from the origin ρ is a stratified submersion on $X_0 \setminus \{0\}$. Consequently we have a homeomorphism

$$\begin{tikzcd} \overline{X}_0 \setminus \{0\} \arrow[r, "\cong"] \arrow[dr, "\rho"'] & X_0 \cap \partial B_\varepsilon \times (0,\varepsilon] \arrow[d] \\ & (0,\varepsilon]. \end{tikzcd}$$

Thus we may write each point $x \in \overline{X}_0 \setminus \{0\}$ as $x = (y,\tau)$ with y in the link and $\tau \in (0,\varepsilon]$. A homotopy of the point map to the identity can now be constructed as follows:

$$\begin{array}{rcl} H : \overline{X}_0 \times [0,1] & \to & \overline{X}_0, \\ (x,t) & \mapsto & \begin{cases} (y, t\cdot\tau) & \text{if } t \neq 0 \text{ and } x = (y,\tau) \in \overline{X}_0 \setminus \{0\}, \\ 0 & \text{otherwise} \end{cases}. \end{array}$$

Because 0 is an isolated zero of ρ on $\mathbb{C}^N$, this map is continuous. □

It is appearent from Lemma 2.1.1 and Corollary 2.1.3 that a Milnor ball B for $(X_0, 0)$ can be chosen arbitrarily small without changing the space $X_0 \cap B$ up to homeomorphism. Therefore $X_0 \cap B$ depends indeed only on the germ $(X_0, 0)$ and not on a chosen representative X_0 or on B.

Now let

$$\begin{tikzcd} (X_0,0) \arrow[r, hook] \arrow[d] & (X,0) \arrow[r, hook] \arrow[d, "\pi"] & (\mathbb{C}^N,0)\times(\mathbb{C}^k,0) \arrow[d, "\pi"] \\ \{0\} \arrow[r, hook] & (Y,0) \arrow[r, hook] & (\mathbb{C}^k,0) \end{tikzcd}$$

be a flat family over a germ $(Y,0)$ with central fiber $\pi^{-1}(\{0\}) \cap (X,0) = (X_0,0) \subset (\mathbb{C}^N,0)$. For chosen representatives $X_0 \hookrightarrow X \xrightarrow{\pi} Y$ in open sets $U \times D \subset \mathbb{C}^N \times \mathbb{C}^k$ we say that the family is a *smoothing* of the singularity $(X_0, 0)$, if there are points $u \in Y$ arbitrary close to 0 such that the fiber $Y_u = \pi^{-1}(\{u\})$ is smooth. It is easy to see that the property of a deformation to be a smoothing does not depend on the chosen representatives.

We can choose a Milnor ball $B \subset U \subset \mathbb{C}^N$ for X_0. Then the space

$$\overline{X}_0 = X_0 \cap B$$

is compact and since X is closed in $U \times D$ the restriction

$$\pi : X \cap (B \times D) \to D$$

is proper. In this case $B \times D$ is called a *Milnor tube* for the deformation $X_0 \hookrightarrow X \xrightarrow{\pi} Y$.

If $u \in Y$ is a point with smooth fiber X_u, then the *Milnor fiber* is supposed to be the space

$$\overline{X}_u := X_u \cap B \times \{u\}.$$

However, in general this notion is not yet well behaved. To make a Milnor fiber an invariant of the given singularity, we need make further assumptions which can differ depending on whether or not the singularity is isolated, whether it admits a versal deformation, whether it is equidimensional and so on.

2.1.1 Milnor Fibers of Isolated Singularities

In this section we shall restrict ourselves to equidimensional isolated singularities $(X_0, 0) \subset (\mathbb{C}^N, 0)$, which admit a smoothing over some $(\mathbb{C}^k, 0)$. This is to avoid unnecessary technicalities arising from deformations over germs $(Y, 0)$, which are singular themselves. We saw such a deformation in Example 1.4.15 ii). But even in this case the definition of Milnor fibers for isolated singularities could be reduced to smoothings over $(\mathbb{C}^k, 0)$ as we will sketch in the end of this section.

Lemma 2.1.4. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated singularity and $X_0 \hookrightarrow X \xrightarrow{\pi} \mathbb{C}^k$ a deformation of $(X_0, 0)$. Again let ρ be the squared distance function from the origin in $\mathbb{C}^N$ and $B = \{\rho = \varepsilon\}$ a Milnor ball for $(X_0, 0)$. There exists $\eta > 0$ and an open ball D in $\mathbb{C}^k$ around the origin such that*

$$(\rho, \pi) : (\rho^{-1}(\varepsilon - \eta, \varepsilon + \eta) \times D) \cap X \to (\varepsilon - \eta, \varepsilon + \eta) \times D$$

is a trivial fiber bundle.

Proof. This is almost a direct consequence of Thom's First Isotopy Lemma A.3.2. We first show that at each point $p \in \partial \overline{X}_0 \subset X$ the space X must also be smooth.

Let $f_i \in \mathcal{O}_N$ be holomorphic functions defining X_0 in a neighborhood U of B around the origin in $\mathbb{C}^N$ and F_i lifts of the f_i to holomorphic functions on $U \times D \subset \mathbb{C}^N \times \mathbb{C}^k$ in the ideal sheaf of the total space X. Since the deformation of $(X_0, 0)$ is flat, also the induced deformation of (X_0, p) at p is flat and hence

$$\dim \mathcal{O}_{X,p} = \dim \mathcal{O}_{X_0,p} + k$$

by Theorem 1.2.23. Therefore X is smooth at p if the jacobian matrix

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial u} \end{pmatrix}$$

has rank $N - \dim(X_0, p)$ at p. But from the equations defining (X_0, p) we see

$$\operatorname{rank} \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial u} \\ 0 & \mathbf{1}_k \end{pmatrix} = \operatorname{rank} \left(\frac{\partial F}{\partial x} \right) + k = N - \dim(X_0, p) + k$$

since at p the derivatives $\partial F / \partial x$ and $\partial f / \partial x$ coincide and X_0 is smooth at p.

Now that we can assume X to be smooth at all points of $\partial \overline{X}_0$, we consider the map (ρ, π) as above. Clearly it is a submersion at all points of $\partial \overline{X}_0$. We can use the compactness of $\partial \overline{X}_0$ to construct a neighborhood as in the statement, on which (ρ, π) is a proper submersion. Now the statement follows from Thom's First Isotopy Lemma. $\square$

The control on the boundary $\partial \overline{X}_0$ in the deformation of a singularity provided by Lemma 2.1.4 is a first ingredient to a well defined Milnor fiber for isolated singularities. The second one is a good control on the singular fibers in a deformation.

Definition 2.1.5. Let $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{u} (\mathbb{C}^k, 0)$ be a deformation of a singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$. The germ in $(\mathbb{C}^k, 0)$ coming from the set

$$\Delta = \{u \in \mathbb{C}^k : X_u \text{ is singular}\}$$

is called the *discriminant* of the deformation.

In general this set can be very badly behaved. However, we have:

Lemma 2.1.6. *If $(X_0, 0) \subset (\mathbb{C}^N, 0)$ as in Definition 2.1.5 is an isolated singularity, then the discriminant $(\Delta, 0)$ is a closed analytic set.*

Proof. Let $S = \{x \in X : \overline{X}_{u(x)} \text{ is singular at } x\} \subset X$ be the *relative singular locus* of the deformation. Clearly S is a closed analytic set since it has a description as the vanishing locus of certain minors of the jacobian of the equations defining $(X, 0)$. Moreover, the projection $u : (S, 0) \to (Y, 0)$ is finite. To see this, observe that $\mathcal{O}_{S,0}$ is a finite $\mathcal{O}_N$-module and $X_0 \cap S$ is just a point. Hence, $\dim_{\mathbb{C}} \mathcal{O}_{S,0}/\langle u \rangle < \infty$ and we can apply the Weierstrass Finiteness Theorem to deduce that $\mathcal{O}_{S,0}$ is a finite $\mathcal{O}_{Y,0}$-module. Now $\Delta = \operatorname{Supp}_{\mathcal{O}_{Y,0}} \mathcal{O}_{S,0}$ is closed analytic and we're done. □

Putting this together with the third ingredient - the existence of a semi-universal deformation - we obtain the following theorem:

Theorem 2.1.7. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity giver by a matrix $A \in \operatorname{Mat}(1, c; \mathcal{O}_N)$ and $\mathbf{A} \in \operatorname{Mat}(1, c; \mathcal{O}_{N+\kappa})$ the matrix describing the semi-universal unfolding of A over $(\mathbb{C}^\kappa, 0)$. Let $B \times D \subset \mathbb{C}^N \times \mathbb{C}^\kappa$ be a Milnor tube for the induced versal determinantal deformation of $(X_0, 0)$ and $\Delta \subset D$ the discriminant. The complement of Δ is nonempty and if D is chosen small enough, then for all points $u \in D \setminus \Delta$ the fibers $\overline{X}_u$ are diffeomorphic.*

Recall from Theorem 1.4.19 that, if they exist, the semi-universal unfolding of A and the semi-universal deformation of $(X_0, 0)$ agree. But existence is clear from Theorem 1.4.13, because $(X_0, 0)$ was assumed to be an isolated singularity.

Proof. We first show that $(\Delta, 0)$ is a proper subset of $(\mathbb{C}^\kappa, 0)$. To this end consider the deformation over $(\mathbb{C}^c, 0)$ given by

$$\tilde{\mathbf{A}} = A + Y$$

where $Y = (y_1, \ldots, y_c) \in \operatorname{Mat}(1, c; \mathcal{O}_c)$ is the matrix, whose entries are just the coordinate functions of $\mathbb{C}^c$. The induced deformation of $(X_0, 0)$ can, of course, be understood as taking fibers of arbitrary points $y \in \operatorname{Mat}(1, c; \mathbb{C})$ under the map A. Thus any representative $X \subset \mathbb{C}^{N+c}$ of the total space of the induced deformation of $(X_0, 0)$ is canonically isomorphic to an open set $U \subset \mathbb{C}^N$. According to Sard's Theorem [60] for any such representative there is a dense set $\Omega \subset \mathbb{C}^c$ of points $q \in \Omega$, such that X_q is smooth.

Because the unfolding over $(\mathbb{C}^\kappa, 0)$ is semi-universal, we get a map $\Phi : (\mathbb{C}^c, 0) \to (\mathbb{C}^\kappa, 0)$ such that $\tilde{\mathbf{A}}$ is equivalent to $\mathbf{A} \circ (\mathrm{id}_{\mathbb{C}^N}, \Phi)$. Hence there must be smooth fibers X_u for u arbitrary close to 0 in $\mathbb{C}^\kappa$, too; and $(\Delta, 0) \subset (\mathbb{C}^\kappa, 0)$ is a proper subset.

Choose a Milnor ball B for the given representative X_0 of $(X_0, 0)$. we show that there exists a neighborhood D of the origin in $\mathbb{C}^\kappa$ such that over the complement $U := D \setminus \Delta$ of Δ in D the projection

$$\pi : \pi^{-1}(U) \cap X \cap B \times D \to U$$

is a fiber bundle.

For the boundary this is just Lemma 2.1.4 for some open ball $D \subset \mathbb{C}^\kappa$. Now if $q \in D \setminus \Delta$ is arbitrary and $\overline{X}_q$ the smooth fiber over u, then the total space X must be smooth at all points $p \in \overline{X}_q$ for the same reasons as in the proof of Lemma 2.1.4. In particular, if the coordinates of $\mathbb{C}^\kappa$ are $u_1, \dots, u_\kappa$, then the functions $u_i - u_i(q)$ are a submersion on X at all points of $\overline{X}_q$ and a submersion on the boundary $\partial \overline{X}$ at all points in $\partial \overline{X}_q$. In other words: If we regard the manifold with boundary $\overline{X}_q$ as a stratified space, then the projection π is a stratified submersion along the compact set $\overline{X}_q$. Since the condition to be a submersion is open and $\overline{X}_q$ is compact, there exists a neighborhood of the form $W \times D' \subset \mathbb{C}^N \times \mathbb{C}^\kappa$ of $\overline{X}_q$ in X such that π is a submersion on $X \cap W \times D$ and we can again apply Thom's First Isotopy Lemma.

Now the claim follows, since the complex analytic set Δ has real codimension at least 2 in D and hence its complement U is connected: All the fibers of a fiber bundle over a connected space are diffeomorphic. □

Definition 2.1.8. The space $\overline{X}_u$ for $u \in D \setminus \Delta$ as in Theorem 2.1.7 is called the *Milnor fiber* of the isolated complete intersection singularity $(X_0, 0)$.

Example 2.1.9. We give two examples to illustrate the difficulties beyond isolated complete intersection singularities.

a) Consider the A_1-line singularity $X_0 = \{xy = 0\} \subset \mathbb{C}^3$, where the coordinates of $\mathbb{C}^3$ are (x, y, z). Clearly the singular locus of X_0 is the whole z-axis. For different $k \in \mathbb{N}$ the deformations over $\mathbb{C}[u]$ are given by the equation
$$xy - u \cdot z^k - u^2 = 0.$$
We shall see in the next section that if we chose a Milnor ball B for X_0 centered at the origin, then for different k the fibers $\overline{X}_u$ over $u \neq 0$ in these families are not diffeomorphic. From this we see that for non-isolated singularities we have to specify the deformation we're interested in.

b) Let $(X_0, 0) \subset (\mathbb{C}^6, 0)$ be the singularity of the generic determinantal variety $M^2_{3,2}$ at the origin given by the 2×2-minors of the matrix
$$\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}.$$
This is an isolated singularity. We know from Theorem 1.4.21 that any deformation of $(X_0, 0)$ is determinantal. But there are only trivial determinantal deformations, since any perturbation of the matrix can be absorbed into an analytic change of coordinates. Therefore $(X_0, 0)$ does not have a Milnor fiber at all. This is an example of a *rigid singularity*.

One can try to define the Milnor fiber for arbitrary isolated singularities. Given the existence of a semi-universal deformation one can show the following.

Theorem 2.1.10. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated singularity and $(Y, 0)$ the base of a semi-universal deformation of $(X_0, 0)$ in the sense of Grauert, Theorem 1.4.13. Then for each of the components of $(Y, 0)$ there is at most one Milnor fiber of $(X_0, 0)$ up to diffeomorphism. In particular, the number of Milnor fibers of an isolated singularity is finite.*

The proof of Theorem 2.1.10 proceeds as the proof of Theorem 2.1.7, but for every one of the finitely many components of the base $(Y, 0)$ seperately. In Pinkham's example, Example 1.4.15 ii), this would work directly, since both components are smooth when considered for themself. But in the example by Rim, Example 1.4.15 iv), the base is irreducible and singular. Consequently the total space X of any deformation, being a subspace of $\mathbb{C}^N \times (Y, 0)$, might not be smooth at points in X_0 and we might not be able to apply the arguments above directly.

One way to make our machinery work is to reduce to a flat family over a smooth base by pulling back the deformation over $(Y, 0)$ to a resolution $\rho : (\hat{Y}, E) \to (Y, 0)$ in the sense of Hironaka [43], [44]. Another way is to use Whitney stratifications of the total space $(X, 0)$ and the base $(Y, 0)$ compatible with the projection π and a chosen Milnor ball.

The second approach also allows one to define an analogue of the Milnor fiber for any given deformation over $(\mathbb{C}, 0)$ of a singularity $(X_0, 0)$ with arbitrary singular locus. This was done for example by Lê in [50]. He proves the following fibration theorem, which we will need in Chapter 4, when we define Milnor fibers for nonisolated singularities.

Theorem 2.1.11. *(Lê, [50]) Let $X \subset U \subset \mathbb{C}^N$ be an analytic subset of an open set U of $\mathbb{C}^N$. Let $f : X \to \mathbb{C}$ be an analytic function. Let $x \in X$ and suppose that $f(x) = 0$. Then if $\varepsilon > 0$ is small enough and $\eta > 0$, $\varepsilon \gg \eta$, then the mapping induced by f:*

$$\Psi_{\varepsilon,\eta} : B_\varepsilon \cap X \cap f^{-1}(D_\eta \setminus \{0\}) \to D_\eta \setminus \{0\}$$

- where B_ε is the closed real ball in $\mathbb{C}^N$ of center x and radius $\varepsilon > 0$, D_η is the open disc of $\mathbb{C}$ centered at 0 and with radius $\eta > 0$, is a topological fibration.

Note that in this theorem the fibers of $\Psi_{\varepsilon,\eta}$ are not necessarily smooth. It applies in our setting above if we take f to be the projection π of a given deformation. In this case we may replace B_ε by a Milnor tube.

2.1.2 Homology Groups of Milnor Fibers

We gather the important theorems concerning the topology of smooth Milnor fibers. The first one is a rather general statement about the intersection of complex analytic submanifolds of $\mathbb{C}^N$.

Theorem 2.1.12 (Lefschetz Hyperplane Theorem)**.** *Let $B \subset \mathbb{C}^N$ be a ball and $X \subset \mathbb{C}^N$ be a locally closed holomorphic embedding of a complex manifold X of complex dimension d such that $\overline{X} := X \cap B$ is compact and $\partial\overline{X} = \partial B \cap X$ is a transversal intersection. Then*

$$H_k(\overline{X}) = 0 \quad \text{for all } k > d$$

and

$$H_k(\overline{X}, \partial\overline{X}) = 0 \quad \text{for all } 0 \leq k < d.$$

For a proof see e.g. [52]. It uses Morse theory and the Levy form of the squared distance function to a point in $\mathbb{C}^N$ to bound the index on critical points. The next theorem marks the starting point of the investigation of the topology of Milnor fibers.

Theorem 2.1.13 (Milnor, [54]). *Let $(X_0, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated hypersurface singularity given by a holomorphic map germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the Milnor fiber $\overline{X}_u$ of $(X_0, 0)$ is homotopic to a bouquet of spheres of real dimension $n = \dim(X_0, 0)$. The number of these spheres, i.e. the middle Betti number of $\overline{X}_u$, is equal to*

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_N / \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} \right\rangle, \tag{2.1}$$

the Milnor number of $(X_0, 0)$.

The formula (2.1) for the computation of the middle Betti number of the Milnor fiber is a remarkable connection between topological invariants of the Milnor fiber and analytic invariants of the singularity itself. However, a priori the *Milnor algebra*

$$M_f := \mathcal{O}_N / \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} \right\rangle,$$

comes from the *map* f and not a from the singularity $(X_0, 0)$. In fact it is the space of infinitesimal unfoldings of the map germ f up to $\mathcal{R}$-equivalence, see e.g. [5]. From the definition of the $T^1_{X_0,0}$ we see that in this case

$$T^1_{X_0,0} \cong \mathcal{O}_N / \left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} \right\rangle,$$

and therefore we have a natural inequality

$$\mu \geq \tau. \tag{2.2}$$

of the Milnor- and the Tjurina number.

Example 2.1.14. Consider the A_k-surface singularity $(X_0, 0)$ given by the equation

$$f = x \cdot y - z^{k+1} = 0$$

in $(\mathbb{C}^3, 0)$. The Milnor algebra is easily computed to be

$$M_f = \mathbb{C}\{x, y, z\} / \langle x, y, z^k \rangle$$

and hence $\mu = k$. Because f is a *quasihomogeneous* polynomial, the Euler relation[1] gives

$$2(k+1) \cdot f = (k+1) \cdot x \cdot \frac{\partial f}{\partial x} + (k+1) \cdot y \cdot \frac{\partial f}{\partial y} + 2 \cdot z \cdot \frac{\partial f}{\partial z}$$

and we see that f is already contained in the ideal generated by the partial derivatives of f. We deduce $T^1_{X_0,0} \cong M_f$ and $\tau = \mu$.

[1]For a definition of quasihomogeneity and the Euler relation see e.g. [38]

A Milnor ball B for this singularity can be chosen to be of arbitrary size - another consequence of quasihomogeneity. For a small $u \in \mathbb{C}$ the space $\overline{X}_u = B \cap \{f = u\} \subset \mathbb{C}^3$ is smooth and homotopic to a bouquet of k spheres of real dimension 2.

From this we see that also the smooth fibers in Example 2.1.9 $i)$ had different homotopy type for different k: For fixed k one can show that they are diffeomorphic to $\overline{X}_u$ from this example.

Milnor's result was generalized to isolated complete intersection singularities by Hamm, Lê and Greuel.

Theorem 2.1.15 (Hamm, [39]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity of codimension d defined by*

$$f = (f_1, \ldots, f_d) : (\mathbb{C}^N, 0) \to (\mathbb{C}^d, 0).$$

Then the Milnor fiber of $(X_0, 0)$ is homotopic to a bouquet of spheres of real dimension $N - d = \dim(X_0, 0)$.

The number $\mu(f)$ of such spheres can be computed by the so called *Lê-Greuel formula*, see [70] and [35]. The key observation leading to this formula is that if $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is an isolated complete intersection singularity of codimension d defined by

$$f = \begin{pmatrix} f_1 & \cdots & f_d \end{pmatrix},$$

then, if we replace f by a general $\mathbb{C}$-linear combination of the f_i, also $(X'_0, 0) = \{f_1 = \cdots = f_{d-1} = 0\}$ is an ICIS given by $f' := (f_1, \ldots, f_{d-1})$. Now if we let

$$D = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_N} \end{pmatrix}$$

be the jacobian of f and $I \subset \mathcal{O}_N$ the ideal defined by the maximal minors of D and $\langle f' \rangle$, then Lê proved in [70]:

$$\mu(f) + \mu(f') = \dim_{\mathbb{C}} \mathcal{O}_N / I. \tag{2.3}$$

This gives a way to compute the middle Betti number of any ICIS inductively. In particular, this can easily be done for any explicit singularity with the help of a computer algebra system. A similar formula was found by Greuel in [35] by quite different methods using relative holomorphic de Rham Cohomology.

There is a more structural reason behind the formula (2.2), as one can guess from Example 2.1.14. The quasihomogeneity of a singularity implies the equality of Milnor- and Tjurina number for isolated complete intersection singularities - another result by Greuel:

Theorem 2.1.16 ([36]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an ICIS of dimension $n \geq 1$ with Milnor number $\mu = b_n(\overline{X}_u)$ and Tjurina number $\tau = \dim_{\mathbb{C}} T^1_{X_0,0}$. We have*

$$\mu \geq \tau.$$

If the equations describing $(X_0, 0)$ are quasihomogeneous, then equality holds.

There was little insight to what happens beyond isolated complete intersection singularities. However, there is one specific estimate on the possible degrees of nonzero reduced homology groups of Milnor fibers of smoothable isolated singularities. We shall not actually need it in what follows, but it played an important role in the development of the research carried out for this thesis. Therefore we state it here.

Theorem 2.1.17 (Greuel, Steenbrink [34]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional isolated singularity of codimension c, which admits a smoothing with Milnor fiber $\overline{X}_u$. Then*

$$\overline{H}_k(\overline{X}_u) = 0 \quad \text{for all } k < N - 2c.$$

Note that the codimension of a singularity is bounded from below by its embedding dimension. In the same article Greuel and Steenbrink also show:

Theorem 2.1.18 (Greuel, Steenbrink [34]). *Let X_t be the Milnor fiber of a smoothing of a normal isolated singularity; then $b_1(X_t) = 0$.*

Being Cohen-Macaulay, determinantal singularities of dimension ≥ 2 fall into this category. However, the statement of Theorem 2.1.18 is about the Betti-number only. It remains an open question, whether there are smoothable determinantal singularities, for which the first homology group of the Milnor fiber is torsion.

2.2 The Milnor Fiber of a Determinantal Singularity

In this section we develop the notion of a determinantal Milnor fiber. While the ideas were present in the literature as for example in [20], [8], [59], [9], [17], it is - to the knowledge of the author - the first attempt to explicitly prove the existence and *uniqueness* of the determinantal Milnor fiber in this generality, using the versal determinantal deformation from $K_{\mathcal{V}}$-equivalence developed in the previous chapter.

There were three ingredients to the uniqueness of the Milnor fiber for isolated complete intersection singularities: The good behavior of the boundary under deformations, the analyticity of the discriminant and the existence of a semi-universal deformation with a smooth base. For a determinantal singularity $(X_0, 0)$ given by a matrix $A \in \mathrm{Mat}(m, n; \mathbb{C}^N)$ the semi-universal deformation might in general have a base with several components as Pinkham's example shows. But the determinantal deformations coming from A and, in case $\dim_{\mathbb{C}} \mathrm{Inf}(A) < \infty$, or probably more general $\dim_{\mathbb{C}} \mathrm{Inf}_t(A) < \infty$, a versal determinantal deformation of $(X_0, 0)$ give a distinct choice of deformations of $(X_0, 0)$. Moreover, we can always assume that the base of such a deformation is smooth. We may therefore in a first attempt make the following definition.

Definition 2.2.1. Let $A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ describe an isolated determinantal singularity $(X_0, 0)$ of type (m, n, t) with a versal determinantal deformation over $(\mathbb{C}^\gamma, 0)$ given by $\mathbf{A} \in \mathrm{Mat}(m, n; \mathcal{O}_{N+\gamma})$. If $\mathbf{A}$ is a smoothing of $(X_0, 0)$, we define the *determinantal Milnor fiber* of $(X_0, 0)$ to be the Milnor fiber of the versal determinantal deformation.

The following examples point out the problems with this definition.

Example 2.2.2. We first give an example of an isolated determinantal singularity, which is not smoothable.

i) *(Frühbis-Krüger, Neumer [25])* For $k \geq 2$ let $(X_0, 0) \subset (\mathbb{C}^6, 0)$ be defined by the 2-minors of the matrix

$$\begin{pmatrix} x & y & v \\ z & w & x+u^k \end{pmatrix}.$$

A versal determinantal deformation over $\mathbb{C}\{t_0, \dots, t_{k-2}\}$ is given by perturbation with

$$t_0 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u^0 \end{pmatrix} + \dots + t_{k-2} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u^{k-2} \end{pmatrix}.$$

Thus, for fixed $t = (t_0, \dots, t_{k-2})$ we will find a polynomial $x + P_t(u) = x + u^k + \sum_{i=0}^{t-2} t_i \cdot u^i$ in the lower right corner. If t is general, we will assume $P_t(u)$ to have k distinct roots $u_1, \dots, u_k$, and at each of these roots we can do an analytic change of coordinates and replace u by $\tilde{u} = x + P_t(u)$. But in this coordinate system the fiber X_t over t just looks like the singularity from example 2.1.9 defined by

$$\begin{pmatrix} x & y & v \\ z & w & \tilde{u} \end{pmatrix}.$$

Thus for general t there will always be at least k singular points on each fiber X_t and the versal determinantal deformation of $(X_0, 0)$ is not a smoothing.

ii) Consider the determinantal singularity $(X_0, 0) \subset (\mathbb{C}^7, 0)$ of type $(2, 3, 2)$ given by the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6^2 + x_1^2 + x_2^2 - x_7^2 \end{pmatrix}.$$

The singular locus of $(X_0, 0)$ is a whole curve given by the equation $x_6^2 - x_7^2 = 0$ in the (x_6, x_7)-plane, so $(X_0, 0)$ is not an isolated singularity. However, direct computations using Singular show that the dimension $\dim_{\mathbb{C}} \operatorname{Inf}(A)$ is equal to 3 and in particular finite. Thus $(X_0, 0)$ admits a versal determinantal unfolding given by perturbations with the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The singularity in the first example did not admit a smoothing. But the remaining singularities for a generic perturbation were rigid, i.e. they did not admit any nontrivial deformations. Therefore, despite being singular, there is hope that the generic fiber in the versal determinantal deformation of $(X_0, 0)$ is a unique topological space up to homeomorphism.

It is reasonable to assume that the same holds for the second example. Again, if we let $\mathbf{A}$ be a matrix defining the semi-universal unfolding of

A and u a generic point in the base close to 0, then the fiber $\overline{X}_u = B \cap \mathbf{A}_u^{-1}(M_{2,3}^2)$ for a chosen Milnor ball B will be singular. But if $p \in \overline{X}_u$ is a singular point in the interior of $\overline{X}_u$, then locally at p the space $\overline{X}_u$ looks like a product

$$(\overline{X}_u, p) \cong (\mathbb{C}, 0) \times (Y_0, 0),$$

where $(Y_0, 0)$ is the rigid singularity in $(\mathbb{C}^6, 0)$ from Example 2.1.9, ii). In the following we will show that the notion of a Milnor fiber for determinantal singularities can be extended to such cases.

2.2.1 Deformations to Stabilizations

In [71] S. Trivedi deals with holomorphic mappings $f : M \to N$ between complex manifolds. Given a countable collection of submanifolds $(\Sigma_\alpha)_{\alpha\in\mathbb{N}}, \Sigma_\alpha \subset N$ (e.g. a stratification of N), one can ask for f to be transversal to all Σ_α along a given subset $K \subset M$.

Definition 2.2.3. A differentiable map $f : M \to N$ between smooth manifolds is transversal to a given submanifold $\Sigma \subset N$ at a point $p \in M$ if either $\dim M < \operatorname{codim}_N \Sigma$ and $f(p) \notin \Sigma$, or, in case $\dim M \geq \operatorname{codim}_N \Sigma$ and $f(p) \in \Sigma$, we have

$$\mathrm{d}f|_p(T_pM) + T_{f(p)}\Sigma = T_{f(p)}N. \tag{2.4}$$

In this case we write $f \pitchfork_p \Sigma$. If f is transversal to Σ for all points in a given subset $K \subset M$, we write $f \pitchfork_K \Sigma$.

Here $\mathrm{d}f|_p$ is the differential of f at p. This definition carries over to holomorphic maps between complex manifolds in the obvious way. We have the following theorem by S. Trivedi:

Theorem 2.2.4 ([71],Theorem 2.1 and Theorem 3.1)**.** *Let M be a Stein manifold, N be an Oka manifold and $(\Sigma_\alpha)_{\alpha\in\mathbb{N}}$ a countable collection of complex submanifolds in N. Then the set*

$$\{f : M \to N \text{ holomorphic } : f \pitchfork_M \Sigma_\alpha \quad \forall \alpha \in \mathbb{N}\}$$

is dense in the weak topology on the set of holomorphic maps between M and N. Moreover, if the Σ_α give a Whitney (a)-regular stratification of the space $\bigcup_{\alpha\in\mathbb{N}} \Sigma_\alpha \subset N$ and $K \subset M$ is any compact subset, then the set

$$\{f : M \to N \text{ holomorphic } : f \pitchfork_K \Sigma_\alpha \quad \forall \alpha \in \mathbb{N}\}$$

is open.

In our setting M will be an open neighborhood U of the origin in $\mathbb{C}^N$ and f will be a representative of a map germ A defining a determinantal singularity $(X_0, 0)$. For a chosen Milnor ball $B \subset U$ the theorem states that there are other matrices A' close to A such that $A' \pitchfork_B M_{m,n}^t \setminus M_{m,n}^{t-1}$ for all t and that once such a map has been found, it is stable. This stability then carries over to the preimage $A'^{-1}(M_{m,n}^t) \cap B$ and we obtain our Milnor fiber. In this sense we shall use Theorem 2.2.4 as a replacement of Sard's Theorem for determinantal singularities. For the rest of this section we will work on making Theorem 2.2.4 available in our setting of unfoldings of maps to $\mathrm{Mat}(m, n; \mathbb{C})$.

In his proof of Theorem 2.2.4 S. Trivedi gives an explicit construction of a deformation $F : M \times (\mathbb{C}^k, 0) \to N$ of a given map $f : M \to N$ such that there exist parameters $u \in \mathbb{C}^k$ arbitrary close to 0 with $F_u : M \to N, x \mapsto F(x, u)$ transversal. Since we will not need the Whitney topology and only work with representatives of complex space germs, it is this construction, which we may extract from Theorem 2.2.4. S. Trivedi attributes the idea to R. Abraham [1].

Lemma 2.2.5. *Suppose $A : U \subset \mathbb{C}^N \to \mathrm{Mat}(m, n; \mathbb{C})$ is a holomorphic map. Consider the unfolding of A over $\mathbb{C}^{m \cdot n} = \mathrm{Mat}(m, n; \mathbb{C})$ given by*

$$\mathbf{A} = A + Y$$

where $Y = (y_{i,j})$ is the matrix, whose entries are the variables $y_{i,j}$ of $\mathcal{O}_{m \cdot n}$. Let $K \subset U$ be a compact subset of U Then the set of points $y \in \mathbb{C}^{m \cdot n}$ for which the map $\mathbf{A}_y = \mathbf{A}(\cdot, y) : U \to \mathrm{Mat}(m, n; \mathbb{C})$ fulfills

$$\mathbf{A}_y \pitchfork_K M^t_{m,n} \setminus M^{t-1}_{m,n}$$

for all $0 \leq t \leq \min\{m, n\}$, is dense.

For a proof see e.g. the proof of [71, Lemma 2.2].

We need some preparations to link the space $\mathrm{Inf}(A)$ to transversality.

Theorem 2.2.6. *$A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ a holomorphic map germ. The following are equivalent:*

i) $\mathrm{Inf}(A) = 0$.

ii) The map A is transversal to all strata $(M^t_{m,n} \setminus M^{t-1}_{m,n})_{t=0}^{\min\{m,n\}}$ of the canonical Whitney stratification of $\mathrm{Mat}(m, n; \mathbb{C})$ at 0.

The next theorem reveals the connection with smoothness.

Theorem 2.2.7. *Let $A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ be a holomorphic map germ and $0 < t \leq \min\{m, n\}$ be arbitrary. Suppose that $(X_0, 0) := (A^{-1}(M^t_{m,n}), 0)$ has expected codimension $(m - t + 1) \cdot (n - t + 1)$. If $(X_0, 0)$ is smooth at p then $\mathrm{rank}\, P = t - 1$ and A is transversal to all strata $(M^s_{m,n} \setminus M^{s-1}_{m,n})_{s=0}^{\min\{m,n\}}$ at 0. In particular $\mathrm{Inf}(A) = 0$.*

We see that if $(X_0, 0) = (A^{-1}(M^t_{m,n}), 0)$ has expected codimension, then smoothness of $(X_0, 0)$ is a sufficient, but in general not a necessary condition for transversality of A at p.

In order to prepare for the proofs of Theorem 2.2.6 and Theorem 2.2.7, we first shift our attention to the stratification $(M^t_{m,n} \setminus M^{t-1}_{m,n})_{t=0}^{\min\{m,n\}}$ of $\mathrm{Mat}(m, n; \mathbb{C})$. Suppose $P \in M^{t+1}_{m,n} \setminus M^t_{m,n}$ is a matrix of rank t. We first change coordinates on $\mathrm{Mat}(m, n; \mathbb{C})$ so that

$$P = \begin{pmatrix} \mathbf{1}_t & 0 \\ 0 & 0 \end{pmatrix}.$$

This can be done by left- and right-multiplication with suitable invertible matrices $F \in \mathrm{GL}(m, \mathbb{C})$ and $G \in \mathrm{GL}(n, \mathbb{C})$. In particular this operation preserves all the strata of $\mathrm{Mat}(m, n; \mathbb{C})$.

Now we may again change coordinates in a non-linear way as follows. Let $U \subset \mathrm{Mat}(t,t;\mathbb{C})$ be a neighborhood of the origin such that the matrix exponential

$$\exp : U \to \mathrm{Mat}(t,t;\mathbb{C}), \quad A \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

is a holomorphic diffeomorphism onto its image. If we let $X \in U$ and $Y \in \mathrm{Mat}(t, n-t;\mathbb{C}), Z \in \mathrm{Mat}(m-t,t;\mathbb{C}), W \in \mathrm{Mat}(m-t,n-t;\mathbb{C})$ be arbitrary, we can write each matrix Q close to P as

$$Q(X,Y,Z,W) = \begin{pmatrix} \exp(X) & Y \\ Z & Z\cdot\exp(-X)\cdot Y + W \end{pmatrix} \in \mathrm{Mat}(m,n;\mathbb{C})$$

In these coordinates given by the entries of X, Y, Z and W around P the stratum $M^{t+1}_{m,n} \setminus M^t_{m,n}$ appears as $\{W = 0\}$.

What about the other strata? Consider the ideal $\langle Q(X,Y,Z,W)^{\wedge t}\rangle$ in the local ring $\mathcal{O}_{\mathbb{C}^{m\cdot n},P}$ of $\mathrm{Mat}(m,n;\mathbb{C})$ at M. As a consequence of Corollary A.1.2 this ideal does not change if we multiply Q from the left or from the right by invertible matrices $F \in \mathrm{GL}(m;\mathcal{O}_{\mathbb{C}^{m\cdot n},P})$ and $G \in \mathrm{GL}(n;\mathcal{O}_{\mathbb{C}^{m\cdot n},P})$ respectively:

$$\langle Q^{\wedge s}\rangle = \langle (F\cdot Q\cdot G)^{\wedge s}\rangle.$$

Hence if we decompose

$$Q = \begin{pmatrix} \mathbf{1}_t & 0 \\ Z\cdot\exp(-X) & \mathbf{1}_{m-t} \end{pmatrix} \cdot \begin{pmatrix} \exp(X) & 0 \\ 0 & W \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_t & \exp(-X)\cdot Y \\ 0 & \mathbf{1}_{n-t} \end{pmatrix} \tag{2.5}$$

we see that

$$\langle Q^{\wedge s}\rangle = \left\langle \begin{pmatrix} \exp(X) & 0 \\ 0 & W \end{pmatrix}^{\wedge s} \right\rangle = \langle W^{\wedge(s-t)}\rangle.$$

This immediately implies the following Lemma.

Lemma 2.2.8. *If $P \in M^{t+1}_{m,n} \setminus M^t_{m,n}$ is a matrix of rank t, then for $s \geq t+1$ locally at P the analytic varieties $M^s_{m,n}$ at P are isomorphic to the products*

$$\begin{aligned} &\left((\mathrm{Mat}(m,n;\mathbb{C}),P) \supset \cdots \supset (M^{t+2}_{m,n},P) \supset (M^{t+1}_{m,n},P)\right) \\ \cong\; &(\mathbb{C}^{m\cdot n-(m-t)\cdot(n-t)},0) \times \left(\cdots \supset M^1_{m-t,n-t},0) \supset (M^0_{m-t,n-t},0)\right). \end{aligned}$$

We are now in the position to proof Theorem 2.2.6.

Proof. (of Theorem 2.2.6) We first show $ii) \Rightarrow i)$. Let $p = 0 \in \mathbb{C}^N$ be the origin. We use the coordinates above around $P = A(p)$ and write $X = X \circ A$, $Y = Y \circ A$ and so on for the composition of A with the local coordinate matrices introduced above.

Let t be the rank of P. Being transversal implies that A does not meet strata of codimension $> N$, so we can assume $\operatorname{codim} M^{t+1}_{m,n} \setminus M^t_{m,n} = (m-t)\cdot(n-t) \leq N$ and $A \pitchfork_p M^{t+1}_{m,n} \setminus M^t_{m,n}$. In the introduced coordinates we see from Lemma 2.2.8 that this is the case if and only if $W \circ A : (\mathbb{C}^N, p) \to (\mathrm{Mat}(m-t,n-t;\mathbb{C}),0)$ is a submersion at p. It follows at once that A is also transversal to all strata $M^s_{m,n} \setminus M^{s-1}_{m,n}$ in a neighborhood of p for all $t+1 \leq s \leq \min\{m,n\}$.

We have to show $\operatorname{Inf}(A) = 0$. So let $H \in \operatorname{Mat}(m, n; \mathcal{O}_{\mathbb{C}^N,p})$ be arbitrary. Write

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} e^X & Y \\ Z & Z \cdot e^{-X} \cdot Y + W \end{pmatrix}$$

with the usual block sizes as above. We need to show that H reduces to the zero matrix in $\operatorname{Mat}(m, n; \mathcal{O}_{\mathbb{C}^N,p})$. This is done in two steps.

As usual let $\langle F \cdot A + A \cdot G \rangle$ denote the submodule of $\operatorname{Mat}(m, n; \mathcal{O}_{\mathbb{C}^N,p})$ generated by left and right-multiplication of A with square matrices F and G. From the decomposition of A into $A = L \cdot A' \cdot R$ as in (2.5) we see that

$$\langle F \cdot A + A \cdot G \rangle = \langle \tilde{F} \cdot A'R + LA' \cdot \tilde{G} \rangle.$$

Direct computation yields

$$A'R = \begin{pmatrix} e^X & Y \\ 0 & W \end{pmatrix} \quad \text{and} \quad LA' = \begin{pmatrix} e^X & 0 \\ Z & W \end{pmatrix}.$$

Using this, we can reduce any given H to

$$\begin{aligned} \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} H_{1,1} e^{-X} & 0 \\ 0 & 0 \end{pmatrix} A'R + \frac{1}{2} LA' \begin{pmatrix} e^{-X} H_{1,1} & 0 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ (H_{2,1} - \frac{1}{2} Z e^{-X} H_{1,1}) e^{-X} & 0 \end{pmatrix} A'R \\ &+ LA' \begin{pmatrix} 0 & e^{-X}(H_{1,2} - \frac{1}{2} H_{1,1} e^{-X} Y) \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & H_{2,2} - (H_{2,1} e^{-X} Y + Z e^{-X} H_{1,2} + Z e^{-X} H_{1,1} e^{-X} Y) \end{pmatrix}. \end{aligned}$$

Note that during this reduction process we only used matrices $\tilde{F}$ and $\tilde{G}$ with lower right block being zero. For the remaining block we find

$$\begin{pmatrix} 0 & 0 \\ 0 & F' \end{pmatrix} A'R = \begin{pmatrix} 0 & 0 \\ 0 & F'W \end{pmatrix}$$

and

$$LA' \begin{pmatrix} 0 & 0 \\ 0 & G' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & WG' \end{pmatrix}.$$

For the second step observe that the same reduction process to an $(m - t) \times (n - t)$-matrix can be done with the differential $\mathrm{d}A$. Modulo $\langle FA + AG \rangle$ we obtain

$$\mathrm{d}A \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{d}W - (\mathrm{d}Z e^{-X} Y + Z e^{-X} \,\mathrm{d}Y + Z e^{-X} (\mathrm{d}e^X) e^{-X} Y) \end{pmatrix}.$$

Denote the lower right entry by Q. Regarding $W = W \circ A$ as a $\mathbb{C}^{(m-t)(n-t)}$-valued function, we see that $W \circ A$ being a submersion at the point $p = 0$ means nothing else but that

$$\mathrm{d}W|_p : p.T_{\mathbb{C}^N} \to P.T_{\mathbb{C}^{(m-t)\cdot(n-t)}}$$

has full rank. This property is measured modulo $\mathfrak{m}$, the maximal ideal of $\mathcal{O}_{\mathbb{C}^N,p}$ at p. But since Y and Z have entries in $\mathfrak{m}$, also $Q|_p : p.T_{\mathbb{C}^N} \to P.T_{\mathbb{C}^{(m-t)\cdot(n-t)}}$ has full rank $(m-t)\cdot(n-t)$ and, hence,

$$Q : \mathcal{O}^N_{\mathbb{C}^N,p} \to \mathcal{O}^{(m-t)(n-t)}_{\mathbb{C}^N,p}$$

gives an epimorphism of free modules by Nakayama's Lemma. Putting these two steps together, we see that every H can first be reduced to an $(m-t)\times(n-t)$-matrix and then successively to zero by Q.

For the other direction $i) \Rightarrow ii)$ we use Lemma 2.2.5. Let $\mathbf{A}$ be the unfolding of A over $(\mathbb{C}^{m\cdot n},0)$ with a dense set of parameters $u \in \mathbb{C}^{m\cdot n}$ with transversal fibers from Lemma 2.2.5.

Since by assumption $\operatorname{Inf}(A) = 0$, every unfolding of A is trivial and hence there exist matrices $F \in \mathrm{GL}(m;\mathcal{O}_{N+m\cdot n})$, $G \in \mathrm{GL}(n;\mathcal{O}_{N+m\cdot n})$ and a germ of an analytic diffeomorphism

$$(\Phi, \mathrm{id}_{\mathbb{C}^{m\cdot n}}) : (\mathbb{C}^N,0)\times(\mathbb{C}^{m\cdot n},0) \to (\mathbb{C}^N,0)\times(\mathbb{C}^{m\cdot n},0)$$

such that $\mathbf{A}(x,y) = F(x,y)\cdot(A\circ\Phi(x,y))\cdot G(x,y)$.

But left- and right multiplication by invertible matrices preserves the stratification of $\mathrm{Mat}(m,n;\mathbb{C})$ and hence for any $y \in \mathbb{C}^{m\cdot n}$ the map $\mathbf{A}_y$ is transversal to the canonical stratification at 0 if and only if $A\circ\Phi(\cdot,y)$ is transversal. Clearly, transversality is independent of the composition with the diffeomorphism $\Phi(\cdot,y)$. If we now choose a closed ball B around the origin of $\mathbb{C}^N$ and a polydisc $D \subset \mathbb{C}^{m\cdot n}$ in the deformation base such that a representative of $\mathbf{A}$ is defined on some open neighborhood of $B\times D$, then we can choose $y \in D$ such that $\mathbf{A}_y$ is transversal at all points $p \in B\times\{y\}$ and deduce that also A must have been. $\square$

Proof. (of Theorem 2.2.7). Let $r = \operatorname{rank} P$ be the rank of $P = A(0)$. Clearly, if $t \le r$, then $(A^{-1}(M^t_{m,n}),0) = \emptyset$, so we can assume $t > r$. We may change coordinates on the target space as above and thus decompose the map A into the block matrices:

$$\begin{aligned} X\circ A &\in \mathrm{Mat}(r,r;\mathbb{C}) \qquad & Y\circ A &\in \mathrm{Mat}(r,n-r;\mathbb{C}) \\ Z\circ A &\in \mathrm{Mat}(m-r,r;\mathbb{C}) \qquad & W\circ A &\in \mathrm{Mat}(m-r,n-r;\mathbb{C}). \end{aligned}$$

As already described in Lemma (2.2.8) we have

$$X_0 = A^{-1}(M^t_{m,n}) = (W\circ A)^{-1}(M^{t-r}_{m-r,n-r})$$

and the equations defining V are the entries of the matrix $(W\circ A)^{\wedge t-r}$.

According to the jacobian criterion $(X_0,0)$ is smooth at 0 if and only if the jacobi matrix

$$\mathrm{d}(W\circ A)^{\wedge t-r} = (\mathrm{d}(W\circ A))\cdot\left(\mathrm{d}(\cdot)^{\wedge t-r}|_{W\circ A}\right)$$

has rank $\operatorname{codim}(V,0)$ at 0. Now $(W\circ A)(0)$ is the zero matrix and hence all its entries lay in the maximal ideal $\mathfrak{m}$ of the ring $\mathcal{O}_N$. It follows from the expression for the differential of minors (1.43) that for $t-r \neq 1$ the second differential $\mathrm{d}(\cdot)^{\wedge t-r}|_{W\circ A}$ always has rank zero at 0. In this case the germ $(X_0,0)$ cannot be smooth, because the rank of $\mathrm{d}(W\circ A)$ at 0 is bounded from above by the rank of $\mathrm{d}(\cdot)^{t-r}|_{W\circ A}$.

Hence $t - r = 1$, or equivalently: $\operatorname{rank} P = t - 1$. The condition $(W \circ A)^{t-r} = W \circ A = 0$ gives exactly

$$c := (m - t + 1)(n - t + 1)$$

equations. This number agrees with the codimension of $(X_0, 0)$ and hence $(X_0, 0)$ is a complete intersection and smooth if and only if $\mathrm{d}(W \circ A)(0)$ has rank c. But this is equivalent to the condition on A to be transversal to all $M^s_{m,n}$ at p.

The assertion about $\operatorname{Inf}(A)$ now follows from Theorem 2.2.6 □

Unlike a singularity a Milnor fiber is not a space germ anymore. It is not a *local*, but a *global* object. In the following we therefore need to shift our point of view and take a global perspective on what we encountered only in the local setting before. One instance of this shift to a global viewpoint is the following. Given $A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), 0)$ we can choose representatives

$$A : U \to \operatorname{Mat}(m, n; \mathbb{C})$$

of A on some open set $U \subset \mathbb{C}^N$. On U the map A induces a coherent analytic sheaves $\mathcal{I}nf(A)$ associated to the presentation

$$\begin{array}{rcl} \operatorname{Mat}(m, n; \mathcal{O}(U)) \oplus \operatorname{Mat}(m, n; \mathcal{O}(U)) \oplus T_{\mathbb{C}^N}(U) & \to & \operatorname{Mat}(m, n; \mathcal{O}(U)), \\ (F, G, \xi) & \mapsto & F \cdot A + A \cdot G + \xi(A). \end{array}$$

By $\mathcal{O}(U)$ we mean the holomorphic functions on $U \subset \mathbb{C}^N$ and $T_{\mathbb{C}^N}(U)$ are, of course, the holomorphic sections in the tangent bundle over U. The germs of those sheaves on any point $p \in U$ are

$$\mathcal{I}nf(A)_p = \operatorname{Inf}(A : (\mathbb{C}^N, p) \to (\operatorname{Mat}(m, n; \mathbb{C}), A(p)).$$

The obvious analogue can be constructed for any unfolding $\mathbf{A}$ of A over $(\mathbb{C}^k, 0)$ and we obtain a coherent analytic sheaf $\mathcal{I}nf^{\mathrm{rel}}(\mathbf{A})$ on an open set $U \times D$ for some $D \subset \mathbb{C}^k$.

This enables us to link the condition $\dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$ to a more common notion in the context of determinantal singularities.

Definition 2.2.9 ([20]). Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$. Then $(X_0, 0)$ is called an *essentially isolated determinantal singularities* (EIDS), if the map

$$A : (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), 0)$$

is transversal to all strata $M^s_{m,n} \setminus M^{s-1}_{m,n}$ of $\operatorname{Mat}(m, n; \mathbb{C})$ in a punctured neighborhood of the origin.

Corollary 2.2.10. *A determinantal singularity $(X_0, 0)$ as in Definition 2.2.9 is an EIDS if and only if $\dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$. In particular the matrix A defining an EIDS admits a semi-universal unfolding and, hence, the singularity $(X_0, 0)$ has a versal determinantal deformation.*

Proof. The condition $\dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$ is equivalent to $\mathcal{I}nf(A)$ being supported only at the origin for any representative of A which is defined on a sufficiently small open neighborhood of the origin. Given the two theorems above, Theorem 2.2.6 and Theorem 2.2.7, we see that $\dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$ if and only if the map A is transversal to all strata $M^s_{m,n} \setminus M^{s-1}_{m,n}$ in a punctured neighborhood of the origin. $\square$

2.2.2 Stabilizations in the Versal Determinantal Unfolding

We now start to prove the existence and uniqueness of the Milnor fiber for an EIDS. First, we give an analogue of 2.1.4.

Lemma 2.2.11. *Let $U \subset \mathbb{C}^N$ be an open domain containing the origin and $A : U \to \operatorname{Mat}(m,n;\mathbb{C})$ be a representative of a holomorphic map germ defining an EIDS $(X_0,0) = A^{-1}(M^t_{m,n}),0) \subset (\mathbb{C}^N,0)$, and let $\mathbf{A}$ be a representative of a semi-universal unfolding of A over some open polydisc $D \subset \mathbb{C}^\kappa$. Denote the projection to $\mathbb{C}^\kappa$ by u.*

Choose a Milnor ball $B = \{\rho = \varepsilon\}$ for $(X_0,0)$ in U and let $X = \mathbf{A}^{-1}(M^t_{m,n}) \subset U \times D$ be a representative of the total space of the deformation. There exist $\eta > 0$ and an open polydisc $D' \subset D$ around the origin in $\mathbb{C}^\kappa$ such that the map

$$(\rho, u) : X \cap \rho^{-1}((\varepsilon - \eta, \varepsilon + \eta)) \cap u^{-1}(D') \to D'$$

is a topological fiber bundle.

Proof. For any point $p \in \partial \overline{X}_0$ we have $\mathcal{I}nf(A)_p = 0$ and hence the induced deformation of (X_0,p) is trivial. In particular, for any $i = 1,\ldots,\kappa$ there exist matrices $F_i \in \operatorname{Mat}(m,m,\mathcal{O}_{\mathbb{C}^{N+\kappa},p})$ and $G_i \in \operatorname{Mat}(n,n,\mathcal{O}_{\mathbb{C}^{N+\kappa},p})$ and a vector field $\xi_i \in T_{\mathbb{C}^{N+\kappa},p}$ with $\mathrm{d}u_i(\xi) = 1$ and $\mathrm{d}u_j(\xi) = 0$ for all $j \neq i$, such that

$$\xi_i(\mathbf{A}) = F_i \cdot \mathbf{A} + \mathbf{A} \cdot G_i.$$

All involved elements are defined on some open neighborhood $W \times D'$ of p. Since $\partial \overline{X}_0$ is compact, we can cover it by finitely many of these neighborhoods:

$$\partial \overline{X}_0 \subset \bigcup_{\alpha=1}^{M} W_\alpha \times D'_\alpha.$$

Taking the minimum of all the radii of the D'_α, we may assume that all D'_α are the same D'. We denote the union of this cover by $W \times D'$.

Now we choose a C^∞ partition of unity $(\varphi_\alpha, W_\alpha \times D')$ subordinate to this cover and glue the local holomorphic sections to differentiable sections of the respective vector bundles:

$$\tilde{F}_i = \sum_{\alpha=1}^{M} \varphi_\alpha \cdot F_i^\alpha \in \operatorname{Mat}(m,m;C^\infty(W,\mathbb{C})),$$
$$\tilde{G}_i = \sum_{\alpha=1}^{M} \varphi_\alpha \cdot G_i^\alpha \in \operatorname{Mat}(n,n;C^\infty(W,\mathbb{C})),$$
$$\tilde{\xi}_i = \sum_{\alpha=1}^{M} \varphi_\alpha \cdot \xi_i^\alpha \in C^\infty(W, T_{\mathbb{C}^{N+\kappa}}).$$

Clearly for all $\lambda \in \mathbb{C}$ we have

$$\lambda \cdot \tilde{\xi}_i(\mathbf{A}) = \lambda \cdot \tilde{F}_i \cdot \mathbf{A} + \lambda \cdot \mathbf{A} \cdot \tilde{G}_i.$$

Note that due to the complex structure on the tangent bundle $T_{\mathbb{C}^N \times \mathbb{C}^\kappa}$, multiplication with complex scalars still makes sense. We can now proceed as in the proof of Theorem 1.4.10 for the 2κ different C^∞-vector fields $\tilde{\xi}_i$ and $\sqrt{-1} \cdot \tilde{\xi}_i$.

Consider the real- and imaginary parts of the coordinates (x, u) of $\mathbb{C}^N \times \mathbb{C}^\kappa$ as real coordinates and let $\tau = \Im u_\kappa$ be the last one of them. We write $(x, \tilde{u}, \tau)$ for this coordinate system, where $\tilde{u}$ consists of all remaining real and imaginary parts of the complex coordinates u.

Since $\partial \overline{X}_0$ is compact, after possibly shrinking W again, there exists a C^∞-flow

$$\Phi_\kappa : W \times (-\varepsilon, \varepsilon) \to \mathbb{C}^N \times \mathbb{C}^\kappa$$

of the vector field $\sqrt{-1} \cdot \tilde{\xi}_\kappa$ and we can do a differentiable change of coordinates on W, which preserves $\tau = \Im u_\kappa$, such that in the new coordinates $\tilde{\xi}_\kappa = \frac{\partial}{\partial \tau}$. Along the flow lines there exist C^∞-solution operators P and Q for the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{A} = \tilde{F} \cdot \mathbf{A} + \mathbf{A} \cdot \tilde{G}.$$

depending smoothly on the initial condition, i.e. the starting point $p \in \partial \overline{X}_0$, such that

$$\mathbf{A}(x, \tilde{u}, \tau) = P(x, \tilde{u}, \tau) \cdot \mathbf{A}(x, \tilde{u}, 0) \cdot Q(x, \tilde{u}, \tau)$$

on some open neighborhood $W \times D'$ of $\partial \overline{X}_0$. Consequently locally around $\partial \overline{X}_0$ the space X is a product in the direction of τ.

We proceed with the next vector field as usual and eventually end up with an isomorphism

$$X \cap W \cong (X_0 \cap W) \times D' \tag{2.6}$$

of Whitney stratified spaces. Note that the stratification is not necessarily complex analytic anymore, although the central fiber $X_0 \cap W$ is canonically stratified by the strata $A^{-1}(M^s_{m,n} \setminus M^{s-1}_{m,n})$ according to Theorem 2.2.6.

It remains to show that also the squared distance function ρ locally defines a fibration on this space. But this is easy given the fact that ρ was a stratified submersion on X_0 around $\partial \overline{X}_0$. From (2.6) it is clear that (ρ, u) defines a proper stratified submersion on X in a neighborhood of $\partial \overline{X}_0$. We can now apply Thom's First Isotopy Lemma A.3.2 to finish the proof. □

Since not all determinantal singularities are smoothable, as we saw in Example 2.2.2, the classical discriminant is not the right object to work with. With a view towards Theorem 2.2.6, we give the following definition.

Definition 2.2.12. Suppose $A : (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), P)$ is a holomorphic map germ and $\mathbf{A} \in \mathcal{O}_{N+k}$ an unfolding of A over $(\mathbb{C}^k, 0)$. Let

$$\mathrm{Inf}^{\mathrm{rel}}(\mathbf{A}) = \mathrm{Mat}(m, n; \mathcal{O}_{N+k}) / \left\langle \frac{\partial \mathbf{A}}{\partial x} + F \cdot \mathbf{A} + \mathbf{A} \cdot G \right\rangle \tag{2.7}$$

be the *relative infinitesimal deformations* of $\mathbf{A}$. The germ

$$(\Delta_{\mathrm{det}}, 0) = \mathrm{Supp}_{\mathcal{O}_k} \mathrm{Inf}^{\mathrm{rel}}(\mathbf{A}) \subset (\mathbb{C}^k, 0)$$

is called the *determinantal discriminant* of the deformation.

The Weierstrass Finiteness Theorem implies that, if A defines an EIDS, i.e. if $\dim_{\mathbb{C}} \operatorname{Inf}(A) < \infty$, then the determinantal discriminant is closed analytic.

For complete intersection singularities the classical discriminant was a proper subset of the base of a versal deformation because of Sard's Theorem. In our case of determinantal singularities we can use Theorem 2.2.6 and the density of transversal maps 2.2.4 to obtain a "generic" and "stable" fiber.

Theorem 2.2.13. *Let $(X_0, 0)$ be an EIDS of type (m, n, t) given by a matrix $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$. Let $(\Delta_{\text{det}}, 0) \subset (\mathbb{C}^\kappa, 0)$ be the determinantal discriminant in the versal determinantal deformation of $(X_0, 0)$ coming from the semi-universal unfolding $\mathbf{A}$ of A. Suppose $0 \in B \subset \mathbb{C}^N$ is a Milnor ball for $(X_0, 0)$. For any open polydisc $D \subset \mathbb{C}^\kappa$ the complement $U = D \setminus \Delta_{\text{det}}$ is nonempty and we can choose D small enough such that for all $u \in U = D \setminus \Delta_{\text{det}}$ the fibers*

$$\overline{X}_u = B \cap \mathbf{A}_u^{-1}(M_{m,n}^t)$$

are isomorphic as Whitney stratified spaces.

If $N < (m - t + 2) \cdot (n - t + 2)$, then $\overline{X}_u$ is smooth. Otherwise let $r \in \mathbb{N}$ be the smallest number such that $N \geq (m - r) \cdot (n - r)$. For $r \leq s < t$ set $\overline{X}_u^{(s)} := B \cap \mathbf{A}_u^{-1}(M_{m,n}^{s+1})$. Then these sets form a Whitney stratification of $\overline{X}_u$

$$\overline{X}_u = \overline{X}_u^{(t-1)} \supsetneq \overline{X}_u^{(t-2)} \supsetneq \cdots \supsetneq \overline{X}_u^{(r)}$$

and locally around a point $p \in \overline{X}_u^{(s)} \setminus \overline{X}_u^{(s-1)}$ we find analytic isomorphisms

$$(\overline{X}_u, p) \cong (\mathbb{C}^{N-(m-s)(n-s)}, 0) \times (M_{m-s,n-s}^{t-s+1}, 0).$$

Note the shift of indices: $\overline{X}_u^{(s)} \setminus \overline{X}_u^{(s-1)}$ is the set of points, on which $\mathbf{A}_u$ has rank *exactly* s while $M_{m,n}^t$ are the matrices of rank $< t$.

Definition 2.2.14. The space $\overline{X}_u$ as in Theorem 2.2.13 is called the *determinantal Milnor fiber* of the singularity $(X_0, 0)$. An unfolding of A like $\mathbf{A}$ above is called a *stabilization* of A, if there are points u arbitrary close to 0 in the base of the unfolding, for which $\mathbf{A}_u$ is transverse to all strata of $\operatorname{Mat}(m, n; \mathbb{C})$.

Proof. The fact that $(\Delta_{\text{det}}, 0) \subset (\mathbb{C}^\kappa, 0)$ is a proper subset follows directly from Theorem 2.2.6 and Lemma 2.2.5: There exists a stabilization of A and hence these must also appear in the semi-universal unfolding. For any set of representatives of the semi-universal unfolding $\mathbf{A}$ of A we then have $\mathcal{I}nf(\mathbf{A}_u)_p = 0$ at all points $p \in \overline{X}_u$ for $u \notin \Delta_{\text{det}}$.

Since the determinantal discriminant $(\Delta_{\text{det}}, 0) \subset (\mathbb{C}^\kappa, 0)$ is closed analytic, it is of real codimension ≥ 2 and hence the space U is connected. To prove uniqueness of $\overline{X}_u$ it is therefore sufficient to show that the restriction of the projection

$$\pi : X \cap B \times U \to U$$

is a fiber bundle over U.

To see this we may proceed as in the proof of Lemma 2.2.11 and construct local holomorphic vector fields ξ_i^α and matrices F_i^α and G_i^α, only this time on a neighborhood $W \times D''$ of the whole space $\overline{X}_u$. We may assume that the polydisc D'' around u in $\mathbb{C}^\kappa$ is chosen small enough such that D'' is contained in the polydisc D' around the origin from Lemma 2.2.11, so that we have a well behaved fibration along the boundary. Now glueing these local sections to a partition of unity, we obtain a local fibration over D'' with the whole space $\overline{X}_u$ as fiber.

Another consequence of $\mathcal{I}nf(\mathbf{A}_u)_p = 0$ for all points $p \in \overline{X}_u$ is that $\mathbf{A}_u \pitchfork_B M_{m,n}^s \setminus M_{m,n}^{s-1}$ for all s. In particular, $\mathbf{A}_u$ does not meet any strata of codimension $> N$. Therefore, if $N < (m-t+2)\cdot(m-t+2)$, then X_u is the preimage of a smooth manifold $M_{m,n}^t \setminus M_{m,n}^t$ under a transversal map $\mathbf{A}_u$ and hence smooth itself.

For the other case, let $p \in \overline{X}_u^{(s)} \setminus \overline{X}_u^{(s-1)}$ for some $r \le s < t$ and $P = \mathbf{A}_u(p) \in \mathrm{Mat}(m,n;\mathbb{C})$. As we saw in the outlines preceeding Lemma 2.2.8, we can choose local coordinates (X,Y,Z,W) on the target space and then $W \cdot \mathbf{A}_u$ is a submersion at p. Hence, the stratum $\overline{X}_u^{(s)}$ has dimension $N-(m-s)\cdot(n-s)$. The higher-dimensional strata appear as preimages of $M_{m-s,n-s}^\bullet$ under the submersion $W \circ \mathbf{A}_u$. Since the preimage of a Whitney regular stratification under a submersion is again Whitney regular, they connect to $\overline{X}_u^{(s)}$ in a Whitney regular way forming the germ of a fiber bundle over $\overline{X}_u^{(s)}$. □

Definition 2.2.15. We call the space $\overline{X}_u$ as in Theorem 2.2.13 the *determinantal Milnor fiber* of the singularity $(X_0, 0)$. The Betti numbers

$$b_i(X_0,0) = \operatorname{rank} H_i(\overline{X}_u)$$

of $\overline{X}_u$ are called the *Betti numbers of* $(X_0, 0)$. Generators of the homology groups $H_\bullet(\overline{X}_u)$ are *vanishing cycles*, and the difference of the Euler-characteristics

$$\nu(X_0,0) := (-1)^{\dim(X_0,0)} \cdot (\chi(\overline{X}_u) - \chi(\overline{X}_0)) = (-1)^{\dim(X_0,0)} \cdot (\chi(\overline{X}_u) - 1)$$

is the *vanishing Euler-characteristic* of the singularity $(X_0, 0)$.

Remark 2.2.16. The idea to use transversality theorems to obtain a "generic" and "stable" fiber of a given determinantal singularity is not new. In fact W. Ebeling and S.M. Gusein-Zade gave the idea of an "essential determinantal smoothing", which in our case is the determinantal Milnor fiber, in their article [20]. However, their arguments were only sketched and not properly put into context in the theory of versal unfoldings. A more detailed treatment is given by J.J. Nuño-Ballesteros, B. Oréfice-Okamoto and J.N. Tomazella in [8], where the authors also take a generic constant perturbation of the defining matrix, to define what we call the determinantal Milnor fiber. But again, the theory of unfoldings and its associated analytic invariants are not presented. The consequent development of the space $\mathrm{Inf}(A)$ and semi-universal unfoldings enable us to present the results of this section in this generality.

Remark 2.2.17. Also, the authors in [8] only treat the smoothable case and make further assumptions in their definition of isolated determinantal singularities (IDS). For them an IDS $(X_0, 0)$ of type (m, n, t) given by A has to fulfill

- $(X_0, 0)$ is an isolated singularity, which admits a determinantal smoothing,
- $\operatorname{rank} A(x) = t - 1$ for all $x \in X$ in a punctured neighborhood of the origin.

Due to Theorem 2.2.7 it is possible to omit the second requirement since it follows from the first one.

Remark 2.2.18. In view of the section on versal determinantal deformations, we would like to point out that all results in this section can be reformulated using $\operatorname{Inf}_t(A)$ instead of $\operatorname{Inf}(A)$ and adapting the proofs in a similar way as one obtains the proof of Theorem 1.4.16 from the one of Theorem 1.4.10.

2.3 The Euler-characteristic of smooth Milnor fibers

We give an overview on different results concerning the vanishing Euler-characteristic of determinantal singularities. As a new contribution to the field we reprove Theorem 2.3.10 using methods from stratified Morse theory. For an alternative proof using methods similar to those exhibited in [6], see [59].

There are at least three methods to compute the Euler characteristic of a determinantal Milnor fiber of a smoothable determinantal singularity. Two of them can effectively be implemented in computer algebra systems to actually perform the computation for any explicitly given singularity.

For the author it was the results by J. Damon and B. Pike in their article [17], which marked the starting point of this work. They developed an algorithm for the computation of the vanishing Euler characteristic and applied it to certain members of the list of simple isolated Cohen-Macaulay codimension 2 singularities from A. Frühbis-Krüger and A. Neumer in [25]. This lead to conjectures concerning the vanishing Euler characteristic of the Milnor fibers in discrete families of simple singularities.

They observed a rather unexpected behaviour for the threefold singularities. Due to the Theorem by Greuel and Steenbrink 2.1.17, dimension 3 and codimension 2 leaves two Betti-numbers b_2 and b_3 to contribute to the vanishing Euler-characteristic. At that point it seems to have been a usual assumption that Cohen-Macaulay codimension 2 singularities behave like isolated complete intersection singularities, i.e. that only the middle Betti-number, b_3 in this case, is nonzero. However, this could not possibly be the case as the following example shows.

Example 2.3.1. i) Consider the so called A_0^+ singularity in $(\mathbb{C}^5, 0)$ of type $(2, 3, 2)$ given by the matrix

$$\begin{pmatrix} x & v & y \\ z & w & x \end{pmatrix}.$$

According to the computations in [17] the vanishing Euler-characteristic of this singularity is

$$\nu(X_0, 0) = -1 = b_3 - b_2$$

and hence b_2 must be nonzero.

ii) The singularity above can be seen as a member of a whole family. Namely for $k \in \mathbb{N}$ the Π_k singularity is described by the matrix

$$\begin{pmatrix} x & v & y \\ z & y - w^k & x \end{pmatrix}.$$

For $k = 1$ we can bring this matrix to the form of the A_0^+ singularity by a linear change of coordinates in $(\mathbb{C}^5, 0)$. The vanishing Euler-characteristic seems not to depend on k since, according to [17], for the first few values $k = 2, \ldots, 4$ it was always equal to -1 again.

2.3.1 Indices of 1-forms on Determinantal Singularities

Some years before, W. Ebeling and S.M. Gusein-Zade published a series of papers including [20], in which they related the vanishing Euler characteristic of an EIDS to indices of 1-forms. In the case of isolated smoothable determinantal singularities they obtain the following formula ([20, Section 3]):

$$\mathrm{ind}_{\mathrm{PH}}(\omega; X_0, 0) = \mathrm{ind}_{\mathrm{rad}}(\omega; X_0, 0) + (-1)^{\dim(X_0,0)}(\chi(\overline{X}_u) - 1). \tag{2.8}$$

Here $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is an isolated determinantal singularity and ω the germ of a continous complex 1-form on $(X_0, 0)$ with no zero on the regular part $X_0 \setminus \{0\}$ in a neighborhood of the origin.

The *radial index* $\mathrm{ind}_{\mathrm{rad}}(\omega; X_0, 0)$ of the 1-form ω on $(X_0, 0)$ is defined as the sum of indices on $\overline{X}_0 \setminus \{0\}$ of a generic perturbation $\tilde{\omega}$ of ω, which coincides with ω in a neighborhood of the boundary $\partial X_0 = \partial B \cap X_0$ for a Milnor ball B. For the definition of the index see e.g. [54].

On the other hand, the *Poincaré-Hopf index* $\mathrm{ind}_{\mathrm{PH}}(\omega; X_0, 0)$ needs the Milnor fiber $\overline{X}_u$ for its definition. Again let $\tilde{\omega}$ be a perturbation of ω, which coincides with ω on a neighborhood of $\partial\overline{X}_u$. This index is then defined as the number of indices of $\tilde{\omega}$ on the interior of $\overline{X}_u$.

A special case, in which the formula (2.8) can be applied, is when $\omega = \mathrm{d}f$ for a holomorphic function $f : (X_0, 0) \to (\mathbb{C}, 0)$. If the differential $\mathrm{d}f$ has no singular point on $X_0 \setminus \{0\}$ in a neighborhood of the origin, then $(Y_0, 0) = \{f = 0\} \cap (X_0, 0)$ is an isolated singularity of dimension $\dim(X_0, 0) - 1$ and the perturbation of f by a constant v gives a smoothing of $(Y_0, 0)$ with Milnor fiber

$$\overline{Y}_v = \{f = v\} \cap X_0 \cap B,$$

where B is a Milnor ball for $(X_0, 0)$ and $(Y_0, 0)$ at the same time and $v \in \mathbb{C}$ is chosen sufficiently close to 0. In a preceeding paper the authors prove the following.

Theorem 2.3.2 ([21], Theorem 3)**.** *In the above setup one has*

$$\mathrm{ind}_{\mathrm{rad}}(\mathrm{d}f; X_0, 0) = (-1)^{\dim(X_0,0)+1}(\chi(\overline{Y}_v) - 1).$$

In fact the theorem holds for arbitrary equidimensional complex space germs $(X_0, 0)$ with an isolated singularity at 0.

If $f = l \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^N, \mathbb{C}) = (\mathbb{C}^N)^\vee$ is a general linear form, then the differential $\mathrm{d}l$ is regular on the interior of $\overline{X}_0 \setminus \{0\}$ and has only isolated critical points on the interior of all smooth fibers $\overline{X}_u$ for a given determinantal smoothing of $(X_0, 0)$. Therefore the formula (2.8) and Theorem 2.3.2 are applicable – even without a perturbation to $\tilde{\omega}$. The Milnor fiber $\overline{Y}_v$ of l on $(X_0, 0)$ is also known as the *complex link* of the singularity $(X_0, 0)$, see e.g. [31].

Comparing the results we see the following:

$$\mathrm{ind}_{\mathrm{PH}}(\mathrm{d}l; X_0, 0) = (-1)^{\dim(X_0,0)}(\chi(\overline{X}_u) - \chi(\overline{Y}_v)). \tag{2.9}$$

Apparently the Poincaré-Hopf index measures the difference of the Euler-characteristics of the Milnor fiber $\overline{X}_u$ of $(X_0, 0)$ and its complex link $\overline{Y}_v = \overline{X}_0 \cap \{l = v\}$. Now observe that, since $\overline{Y}_v$ is a smooth manifold with boundary, it is, for u small enough, diffeomorphic to the hyperplane section $\overline{X}_u \cap \{l = v\}$ by Ehresmann's Fibration Theorem. Thus we are in fact dealing with the difference of Euler-characteristics of the Milnor fiber $\overline{X}_u$ and a smooth hyperplane section thereof.

A geometric interpretation of this fact can be found in [8] and also in [59]. Roughly speaking, the machinery works as follows. The function l has only isolated non-degenerate critical points on the interior of $\overline{X}_u$, i.e. it is a complex Morse function. Then the real part $\Re l$ is a real Morse function, whose index is always exactly $n := \dim_{\mathbb{C}} \overline{X}_u$ at the same critical points. Since each one of these points contributes with index 1 to the Poincaré-Hopf index, the number of critical points is equal to $e := \mathrm{ind}_{\mathrm{PH}}(\mathrm{d}l; X_0, 0)$. Thus we obtain $\overline{X}_u$ from $\overline{Y}_v = \overline{X}_u \cap \{l = v\}$ by attaching e cells of real dimension n. The formula (2.9) now easily follows from the long exact sequence of the pair $(\overline{X}_u, \overline{Y}_v)$.

But there is one detail that has been forgotten in these outlines: The space $\overline{X}_u$ is a smooth manifold *with boundary*. Hence, applying classical Morse theory to the function $\Re l$ on $\overline{X}_u$, does not work directly. One has to switch to stratified Morse theory as for example exhibited by Goresky and MacPherson in [31]. This means, we also have to take into account critical points of $\Re l$ on the boundary $\partial \overline{X}_u$.

In [8] the authors work with open Milnor balls and, hence, with open Milnor fibers without boundary. Unfortunately they based there original results on a theorem that apparently turned out to be false. In the following we will describe a way to make the Morse theoretic arguments work in our setting with closed Milnor balls. During the prepraration of this work, the authors of [8] independently came up with an erratum, which uses quite similar methods also based on stratified Morse theory.

2.3.2 Polar Varieties and the Scanning Process

Of course, for the machinery sketched above to work, the smoothing of $(X_0, 0)$, the linear form l and the Milnor ball B have to be chosen in a compatible way. This leads us to the realm of polar curves and multiplicities.

Definition 2.3.3. Let M be a manifold of real dimension n and $f : M \to \mathbb{R}$ a C^∞-function on M. For another function $g : M \to \mathbb{R}$ the *polar locus* of g with respect to f is defined as

$$\Gamma(g,f) = \overline{\left\{x \in M \setminus \operatorname{Crit}(f) : \mathrm{d}g(x) \in \langle \mathrm{d}f(x)\rangle_{\mathbb{R}} \subset x.\Omega^1_M\right\}}.$$

The notion of polar locus carries over in the obvious way to Whitney stratified spaces and the analytic category. In the latter case, $\bar{\cdot}$ denotes the analytic closure.

Given this definition one can use Bertini-Sard type methods to prove the following lemma.

Lemma 2.3.4. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional singularity of pure dimension n with a strictly descending sequence of analytic subspaces*

$$X_0 \supset \cdots \supsetneq X_0^{(2)} \supsetneq X_0^{(1)} \supset \{0\}$$

giving rise to a complex analytic Whitney stratification with strata $X_0^{(i)} \setminus X_0^{(i-1)}$. Let $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{u} (\mathbb{C}, 0)$ be a smoothing of $(X_0, 0)$ and $(X, 0) \subset (\mathbb{C}^N, 0) \times (\mathbb{C}, 0)$ the total space. Denote by u the projection to the deformation base and let p be the projection to $(\mathbb{C}^N, 0)$. There exists a dense set Ω of linear forms l on $\mathbb{C}^N$ with complement of measure zero such that

i) the closure of the polar locus $\Gamma(p^*l, u) \subset X \setminus X_0$ is either empty or a branched covering over $(\mathbb{C}, 0)$ via u.

ii) the linear form l does not have any critical points on $(X_0, 0)$ in the Whitney stratified sense.

Proof. We first show $i)$ and define

$$M = \overline{\{(x,l) \in X \setminus X_0 \times (\mathbb{C}^N)^\vee : x.T_X \subset \ker p^*l\}} \subset X \times (\mathbb{C}^N)^\vee.$$

Since u is a submersion on $X \setminus X_0$, the space $X \setminus X_0$ is a complex submanifold of $\mathbb{C}^{N+1}$. At each point $x \in X \setminus X_0$ the natural pairing $p^*(\mathbb{C}^N)^\vee \times x.T_X \to \mathbb{C}$ induces a surjection $(\mathbb{C}^N)^\vee \twoheadrightarrow (\ker \mathrm{d}u)^\vee$. From this it is easy to see that $M^* := M \setminus (M \cap X_0)$ is a complex manifold of dimension

$$n + 1 + N - n = N + 1.$$

Let $l_0 \in (\mathbb{C}^N)^\vee$ be a regular value of p restricted to M^*. Either $C := p^{-1}(\{l_0\}) \cap M$ is empty, or it consists of analytic curves, which are smooth outside points in X_0. Right now we are only interested in the polar locus outside X_0, so no harm is done if we replace C by $\overline{C \setminus X_0}$. Now $C = \Gamma(l_0, u)$ is the polar locus and clearly the restriction of u to C gives a branched covering over $(\mathbb{C}, 0)$.

The set of such regular values of $p|_{M^*}$ in $(\mathbb{C}^N)^\vee$ is a first candidate for Ω.

To show $ii)$ we proceed in a similar way. Let $X_0^{(i)} \setminus X_0^{(i-1)}$ be one of the strata of X_0 with its reduced structure as a complex manifold. The set

$$M^{(i)} = \overline{\{(x,l) \in X_0^{(i)} \setminus X_0^{(i-1)} \times (\mathbb{C}^N)^\vee : x.T_{X_0^{(i)}} \subset \ker l\}}$$

is a proper complex analytic subset of $X_0^{(i)} \times (\mathbb{C}^N)^\vee$ and over all points x outside $X_0^{(i-1)}$ it is a smooth manifold of dimension N. Again let $p : X_0^{(i)} \times (\mathbb{C}^N)^\vee \to (\mathbb{C}^N)^\vee$ be the second projection. For a regular value l of $p|_{X_0^{(i)} \setminus X_0^{(i-1)}}$ the preimage $K = p^{-1}(\{l\}) \subset X_0^{(i)} \setminus X_0^{(i-1)}$ is smooth of dimension 0. In particular all the points belonging to K are isolated and $\overline{K} \subset X_0^{(i)}$ is complex analytic.

Now if 0 was in the closure of K, then there would be a real analytic curve $\gamma : [0, \varepsilon) \to \overline{K}$ with $\gamma(0) = 0$ and $\gamma(t) \in K \subset X_0^{(i)}$ for all $t > 0$. But this would contradict the fact that K consists only of isolated points.

Since this holds for all of the finitely many strata, we may replace Ω by the intersection of all the regular values for all constructions in this proof at once. □

Definition 2.3.5. The multiplicity of the branched covering $u : \Gamma(l_0, u)$ in the setting of Lemma 2.3.4 is the *polar multiplicity* of l_0 on the given smoothing.

Remark 2.3.6. The definitions of polar multiplicities we give here might seem quite unusual. In fact there is a more general notion of *polar varieties* and their multiplicities. It has recently seen a lot of interest in the context of determinantal singularities, see e.g. [28], [29], [27].

In view of Lemma 1.2.25, we also want to make sure that a linear form l can be chosen in such a way that any deformation of $(X_0, 0)$ induces a deformation of $(X_0, 0) \cap \{l = 0\}$, i.e. the induced family is indeed flat.

Lemma 2.3.7. *Let $\mathcal{O}_{X_0,0} = \mathcal{O}_N/I$ be an analytic quotient of $\mathcal{O}_N$. If $\mathcal{O}_{X_0,0}$ not Artinian, then the set of zero divisors on $\mathcal{O}_{X_0,0}$ in $(\mathbb{C}^N)^\vee$ has measure zero.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X_0,0}$ and

$$R = \operatorname{gr}_{\mathfrak{m}} \mathcal{O}_{X_0,0} = \mathcal{O}_{X_0,0}/\mathfrak{m} \oplus \bigoplus_{d \in \mathbb{N}} \mathfrak{m}^d/\mathfrak{m}^{d+1}$$

the associated *graded ring* (see e.g. [22]). Since any nonzerodivisor on R is automatically a nonzerodivisor on $\mathcal{O}_{X_0,0}$, and R is Artinian if and only if $\mathcal{O}_{X_0,0}$ is, it is sufficient to show the existence of Ω for R.

But this is clear since, unless R is Artinian, for all $d \in \mathbb{N}$ the set

$$\Omega_d = \left\{ l \in (\mathbb{C}^N)^\vee : \mathfrak{m}^d/\mathfrak{m}^{d+1} \xrightarrow{\cdot l} \mathfrak{m}^{d+1}/\mathfrak{m}^{d+2} \text{ is injective} \right\}$$

is a proper open algebraic set in $(\mathbb{C}^N)^\vee$ and therefore in particular dense with complement of measure zero. But then also the intersection $\Omega = \bigcap_{d \in \mathbb{N}} \Omega_d$ has a complement of measure zero. For any $l \in \Omega$ also the multiplication by l on R must be injective and such an l can not be a zerodivisor on $\mathcal{O}_{X_0,0}$. □

Given a smoothing $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{u} (\mathbb{C}, 0)$ of a determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of dimension d, we can choose a linear form $l_1 \in (\mathbb{C}^N)^\vee$ satisfying all the requirements of Lemma 2.3.4 and Lemma 2.3.7. After a linear change of coordinates we may assume that $l = x_N$ is just the last coordinate. Thus we obtain a second isolated determinantal singularity $(Y_0^1, 0) \subset (\mathbb{C}^{N-1}, 0)$ from the intersection $Y_0^1 = X_0 \cap \{l_1 = 0\}$.

We can proceed inductively and define hyperplanes $l_2, \dots, l_d$ until we finally end up with the analytic point-scheme $Y_0^d = X_0 \cap \{l_1 = l_2 = \dots = l_d = 0\}$. We also set $Y_0^0 = X_0$. Having done this, we can choose a Milnor ball B for all singularities at once and deform by u just as much that for all induced deformations of the Y_0^i the intersection $Y_0^i \cap \partial B$ stays transversal.

Now consider the function l_1 on $\overline{X}_u$. If 0 is not a regular value of l_1, then we can replace l_1 by $l_1 - c$ for some regular value c close to 0. In this situation $\overline{Y}_u^1 = \overline{X}_u \cap \{l_1 = 0\}$ is a smooth manifold with boundary, the Milnor fiber of $(Y_0^1, 0)$. Again, we can proceed inductively and we obtain a chain of Milnor fibers

$$\overline{Y}_u^d \subset \overline{Y}_u^{d-1} \subset \dots \subset \overline{Y}_u^1 \subset \overline{X}_u.$$

The topology of $\overline{Y}_u^d$ is simple: It is just a collection of smooth points, whose number is equal to the multiplicity e of the singularity $(X_0, 0)$ at 0. This follows from the flatness of the induced deformation of $(Y_0^d, 0)$ guaranteed by Lemma 2.3.7 and Lemma 1.2.25. In particular

$$\chi(\overline{Y}_u^d) = e.$$

At this point the hope is that we can rebuild the Milnor fiber $\overline{X}_u$ stepwise from $\overline{Y}_u^d$ as follows. We obtain $\overline{Y}_u^{d-1}$ from $\overline{Y}_u^d$ by attaching m_1 cells of dimension 1 where m_1 is the polar multiplicity of l_d on the smoothing of $(Y_0^{d-1}, 0)$. This allows us to compute the Euler-characteristic of $\overline{Y}_u^{d-1}$:

$$\chi(\overline{Y}_u^{d-1}) = -m_1 + e.$$

We may now proceed inductively and finally recover the Euler-characteristic of $\overline{X}_u$ from the polar multiplicities and the multiplicity e of $(X_0, 0)$ at 0. But for this to work we have to prove that, in fact, the polar multiplicity alone determines, which cells are attached.

Instead of the real part $\Re l_i$ we will consider $|l_i|^2$ as a Morse function on $\overline{Y}_u^{i-1}$. The particular problem with this does not lay in the interior of $\overline{X}_u$, and the $\overline{Y}_u^i$, where $|l_i|^2$ has critical points precisely at the critical points of l, but at their boundaries: We have to show that they can be neglected. To do so, we first prove a technical lemma. It is inspired by the work of Y. Yomdin [47] and his "lemma on $\operatorname{grad} h_1$ and $\operatorname{grad} h_2$".

Lemma 2.3.8. *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a complex space, X a representative and*

$$\{0\} \subset X^{(1)} \subsetneq X^{(2)} \subsetneq \dots \subsetneq X^{(r)} = X$$

a strictly ascending chain of analytic subspaces, which give rise to a complex analytic Whitney stratification as usual. Suppose we're given two real analytic functions $g, h : X \to \mathbb{R}_{\geq 0}$ taking values only in the nonnegative real numbers with $g(0) = h(0) = 0$. Then there exists a neighborhood U of the origin in $\mathbb{C}^N$, such that both f and g have no critical points on $X \cap U$ in the Whitney stratified sense outside $\{f = 0\}$ and $\{g = 0\}$ respectively. Furthermore, if $x \in (X \cap U) \setminus (\{f = 0\} \cup \{g = 0\})$ is a point, $\Sigma^{(i)} = X^{(i)} \setminus X^{(i-1)}$ the stratum containing x and $\mathrm{d}g$ and $\mathrm{d}h$ are linearly dependent in $x.\Omega^1_{\Sigma^{(i)}}$ over $\mathbb{R}$, say

$$\mathrm{d}g(x) = \lambda \cdot \mathrm{d}h(x) \quad \text{in} \quad x.\Omega^1_{\Sigma^{(i)}},$$

then $\lambda \geq 0$.

Proof. Consider X and its Whitney stratification with its reduced structure. The existence of a neighborhood U of the origin, such that neither f nor g have a critical point on $X \cap U$ outside their zero set, follows just like the existence of Milnor balls.

For $0 < i \leq r$ we set

$$K^{(i)} = \{x \in \Sigma^{(i)} : \mathrm{d}g(x) \text{ and } \mathrm{d}h(x) \text{ linearly dependent over } \mathbb{R} \text{ in } x.\Omega^1_{\Sigma^{(i)}}\},$$

where we consider $\mathrm{d}g$ and $\mathrm{d}h$ as real analytic sections in the cotangent bundle of $\Sigma^{(i)}$.

We would like to single out those points x in $K^{(i)}$, for which we have $\mathrm{d}g(x) = \lambda \cdot \mathrm{d}h(x)$ for some $\lambda < 0$ in $\mathbb{R}$. To do so, let $\pi : Y \to X^{(i)}$ be the generalized Nash-blowup of $X^{(i)}$ along its cotangent sheaf $\Omega^1_{X^{(i)}}$. For the definition of generalized Nash-blowups see the appendix, Definition A.2.5. Let $\overline{K}^{(i)} \subset Y$ be the analytic closure of $K^{(i)}$ in Y. Clearly, $\overline{K}^{(i)} \cap \pi^{-1}(\Sigma^{(i)}) \cong K^{(i)}$. Now let g be any real analytic metric on the Nash bundle Q over Y. We set

$$K^{(i)}_- := \{y \in \overline{K}^{(i)} \setminus \pi^{-1}(X^{(i-1)}) : g(\overline{\pi}_*(\mathrm{d}g), \overline{\pi}_*(\mathrm{d}h)) < 0\}.$$

Here, $\overline{\pi}_* : \Omega^1_{X^{(i)}} \to \pi_* Q$ is the Nash homomorphism. The set $K^{(i)}_-$ is real semi-analytic and its semi-analytic closure $\overline{K}^{(i)}_-$ in Y suffices $\overline{K}^{(i)}_- \cap \Sigma^{(i)} = K^{(i)}_-$.

Suppose there is a point $y \in \overline{K}^{(i)}_-$ such that $\pi(y) = 0$. Then according to the Curve Selection Lemma there exists a real analytic curve

$$\gamma : [0, \varepsilon) \to \overline{K}^{(i)}_-, \quad \gamma(0) = y$$

such that $\gamma(t) \in K^{(i)}_-$ for all $t > 0$.

We can integrate $\mathrm{d}g$ and $\mathrm{d}h$ along γ. For $t > 0$ we find

$$g(\gamma(t)) = \int_0^t \mathrm{d}g(\gamma(t)) \cdot \dot{\gamma}(\tau)\, \mathrm{d}\tau \tag{2.10}$$

and the density $\mathrm{d}g(\gamma(t)) \cdot \dot{\gamma}(\tau)$ is real analytic. Hence, there exists a nonzero initial term $a \cdot t^n$ for some $n \in \mathbb{N}$. The coefficient a must be > 0, since, if it wasn't, then for τ small enough, the integrand would be negative and, hence, also $g(\gamma(t))$ for some small t. The same holds for the function h and the coefficient of the initial term of $\mathrm{d}h(\gamma(t))$.

By construction, $\mathrm{d}g$ and $\mathrm{d}h$ are linearly dependent along γ for $t > 0$, i.e. we have $\mathrm{d}g(t) = \lambda(t) \cdot \mathrm{d}h(t)$ in $\gamma^*\Omega^1_{\Sigma^{(i)}}$ with $\lambda(t) < 0$ for all $t > 0$. Now along γ the coefficient $\lambda = \lambda(t)$ is a quotient of real analytic functions

$$\lambda(t) = \frac{\mathrm{d}g(\gamma(t)) \cdot \dot{\gamma}(t)}{\mathrm{d}h(\gamma(t)) \cdot \dot{\gamma}(t)}.$$

Depending on the order in t of numerator and denominator either $\lambda(t)$ or $1/\lambda(t)$ exists as an analytic function in t at $t = 0$. But then this function must have a positive leading coefficient contradicting the assumption $\lambda(t) \leq 0$ for all $t > 0$.

We deduce that such a point $y \in \overline{K}_-^{(i)} \cap \pi^{-1}(\{0\})$ can not exist. Consequently $0 \in X$ can not be in the image $\pi(\overline{K}_-^{(i)})$. Since $\pi : Y \to X^{(i)}$ is an isomorphism outside $X^{(i-1)}$, the result follows. □

The generalized Nash-blowup in the preceeding proof was necessary in order for the real analytic metric to exist. There is no canonical way to put a metric on the fibers of the coherent sheaf $\Omega^1_{\Sigma^{(i)}}$ and we need to replace it by a vector bundle.

When we encounter functions g and h as in Lemma 2.3.8 in the following, we will say that the (real) differentials $\mathrm{d}g$ and $\mathrm{d}h$ "point in the same direction". This notion originally comes from the gradients and not from differentials, but we shall adapt it here since mathematically the formulations in terms of gradients and differentials are equivalent.

Theorem 2.3.9. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$, $l \in (\mathbb{C}^N)^\vee$ and $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{u} (\mathbb{C}, 0)$ be as in Lemma 2.3.4, and Lemma 2.3.7. We set $(Y_0, 0) = (X_0, 0) \cap \{l = 0\}$. Let m_d be the polar multiplicity of l on the given smoothing. The Milnor fiber $\overline{X}_u$ of $(X_0, 0)$ is obtained from the Milnor fiber $\overline{Y}_v$ of $(Y_0, 0)$ by attaching m_d cells of dimension $n = \dim(X_0, 0)$.*

Proof. As usual let $\rho(x) = |x|^2$. We set $q(x) = |l|^2$. According to Lemma 2.3.8, there exists a neighborhood U of the origin, on which $\mathrm{d}\rho$ and $\mathrm{d}q$ point in the same direction on all points of $(X_0 \cap U) \setminus (\{\rho = 0\} \cup \{q = 0\})$. Choose a Milnor ball B for $(X_0, 0)$ and $(Y_0, 0)$, which is properly contained in U.

We would like to deform by u, but there are certain further restrictions on the choice of this parameter. Let $K \subset \partial\overline{X}_0$ be the set of critical points of q on the boundary of $\overline{X}_0$ outside $\partial\overline{Y}_0$. Clearly, K is a compact subset of $\mathbb{C}^N$. We can choose a real analytic metric g on the real vector bundle $\Omega^1_{X_0}|_{\partial\overline{X}_0}$. From this we obtain an expression

$$\lambda(x) = \frac{g(\mathrm{d}\rho(x), \mathrm{d}q(x))}{g(\mathrm{d}\rho(x), \mathrm{d}\rho(x))}$$

on $\partial\overline{X}_0$. Along K the function $\lambda(x)$ is just the coefficient of the linear dependence $\mathrm{d}q(x) = \lambda(x) \cdot \mathrm{d}\rho(x)$. According to Lemma 2.3.8, the function $\lambda(x)$ is positive on K and, hence, bounded from below away from zero on a neighborhood V of the compact set K in the representative X_0.

A perturbation by u induces a family of diffeomorphisms

$$\Phi_u : (X_0, \partial\overline{X}_0) \to (X_u, \partial\overline{X}_u)$$

depending smoothly on u. So does $\lambda(\Phi_u(x))$ as a function of u. We choose u small enough such that λ is still bounded away from zero on the open set $V' = \Phi_u(V) \subset X_u$ and such that outside V the differentials $\mathrm{d}q$ and $\mathrm{d}\rho$ stay linearly independent.

Consider q as a function on $\overline{X}_u$. By construction, the intersection $\{l = 0\} \cap \partial\overline{X}_u$ was transversal and for u small enough, this will be preserved. If $\{l = 0\} \cap \overline{X}_u$ is not a smooth manifold with boundary, we may replace l by $l - c$ for some constant $c \in \mathbb{C}$ close to zero. A direct calculation shows that on the interior of $\overline{X}_u$, the real valued function q has nondegenerate critical points of index n precisely at the critical points of the complex valued function l.

On the boundary $\partial\overline{X}_u$ we only know that the critical points of q are contained in V'. We may approximate q by a Morse function $\tilde{q}$ on both, the interior and the boundary of $\overline{X}_u$. On the interior, this does not change the number of critical points or their indices. On the boundary the critical points $(p_i)_{i=1}^m$ of $\tilde{q}$ also only appear in V'. If we now use $\tilde{q}$ as a Morse function on the manifold with boundary $\tilde{X}_u$, then the following happens.

At each critical point x of $\tilde{q}$ in the interior of $\overline{X}_u$ we attach a cell of real dimension n.

For a critical point y of $\tilde{q}$ on the boundary $\partial\overline{X}_u$ the homotopy type does not change due to Theorem A.3.6, because we have a linear dependence $\mathrm{d}\tilde{q}(y) = \lambda \cdot \mathrm{d}\rho(y)$ for some $\lambda > 0$ and, hence, "the gradient of $\tilde{q}$ is pointing outwards". □

Theorem 2.3.10. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a smoothable isolated determinantal singularity of dimension d and type (m, n, t) given by a matrix $A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ such that $\dim_{\mathbb{C}} \mathrm{Inf}(A) < \infty$. Then the vanishing Euler-characteristic of $(X_0, 0)$ can be computed as*

$$\nu(X_0, 0) = (-1)^d \cdot \left(e + \sum_{i=1}^{d} (-1)^i \cdot m_i \right), \tag{2.11}$$

where e is the multiplicity of $(X_0, 0)$ at 0 and the m_i are the polar multiplicities of successive general hyperplane sections of $(X_0, 0)$.

Proof. This directly follows from what has been said before Lemma 2.3.8 and Theorem 2.3.9. □

Remark 2.3.11. Note that the polar multiplicities are purely algebraic objects. This enables us to use Theorem 2.3.10 to effectively compute the vanishing Euler-characteristic for any smoothing of an isolated determinantal singularity with the help of computer algebra systems like Singular.

Remark 2.3.12. In [20], the authors work out formulas for the vanishing Euler characteristic for arbitrary EIDS, i.e. also those, which are not smoothable. Using methods of stratified Morse theory, it should be possible to give the analogous theorems in terms of polar multiplicities, i.e. to extend Theorem 2.3.10 in a generalized way also to these cases.

Chapter 3

Tjurina Transformations

This chapter summarizes and extends the results from [26]. They are new and were obtained in a joint work by A. Frühbis-Krüger and the author. The simple isolated Cohen-Macaulay codimension 2 singularities - especially the threefolds - were the main ground for development and verification of hypotheses. We will therefore mainly be concerned with singularities that fall into the same category concerning their describing matrices. Most of the stated theorems, however, are valid not only for simple singularities. The key tool for the topological considerations is the Tjurina modification, an idea that has been around for many years. But it seems that [26] is the first account of its systematic usage for the explicit computation of vanishing cycles.

What we today call Tjurina modifications is a blowup construction, which was first used by G. Tjurina in [69] in her studies of rational triple point singularities, where she applied it to determinantal surface singularities of type $(2, 3, 2)$. Later, also D. van Straten used them in his PhD-thesis [68]. The Tjurina modification has a natural generalization to determinantal singularities, which was already used by W. Ebeling and S.M. Gusein-Zade in [20]. But they applied it only to the remaining singularities in determinantal Milnor fibers, not to the original singularities themselves. Another recent instance of the usage of Tjurina modifications can be found in [28] by T. Gaffney and A. Rangachev.

We apply Tjurina modifications to the total space of determinantal deformations. As we will see below, under certain conditions, deformation and modification are compatible and we can reduce questions about the vanishing topology of a determinantal singularity $(X_0, 0)$ to questions about isolated complete intersection singularities - a world which is understood much better.

This compatibility must have been observed already by other mathematicians since traces of it can be found in the literature concerning the simultaneous resolution of surface singularities and cones over rational normal curves. Nevertheless, it seems that a systematic study of this interplay as we present it here for determinantal singularities has not been done before.

The construction of the Tjurina modification has also been carried out recently by H. Møller Pedersen in [55]. He uses a slightly different definition and applies it to determinantal singularities of type (d, d, d) to construct resolutions.

3.1 Tjurina Modification

3.1.1 ...for the Generic Determinantal Varieties

Definition 3.1.1. Let $t \leq m < n$ be positive integers and $Y = (y_{i,j})_{0<i\leq m,\,0<j\leq n}$ the generic matrix with entries $y_{i,j}$ over the ring $\mathbb{C}[\underline{y}]$. The *Tjurina modification* of the generic determinantal variety $M^t_{m,n} \subset \mathrm{Mat}(m,n;\mathbb{C})$ is defined as the generalized Nash-blowup of $M^t_{m,n}$ along the coherent sheaf $\mathcal{G}$ presented by the matrix $Y|_{M^t_{m,n}}$. The strict transform of $M^t_{m,n}$ is called the *Tjurina transform* and denoted by $W^t_{m,n} \subset M^t_{m,n} \times \mathrm{Grass}(t-1,m)$.

The transpose Tjurina modification is the modification coming from the generalized Nash-blowup of the sheaf presented by $Y^T|_{M^t_{m,n}}$, the transpose of Y restricted to $M^t_{m,n}$.

The Tjurina modification for the generic determinantal varieties has an easy explicit description. Consider the following commutative diagram.

$$\begin{tikzcd}
W^t_{m,n} \arrow[r, hook] \arrow[d, "\Pi"'] & \mathrm{Mat}(m,n;\mathbb{C}) \times \mathrm{Grass}(t-1,m) \arrow[d] \arrow[dr, "\hat{L}"] & \\
M^t_{m,n} \arrow[r, hook] & \mathrm{Mat}(m,n;\mathbb{C}) \arrow[r, dashed, "L"'] & \mathrm{Grass}(t-1,m)
\end{tikzcd} \tag{3.1}$$

Here, Π is the map induced from the first projection to $\mathrm{Mat}(m,n;\mathbb{C})$, L is the rational map taking a point $A \in M^t_{m,n}$ to span A (cf. (A.24) in the appendix) and $\hat{L}$ is its natural prolongation over the blowup.

The locus on which L is not defined is precisely $M^{t-1}_{m,n}$. In the definition of the generalized Nash blowup A.2.5 we find equations (A.26), which have to vanish on $W^t_{m,n}$ on each of the standard charts U_I of $\mathrm{Mat}(m,n;\mathbb{C}) \times \mathrm{Grass}(t-1,m)$:

$$\Theta^I \cdot A = 0 \in \mathcal{O}_{m\cdot n}[Z^I] \tag{3.2}$$

Lemma 3.1.2. *Let $W^t_{m,n} \subset \mathrm{Mat}(m,n;\mathbb{C}) \times \mathrm{Grass}(t-1,m)$ be the Tjurina transform of $M^t_{m,n}$ as in (3.1).*

i) Π is a isomorphism over $M^t_{m,n} \setminus M^{t-1}_{m,n}$.

ii) The equations (3.2) already determine $W^t_{m,n}$ in each chart. If $T \to \mathrm{Grass}(t-1,m)$ is the tautological bundle over $\mathrm{Grass}(t-1,m)$, then $W^t_{m,n}$ is canonically isomorphic to the total space of $\bigoplus_{i=1}^n T$. In particular, Π is a resolution of the singularities of $M^t_{m,n}$.

iii) For all $0 \leq r < t$ and each point $A \in M^r_{m,n} \setminus M^{r-1}_{m,n}$ we find the fiber $\Pi^{-1}(\{A\}) \cong \mathrm{Grass}(t-r, m-r+1)$.

Proof. Part $i)$ directly follows from the definition A.2.5 and the properties of the generalized Nash-blowup, Lemma A.2.7.

To see $ii)$ first observe that each column of A gives one copy of T in the description of the tautological bundle over $\mathrm{Grass}(t-1,m)$, Lemma A.2.4. Thus the space W' defined by these equations is smooth and irreducible and contains $W^t_{m,n}$. On the other hand

$$\Gamma_L(M^t_{m,n} \setminus M^{t-1}_{m,n}) \subset W' \subset W^t_{m,n}$$

and $M^t_{m,n} \setminus M^{t-1}_{m,n}$ is an open dense subset of $M^t_{m,n}$, since $M^t_{m,n}$ is irreducible. We deduce that $W' = W^t_{m,n}$.

For part $iii)$ let $W := \operatorname{span} A \subset \mathbb{C}^m$ be the r-dimensional subspace spanned by A. Then as above $(A, p) \in W^t_{m,n}$ if and only if $\operatorname{span} A$ is contained in the linear subspace represented by p and hence

$$\begin{aligned}\Pi^{-1}(\{A\}) &= \{W \subset V \subset \mathbb{C}^m : \dim V = t-1\} \\ &\cong \{\{0\} \subset V' \subset \mathbb{C}^{m-r+1} : \dim V = t-r\} \\ &= \operatorname{Grass}(t-r, m-r+1)\end{aligned}$$

by passing to $\mathbb{C}^m / W \cong \mathbb{C}^{m-r+1}$. □

3.1.2 . . . for Determinantal Singularities

In Chapter 1 we saw that every determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, t) inherits a free resolution of $\mathcal{O}_{X_0,0}$ over $\mathcal{O}_N$ from the graded free resolution associated to the generic determinantal variety $M^t_{m,n}$ via the defining matrix A. The Tjurina modification of $(X_0, 0)$ is also defined as the "modification inherited from $M^t_{m,n}$".

Definition 3.1.3. The Tjurina modification of $(X_0, 0)$ is defined by the following commutative diagram:

$$\begin{tikzcd} X_0 \times_{M^t_{m,n}} W^t_{m,n} \arrow[r] \arrow[d] & W^t_{m,n} \arrow[d, "\Pi"'] \arrow[dr, "\hat{L}"] & \\ X_0 \arrow[r, "A"] & M^t_{m,n} \arrow[r, dashed, "L"'] & \operatorname{Grass}(t-1, m) \end{tikzcd} \quad . \qquad (3.3)$$

Usually we will consider the Tjurina transform of $(X_0, 0)$ as a germ in $(\mathbb{C}^N, 0) \times \operatorname{Grass}(t-1, m)$ along the compact subset $E := \{0\} \times \operatorname{Grass}(t-1, m)$ and denote it by (Y_0, E). The natural projection $(Y_0, E) \to (X_0, 0)$ will be denoted by π_0.

It is clear that $\pi_0 : (Y_0, E) \to (X_0, 0)$ is an isomorphism outside the set $A^{-1}(M^{t-1}_{m,n}) \subset X_0$. From Corollary 2.2.7 we obtain the following result:

Lemma 3.1.4. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated determinantal singularity of type (m, n, t) defined by $A \in \operatorname{Mat}(m, n; \mathcal{O}_N)$ and (Y_0, E) its Tjurina transform. Then*

$$\pi_0 : (Y_0, E) \to (X_0, 0)$$

is an isomorphism over $X_0 \setminus \{0\}$.

Proof. Let X_0 be a representative and $p \in X_0 \setminus \{0\}$ an arbitrary point. Since (X_0, p) has expected codimension under the map

$$A : (\mathbb{C}^N, p) \to (\operatorname{Mat}(m, n; \mathbb{C}), A(p))$$

and X_0 is smooth at p, we deduce from 2.2.7, that $\operatorname{rank} A(p) = t-1$, so $p \notin A^{-1}(M^{t-1}_{m,n})$. Now the claim follows from the fact that the Tjurina modification of the generic determinantal variety $M^t_{m,n}$ is an isomorphism away from $M^{t-1}_{m,n}$, Lemma 3.1.2 $i)$. □

The given definition of the Tjurina transform has the advantage that we can easily give explicit equations. We describe them in the next lemma.

Lemma 3.1.5. *Let $I \subset \{1, \dots, m\}$ be an ordered multiindex of order $\#I = t-1$ and U_I the standard chart of the Grassmannian $\mathrm{Grass}(t-1, m)$. The equations for the Tjurina transform $(Y_0, E) \subset (\mathbb{C}^N, 0) \times \mathrm{Grass}(t-1, m)$ in this chart are given by*

$$\Theta^I \cdot A = 0 \in \mathrm{Mat}(m-t+1, n; \mathcal{O}_N[Z^I]),$$

where Θ^I is the standard representative matrix and Z^I the standard affine coordinates for this chart as defined in (A.14) and (A.18).

Proof. It is clear from the construction of the fiber product that these equations have to vanish on (Y_0, E). We have to show that also the equations defining $(X_0, 0)$ in $\mathbb{C}^N$ are already contained in the ideal generated by the entries of $\Theta^I \cdot A$.

But if we let $J \subset \{1, \dots, m\}$ be the multiindex complementary to I, $L = (1, \dots, m-t+1)$ and $K = (1, \dots, n)$, then from the definition of Θ^I we have

$$\Theta^I \cdot A = 0 \quad \Leftrightarrow \quad \Theta^I_{L,I} \cdot A_{I,K} = A_{J,K}.$$

From this we see that the rows of A in J can be expressed by the $t-1$ rows in I modulo $\langle \Theta^I \cdot A \rangle$. Consequently, all t-minors of A considered as elements in $\mathcal{O}_N[Z^I]$ are already contained in $\langle \Theta^I \cdot A \rangle$. $\square$

In the definition of the Tjurina transform of a determinantal singularity, we do *not* blow up the rational map

$$L \circ A : X_0 \dashrightarrow \mathrm{Grass}(t-1, m).$$

Therefore in general the Tjurina transform (Y_0, E) has a decomposition

$$Y_0 = \overline{Y_0 \setminus X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n}} \cup X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n}, \tag{3.4}$$

where the space $\overline{Y_0 \setminus X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n}}$ is the strict transform [1] of X_0 under the blowup of $L \circ A$. In particular it has the same dimension as $(X_0, 0)$ along E.

To clearify what we mean by this, we define the dimension of a germ germ (Y, E) along a compact set E as

$$\dim(Y, E) = \sup_{p \in E} \quad \dim(Y, p). \tag{3.5}$$

Corollary 3.1.6. *The Tjurina transform (Y_0, E) is a local complete intersection if and only if*

$$\dim(X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n}, E) \leq \dim(X_0, 0). \tag{3.6}$$

If $(X_0, 0)$ is an isolated singularity, then

$$X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n} = \{0\} \times \mathrm{Grass}(t-1, m) \tag{3.7}$$

and (3.6) becomes

$$N \geq n \cdot (m-t+1). \tag{3.8}$$

[1] In fact in [55], and [28] the Tjurina transform is defined to be this strict transform.

Proof. The ambient space of (Y_0, E) has dimension

$$N + \dim \mathrm{Grass}(t-1, m) = N + (t-1)\cdot(m-t+1).$$

Let $p \in \mathbb{C}^N \times \mathrm{Grass}(t-1, m)$ be an arbitrary point in E and $X \times U_I$ a chart containing it. According to Lemma 3.1.5, there are $n \cdot (m-t+1)$ equations defining (Y_0, p). Therefore the expected dimension of (Y_0, p) as a complete intersection would be

$$N + (t-1)\cdot(m-t+1) - n\cdot(m-t+1) = N - (n-t+1)\cdot(m-t+1).$$

But this is equal to the dimension of $(X_0, 0)$. From the decomposition (3.4) we see that

$$\dim(Y_0, p) = \max\{\dim(X_0, 0), \dim(X_0 \times_{M^t_{m,n}} W^{t-1}_{m,n}, p)\}.$$

Thus if (3.6) holds, then (Y_0, p) has codimension $N-(n-t+1)(m-t+1)$ in a Cohen-Macaulay ring and its ideal is generated by the same number of elements. According to Theorem 1.2.15 (Y_0, p) must be a complete intersection at p.

If $(X_0, 0)$ has an isolated singularity, then according to Corollary 2.2.7 $A(p) \notin M^{t-1}_{m,n}$ for all $p \in X_0 \setminus \{0\}$. On the other hand due to the minimality condition on A we may assume that all entries $a_{i,j} \in \mathfrak{m}$ - the maximal ideal of $\mathcal{O}_N$. Hence $A(0) = 0$ and with the given equations for the Tjurina transform from Lemma 3.1.5 we find no conditions on the affine coordinates of $\mathrm{Grass}(t-1, m)$ in each chart in the fiber over 0. Putting this together, we obtain (3.7). The inequality (3.6) is now a simple consequence from counting dimensions. $\square$

3.1.3 ...in Family

Definition 3.1.7. Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix $A \in \mathrm{Mat}(m, n; \mathcal{O}_N)$ and $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{u} (\mathbb{C}^k, 0)$ be a determinantal deformation described by $\mathbf{A} \in \mathrm{Mat}(m, n; \mathcal{O}_{N+k})$. The *Tjurina modification in family* is defined by the following commutative diagram:

$$\begin{tikzcd}
X_0 \times_{M^t_{m,n}} W^t_{m,n} \arrow[r, hook] \arrow[d, "\pi_0"'] & X \times_{M^t_{m,n}} W^t_{m,n} \arrow[r] \arrow[d, "\pi"'] & W^t_{m,n} \arrow[d, "\Pi"'] \arrow[dr, "\hat{L}"] & \\
X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, "\mathbf{A}"] \arrow[d, "u"] & M^t_{m,n} \arrow[r, dashed, "L"'] & \mathrm{Grass}(t-1, m) \\
\{0\} \arrow[r, hook] & \mathbb{C}^k & &
\end{tikzcd} \tag{3.9}$$

We usually think of the space $X \times_{M^t_{m,n}} W^t_{m,n}$ sitting over X as a germ in $(\mathbb{C}^N, 0) \times \mathrm{Grass}(t-1, m) \times (\mathbb{C}^k, 0)$ along the set $\{0\} \times \mathrm{Grass}(t-1, m) \times \{0\} \cong E$ and denote it by (Y, E).

By the same reasoning as in the preceeding section for the determinantal singularities, the equations for (Y,E) in the chart $\mathbb{C}^N \times U_I$ are given by

$$\Theta^I \cdot \mathbf{A} = 0 \tag{3.10}$$

For $u \in \mathbb{C}^k$ we denote the fibers over it by X_u and $Y_u := \pi^{-1}(X_u)$ respectively. The restriction of π to Y_u will be named π_u. In the whole setup of Definition 3.1.7, we consider representatives and we may choose a Milnor Ball B for $(X_0, 0)$ and a neighborhood $D \subset \mathbb{C}^k$ of the origin compatible with the choice of B and the given deformation.

Lemma 3.1.8. *Consider the Tjurina modification in family as in (3.9).*

i) If $\overline{X}_u$ is a determinantal Milnor fiber over some $u \in D$, then

$$\pi_u : \overline{Y}_u \to \overline{X}_u$$

is a resolution of the singularities of $\overline{X}_u$. If $\overline{X}_u$ is smooth, then π_u is an isomorphism.

ii) If (Y_0, E) is a local complete intersection, then the family

$$u \circ \pi : (Y, E) \to (\mathbb{C}^k, 0) \tag{3.11}$$

is flat at all points $p \in E$.

Proof. Part $i)$ directly follows from the description of the singularities of a determinantal Milnor fiber in Theorem 2.2.13 and the fact that Tjurina modification gives a resolution of the singularities of the generic determinantal varieties, Lemma 3.1.2.

For part $ii)$ we only need Theorem 1.3.1, because the equations given by $\Theta^I \cdot \mathbf{A}$ can be regarded as perturbations of the equations $\Theta^I \cdot A$ in all charts. □

The direction in which we want to go from here is already revealed in the formulation of Lemma 3.1.8. We would like to study the Tjurina modification in family for stabilizations of determinantal singularities. If the Tjurina transform (Y_0, E) is a local complete intersection, then we can study the vanishing cycles in the induced deformation of (Y_0, p) at points $p \in E$, which occur as we pass to a deformed fiber $\overline{Y}_u$. Then we use the map π_u to compare $\overline{Y}_u$ with $\overline{X}_u$, our object of interest.

Remark 3.1.9. For Cohen-Macaulay codimension 2 singularities (i.e. determinantal singularities of type $(t, t+1, t)$, cf. Theorem 1.4.20), the Tjurina transform takes a particularly simple form. The Grassmannian in question is always

$$\mathrm{Grass}(t-1, t) \cong \mathrm{Grass}(t-1,t)^\vee = \mathrm{Grass}(1,t) = \mathbb{P}^{t-1}.$$

If we let $s = (s_1 : \cdots : s_t)$ be the homogeneous coordinates of $\mathbb{P}^{t-1}$ and $\mathbf{A}$ a matrix defining a deformation of a Cohen-Macaulay codimension 2 singularity $(X_0, 0)$, then the equations for the Tjurina modification in family are

$$\begin{pmatrix} s_1 & \cdots & s_t \end{pmatrix} \cdot \mathbf{A} = 0. \tag{3.12}$$

3.2 Topology of Simple ICMC2 Singularities

As remarked earlier, the starting point of the research carried out for this thesis was the observations by J. Damon and B. Pike in [17] concerning the vanishing Euler-characteristic of simple Cohen-Macaulay codimension 2 threefold singularities. These were classified by A. Frühbis-Krüger and A. Neumer in [25]. Simple isolated Cohen-Macaulay codimension 2 singularities occur in dimensions from 0 to 4.

For fat point singularities, topological questions are rather trivial. The topology of curves could be treated already using methods which were not specific to determinantal singularities, see [14], and [37]. In dimension 2, there is always only one Betti number of the determinantal Milnor fiber, which contributes to the vanishing Euler-characteristic due to Theorem 2.1.18. This Betti number can therefore be determined using Theorem 2.3.10.

It is dimension 3, where for the first time we do have two Betti numbers besides b_0, namely b_3 and b_2, which can be nonzero. J. Damon and B. Pike showed with their computation of the vanishing Euler-characteristic that among the singularities listed in [25], see also Table 3.1, there are members, for which b_2 must be nonzero and others, for which b_3 is not zero. However, they were unable to determine them individually. The starting point of this work was, to compute those numbers and explain their behaviour and growth within the discrete families of singularities listed in Table 3.1.

This task could finally be achieved in [26], where, together with A. Frühbis-Krüger, we announced the outcomes of those computations. The methods developed for this purpose can, of course, be applied to more general singularities, i.e. being simple is not a requirement of the involved theorems. Nevertheless, the conditions in the formulated theorems would seem rather unmotivated and restrictive if one did not have in mind the original lists of simple singularities from the classification. In this chapter, we therefore carry them along to explain, in which sense the exhibited theory is natural. The results in the individual dimensions are then presented in the order in which they were found. For fourfolds we only give an idea, how it should be possible to generalize the methods to non-smoothable singularities.

First of all, the simple Cohen-Macaulay codimension 2 singularities considered in [25] are always isolated. Since the type of a Cohen-Macaulay codimension 2 singularity is always $(t, t+1, t)$ for some $t \in \mathbb{N}$, we know from Corollary 3.1.6 and Remark 3.1.9 that if $(X_0, 0) \subset (\mathbb{C}^N, 0)$ is isolated, then

$$E = \pi_0^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{t-1}$$

is the exceptional set of the Tjurina transform (Y_0, E). A simple computation shows that (Y_0, E) is a local complete intersection if and only if $N \geq t+1$, i.e. $\dim(X_0, 0) = N - 2 \geq t - 1$.

Isolated simple Cohen-Macaulay codimension 2 singularities, which are not complete intersections, occur only as determinantal singularities of type $(2, 3, 2)$, as shown in [25]. Their classification proceeds by roaming through the space of jets of defining matrices and demarcating those leading to non-simple singularities.

For any integer r the r- *jet* of a holomorphic function $f \in \mathcal{O}_N$ can be defined as the equivalence class

$$j^r f = f + \mathfrak{m}^{r+1} \in \mathcal{O}_N/\mathfrak{m}^{r+1} \tag{3.13}$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_N$. In particular we have $\mathrm{d}f(0) \neq 0$ if and only if $j^1 f \neq 0$. This notion carries over to matrices $A \in \mathrm{Mat}(m,n;\mathcal{O}_N)$ in the obvious way.

We can use the jets to classify matrices A defining a determinantal singularity $(X_0, 0)$. Suppose there is one entry $a_{i,j}$ with nonzero 1-jet. Without loss of generality we may assume $(i,j) = (1,1)$. After a change of coordinates on the ambient space $(\mathbb{C}^N, 0)$, we may assume that $a_{1,1} = x_1$ is the first coordinate. Using row and column operations, we can eliminate all occurences of terms involving x_1 in the first row and the first column and thus bring A to the form

$$\begin{pmatrix} x_1 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & * & \cdots & * \end{pmatrix}$$

where neither of the $a_{i,1}$ nor of the $a_{1,j}$ involves x_1 at all. If now any of the entries $a_{i,j}$ outside the first column has a 1-jet, which is linealy independent of $j^1 a_{1,1} = x_1$, we can add λ times the i-th row to the first for a sufficiently general $\lambda \in \mathbb{C}$ and replace the coordinate x_1 by $x_1 - \lambda \cdot a_{i,j}$. Then without loss of generality we can assume that it is the entry $a_{1,2}$, whos 1-jet is independent of x_1. Now we can repeat the procedure: After another change of coordinates in $(\mathbb{C}^N, 0)$, which can be chosen to preserve x_1, we can assume $a_{1,2} = x_2$ and then use column operations on A to eliminate any further occurence of x_2 among the $a_{1,j}$ for $j > 2$. Note, however, that if we tried to eliminate x_2 also from the other entries of the second column, we would in general ruin the elimination of x_1-terms in the first column. Hence this reduction is a privilege of the first step only. If we continue looking for 1-jets of entries outside the first two columns, which are linearly independent of x_1 and x_2, we eventually end up with a matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_d & a_{1,d+1} & \cdots & a_{1,n} \\ a_{2,1} & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & * & \cdots & * & * & \cdots & * \end{pmatrix}, \tag{3.14}$$

where d is the maximal number of linearly independent 1-jets in the first row and for all the entries $a_{i,j}$ in the last $n-d$ columns we can assume $j^1 a_{i,j} = 0$, i.e. $a_{i,j} \in \mathfrak{m}^2$.

As it turns out in [25], for all simple ICMC2 singularities of dimension > 1 and those cases bounding the simple ones in the classification, the defining matrices $A \in \mathrm{Mat}(2,3;\mathcal{O}_N)$ are all equivalent to matrices, whos 1-jet takes the following form

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ * & * & * \end{pmatrix} \tag{3.15}$$

For all such singularities, the Tjurina modification is particularly well behaved.

Lemma 3.2.1 ([26]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated determinantal singularity of type $(2, 3, 2)$ and of dimension > 0. Then the Tjurina transform (Y_0, E) has at most isolated singularities, iff the defining matrix A for $(X_0, 0)$ is equivalent to a matrix of the form (3.15).*

Proof. Tjurina modification is an isomorphism outside the singular locus, which implies that the singular locus of Y_0 is contained in $E = \pi_0^{-1}(\{0\}) \cong \mathbb{P}^1$. Because E is irreducible, the singular locus of Y_0 is either a finite number of points or the whole $\mathbb{P}^1$.
From the type of the singularity we deduce that (Y_0, E) is a local complete intersection.

We first show that (Y_0, E) has isolated singularities if the defining matrix is of the form (3.15). Consider the second chart $U_{(2)}$ of $\mathbb{P}^1 = \text{Grass}(1, 2)^\vee$. The local equations for (Y_0, E) are

$$\begin{pmatrix} 1 & z^{(2)} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\ a & b & c \end{pmatrix} = 0,$$

where $a, b, c \in \mathfrak{m}$ are arbitrary entries in the maximal ideal. The jacobian matrix of these three defining equations takes the form

$$\begin{pmatrix} 1 + z^{(2)} \frac{\partial a}{\partial x_1} & z^{(2)} \frac{\partial a}{\partial x_2} & z^{(2)} \frac{\partial a}{\partial x_3} & \cdots \\ z^{(2)} \frac{\partial b}{\partial x_1} & 1 + z^{(2)} \frac{\partial b}{\partial x_2} & z^{(2)} \frac{\partial b}{\partial x_3} & \cdots \\ z^{(2)} \frac{\partial c}{\partial x_1} & z^{(2)} \frac{\partial c}{\partial x_2} & 1 + z^{(2)} \frac{\partial c}{\partial x_3} & \cdots \end{pmatrix} \tag{3.16}$$

Clearly, the first minor of this matrix is a unit in $\mathcal{O}_{\mathbb{C}^N \times \mathbb{C}, 0}$, the local ring at the origin of this chart. Hence $\text{Sing}(Y_0, E)$ can not be the whole exceptional set $E = \{0\} \times \mathbb{P}^1$ and must therefore consist of finitely many points.

On the other hand, suppose that the defining matrix A of $(X_0, 0)$ can not be brought to the form (3.15). Then according to the normal form (3.14), we can assume that the entries of the last column of A are all in $\mathfrak{m}^2$. But then one row of the jacobian has all entries in $\mathfrak{m}$ and hence also all all the maximal minors of the jacobian will be in $\mathfrak{m}$. Since the vanishing locus of $\mathfrak{m} = \langle \underline{x} \rangle$ on (Y_0, E) is exactly the exceptional set $E = \{0\} \times \mathbb{P}^1$, it must therefore consist of singular points of Y_0 in this case. $\square$

Lemma 3.2.2. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an isolated determinantal singularity of type (m, n, t) and*

$$\pi_0 : (Y_0, E) \to (X_0, 0)$$

the projection of the Tjurina modification of $(X_0, 0)$. Then for a suffiently small representative E is a deformation retract of Y_0.

Proof. Since $(X_0, 0)$ is an isolated singularity, the variety $E = \{0\} \times \text{Grass}(t-1, m)$ is closed and projective, hence compact. It follows from [51], that E is a Euclidean Neighborhood Retract of an open neighborhood U of E in Y_0. But outside E the map π_0 is an isomorphism, so $\pi_0(U) \subset X_0$ is open. We may replace X_0 by $X_0 \cap \pi_0(U)$ and Y_0 by $\pi_0^{-1}(X_0)$. $\square$

For ICMC2 singularities of type $(t, t+1, t)$ this means that the homotopy type of Y_0 is completely determined by the exceptional set $E \cong \mathbb{P}^{t-1}$ of the

Tjurina transform. In particular we find

$$H_i(Y_0) \cong H_i(\mathbb{P}^{t-1}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 2(t-1) \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

For the statement of the next theorem compare with Theorem 2.2.13 and Corollary 3.1.6.

Theorem 3.2.3. *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an ICMC2 singularity of type $(t, t+1, t)$ for $t + 1 \leq N < 6$, so that the Tjurina transform (Y_0, E) is a local complete intersection and $(X_0, 0)$ has a determinantal smoothing. Let $d = N - 2$ be the dimension of $(X_0, 0)$. Suppose the singular set $\Sigma \subset E$ of (Y_0, E) consists only of isolated points and let F_q be the Milnor fiber of the isolated singularity (Y_0, q).*

There is a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H_{d+1}(\mathbb{P}^{t-1}) \longrightarrow \bigoplus_{q\in\Sigma} H_d(F_q) \longrightarrow H_d(\overline{X}_u) \longrightarrow \\ \longrightarrow H_d(\mathbb{P}^{t-1}) \longrightarrow 0 \longrightarrow \cdots \\ \cdots \longrightarrow 0 \longrightarrow H_2(\overline{X}_u) \longrightarrow \\ \longrightarrow H_2(\mathbb{P}^{t-1}) \longrightarrow 0 \longrightarrow H_1(\overline{X}_u) \longrightarrow 0 \end{array} \tag{3.18}$$

and $\overline{X}_u$ is connected.

Note that the way in which (3.18) is presented is probably a bit misleading. We attempt to make a general statement, but in fact the condition $t + 1 \leq N \leq 6$ narrows the possible configurations drastically. As it turns out, we can observe very different outcomes for the topology of $\underline{X}_u$ depending for different choices of t and N within these ranges. This will be discussed below in further sections.

Proof. Choose a Milnor ball B for $(X_0, 0)$ and let $X_0 \hookrightarrow X \xrightarrow{u} \mathbb{C}$ be a smoothing. For some disc $D \subset \mathbb{C}$ around the origin, over which the smoothing is well behaved for the chosen Milnor ball, we consider the Milnor tube $B \times D \subset \mathbb{C}^N \times \mathbb{C}$ and its preimage

$$T = \pi^{-1}(B \times D) \subset \mathbb{C}^N \times \mathbb{P}^{t-1} \times \mathbb{C}$$

We set $\overline{Y} = Y \cap T$ and denote the fiber over u by $\overline{Y}_u$.

At all singular point $q \in E$ of Y_0, we can choose Milnor balls B_q for the singularities (Y_0, q), which are disjoint from the boundary $\partial \overline{Y}_0 \cong \partial \overline{X}_0$. Since the underlying family for $(X_0, 0)$ is a smoothing, the projection

$$\pi_u : \overline{Y}_u \to \overline{X}_u$$

is an isomorphism for small $u \neq 0$ by Lemma 3.1.8, the induced deformations of all singularities (Y_0, q) are smoothings as well.

Now the (Y_0, q) are isolated complete intersection singularities and hence their Milnor fiber is known due to Theorem 2.1.13, Theorem 2.1.15, and the

Lê-Greuel formula (2.3). Outside the singular points we have a smooth manifold with boundary

$$Z_0 := \overline{Y}_0 \setminus \bigcup_{p\in\Sigma} (Y_0 \cap \overset{\circ}{B}_q),$$

where as usual $\circ$ denotes the interior. The boundary of Z_0 decomposes as

$$\partial Z_0 = \partial_1 Z_0 \cup \partial_2 Z_0 = \partial Y_0 \cup \bigcup_{p\in\Sigma} \partial B_q \cap Y_0.$$

We may now choose u small enough such that the associated deformation of Z_0 is trivial in the differentiable category and for all singularities (Y_0, q) we pass to a Milnor fiber $F_q := B_q \cap \overline{Y}_u$. From the pair of spaces $(\overline{Y}_u, \bigcup_{q\in\Sigma} F_q)$ we obtain a long exact sequence

$$\cdots \longrightarrow \bigoplus_{q\in\Sigma} H_i(F_q) \longrightarrow H_i(\overline{Y}_u) \longrightarrow H_i(\overline{Y}_u, \textstyle\bigcup_{q\in\Sigma} F_q) \longrightarrow \cdots \tag{3.19}$$

We can compute the terms $H_i(\overline{Y}_u, \bigcup_{p\in\Sigma} F_q)$ as follows. By excision we have isomorphisms

$$H_i(\overline{Y}_u, \bigcup_{p\in\Sigma} F_q) \cong H_i(Z_u, \partial_2 Z_u) \cong H_i(Z_0, \partial_2 Z_0) \cong H_i(\overline{Y}_0, B_q \cap Y_0) \cong H_i(\overline{Y}_0)$$

for all $i \geq 0$. The last step is explained as follows. The spaces $B_q \cap Y_0$ are contractible according to their conical structure and the properties of the Milnor ball and hence $H_i(B_q \cap Y_0) = 0$ for all $i > 0$. Thus the isomorphism follows from the long exact sequence of the pair $(\overline{Y}_0, B_q \cap Y_0)$.

Since according to Lemma 3.2.2 Y_0 is homotopy equivalent to the exceptional set $E \cong \mathbb{P}^{t-1}$, we can replace any occurence of $H_i(\overline{Y}_u, \bigcup_{p\in\Sigma} F_q)$ in (3.19) by $H_i(\mathbb{P}^{t-1})$. If we also take into account the isomorphism $\overline{Y}_u \cong \overline{X}_u$ and the result by Hamm on the F_q, we obtain (3.18). □

The following example illustrates how one can use the exact sequence (3.18) to compute the Betti numbers of the Milnor fiber $\overline{X}_u$ of a determinantal singularity.

Example 3.2.4. Let us consider the ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^5, 0)$ given by the 3-minors of the matrix

$$A = \begin{pmatrix} x & y & z & 0 \\ y-v & z-v & 0 & u \\ y+z & x+u & x-u & v \end{pmatrix}.$$

Let $(s_1 : s_2 : s_3)$ be the projective coordinates of $\mathbb{P}^2 = \text{Grass}(2,3)$. Then we obtain $Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^2$ as the zero locus of the equations

$$\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} \begin{pmatrix} x & y & z & 0 \\ y-v & z-v & 0 & u \\ y+z & x+u & x-u & v \end{pmatrix} = 0. \tag{3.20}$$

The Tjurina transform Y_0 is still singular at 10 distinct points in $\{0\} \times \mathbb{P}^2 \subset \mathbb{C}^5 \times \mathbb{P}^2$. But there we only find 3-dimensional A_1 singularities embedded in higher dimensional space. Thus compared to the singularity $(X_0, 0)$, the situation became much simpler. Consider e.g. the singularity at the point $p = (0, (1 : 0 : 0))$ in the chart $s_1 \neq 0$: The first three lines of the system (3.20) define a smooth variety H of dimension 4 around p. Inside H the equation

$$s_2^{(1)} \cdot u + s_3^{(1)} \cdot v = 0$$

in the last line provides the A_1 singularity.

Consider the deformation with a parameter ε given by the matrix

$$\mathbf{A} = \begin{pmatrix} x & y & z & 3\varepsilon \\ y - v & z - v & 3\varepsilon & u \\ y + z & x + u & x - u & v \end{pmatrix}$$

Let $X \subset \mathbb{C}^5 \times \mathbb{C}$ be the total space of the deformation

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "\varepsilon"] \\ \{0\} \arrow[r] & \mathbb{C} \end{tikzcd}$$

The Tjurina modification in family in $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^2$ is now described by the equations

$$\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} \cdot \mathbf{A} = 0$$

As a direct computation shows, all fibers of this family except the one over $\varepsilon = 0$ are smooth. The long exact sequence (3.18) splits into several parts:

$$0 \longrightarrow H_4(\mathbb{P}^2) \longrightarrow \bigoplus_{q \in \Sigma} H_3(F_q) \longrightarrow H_3(\overline{X}_u) \longrightarrow H_3(\mathbb{P}^2) \tag{3.21}$$

$$0 \longrightarrow H_2(\overline{X}_u) \longrightarrow H_2(\mathbb{P}^2) \longrightarrow 0 \tag{3.22}$$

$$0 \longrightarrow H_1(\overline{X}_u) \longrightarrow H_1(\mathbb{P}^2) \longrightarrow 0 \tag{3.23}$$

Concerning (3.21): Recall from (3.17) that the homology of $\mathbb{P}^2$ vanishes in odd degrees and we therefore have a zero on the right. Now it is well known that $H_3(F_q) = \mathbb{Z}$ for the Milnor fiber F_q of an A_1-threefold singularity. Since there are 10 of them we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{10} \longrightarrow H_3(\overline{X}_u) \longrightarrow 0$$

and hence $b_3(\overline{X}_u) = 9$. A closer observation with explicit coordinates in fact shows that $H_3(\overline{X}_u)$ does not have torsion.

From the other two equations (3.22) and (3.23) we directly obtain isomorphisms

$$H_2(\overline{X}_u) \cong \mathbb{Z}, \quad H_1(\overline{X}_u) = 0.$$

In total we find

$$b_i(X_0, 0) = \begin{cases} 9 & \text{if } i = 3 \\ 1 & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

for the Betti numbers of the Milnor fiber of the original determinantal singularity $(X_0, 0)$.

Depending on the choices of N and t in Theorem 3.2.3 we can observe very different behaviour. However, in all cases we will recognize phenomena, which could already be observed in the above example.

3.2.1 Betti Numbers for the Threefolds

Theorem 3.2.5. *Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be an ICMC2 singularity of type $(3, 4, 3)$ such that the Tjurina transform (Y_0, E) has only isolated singularities (Y_0, q) at points $q \in E$. If we let $b_3(q)$ be the middle Betti number of the ICIS (Y_0, q) at q, then the Betti numbers of $(X_0, 0)$ are*

$$b_i(X_0, 0) = \begin{cases} \left(\sum_{q \in \mathrm{Sing}(Y_0, E)} b_3(q)\right) - 1 & \text{if } i = 3 \\ 1 & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases} .$$

All homology groups are torsion free.

Proof. Directly along the lines of example 3.2.4. It remains to show that $H_3(X_0, 0)$ is free in general. But this can be seen from the attachment process leading to Theorem 2.3.10: We obtain the Milnor fiber $\overline{X}_u$ from a generic hyperplane section $H \cap \overline{X}_u$ of it, by attaching cells of real dimension 3. The corresponding part from the long exact sequence of the pair $(\overline{X}_u, H \cap \overline{X}_u)$ is

$$0 \longrightarrow H_3(\overline{X}_u) \longrightarrow H_3(\overline{X}_u, H \cap \overline{X}_u) \longrightarrow H_2(H \cap \overline{X}_u) ,$$

since the dimension of $H \cap \overline{X}_u)$ is 2 and hence its third homology group is zero due to the Lefschetz Hyperplane Theorem. Now the relative homology group $H_3(\overline{X}_u, H \cap \overline{X}_u)$ is just $\mathbb{Z}^r$, where r is the number of attached cells. Being a submodule of a torsion free module, $H_3(\overline{X}_u)$ can not have torsion itself. □

Theorem 3.2.6. *Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be an ICMC2 singularity of type $(2, 3, 2)$ such that the Tjurina transform (Y_0, E) has only isolated singularities (Y_0, q) at points $q \in E$. If we let $b_3(q)$ be the middle Betti number of the ICIS (Y_0, q) at q, then the Betti numbers of $(X_0, 0)$ are*

$$b_i(X_0, 0) = \begin{cases} \sum_{q \in \mathrm{Sing}(Y_0, E)} b_3(q) & \text{if } i = 3 \\ 1 & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases} .$$

All homology groups are torsion free.

Proof. The proof is the same as for Theorem 3.2.5, only that the term $H_4(\mathbb{P}^1)$ appearing in (3.18) is zero. □

We used this theorem to compute the homology groups of all simple ICMC2 threefold singularities. In the following table we do not only list the Betti numbers of the Milnor fibers, but also the types of the singularities appearing in the Tjurina transform (we adapt the classical names by Arnold [4]), and the Tjurina number

$$\tau = \dim_{\mathbb{C}} T^1_{X_0,0}$$

of $(X_0, 0)$. Recall that, as a consequence of Schaps result, Theorem 1.4.21 and the description of the infinitesimal deformation space in Lemma 1.4.22, this number is equal to $\dim_{\mathbb{C}} \mathrm{Inf}(A)$ for the presentation matrix A of the singularity.

A	τ	sing. in Y_0	b_2	b_3
$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$	1	-	1	0
$\begin{pmatrix} x & y & z \\ v & w & x^{k+1}+y^2 \end{pmatrix}$	$k+2$	A_k	1	k
$\begin{pmatrix} x & y & z \\ v & w & xy^2+x^{k-1} \end{pmatrix}$	$k+2$	D_k	1	k
$\begin{pmatrix} x & y & z \\ v & w & x^3+y^4 \end{pmatrix}$	8	E_6	1	6
$\begin{pmatrix} x & y & z \\ v & w & x^3+xy^3 \end{pmatrix}$	9	E_7	1	7
$\begin{pmatrix} x & y & z \\ v & w & x^3+y^5 \end{pmatrix}$	10	E_8	1	8
$\begin{pmatrix} w & y & x \\ z & w & y+v^k \end{pmatrix}$	$2k-1$	-	1	0
$\begin{pmatrix} w & y & x \\ z & w & y^k+v^2 \end{pmatrix}$	$k+2$	A_{k-1}	1	$k-1$
$\begin{pmatrix} w & y & x \\ z & w & yv+v^k \end{pmatrix}$	$2k$	A_1	1	1
$\begin{pmatrix} w+v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k+1$	A_1	1	1
$\begin{pmatrix} w+v^2 & y & x \\ z & w & y^2+v^k \end{pmatrix}$	$k+3$	A_{k-1}	1	$k-1$
$\begin{pmatrix} w & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	7	A_2	1	2
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & v^2+y^l \end{pmatrix}$	$k+l+1$	A_{k-1}, A_{l-1}	1	$k+l-2$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & yv \end{pmatrix}$	$k+4$	A_{k-1}, A_1	1	k
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & y^2+v^l \end{pmatrix}$	$k+l+2$	A_{k-1}, A_{l-1}	1	$k+l-2$
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv+v^k \end{pmatrix}$	$2k+1$	A_1, A_1	1	2
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k+2$	A_1, A_1	1	2
$\begin{pmatrix} wv+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	8	A_1, A_2	1	3
$\begin{pmatrix} wv & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	9	A_1, A_2	1	3
$\begin{pmatrix} w^2+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	9	A_2, A_2	1	4

$\begin{pmatrix} z & y & x \\ x & w & v^2+y^2+z^k \end{pmatrix}$	$k+4$	D_{k+1}	1	$k+1$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^k w \end{pmatrix}$	$2k+5$	A_{2k+2}	1	$2k+2$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^{k+1} \end{pmatrix}$	$2k+4$	A_{2k+1}	1	$2k+1$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yw+z^2 \end{pmatrix}$	8	D_5	1	5
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^3+z^2 \end{pmatrix}$	9	E_6	1	6
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vy+z^2 \end{pmatrix}$	7	D_3	1	3
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vz+y^2 \end{pmatrix}$	8	A_4	1	4
$\begin{pmatrix} z & y & x+v^2 \\ x & w & z^2+y^2 \end{pmatrix}$	9	D_5	1	5

TABLE 3.1: Homology of Milnor fibers of simple ICMC2 threefold singularities

The classification of simple ICMC2 singularities provided a further set of examples, for which these computations could be done. The result is the following table.

A	τ	sing. in Y_0	b_2	b_3
$\begin{pmatrix} x & y & z \\ w & v & x^4+y^4 \end{pmatrix}$	11	X_9	1	9
$\begin{pmatrix} x & y & z \\ w & v & x^3+y^6 \end{pmatrix}$	12	J_{10}	1	10
$\begin{pmatrix} w+v^2 & y & x \\ z & w & y^3+v^3 \end{pmatrix}$	8	D_4	1	4
$\begin{pmatrix} w+v^3 & y & x \\ z & w & y^2+v^4 \end{pmatrix}$	9	A_3	1	3
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^3+z^3 \end{pmatrix}$	11	$T_{3,3,3}$	1	8
$\begin{pmatrix} z & y & x \\ x & w & v^3+y^2+z^3 \end{pmatrix}$ [2]	13	$T_{3,3,3}$	1	8
$\begin{pmatrix} z & y & x \\ x & w & v^3+y^3+z^2 \end{pmatrix}$	17	U_{12}	1	12
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^4+z^2 \end{pmatrix}$	12	X_9	1	9
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vz+yz+vw \end{pmatrix}$	10	D_6	1	6
$\begin{pmatrix} z & y & x+v^3 \\ x & w & vy+z^2 \end{pmatrix}$	9	A_3	1	3

[2] There is a typesetting error in this matrix in [25]. The right-hand lower entry here is the correct one.

$\begin{pmatrix} z & y & x+v^3 \\ x & w & y^2+yz+z^2 \end{pmatrix}$	15	X_9	1	9
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vy+yz+z^3 \end{pmatrix}$	8	D_4	1	4

TABLE 3.2: Homology of Milnor fibers for the bounding non-simple threefold singularities

3.2.2 Decomposition of Infinitesimal Deformations

For simple ICMC2 singularities $(X_0, 0) \subset (\mathbb{C}^5, 0)$ there are only simple hypersurface singularities occuring in the Tjurina transform (Y_0, E). There is a reason behind this, which becomes appearent if one considers the infinitesimal deformations of $(X_0, 0)$, i.e. the space $T^1_{X_0,0} \cong \mathrm{Inf}(A)$, and the induced infinitesimal deformations for the isolated singularities (Y_0, q) in the Tjurina transform.

In this section we aim to prove the following theorem:

Theorem 3.2.7. *Let $N \geq 3$ and $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be an ICMC2 singularity of type $(2, 3, 3)$ such that the Tjurina transform (Y_0, E) has at most isolated singularities (cf. Lemma 3.2.1). Furthermore let $X_0 \hookrightarrow X \longrightarrow \mathbb{C}^\tau$ be a semi-universal deformation of X_0 in the sense of Grauerts Theorem 1.4.13. There is a decomposition*

$$T^1_{X_0,0} \cong H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{q \in \mathrm{Sing}(Y_0,E)} T^1_{Y_0,q}, \tag{3.24}$$

where T_{Y_0} denotes the tangent sheaf $\mathrm{Hom}_{\mathcal{O}_{Y_0}}(\Omega^1_{Y_0}, \mathcal{O}_{Y_0})$ of (Y_0, E). In particular, the induced family $(Y_0, q) \hookrightarrow (Y, q) \longrightarrow \mathbb{C}^\tau$ is again versal for each of the arising singularities.

We first discuss some implications of this theorem with a view towards the question, to which extend there are analogues of Milnor's formula in Theorem 2.1.13, the inequality $\mu \geq \tau$, (2.2), and Theorem 2.1.16 for the computations of the Betti numbers of the Milnor fiber.

Corollary 3.2.8. *Let $(X_0, 0)$ be as in Theorem 3.2.7 and of dimension 3. If all the isolated singularities (Y_0, p) in the Tjurina transform are quasihomogeneous, then we have an equality*

$$\tau = h^1(Y_0, T_{Y_0}) + b_3(X_0, 0),$$

where $\tau = \dim_{\mathbb{C}} T^1_{X_0,0}$ is the Tjurina number.

Proof. From Theorem 2.1.16 it follows that the local Milnor numbers of the ICIS (Y_0, p) in the Tjurina transform are all equal to the local Tjurina numbers $\tau(Y_0, p)$. The formula is now deduced from the decomposition (3.24) and Theorem 3.2.6. □

For simple ICMC2 threefold singularities, the versality of the induced deformations for all (Y_0, p) explains, why we only find simple singularities in the Tjurina transform. As one can observe from the Table 3.1, they are all hypersurfaces and therefore members of the original classification by Arnold [4]. Since all these singularities are quasihomogeneous, Corollary 3.2.8 holds in particular for all singularities in Table 3.1.

Remark 3.2.9. There are families in Table 3.1, for which the term $h^1(Y_0, T_{Y_0})$ grows linearly with k. For example the seventh entry, the so called Π_k family, as well as the 17th and 18th entry. The deformation parameters corresponding to the term $H^1(Y_0, T_{Y_0})$ in the decomposition (3.24) of the $T^1_{X_0,0}$ are not reflected in the vanishing topology of the singularity at all. This phenomenon is contrary to anything that could be observed for isolated complete intersection singularities before.

We prepare for the proof of Theorem 3.2.7. As usual let $\underline{x} = (x_1, \dots, x_N)$ be the affine coordinates of $\mathbb{C}^N$ at 0 and $\underline{s} = (s_1 : s_2)$ the homogeneous coordinates of $\mathbb{P}^1$. The space of embedded first order deformations for the Tjurina transform $\iota : (Y_0, E) \hookrightarrow \mathbb{C}^5 \times \mathbb{P}^1$ can be described as follows (see e.g. [64], [41]). Let $\mathcal{I}$ be the ideal sheaf defining (Y_0, E) in $(\mathbb{C}^5 \times \mathbb{P}^1, E)$. We take global sections of the normal bundle

$$N_{Y_0} = H^0(Y_0, \mathcal{H}om_{\mathcal{O}}(\mathcal{I}, \mathcal{O}_{Y_0}))$$

and divide by those deformations coming from global sections of the tangent bundle $H^0(Y_0, \iota^* T_{\mathbb{C}^5 \times \mathbb{P}^1})$. The resulting quotient will be denoted by

$$N' := N_{Y_0} / H^0(Y_0, \iota^* T_{\mathbb{C}^5 \times \mathbb{P}^1}). \tag{3.25}$$

Note that the global section functor takes coherent sheaves to finitely generated $\mathbb{C}\{\underline{x}\}$-modules. In fact N' is naturally a $\mathbb{C}\{\underline{x}\}$-module with support in the point 0 and hence a finite dimensional vector space over $\mathbb{C}$. To see this, observe that outside the singular locus $0 \in X_0$ (and outside $E \subset Y_0$ respectively), the space Y_0 is described as a graph over X_0 and we therefore have a natural splitting of the normal bundle

$$N_{Y_0} = N_{X_0} \oplus T_{\mathbb{P}^1}|_{Y_0}.$$

Because the tangent bundle of $\mathbb{P}^1$ is globally generated, the second summand is killed when forming the quotient N'. But the first summand cancels on the smooth locus anyway.

It is clear from the construction that every deformation of $(X_0, 0)$ induces a deformation of $(Y_0, \mathbb{P}^1 \times \{0\})$. Let $(X_0, 0)$ be given by the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ a & b & c \end{pmatrix} \in \mathrm{Mat}(2, 3; \mathbb{C}\{x_1, \dots, x_N\}).$$

and let

$$H_1 = s_1 \cdot a + s_2 \cdot x_1, \quad H_2 = s_1 \cdot b + s_2 \cdot x_2, \quad H_3 = s_1 \cdot c + s_2 \cdot x_3 \in \mathbb{C}\{\underline{x}\}[s_1, s_2]$$

be the three equations defining the Tjurina transform Y_0 in $\mathbb{C}^5 \times \mathbb{P}^1$, which are homogeneous in $\underline{s}$. On the level of equations there is a map

$$\begin{tikzcd} \{\text{Perturbations of } A\} \arrow[r, "\Lambda"] \arrow[d, dash, "1:1"'] & \{\text{Perturbations of } \underline{H}\} \arrow[d, dash, "1:1"] \\ \mathrm{Mat}(2,3;\mathbb{C}\{\underline{x}\}) \arrow[r, "\Lambda", "E^{(2,3)}_{i,j} \mapsto e_j s_i"'] & \left((\mathbb{C}\{\underline{x}\}[s_1, s_2])^3\right)_{(1)}, \end{tikzcd} \tag{3.26}$$

where the e_i denote the generators of the free module on the right hand side and $E_{i,j}^{(r,s)}$ denote the $r \times s$ matrices possessing only one non-zero entry of value 1 at position i, j. The lower index (1) signifies that we only consider the homogeneous part of degree 1 in $\underline{s}$.

Lemma 3.2.10. *The map Λ induces an isomorphism of first order deformations of $(X_0, 0)$ and (Y_0, E), i.e. an isomorphism of $\mathbb{C}\{\underline{x}\}$-modules*

$$\Lambda : T^1_{X_0,0} \xrightarrow{\cong} N'.$$

Proof. We have already obtained the isomorphism Λ between $\mathrm{Mat}(2, 3; \mathbb{C}\{\underline{x}\})$ and $(\mathbb{C}\{\underline{x}\}[s_1, s_2])^3_{(1)}$. From the description of the $T^1_{X_0,0}$ in Lemma 1.4.22 and the definition of N' we know the relations on both sides. It hence remains to prove that the modules

$$K := \left\langle \frac{\partial A}{\partial \underline{x}} \right\rangle + \langle F \cdot A + A \cdot G \rangle$$

from the description of $\mathrm{Inf}(A) \cong T^1_{X_0,0}$ and $(J_H + I_H)_{(1)}$ are isomorphic. Here $I_H = \langle H_1, H_2, H_3 \rangle \mathbb{C}\{\underline{x}\}^3$ and J_H is generated by the columns of the Jacobian matrix of the H_i defining Y_0.
By construction of $\underline{H}$, we see immediately

$$\Lambda(\frac{\partial A}{\partial x_i}) = \frac{\partial \underline{H}}{\partial x_i},$$

$$\Lambda(E_{i,j}^{(2,2)} \cdot A) = s_i \frac{\partial \underline{H}}{\partial s_j}$$

and

$$\Lambda(A \cdot E_{i,j}^{(2,3)}) = H_i e_j.$$

This provides a $1:1$ correspondence of the generators of these two modules and hence proves the claim about the cokernels:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\Lambda"', "\cong"] & \mathrm{Mat}(2, 3; \mathbb{C}\{\underline{x}\}) \arrow[r] \arrow[d, "\Lambda"', "\cong"] & T^1_{X_0,0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (J_H + I_H)_{(1)} \arrow[r] & (\mathbb{C}\{\underline{x}\}[s_1, s_2])^3_{(1)} \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

□

There is a splitting of the module N' coming from the local-to-global spectral sequence of the exact sequence of sheaves

$$0 \longrightarrow T_{Y_0} \longrightarrow \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow N_{Y_0} \longrightarrow T^1_{Y_0} \longrightarrow 0, \qquad (3.27)$$

which can be explicitly described as follows.

We first split the exact sequence (3.27) into short exact sequences

$$0 \longrightarrow T_{Y_0} \longrightarrow \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow \mathcal{K} \longrightarrow 0 \qquad . \tag{3.28}$$

$$0 \longrightarrow \mathcal{K} \longrightarrow N_{Y_0} \longrightarrow T^1_{Y_0} \longrightarrow 0$$

The long exact sequences in cohomology both have to finish after the degree one terms, because the underlying scheme is covered by two affine charts.

Let again $\mathcal{I}$ be the ideal sheaf of $(Y_0, \mathbb{P}^1)$. If we tensor the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow \mathcal{O}_{Y_0} \longrightarrow 0$$

with the locally free sheaf $T_{\mathbb{P}^1 \times \mathbb{C}^5}$ and take the long exact sequence in cohomology, we see that

$$H^1(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5}) = 0.$$

Looking at the first long exact sequence in cohomology of (3.28), we deduce that

$$\operatorname{coker}\left(H^0(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5}) \to H^0(Y_0, \mathcal{K})\right) \cong H^1(Y_0, T_{Y_0})) \tag{3.29}$$

and

$$H^1(Y_0, \mathcal{K}) = 0. \tag{3.30}$$

Combining these results with the second long exact sequence of (3.28) and recalling that $N' = N_{Y_0}/H^0(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5})$, we obtain a short exact sequence

$$0 \longrightarrow H^1(Y_0, T_{Y_0}) \longrightarrow N' \longrightarrow H^0(Y_0, T^1_{Y_0}) \longrightarrow 0 \,,$$

the middle term of which is a finite dimensional vector space over $\mathbb{C}$. Any choice of a splitting gives us

$$N' = H^1(Y_0, T_{Y_0}) \oplus H^0(Y_0, T^1_{Y_0}). \tag{3.31}$$

The sheaf underlying the right hand side summand is supported only in the singular points and hence affine. Thus if we let $\Sigma(Y_0)$ be the set of singular points of Y_0 we can rewrite (3.31) as

$$N' = H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{p \in \Sigma(Y_0)} T^1_{Y_0,p} \tag{3.32}$$

which is the same as (3.24), given the identification $T^1_{X_0,0} \cong N'$ from Lemma 3.2.10.

In particular for any $q \in \Sigma(Y_0)$ we get a surjective map from $T^1_{X_0,0}$ onto $T^1_{Y_0,q}$ by the composition

$$T^1_{X_0,0} \cong N' \cong H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{p \in \Sigma(Y_0)} T^1_{Y_0,p} \longrightarrow T^1_{Y_0,q},$$

where the last map is the projection to the summand for q. This proves Theorem 3.2.7.

Note that the induced local deformations for the isolated singularities of Y_0 do not need to be semi-universal, i.e. τ might not be minimal.

3.2.3 The surface case

We state the analogue of Theorem 3.2.6 for surfaces.

Theorem 3.2.11. *Let $(X_0, 0) \subset (\mathbb{C}^4, 0)$ be an ICMC2 singularity of type $(2, 3, 2)$ such that the Tjurina transform (Y_0, E) has only isolated singularities (Y_0, q) at points $q \in E$. If we let $b_2(q)$ be the middle Betti number of the ICIS (Y_0, q) at q, then the Betti numbers of $(X_0, 0)$ are*

$$b_i(X_0, 0) = \begin{cases} \left(\sum_{q \in \mathrm{Sing}(Y_0, E)} b_2(q)\right) + 1 & \text{if } i = 2 \\ 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} .$$

All homology groups are torsion free and there is a splitting of the second homology group

$$H_2(\overline{X}_u) \cong \left(\bigoplus_{q \in \mathrm{Sing}(Y_0, E)} H_2(F_q) \right) \oplus \mathbb{Z}, \tag{3.33}$$

where F_q is the Milnor fiber of the singularity (Y_0, q).

In particular $H_1(\overline{X}_u)$ is zero, not just of rank zero. This is stronger then the result by Greuel and Steenbrink on normal surface singularities [34].

Proof. The long exact sequence (3.18) splits into

$$0 \longrightarrow \bigoplus_{q \in \mathrm{Sing}(Y_0, E)} H_2(F_q) \longrightarrow H_2(\overline{X}_u) \longrightarrow H_2(\mathbb{P}^1) \longrightarrow 0$$

and

$$0 \longrightarrow H_1(\overline{X}_u) \longrightarrow 0$$

according to (3.17). The desired splitting exists, because $H_2(\mathbb{P}^1) = \mathbb{Z}$ is torsion free. This concludes the proof. □

For the ICMC2 surface singularities of type $(3, 4, 3)$, we obtain a surprising corollary.

Corollary 3.2.12. *There are no ICMC2 surface singularities of type $(3, 4, 3)$, which have only isolated singularities in the Tjurina transform.*

Proof. Suppose this was the case. Then we would find the following part of the long exact sequence (3.18):

$$0 \longrightarrow H_4(\mathbb{P}^2) \longrightarrow H_3(\overline{X}_u) \longrightarrow 0 .$$

But according to the Lefschetz Hyperplane Theorem, $\overline{X}_u$ can not have a nonzero fourth homology group. □

Also for surfaces we can discuss the decomposition of $T^1_{X_0,0}$ in the Tjurina modification form Theorem 3.2.7.

Corollary 3.2.13. *Let $(X_0, 0) \subset (\mathbb{C}^4, 0)$ be as in Theorem 3.2.7 and of dimension 2. If all the isolated singularities in the Tjurina transform are quasihomogeneous, then*

$$\tau = h^1(Y_0, T_{Y_0}) + b_2(X_0, 0) - 1.$$

Proof. This is basically the same as in the proof of Corollary 3.2.8, only that this time we have a correction term -1 from the decomposition of $H_2(\overline{X}_u)$ in (3.33). $\square$

Remark 3.2.14. In [24], A. Frühbis-Krüger uses Corollary 3.2.13 for ICMC2 surface singularities with isolated singularities in the Tjurina transform, to prove a conjecture by Wahl [72] in this special case. The conjecture says that for a non-Gorenstein surface singularity $(X_0, 0) \subset (\mathbb{C}^4, 0)$ we have an inequality

$$b_2(X_0, 0) = \mu \geq \tau - 1,$$

with equality if and only if $(X_0, 0)$ is quasihomogeneous.

In the proof, A. Frühbis-Krüger shows that for ICMC2 surface singularities with isolated singularities in the Tjurina transform the space $H^1(Y_0, T_{Y_0})$ in the decomposition (3.24) always has dimension 2 for quasihomogeneous singularities. This is contrary to what we observed for the threefolds in Remark 3.2.9. For surfaces, there seems to be a much stricter correspondence between the degrees of freedom for deformation in a semi-universal deformation of the singularity and its vanishing topology. In the case of nonisolated singularities in the Tjurina transform, however, the conjecture is still open.

3.2.4 Topology of Space Curves

ICMC2 singularities of dimension 1, which meet the requirements of Theorem 3.2.3 are *space curves*, i.e. curve singularities in $(\mathbb{C}^3, 0)$. Independently of the development of the methods exhibited here, J. Kass worked out the Tjurina modification in this case for the simple singularities from the list in [25]. He gave a talk about his results in Hannover in June 2014, but, as of this writing, did not yet publish on the subject. We will formulate the analogue of Theorem 3.2.3 for space curves.

Theorem 3.2.15. *Let $(X_0, 0) \subset (\mathbb{C}^3, 0)$ be an ICMC2 singularity of type $(2, 3, 2)$ such that the Tjurina transform (Y_0, E) has only isolated singularities (Y_0, q) at points $q \in E$. If we let $b_1(q)$ be the middle Betti number of the ICIS (Y_0, q) at q, then the Betti numbers of $(X_0, 0)$ are*

$$b_i(X_0, 0) = \begin{cases} \left(\sum_{q \in \mathrm{Sing}(Y_0, E)} b_1(q)\right) - 1 & \text{if } i = 2 \\ 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

All homology groups are torsion free.

After all that has been said in the previous sections, the proof is left to the reader.

3.2.5 Simple Fourfold Singularities

For ICMC2 singularities dimension 4 is special, because according to Theorem 2.2.13 the singularities will not be smoothable anymore. Nevertheless we can apply the machinery provided above, to ask about the topology of the determinantal Milnor fiber.

We start by observing that Theorem 3.2.3 is still true, if we replace $\overline{X}_u$ by $\overline{Y}_u$ in (3.18). This means, we can use it to compute the homology groups of the Tjurina transform $\overline{Y}_u$ of the determinantal Milnor fiber $\overline{X}_u$.

If $(X_0, 0) \subset (\mathbb{C}^6, 0)$ is of type $(2, 3, 2)$, we obtain

$$H_i(\overline{Y}_u) = \begin{cases} \bigoplus_{q \in \mathrm{Sing}(Y_0, E)} H_4(F_q) & \text{if } i = 4 \\ \mathbb{Z} & \text{if } q = 2 \\ 0 & \text{otherwise} \end{cases} \tag{3.34}$$

The problem for the fourfolds is, that the Tjurina transform on the determinantal Milnor fiber

$$\pi_\delta : \overline{Y}_\delta \to \overline{X}_\delta$$

is not an isomorphism anymore, but a resolution of singularities. Recall from Example 2.2.2 i) the singularity $(X_0, 0) \subset (\mathbb{C}^6, 0)$ defined by the matrix

$$A = \begin{pmatrix} x & y & v \\ z & w & x + u^k \end{pmatrix}.$$

for some $k \in \mathbb{N}$. Direct computations show that the Tjurina transform $(Y_0, E) \subset (\mathbb{C}^4, 0) \times \mathbb{P}^1$ is smooth. Now consider the stabilization of A given by the perturbation with the matrix

$$\delta \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this dimension, the degeneracy locus of A defined by $\langle A^{\wedge 1} \rangle$ is a complete intersection:

$$\langle A^{\wedge 1} \rangle = \langle x, y, z, v, w, u^k \rangle.$$

Any determinantal deformation of $(X_0, 0)$ therefore also leads to a deformation of the degeneracy locus. In this case, the perturbation by δ results in k distinct smooth points $p_1, \ldots, p_k \in \overline{X}_\delta$, over which π_δ is not an isomorphism anymore.

We know from Theorem 2.2.13 and Lemma 3.1.8 what happens over these points: We find the resolution of singularities from the generic determinantal varieties. Thus over every point p_i there is an exceptional set

$$E_i = \mathbb{P}^1 \times \{p_i\} \subset \overline{Y}_\delta$$

sitting over it. In this particular example one can see that if we consider the exceptional set $E \subset Y_0$ as the preimage of the fat point scheme defined by $\langle A^{\wedge 1} \rangle$, then the embedded deformation of E in Y_0 induced from the perturbation of A by δ splits E into the E_i above. From this, it is easy to see that the fundamental cycles of these E_i in homology are all homologous to the generator of $H_2(\overline{Y}_\delta)$.

Now for fixed $\delta \neq 0$ let $(B_i)_{i=1}^k$ be Milnor balls for the singularities $(\overline{X}_\delta, p_i)$. If we now consider the long exact sequence of the pair $(\overline{Y}_\delta, \bigcup_{i=1}^k \pi_\delta^{-1}(B_i \cap$

$\overline{X}_\delta))$, we obtain

$$H_i(\overline{Y}_\delta, \bigcup_{i=1}^{k} \pi_\delta^{-1}(B_i \cap \overline{X}_\delta)) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}^{k-1} & \text{if } i = 3, \\ 0 & \text{otherwise} \end{cases}$$

But these groups are canonically isomorphic to the reduced homology groups of $\overline{X}_\delta$, since all the $B_i \cap \overline{X}_\delta$ are contractible. We therefore obtain

$$H_i(\overline{X}_\delta) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}^{k-1} & \text{if } i = 3, \\ 0 & \text{otherwise} \end{cases} . \tag{3.35}$$

It should be possible to obtain a general result for the topology of the determinantal Milnor fibers of ICMC2 fourfold singularities along these lines. The difficulty is to show that the exceptional sets over the isolated singularities in $\overline{X}_\delta$ are homologous to the generator of the second homology group of the Tjurina transform in general.

Chapter 4

Line Singularities in the Tjurina Transform

The results presented in this chapter are the exclusive work of the author. In large parts it coincides with the article [73]. Central to our considerations is the theory of vanishing cycles for nonisolated singularities as developed by Lê, Siersma, Tibăr, Yomdin and others. Using the original fibration theorem by Milnor [53] and Hamm [39], we generalize results from D. Siersma [66] about hypersurfaces to complete intersections. Then we pick up the considerations by D. Siersma and M. Tibăr in [67] concerning the vanishing topology of projective hypersurfaces with nonisolated singularities and apply it to isolated Cohen-Macaulay codimension 2 (ICMC2) singularities with nonisolated singularities in the Tjurina transform.

4.1 Characteristic Vanishing Cycles

In the computations of the homology groups of Milnor fibers of smoothable ICMC2 singularities in the preceeding chapter we can observe, how the cycles of the Tjurina transform $(Y_0, E) \cong \mathbb{P}^{t-1}$ contribute in (3.18). In some cases they "survive" and in other cases they lead to relations among the local vanishing cycles of the singularities (Y_0, q) in the Tjurina transform, depending on the degree of the homology group in question and the dimension of $(X_0, 0)$.

Suppose $(X_0, 0)$ is a smoothable ICMC2 singularity of type $(2, 3, 2)$, for which the Tjurina transform is smooth. Then the Milnor fiber is diffeomorphic to (Y_0, E), because the deformation of (Y_0, E) induced from a smoothing of $(X_0, 0)$ is trivial in the differentiable category. Consequently if we let

$$L : \overline{X}_u \to \mathbb{P}^1, \quad x \mapsto \text{span}\, A_u(x),$$

be the regular map on the Milnor fiber given by the deformed matrix A_u, then a generator of $H_2(\overline{X}_u)$ is given by the fundamental class of a differentiable section $l : \mathbb{P}^1 \to X_\varepsilon$ of L, i.e. a map l such that $L \circ l = \mathrm{Id}_{\mathbb{P}^1}$.

In general the existence of such a section is hard to prove. But from the proof of the Theorems 3.2.6 and 3.2.11 it is evident that the generator of the second summand of the splitting (3.33), or just the generator of the second homology group in Theorem 3.2.6, "is coming from" the exceptional set. To make this more precise, we give the following definition.

Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix A and

$$\mathbf{A}_u : B \to M_{m,n}$$

the fiber of a a stabilization $\mathbf{A}$ of A defined on some Milnor ball $B \subset \mathbb{C}^N$ for $(X_0, 0)$. Because A_u is transverse to all the strata $M^s_{m,n}$ of $\mathrm{Mat}(m, n; \mathbb{C})$, the Tjurina transform $\overline{Y}_u \subset B \times \mathrm{Grass}(t-1, m)$ of $\overline{X}_u = \mathbf{A}_u^{-1}(M^t_{m,n})$ is a smooth compact manifold with corners (recall that Y_ε is isomorphic to X_ε in case $(X, 0)$ is smoothable). By abuse of notation, let

$$L : \overline{Y}_u \subset B \times \mathrm{Grass}(t-1, m) \to \mathrm{Grass}(t-1, m)$$

be the projection to the Grassmannian. Consider the image $G \subset H^\bullet(\overline{Y}_u)$ of the induced map

$$L^* : H^\bullet(\mathrm{Grass}(t-1, m)) \to H^\bullet(\overline{Y}_u)$$

in cohomology.

Definition 4.1.1. A cycle $[\sigma] \in H_\bullet(\overline{Y}_u)$ is said to be *horizontal* if the cap product $g \cap [\sigma]$ is zero for all $g \in G = L^*(H^\bullet(\mathrm{Grass}(t-1, m)))$. We write

$$[\sigma] \in G^\perp.$$

All other cycles in $H_\bullet(\overline{Y}_u)$ are called *vertical* or *characteristic vanishing cycles* of $(X_0, 0)$. We also say they are *sitting over* the Grassmannian.

Corollary 4.1.2. *Let $\overline{X}_u$ be the Milnor fiber of an ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and of type $(2, 3, 2)$ with only isolated singularities in the Tjurina transform. Then the homology of $\overline{X}_u$ splits into*

$$H_\bullet(\overline{X}_u) \cong G^\perp \oplus \mathbb{Z}$$

where the second summand lives in degree 2, and the cap product with $L^*(H^2(\mathbb{P}^1))$ gives a perfect pairing with $H^2(\mathbb{P}^1) \cong \mathbb{Z}$.

4.1.1 Main Theorem for Line Singularities in the Tjurina Transform

The main goal of this section is to extend Theorem 3.2.6, Theorem 3.2.11 and Corollary 4.1.2 to the case of arbitrary ICMC2 singularities of type $(2, 3, 2)$ and dimension 2 or 3, i.e. we also allow nonisolated singularities in the Tjurina transform.

Theorem 4.1.3. *Let $\overline{X}_u$ be the Milnor fiber of an ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and type $(2, 3, 2)$ given by a matrix $A \in \mathrm{Mat}(2, 3; \mathcal{O}_N)$. Let $\mathbf{A} \in \mathrm{Mat}(2, 3; \mathcal{O}_{N+1})$ be a stabilization of A in the parameter u. For $u \neq 0$ let*

$$L : \overline{X}_u \to \mathbb{P}^1, \quad x \mapsto \mathrm{span}\, \mathbf{A}_u(x).$$

The Milnor fiber $\overline{X}_u$ has $H_1(\overline{X}_u) = 0$ and the homology of $\overline{X}_u$ splits into

$$H_\bullet(\overline{X}_u) \cong \left(L^* H^2(\mathbb{P}^1)\right)^\perp \oplus \mathbb{Z}.$$

The cap product with $L^*(H^2(\mathbb{P}^1))$ gives a perfect pairing of the vertical cycles with $H^2(\mathbb{P}^1) \cong \mathbb{Z}$. If $n = 3$, then $H_2(\overline{X}_u) \cong \mathbb{Z}$ consists of the vertical cycles only.

Since for any given example the Euler characteristic $\nu(X_0, 0)$ can be computed with Theorem 2.3.10, we obtain the following corollary.

Corollary 4.1.4. *Let $\overline{X}_u$ be the Milnor fiber of an ICMC2 threefold singularity of type $(2, 3, 2)$. The Betti numbers of $\overline{X}_u$ can be computed as*

$$b_i(X_0, 0) = \begin{cases} 1 & \text{if } i = 0, 2 \\ \nu(X_0, 0) - 1 & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases} .$$

4.1.2 An Example and Outline of the Proof

To illustrate the ideas of the proof of Theorem 4.1.3, we give an example of an ICMC2 threefold singularity with non-isolated singular locus in the Tjurina transform.

Example 4.1.5. Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be given by the matrix

$$\begin{pmatrix} v & x \\ w & y \\ -2xy & v^2 + w^2 + z^2 \end{pmatrix} \tag{4.1}$$

and consider the smoothing obtained by perturbing the lower left entry with a constant δ. We denote the homogeneous coordinates of $\mathbb{P}^1$ by $(s_1 : s_2)$. Then the equations for the Tjurina transform $(Y_0, V) \subset (\mathbb{C}^5 \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ and its deformation by δ are

$$\begin{pmatrix} v & x \\ w & y \\ -2xy - \delta & v^2 + w^2 + z^2 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0. \tag{4.2}$$

Let us look at the first chart $\{s_1 \neq 0\}$. We write $s = s_2/s_1$ for the corresponding standard affine coordinate. The equations from the first two rows read

$$v = -s \cdot x, \quad w = -s \cdot y.$$

Substituting this in the equation from the last row, we obtain a hypersurface

$$h = s^3 \cdot x^2 + s^3 \cdot y^2 - 2xy + s \cdot z^2,$$

which is perturbed by a constant δ. We can interpret this as a quadratic form Q_s in (x, y, z) parametrized by s and write it in the standard matrix form:

$$h = Q_s(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} s^3 & -1 & 0 \\ -1 & s^3 & 0 \\ 0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \delta.$$

Any quadratic form should of course be diagonalized. To do this, we introduce new coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

in which our hypersurface equation takes the form

$$h = Q_s(\tilde{x}, \tilde{y}, \tilde{z}) = \begin{pmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix} \cdot \begin{pmatrix} s^3+1 & 0 & 0 \\ 0 & s^3-1 & 0 \\ 0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \delta. \tag{4.3}$$

This is a family of A_1-surface singularities, which degenerates as s approaches one of the seven values

$$s \in \sqrt[6]{1} \cup \{0\}.$$

Now it is clear that in this chart the Tjurina transform Y_0 is singular along the whole exceptional set V, the s-axis in this chart.

Let $L : \mathbb{C}^5 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the standard projection, i.e. in this chart the projection to the s-axis. If we restrict h to a general transversal slice to V given by the hypersurface $\{L = c\}$ for a general $c \in \mathbb{C}$, we obtain the *transversal singularity* denoted by $Y_0^{\pitchfork}$:

$$h|_{\{L=c\}} = (c^3+1)\tilde{x}^2 + (c^3-1)\tilde{y}^2 + c\tilde{z}^2 = \delta$$

and a smoothing induced by the perturbation with the constant δ. This transversal singularity is isolated and of type A_1.

For $\delta \neq 0$ we see a vanishing cycle $[\sigma]$ in the Milnor fiber

$$Y_\delta^{\pitchfork} = \{h = \delta\} \cap \{L = c\}$$

of the transversal singularity. It lives in the second homology group $H_2(Y_\delta^{\pitchfork})$ and can be represented by a 2-sphere. This is a candidate for further contributions of the second homology group of

$$Y_\delta \subset \mathbb{C}^5 \times \mathbb{P}^1,$$

the fiber over δ in the given deformation and, hence, for the Milnor fiber X_ε of $(X_0, 0)$. Whether or not $[\sigma]$ is nonzero as an element of $H_2(Y_\delta)$ depends on the inclusion

$$Y_\delta^{\pitchfork} \subset Y_\delta.$$

To shed some light on this question, let us observe the behaviour close to the degeneracy points

$$K := \left\{ (\tilde{x}, \tilde{y}, \tilde{z}, s) : \tilde{x} = \tilde{y} = \tilde{z} = 0, s \in \sqrt[6]{1} \cup \{0\} \right\}.$$

The analytic type of the singularity h at either of these points is what D. Siersma calls the D_∞-singularity, a.k.a. the *Whitney umbrella*. For any $p \in K$ we can choose a Milnor ball $B = B(p)$ for the singularity of h around p and a value $c \in \mathbb{C}$ for the transversal singularity sufficiently close to $s(p)$ such that the intersection B with the hyperplane $\{L = c\}$ is nonempty. D. Siersma shows in [65, Proposition 3.8]:

For the D_∞ singularity of dimension n the pair of Milnor fibers $(Y_\delta \cap B, Y_\delta^{\pitchfork} \cap B)$ is homotopy equivalent to the pair of spheres

$$(S^n, S^{n-1})$$

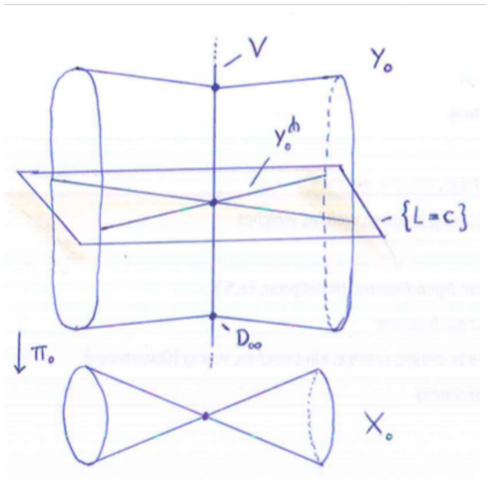

FIGURE 4.1: The Tjurina transform with non-isolated singularities.

where $S^{n-1} \hookrightarrow S^n$ is the standard equatorial embedding.

Let $W \subset \mathbb{C}$ be the complement of some small discs around the special points in $K \subset \mathbb{C}$. Then for $\delta > 0$ small enough

$$L : Y_\delta \cap L^{-1}(W)) \to W \tag{4.4}$$

is a fiber bundle with fiber $Y_\delta^{\pitchfork}$. This means we can freely move the equator of all the vanishing cycles coming from the seven D_∞ points and connect all half spheres globally. The affine part of Y_δ is therefore homotopic to a bouquet of spheres:

$$Y_\delta \setminus \{s_1 = 0\} \cong \underbrace{S^3 \vee \cdots \vee S^3}_{2\cdot 7-1 \text{ times}} \tag{4.5}$$

with each of their equators being homologous to the vanishing cycle S^2 of any of the transversal Milnor fibers.

To complete the picture, let us look at the other chart $\{s_2 \neq 0\}$. We denote the corresponding affine coordinate of $\mathbb{P}^1$ by $t = s_1/s_2$. Again the equations for the first two rows of the matrix allow us to substitute in the equation of the third row and we obtain the perturbation of a hypersurface equation:

$$h := v^2 + w^2 - 2t^3 \cdot vw + z^2 = \delta \cdot t.$$

Regarding this as a quadratic form Q_t in (v, w, z) parametrized by t and diagonalizing as before, we obtain

$$\begin{pmatrix} \tilde{v} & \tilde{w} & \tilde{z} \end{pmatrix} \cdot \begin{pmatrix} 1+t^3 & 0 & 0 \\ 0 & 1-t^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} = \delta \cdot t.$$

We do recover the six degeneracy values for t of the quadratic form at the six roots of unity. However, Q_t does not degenerate at the point $(0, \infty) \in \mathbb{C}^5 \times \mathbb{P}^1$, the origin in this chart. Hence, we can make an analytic change of coordinates around this point such that the local equation h for Y_0 at $(0, \infty)$

is just an A_∞ singularity:

$$h = x^2 + y^2 + z^2 : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0).$$

But note that we do not perturb by a constant, but by $\delta \cdot t$. This means the transversal slice over $\{t = 0\}$ does not deform! We have

$$Y_\infty^\pitchfork := Y_0 \cap \{t = 0\} = Y_\delta \cap \{t = 0\}.$$

The set $Y_\infty^\pitchfork$ is what we call an *axis of the deformation*. The point $(0, \infty)$, in which the axis intersects the exceptional set V, is called the *axis point*.

Being a representative of the germ of an isolated singularity, $Y_\infty^\pitchfork$ is a contractible fiber in the family given by $L : Y_\delta \to \mathbb{P}^1$. If we let $L^{-1}(D) \supset Y_\infty^\pitchfork$ be the preimage of a sufficiently small disk $D \subset \mathbb{P}^1$ around ∞, then we can assume that $L^{-1}(D)$ is also contractible. We compute the Mayer-Vietoris sequence for the two patches $L^{-1}(W)$ and $L^{-1}(D)$.

The intersection $L^{-1}(W \cap D)$ is homotopic to a fiber bundle over the circle S^1 with fiber $Y_\delta^\pitchfork$, which in turn is homotopic to a 2-sphere S^2. The topology of $L^{-1}(W \cap D)$ is thus determined by the Wang sequence of the fibration and we get

$$H_3(L^{-1}(W \cap D)) = 0, \qquad H_2(L^{-1}(W \cap D)) = \mathbb{Z}/2\mathbb{Z}, \tag{4.6}$$
$$H_1(L^{-1}(W \cap D)) = \mathbb{Z}, \qquad H_0(L^{-1}(W \cap D)) = \mathbb{Z}. \tag{4.7}$$

The second homology group is generated by the transversal vanishing cycle in $Y_\delta^\pitchfork$ and the first one by a continous section of the projection to S^1.

Putting all this together, we obtain the following exact sequence for the H_3-term:

$$0 \longrightarrow \mathbb{Z}^{13} \longrightarrow H_3(Y_\delta) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0\ .$$

We may deduce that $b_3(Y_\delta) = 13$. But from this it is not clear whether or not there is a further torsion part in $H_3(Y_\delta)$. For the second homology group the connecting homomorphism in the long exact sequence is an isomorphism:

$$0 \longrightarrow H_2(Y_\delta) \longrightarrow H_1(L^{-1}(W \cap D)) \longrightarrow 0\ .$$

The generator therefore is the difference of two relative cycles for the pairs $(L^{-1}(D), L^{-1}(W \cap D))$ and $(L^{-1}(W), L^{-1}(W \cap D))$, whose common boundary is the fundamental class $[l]$ of a section $l : S^1 = \partial D \subset \mathbb{P}^1 \to Y_\delta$ of L. We can construct the generator of $H_2(Y_\delta)$ by extending l to both D and W as a section of L. Away from the special points, i.e. where L is a fiber bundle, this section exists and is unique up to homotopy by general obstruction theory and the connectivity of the base and fiber. At the D_∞-points we can change coordinates and reduce to the normal form

$$s \cdot x^2 + y^2 + z^2 = \delta.$$

Here l can be extended by

$$s \mapsto (s, x, y, z) = (s, 0, 0, \sqrt{\delta}).$$

It is evident that

$$H^2(\mathbb{P}^1) \times H_2(Y_\delta) \to \mathbb{Z}, \quad (\omega, [\sigma]) \mapsto L^*\omega \cap [\sigma]$$

is a perfect pairing of free $\mathbb{Z}$-modules.

Since the deformation we started with was a smoothing of $(X_0, 0)$, the spaces Y_δ and X_δ are naturally isomorphic and we are done with the determination of the homology groups of the Milnor fiber of $(X_0, 0)$.

We will now outline the proof of the main Theorem 4.1.3 using this example. It is widely inspired by the work of D. Siersma and M. Tibăr on the vanishing topology of projective hypersurfaces [67], in the way we piece together the global picture from local computations and the role played by the axis point.

Step I: Study line singularities, which are local complete intersections. In general the singular locus $V = \{0\} \times \mathbb{P}^1$ of the Tjurina transform $Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^1$ will consist of a Zariski open set U, over which the projection L to $\mathbb{P}^1$ is the germ of a submersion along the exceptional set, i.e. it locally induces the structure of a fiber bundle with fiber $Y_0^\pitchfork$, the transversal singularity. Its Milnor fiber $Y_\delta^\pitchfork$ is well defined up to diffeomorphism. This is done in section 4.2.4.

Then we will treat the special points, i.e. the complement of U. In the above example we saw that the vanishing cycle $[\sigma]$ of the transversal singularity became homologous to zero in the local Milnor fibers of the D_∞ singularities. But in the general case of arbitrary line singularities which are complete intersections, there is no reason for this to hold. Consider for example the F_1A_3 singularity from De Jongs list [48]:

$$f = xz^2 + y^2z = z \cdot (xz + y^2).$$

He shows that its Milnor fiber F is homotopy equivalent to S^1. If we find such a singularity in the Tjurina transform of an ICMC2 surface singularity or a double suspension of it in the Tjurina transform of a threefold, then there are cycles of the transversal Milnor fiber $F^\pitchfork$, which are not homologous to zero in F.

It turns out that the important property we need is the fact that any vanishing cycle of degree $(n-1)$ of the Milnor fiber F of a complete intersection line singularity can be represented by a cycle in the transversal Milnor fiber $F^\pitchfork$. This is done in section 4.2.5, where we give a description of how the local Milnor fiber of those singularities is connected to its transversal Milnor fiber (Corollary 4.2.9 and Theorem 4.2.11 for the threefolds and respectively 4.2.10 and 4.2.13 for the surface case).

Step II: The role of the axis point. In section 4.3.1 we show that for deformations of an ICMC2 singularity $(X_0, 0)$ of dimension n and its Tjurina transform (Y_0, V) coming from a perturbation of the defining matrix A with a general constant matrix B of rank 1, a *generic rank 1 perturbation*, we always have an axis $Y_\infty^\pitchfork$ and an axis point $(0, \infty) \in V$. For the fiber Y_δ of the Tjurina transform in such a deformation, the connectivity of local Milnor fibers F of complete intersection line singularities with their transversal Milnor fibers $F^\pitchfork$ will imply that all homology of degree $n-1$ of $Y_\delta \setminus Y_\infty^\pitchfork$ is concentrated in the transversal Milnor fiber $Y_\delta^\pitchfork$. When gluing in the fiber $Y_\infty^\pitchfork$ of L over ∞, all the cycles in $Y_\delta^\pitchfork$ collapse.

Step III: Putting together the global picture. In the last part, section 4.3.3, we use Mayer-Vietoris arguments to compute the homology groups of Y_δ for a general rank 1 perturbation (Theorem 4.3.4): While the vanishing cycles of $Y_\delta^\pitchfork$ and, hence, also all $(n-1)$-cycles of the local Milnor fibers of the special points are homologous to zero in Y_δ, adding $Y_\infty^\pitchfork$ to $Y_\delta \setminus Y_\infty^\pitchfork$ will also give rise to a new 2-cycle sitting over $\mathbb{P}^1$ in the sense of Definition 4.1.1. This finally leads to the proof of Theorem 4.1.3, in which we pass from a general rank 1 perturbation, for which neither Y_δ nor X_δ are necessarily smooth, to a smoothing of $(X_0, 0)$.

4.2 Line Singularities in the Tjurina Transform

4.2.1 Complete Intersection Line Singularities

Definition 4.2.1. A singularity $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ is called a *line singularity* if the singular locus $V = \operatorname{Sing}(Y_0)$ is the germ of a line in $\mathbb{C}^N$ at 0.

Curves cannot have line singularities unless they are a multiple line themselves. In this section, we will therefore always assume $n = \dim(Y_0, 0) \geq 2$.

Let $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ be a line singularity, which is a complete intersection of codimension d given by the equations $f_1 = \cdots = f_d = 0$. For line singularities there is in general no unique smoothing, as we saw in Example 2.1.9. For hypersurfaces one can, however, consider the "perturbation by a generic constant" and use the Fibration Theorem by Lê, Theorem 2.1.11.

For complete intersections, there is a well-known trick to reduce to a constant perturbation of one holomorphic function on a controlled ambient space, see e.g. [39]. Let

$$\vec{f} : (\mathbb{C}^N, 0) \to (\mathbb{C}^d, 0),$$

be the map defining the line singularity $(Y_0, 0)$ and $0 \in U \subset \mathbb{C}^N$ a neighborhood of the origin on which all the f_i are defined. Consider the map

$$\mathbb{P}\vec{f} : U \setminus Y_0 \to \mathbb{P}^{d-1}, \quad x \mapsto (f_1(x) : \cdots : f_d(x))$$

and choose a regular value $p \in \mathbb{P}^{d-1}$ for $\mathbb{P}f$. After a change of coordinates of $\mathbb{P}^{d-1}$, which corresponds to a new $\mathbb{C}$-linear combination of the generators f_i, we can assume that $p = (1 : 0 : \cdots : 0)$. Then the closure of its preimage in $U \subset \mathbb{C}^N$ is given by

$$Y^* = \{x \in U : f_2(x) = \cdots = f_d(x) = 0\}. \tag{4.8}$$

Lemma 4.2.2. *The singular locus of Y^* is contained in the singular locus of Y.*

Proof. (cf. [39, Lemma 1.1 or Lemma 2.2]) Outside Y_0 the space Y^* is already smooth. If Y^* had a singular point $p \in Y_0$, this means that the jacobian of $(f_2, \ldots, f_d)$ would not have full rank at p. But then also the jacobian of $(f_1, f_2, \ldots, f_d)$ could not have full rank and thus p would be a singular point of Y_0 as well. ☐

We rename the first function f_1 to f. Without loss of generality we can assume that the singular line $(V, 0) = (\mathrm{Sing}(Y), 0)$ is just the germ of the first coordinate axis of $\mathbb{C}^N$. This will be the *standard situation*, from which we will proceed for this section:

$$f : (Y^*, 0) \subset (\mathbb{C}^N, 0) \to (\mathbb{C}, 0), \quad Y_0 := Y^* \cap \{f = 0\}, \tag{4.9}$$

$$\mathrm{Sing}(Y^*, 0) \subset \mathrm{Sing}(Y_0, 0) = (\{x_2 = \cdots = x_N = 0\}, 0). \tag{4.10}$$

Since we are primarily interested in topological questions about the singularity, we will use Whitney stratifications to provide the setup for applications of the first Thom isotopy Lemma. We may assume that Y^* admits a Whitney stratification by the strata

$$(Y^* \setminus Y_0, Y_0 \setminus V, V \setminus \{0\}, \{0\}) \tag{4.11}$$

sufficiently close to the origin. The last stratum $\{0\}$ might, however, be optional.

If we apply Lê's Fibration Theorem 2.1.11 in this setting for some sufficiently small Milnor ball B, we obtain a smooth fiber

$$F_u = B \cap f^{-1}(\{u\}) \cap Y^*$$

for $u \in \mathbb{C}$ sufficiently close to 0. This is what we refer to as the *Milnor fiber of the complete intersection line singularity* $(Y_0, 0)$.

4.2.2 The Polar Curve

Besides the Whitney stratification there is one more thing we need to take into account. Let

$$L : \mathbb{C}^N \to \mathbb{C}, (x_1, \ldots, x_N) \mapsto x_1$$

be the projection to the first coordinate axis. Consider the polar locus of f with respect to L on Y^*

$$\Gamma(f, L) = \overline{\{x \in Y^* \setminus Y_0 : \mathrm{d}L(x), \mathrm{d}f(x) \text{ are linearly dependent in } x.\Omega^1_{Y^*}\}}, \tag{4.12}$$

where $\bar{\cdot}$ denotes the closure. The polar locus can be very nasty. However, for most choices of the projection L we get reasonable control over $\Gamma(f, L)$ in the usual way.

Lemma 4.2.3. *For $a = (a_2, \ldots, a_N)$ let*

$$L_a : \mathbb{C}^N \to \mathbb{C}, \quad (x_1, \ldots, x_N) \mapsto x_1 - \sum_{i=2}^{N} a_i \cdot x_i$$

be the projection bent by a. There exists a dense set $\Omega \subset \mathbb{C}^{N-1}$ of values for a such that for $a \in \Omega$ the polar locus $\Gamma(f, L_a) \subset Y^*$ is either empty or an analytic curve, which is smooth outside Y_0.

Proof. This is a Bertini-type theorem. Consider the following incidence space

$$N^* := \{(x, a) \in Y^* \times \mathbb{C}^{N-1} : \mathrm{d}L_a(x), \mathrm{d}f(x) \text{ are linearly dependent in } x.\Omega^1_{Y^*}\}.$$

It comes along with the two natural projections

$$\begin{tikzcd} & N^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ Y^* & & \mathbb{C}^{N-1}. \end{tikzcd}$$

Over $Y^* \setminus Y_0$ the function f has a full rank differential on T_{Y^*} and, therefore, $N^* \setminus pr_1^{-1}(Y_0)$ is a smooth manifold of complex dimension

$$\dim N^* = \dim Y^* + (N-1) - (\dim Y^* - 1) = N.$$

Let a be a regular value of the projection pr_2 restricted to $N^* \setminus pr_1^{-1}(Y_0)$. Its preimage

$$pr_2^{-1}(\{a\}) \subset Y^* \times \{a\}$$

is either empty or an analytic curve which is smooth outside $Y_0 \times \{a\}$. □

We will, in the following, assume that L_a has been chosen according to lemma 4.2.3. Then we readjust the coordinate system of $\mathbb{C}^N$ in a way that $L_a = L = x_1$ is just the first coordinate function, i.e. the projection to the first axis.

Corollary 4.2.4. *Passing to a smaller representative of Y_0 if necessary, we can furthermore assume that the polar curve meets Y_0 only at points in V.*

4.2.3 The Choice of a Milnor Ball

Let $\rho : \mathbb{C}^N \to \mathbb{R}$ be the squared distance function from the origin and set $B_\varepsilon := \{\rho \leq \varepsilon\}$. For sufficiently small $\varepsilon > 0$ we may assume that

- ρ is a Whitney stratified submersion on $Y^* \cap B_\varepsilon$ with respect to the standard stratification (4.11).
- (cf. [39, Korollar 3.2]) the function

  $$\arg f : Y^* \setminus Y_0 \to S^1$$

  has a differential, which is linearly independent from $\mathrm{d}\rho$ over $\mathbb{R}$ in the real cotangent bundle T^*Y^* of Y^* along $B_\varepsilon \cap (Y^* \setminus Y_0)$.
- the function f has no critical points on $B_\varepsilon \cap Y^*$ away from Y_0.
- the polar curve Γ is either empty or, if it intersects $B_\varepsilon \cap V$, it does so only at the origin.

4.2.4 The Milnor Fiber in the Product Case

In this section we will treat the case that

$$(Y^* \setminus Y_0, Y_0 \setminus V, V)$$

is already a Whitney stratification of Y^* at 0 and the polar curve $\Gamma(f, L)$ is empty.

Thom's first isotopy Lemma yields that Y_0 is a product over V, that is

$$(Y_0, 0) \cong (Y_0^\pitchfork \times V, 0), \tag{4.13}$$

where $(Y_0^\pitchfork, 0)$ is the germ of the *transversal singularity*. This is an isolated singularity obtained from Y_0 by intersecting it with a hyperplane in general position, i.e. transversal to all strata at 0. Lemma 4.2.3 and Corollary 4.2.4 show us how to choose the equation for such a hyperplane.

We will show that the product structure (4.13) also holds for the Milnor fiber. To do so, it is more convenient to have a polydisc rather than a Milnor ball. Assume that the projection $L = x_1$ to the first axis is general and let

$$q = \sum_{i=2}^{N} \overline{x}_i \cdot x_i : \mathbb{C}^N \to \mathbb{R}$$

be the squared distance from V. According to [30, Lemma 2.3], the map

$$(L, q) : Y_0 \setminus V \to V \times \mathbb{R}$$

is a submersion on a neighborhood U of the origin. We may choose $\alpha, \beta \in \mathbb{R}_{>0}$ small enough such that the polydisc

$$\Delta_{\alpha\beta} := \{q \leq \alpha^2\} \cap \{|L| \leq \beta\}$$

is contained in U.

Theorem 4.2.5. *In the above setup for fixed α and β there exists a $\delta > 0$ such that the map*

$$(f, L) : Y^* \cap \Delta_{\alpha\beta} \cap f^{-1}(D_\delta) \to D_\delta \times D_\beta \tag{4.14}$$

is a fiber bundle away from $Y_0 = Y^* \cap \{f = 0\}$.

Definition 4.2.6. The fiber of (4.14) over a general point is called the *transversal Milnor fiber* and denoted by $F^\pitchfork$.

Clearly for fixed $\delta > 0$ we have $F \cong F^\pitchfork \times D_\beta$.

Proof. (of Theorem 4.2.5) Since (L, q) was a submersion on $Y_0 \cap \Delta_{\alpha\beta}$, the horizontal part of the boundary

$$\partial_h(Y_0 \cap \Delta_{\alpha\beta}) := Y_0 \cap \{q = \alpha^2\} \cap \{|L| \leq \beta\}$$

is a fiber bundle over the closed disc D_β. Because it is compact, this property is preserved under small perturbations of f. Hence, we can assume that

$$(f, L) : Y^* \cap f^{-1}(D_\delta) \cap \{q = \alpha^2\} \cap L^{-1}(D_\beta) \to D_\delta \times D_\beta$$

is a fiber bundle.

The absence of the polar curve assures that away from Y_0 we also find no critical points of (f, L) in the interior of $Y^* \cap \Delta_{\alpha\beta}$. Therefore (4.14) is a proper submersion away from Y_0 and, hence, a fiber bundle by Ehresmann's Fibration Theorem. □

4.2.5 The Milnor Fiber at a Special Point

We now treat the general case; i.e. we have a Whitney stratification of $Y^* \subset \mathbb{C}^N$ by the strata

$$(Y^* \setminus Y_0, Y_0 \setminus V, V \setminus \{0\}, \{0\})$$

and a possibly nonempty polar curve $\Gamma \subset Y^*$, which meets Y_0 at $\{0\}$. By passing to smaller representatives, if necessary, we can always reduce to this setup.

Let B be a Milnor ball for Y_0 at 0. When we investigate the topology at the special points in the setting of the Tjurina modification of an ICMC2 singularity, it is the part of the boundary $\Sigma = Y_0 \cap \partial B$, which is close to V, along which Y_0 connects to the remaining space. Therefore we will study mainly two objects in this section: The topology of the second boundary

$$\partial_2 F \subset \partial F,$$

which is the part of the boundary of the Milnor fiber F close to V, and the relative homology groups $H_q(F, \partial_2 F)$, which determine how F is connected to $\partial_2 F$. The precise definition of the second boundary $\partial_2 F$ is given below.

The Second Boundary

In this section we will denote the boundaries of the spaces in question by

$$\Sigma^* := Y^* \cap \partial B, \quad \Sigma := Y_0 \cap \partial B, \quad S := V \cap \partial B.$$

Along the points of S we find the product situation of the preceding section for Y_0. Thus, Theorem 4.2.5 is applicable along the whole circle. However, we do need the slight modification to change L to

$$\tilde{L} : B \to \mathbb{C}, \quad x \mapsto \sqrt{\rho(x)} \cdot \exp(\sqrt{-1} \cdot \arg L),$$

with $\rho = |L|^2 + q$ the squared distance from the origin. The function $\tilde{L}$ is not holomorphic, but approximates L as a differentiable function close to S. Repeating the arguments in the setup and proof of Theorem 4.2.5 along the compact manifold S we obtain:

Corollary 4.2.7. *There exist $\alpha > 0$, and $\delta > 0$ sufficiently small with respect to α such that*

$$(f, \arg L) : \Sigma^* \cap \{q \le \alpha^2\} \cap f^{-1}(D_\delta) \to D_\delta \times S^1 \tag{4.15}$$

is a smooth fiber bundle away from $\{f = 0\}$.

It is easy to see that the fiber of this fiber bundle is canonically diffeomorphic to the transversal Milnor fiber $F^{\pitchfork}$.

Definition 4.2.8. For α and δ as in Corollary 4.2.7 the space

$$\partial_2 F := \Sigma^* \cap \{q \le \alpha^2\} \cap \{f = \delta\}$$

is called the *second boundary* of the Milnor fiber F and the monodromy $\mathbf{T}_\bullet$ from the fibration

$$\arg L : \partial_2 F \to S^1 \tag{4.16}$$

the *vertical monodromy*.

The topology of $\partial_2 F$ is completely determined by the topology of $F^{\pitchfork}$ and the Wang sequence of (4.16). The transversal Milnor fiber $F^{\pitchfork}$ comes from an ICIS of dimension $n-1$, so it is $(n-2)$-connected. For $n \ge 3$ the

Wang sequence splits into two parts:

$$0 \longrightarrow H_n(\partial_2 F) \longrightarrow H_{n-1}(F^{\pitchfork}) \xrightarrow{\mathbf{T}_{n-1}-\mathbf{1}} H_{n-1}(F^{\pitchfork}) \longrightarrow H_{n-1}(\partial_2 F) \longrightarrow 0 \tag{4.17}$$

and

$$0 \longrightarrow H_1(\partial_2 F) \longrightarrow H_0(F^{\pitchfork}) \xrightarrow{\mathbf{T}_0-\mathbf{1}} H_0(F^{\pitchfork}) \longrightarrow H_0(\partial_2 F) \longrightarrow 0\,, \tag{4.18}$$

where $\mathbf{T}_\bullet$ is the monodromy operator of (4.16). Clearly, $\mathbf{T}_0 - \mathbf{1}$ in (4.18) is the zero map. Thus, we proved the following:

Corollary 4.2.9. *Let $n = \dim(Y_0, 0) \geq 3$. The homology groups of $\partial_2 F$ have the following properties:*

1. *$H_n(\partial_2 F)$ is a free subgroup of $H_{n-1}(F^{\pitchfork})$.*
2. *Every cycle in $H_{n-1}(\partial_2 F)$ can be represented by a cycle in $H_{n-1}(F^{\pitchfork})$.*
3. *$H_1(\partial_2 F)$ is free abelian of rank 1 and generated by a section of $\arg L$.*
4. *$\partial_2 F$ is connected.*
5. *All other homology groups are zero.*

If $n = \dim(Y_0, 0) = 2$, the terms $H_{n-1}(\partial_2 F)$ from (4.17) and $H_1(\partial_2 F)$ from (4.18) come together. But the kernel of $\mathbf{T}_0 - \mathbf{1}$ is still free of rank 1 and hence there is a (non-canonical) splitting

$$H_1(\partial_2 F) \cong H_1' \oplus \mathbb{Z} = \operatorname{coker}(T_1 - \mathbf{1}) \oplus \ker(T_0 - \mathbf{1}). \tag{4.19}$$

We call $H_1' = \operatorname{coker} T_1 - \mathbf{1}$ the *transversal* or *horizontal* and the other summand $\mathbb{Z} = \ker T_0 - \mathbf{1}$ the *vertical* cycles of the second boundary $\partial_2 F$.

Corollary 4.2.10. *The homology groups of the second boundary $\partial_2 F$ of the Milnor fiber F of a complete intersection line singularity $(Y_0, 0)$ of dimension 2 have the following properties:*

1. *$H_2(\partial_2 F)$ is a free subgroup of $H_1(F^{\pitchfork})$.*
2. *Every cycle in $H_1(\partial_2 F)$ can be represented by a transversal cycle in $H_1(F^{\pitchfork})$.*
3. *$H_1(\partial_2 F)$ splits into transversal and vertical cycles (4.19) and a generator of the latter is given by the fundamental class of a section of $\arg L$.*
4. *$\partial_2 F$ is connected.*
5. *All other homology groups are zero.*

Connectivity with the Second Boundary

Having described the topology of the second boundary, we now turn to the question how it connects with the Milnor fiber. We will first treat the case $n = \dim(Y_0, 0) \geq 3$ and modify the arguments for the surface case in the next section.

Theorem 4.2.11. *Let $n = \dim Y_0 \geq 3$. Then we have*

$$H_q(F, \partial_2 F) \cong \begin{cases} 0 & 2 < q < n \\ H_{q-1}(\partial_2 F) & q = 2 \\ 0 & 0 \leq q \leq 1 \end{cases} \tag{4.20}$$

where the isomorphisms are induced from the long exact sequence of the pair of spaces $(F, \partial_2 F)$.

The proof of Theorem 4.2.11 follows closely the ideas of Dirk Siersma in his paper [66]. He proved the theorem in the case of hypersurfaces with possibly even more complicated singular locus as the corollary of Lemma 3.8, his "second variation sequence" [1]. It picks up the idea of the original fibration by Milnor and Hamm

$$\arg f : \Sigma^* \setminus \Sigma \to S^1 \tag{4.21}$$

where, as before, $\Sigma^* = Y^* \cap \partial B$ with Σ being the boundary of Y_0. Hamm shows in [39, Satz 1.6], that this is a C^∞-fiber bundle with open fibers. Moreover, he proves that for $\delta > 0$ sufficiently small, (4.21) is in fact fiberwise diffeomorphic to

$$\frac{f}{\delta} : \{|f| = \delta\} \cap Y^* \cap \mathring{B} \to S^1. \tag{4.22}$$

The proof proceeds by construction of an outward pointing vector field on $Y^* \setminus Y_0$, whose flow takes $\{|f| = \delta\} \cap Y^* \cap B$ fiberwise onto $\Sigma^* \setminus \{|f| \geq \delta\}$. For two chosen single fibers we can then establish an isomorphism.

Unlike in the case of an ICIS it is not so easy to see that, if we pass to the closure in B, we still get a fibration.

Lemma 4.2.12. *For sufficiently small $\delta > 0$ the map*

$$\frac{f}{\delta} : \{|f| = \delta\} \cap Y^* \cap B \to S^1 \tag{4.23}$$

is a C^∞ fiber bundle with closed fibers

$$F = \{f = \delta\} \cap Y^* \cap B.$$

Proof. By choice of the Milnor ball there are no critical points of f on $(Y^* \setminus Y_0) \cap B$. Hence, we only have to check that $f/|\delta|$ is a submersion at the boundary

$$\{|f| = \delta\} \cap \Sigma^*.$$

This can be achieved by first using the Curve Selection Lemma to show that f has no critical points on $\Sigma^* \setminus \Sigma$ on a neighborhood U of Σ. In a second step we can exploit the compactness of Σ: For sufficiently small δ the set $\{|f| \leq \delta\} \cap \Sigma^*$ will be contained in U. $\square$

To create the setup to prove Theorem 4.2.11, we first choose $\alpha > 0$ such that

[1]There is a typo in [66]: The third case in the mentioned corollary is $2 < q \leq n - 1$.

- all requirements of Corollary 4.2.7 are fulfilled so that we will have a fibration of the second boundary.
- the space
$$N_\alpha := \Sigma^* \cap \{q \leq \alpha^2\}$$
has $S = V \cap \Sigma$ as a strong deformation retract in Σ^*.

After that we choose $\delta > 0$ sufficiently small with respect to α such that

- again the assumptions of Corollary 4.2.7 are met.
- Lemma 4.2.12 holds and we get a Milnor fibration by f.
- we have Σ as a strong deformation retract of the space
$$\Sigma_{\leq\delta} := \Sigma^* \cap \{|f| \leq \delta\}$$
and the retraction takes the subset $\partial N_\alpha \cap \Sigma_{\leq\delta}$ into itself.

This last space now decomposes as

$$\Sigma_{\leq\delta} = (\Sigma_{\leq\delta} \cap \overline{(\Sigma^* \setminus N_\alpha)}) \cup (\Sigma_{\leq\delta} \cap N_\alpha) =: T_1 \cup T_2.$$

The attentive reader may recognize T_2 from Corollary 4.2.7. The other part, T_1, has a natural structure as a trivial disc bundle over Σ as in the case of isolated singularities since $\Sigma \setminus N_\alpha$ was compact and smooth.

Now we can decompose the space Σ^* as

$$\begin{aligned} \Sigma^* &\cong (\Sigma^* \cap \{|f| \leq \delta\}) \cup (\Sigma^* \cap \{|f| \geq \delta\}) \\ &\cong (\Sigma_{\leq\delta}) \cup (Y^* \cap B \cap f^{-1}(\partial D_\delta)) \end{aligned}$$

according to Hamm's computations where the second part is a smooth fiber bundle over the circle by Lemma 4.2.12.

Proof. (of Theorem 4.2.11) Consider the triple of spaces $(\Sigma^*, F \cup T_2, T_2)$. We have the following isomorphisms for the relative homology groups.

$$\begin{aligned} H_q(\Sigma^*, F \cup T_2) &\cong H_q(\Sigma^*, F \cup T_2 \cup T_1) && (4.24) \\ &\cong H_q(Y^* \cap B \cap f^{-1}(\partial D_\delta), F \cup (\Sigma^* \cap f^{-1}(\partial D_\delta))) && (4.25) \\ &\cong H_q(F \times [0,1], \partial(F \times [0,1])) && (4.26) \\ &\cong H_{q-1}(F, \partial F) \otimes H_1(I, \partial I) = H_{q-1}(F, \partial F) && (4.27) \end{aligned}$$

for $q > 0$ and $H_0(\Sigma^*, F \cup T_2) = 0$. The first line (4.24) holds because T_1 retracts onto the part of the boundary of F outside N_α. By excision we get (4.25), and (4.26) comes from the fibration (Lemma 4.2.12). We deduce (4.27) from the Künneth formula. Furthermore,

$$\begin{aligned} H_q(\Sigma^*, T_2) &\cong H_q(\Sigma^*, N_\alpha \cap \Sigma) && (4.28) \\ &\cong H_q(\Sigma^*, S) && (4.29) \end{aligned}$$

because by assumption T_2 retracts onto $N_\alpha \cap \Sigma$, which in turn retracts onto $S = V \cap \Sigma$. Finally, by excision we deduce

$$H_q(F \cup T_2, T_2) \cong H_q(F, \partial_2 F). \qquad (4.30)$$

With these identifications the long exact sequence from the triple reads

$$\begin{tikzcd}
\cdots \arrow[r] & H_{q+1}(\Sigma^*, F\cup T_2) \arrow[r] \arrow[d, "\cong"] & H_q(F\cup T_2, T_2) \arrow[r] \arrow[d, "\cong"] & H_q(\Sigma^*, T_2) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_q(F, \partial F) \arrow[r] & H_q(F, \partial_2 F) \arrow[r] & H_q(\Sigma^*, S) \arrow[r] & \cdots .
\end{tikzcd} \tag{4.31}$$

Recall that according to the Lefschetz Hyperplane Theorem 2.1.12, the relative homology groups $H_q(F, \partial F)$ vanish for $q < n$. Thus we find isomorphisms

$$H_q(F, \partial_2 F) \cong H_q(\Sigma^*, S) \qquad \text{for } q < n. \tag{4.32}$$

To determine the connectivity of the pair $(F, \partial_2 F)$, we are therefore left with the computation of the relative homology groups $H_q(\Sigma^*, S)$.

The rest of the proof will split into three cases. In any of these we will show that from the long exact sequence in homology of the pair (Σ^*, S) we get

$$H_q(\Sigma^*, S) \cong \begin{cases} 0 & \text{for } 2 < q < n \\ H_{q-1}(S) = \mathbb{Z} & \text{for } 0 < q \leq 2 \end{cases} . \tag{4.33}$$

Case I: Y^* is smooth.
This has been done by Dirk Siersma in [66]. The pair (Σ^*, S) is just (S^{2n-1}, S^1) with the usual equatorial embedding. Clearly (4.33) holds and (4.20) follows for the case $0 \leq q < n, q \neq 2$. For $q = 2$ consider the following commutative diagram

$$\begin{tikzcd}
H_2(F, \partial_2 F) \arrow[r, "\cong"] \arrow[d] & H_2(F\cup T_2, T_2) \arrow[r, "\cong"] \arrow[d] & H_2(\Sigma^*, T_2) \arrow[r, "\cong"] \arrow[d] & H_2(\Sigma^*, S) \arrow[d, "\cong"] \\
H_1(\partial_2 F) \arrow[r, "\cong"] & H_1(T_2) \arrow[r, "\cong"] & H_1(T_2) \arrow[r, "\cong"] & H_1(S)
\end{tikzcd} \tag{4.34}$$

All horizontal maps are isomorphisms. In the lower row they are induced by the inclusion $\partial_2 F \hookrightarrow T_2$ and the retraction of T_2 onto S. The vertical map on the left clearly is an isomorphism, too. This concludes the proof in case I.

Case II: Y^* has an isolated singular point at the origin.
In this case Σ^* is a smooth compact manifold. Let F^* be the Milnor fiber of the isolated complete intersection singularity $(Y^*, 0)$. The dimension of F^* is $n+1$ and according to Theorem 2.1.15 it is homotopic to a bouquet of $(n+1)$-dimensional spheres. The Lefschetz Hyperplane Theorem asserts that one can obtain F^* from Σ^* by attaching cells of dimension $\geq n+1$. Then clearly Σ^* must be $(n-1)$-connected and (4.33) follows from the long exact sequence of the pair (Σ^*, S). The proof is finished with the same arguments as in case I.

Case III: Y^* is also singular along V.
Here S denotes the singular part of the boundary Σ^* of Y^*. For α sufficiently small the pair (Σ^*, S) is homotopic to the pair (Σ^*, N_α). Let again F^* be the Milnor fiber in a smoothing of Y^* and consider the triple $(F^*, \partial F^*, \partial_2 F^*)$.

By excision we clearly have isomorphisms

$$H_q(\Sigma^*, S) \cong H_q(\Sigma^*, N_\alpha) \cong H_q(\partial F^*, \partial_2 F^*)$$

for all q. The long exact sequence for the triple reads

$$\cdots \longrightarrow H_{q+1}(F^*, \partial F^*) \longrightarrow H_q(\partial F^*, \partial_2 F^*) \longrightarrow H_q(F^*, \partial_2 F^*) \longrightarrow \cdots$$

and for $q+1 < n+1 = \dim F^*$ the terms $H_{q+1}(F^*, \partial F^*)$ vanish. Thus for all $0 < q < n$ we have isomorphisms

$$H_q(F, \partial_2 F) \cong H_q(\Sigma^*, S) \cong H_q(\partial F^*, \partial_2 F^*) \cong H_q(F^*, \partial_2 F^*). \tag{4.35}$$

The claim now follows by induction on the codimension of Y^*. For $0 \le q < n, q \neq 2$ the right hand term of (4.35) is zero and in case $q = 2$ we can extend the diagram (4.34) by one more column to obtain

$$\begin{tikzcd} H_2(F, \partial_2 F) \arrow[r, "\cong"] \arrow[d] & \cdots \arrow[r, "\cong"] & H_2(\Sigma^*, S) \arrow[r, "\cong"] \arrow[d] & H_2(F^*, \partial_2 F^*) \arrow[d, "\cong"] \\ H_1(\partial_2 F) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & H_1(S) \arrow[r, "\cong"] & H_1(\partial_2 F^*) \end{tikzcd} \quad . \tag{4.36}$$

□

The Surface Case

We already saw in Section 4.2.5, Corollary 4.2.10 that surfaces need special treatment. The reason for this is that the horizontal and the vertical cycles of the second boundary $\partial_2 F$ do not live in distinct homology groups anymore. In view of its applications for ICMC2 singularities in the next section we will formulate a different connectivity result for the pair $(F, \partial_2 F)$ in the case $n = 2$.

Theorem 4.2.13. *Let $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ be a complete intersection line singularity of dimension $n = 2$. Recall the (non-canonical) decomposition*

$$H_1(\partial_2 F) \cong H_1' \oplus \mathbb{Z}$$

into horizontal and vertical cycles (4.19) for the second boundary $\partial_2 F$ of the Milnor fiber F of Y_0. With these identifications the natural map $\iota_1 : H_1(\partial_2 F) \to H_1(F)$ is surjective and factors via

$$\begin{tikzcd} H_1' \oplus \mathbb{Z} \arrow[rr, two heads, "\iota_1"] \arrow[dr] & & H_1(F) \\ & H_1' \arrow[ur] & \end{tikzcd} \quad . \tag{4.37}$$

In other words: The vertical cycles are homologous to zero in F while every remaining 1-cycle of F comes from a cycle in $\partial_2 F$.

Proof. We can literally copy the setup and the beginning of the proof of Theorem 4.2.11 up to the point where we deduce the isomorphisms (4.32). From this point onwards the proof of Theorem 4.2.13 becomes an investigation of the following part of the long exact sequence from the pair $(F, \partial_2 F)$

$$H_2(F, \partial_2 F) \longrightarrow H_1(\partial_2 F) \longrightarrow H_1(F) \longrightarrow H_1(F, \partial_2 F) \longrightarrow 0$$

(the space $\partial_2 F$ is connected, as can be seen from the Wang sequence; this gives the zero on the right).

Let $l : S^1 \to \partial_2 F$ be a section of $\arg L$ representing the homology class $[l]$ of the generator of the vertical cycles in $H_1(\partial_2 F)$. Consider the commutative diagram

$$\begin{tikzcd}
H_2(F, \partial_2 F) \arrow[r, "\cong"] \arrow[d] & H_2(F \cup T_2, T_2) \arrow[r, "\cong"] \arrow[d] & H_2(\Sigma^*, T_2) \arrow[r, "\cong"] \arrow[d] & H_2(\Sigma^*, S) \arrow[d, "\delta"] \\
H_1(\partial_2 F) \arrow[r, "\kappa"] & H_1(T_2) \arrow[r, "\cong"] & H_1(T_2) \arrow[r, "\cong"] & H_1(S)
\end{tikzcd} \tag{4.38}$$

where the column maps are the natural ones from the corresponding pairs of spaces. Contrary to the higher dimensions the map κ coming from the inclusion $\partial_2 F \hookrightarrow T_2$ is not necessarily an isomorphism anymore. But clearly, it maps $[l]$ into the homology class of the generator of $H_1(S) \cong \mathbb{Z}$.

The factorization (4.37) would follow from δ on the right being surjective. Surjecitivity of ι_1 in (4.37) directly follows from $H_1(F, \partial_2 F)$ being zero.

Case I: Y^* is smooth (cf. [66]).
The pair (Σ^*, S) is nothing but a pair of spheres (S^5, S^1) with the standard equatorial embedding. Clearly δ in (4.38) is surjective and from (4.32) we get

$$H_1(F, \partial_2 F) \cong H_1(S^5, S^1) = 0.$$

Case II: Y^* has an isolated singularity at the origin.
The Milnor fiber F^* of Y^* is a bouquet of spheres of dimension 3 and the pair $(F^*, \partial F^*)$ is 2-connected. We consider a smoothing of $(Y^*, 0)$ compatible with the constructions made for $(Y_0, 0)$. The term

$$H_1(F, \partial_2 F) \cong H_1(\Sigma^*, S) \cong H_1(\partial F^*, \partial_2 F^*)$$

appears in the long exact sequence of the triple $(F^*, \partial F^*, \partial_2 F^*)$:

$$\cdots \longrightarrow H_2(F^*, \partial F^*) \longrightarrow H_1(\partial F^*, \partial_2 F^*) \longrightarrow H_1(F^*, \partial_2 F^*) \longrightarrow \cdots$$

The term on the left is zero because of the connectivity of the pair $(F^*, \partial F^*)$. The one on the right also appears in the long exact sequence of the pair $(F^*, \partial_2 F^*)$:

$$H_1(F^*) \longrightarrow H_1(F^*, \partial_2 F^*) \longrightarrow H_0(\partial_2 F^*) \xrightarrow{\iota_0} H_0(F^*) \,.$$

Since $\partial_2 F^*$ is clearly connected, the map ι_1 is injective. On the other hand $H_1(F^*) = 0$, so we deduce $H_1(F^*, \partial_2 F^*) = 0$. Tracing this back we have shown $H_1(F, \partial_2 F) = 0$ as desired.

The space $\Sigma^* = \partial F^*$ is 1-connected, for if it wasn't, according to the connectivity with its boundary due to the Lefschetz Hyperplane Theorem,

F^* couldn't be a bouquet of 3-spheres. This shows surjectivity of δ in (4.38).

Case III: Y^* is singular along V.
For the surjectivity of δ in (4.38) we apply the same argument as in the higherdimensional case. From the long exact sequence of the triple $(F^*, \partial F^*, \partial_2 F^*)$, the connectivity of $(F^*, \partial F^*)$ and Theorem 4.2.13, we deduce surjectivity of the natural map

$$H_2(\Sigma^*, S) \cong H_2(\partial F^*, \partial_2 F^*) \to H_2(F^*, \partial_2 F^*) \cong H_1(\partial_2 F).$$

Thereby we can extend the diagram (4.38) to the right by the column

$$\begin{tikzcd} H_2(F, \partial_2 F) \arrow[r, "\cong"] \arrow[d] & \cdots \arrow[r, "\cong"] & H_2(\Sigma^*, S) \arrow[r, two heads] \arrow[d, "\delta"] & H_2(F^*, \partial_2 F^*) \arrow[d, "\cong"] \\ H_1(\partial_2 F) \arrow[r, "\kappa"] & \cdots \arrow[r, "\cong"] & H_1(S) \arrow[r, "\cong"] & H_1(\partial_2 F^*) \end{tikzcd} \quad (4.39)$$

Furthermore, we get

$$H_1(F, \partial_2 F) \cong H_1(\Sigma^*, S) \cong H_1(F^*, \partial_2 F^*) = 0$$

from the connectivity of $(F^*, \partial_2 F^*)$. $\square$

4.3 Application to Determinantal Singularities

Let $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ be an ICMC2 singularity of Cohen-Macaulay type $t = 2$ described by the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix},$$

so that the Tjurina transform $(Y_0, \{0\} \times \mathbb{P}^1) \subset (\mathbb{C}^{n+2} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ is given by the equations

$$\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} := \begin{pmatrix} s_1 & s_2 \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} = 0.$$

Assume that Y_0 is singular along the exceptional set $V = \{0\} \times \mathbb{P}^1$. We may choose a Whitney stratification for Y_0 by strata

$$(Y_0 \setminus V, V \setminus \{p_1, \dots, p_N\}, \{p_1, \dots, p_N\}).$$

The first part of this section is devoted to creating a setup, in which the conditions for the methods and results of section 2 are met.

First we construct the space Y^* globally by the same arguments. Let $X_0 \subset U \subset \mathbb{C}^{n+2}$ be a representative of $(X_0, 0)$ in some open neighborhood U of the origin and $Y_0 \subset U \times \mathbb{P}^1$ its Tjurina transform. Consider

$$\mathbb{P}\vec{f} : U \times \mathbb{P}^1 \setminus Y_0 \to \mathbb{P}^2, \quad (x, s) \mapsto (f_1(x,s) : f_2(x,s) : f_3(x,s)).$$

This is a well defined map, even though the f_i are not functions. Choose a regular value $z \in \mathbb{P}^2$ and define

$$Y^* := \overline{\mathbb{P}\vec{f}^{-1}(\{z\})} \subset U \times \mathbb{P}^1.$$

After a change of coordinates of $\mathbb{P}^2$ sending z to $(0:0:1)$, which naturally translates to row operations on A, we can assume that Y^* is given by the equations $f_1 = f_2 = 0$ and that

$$Y_0 = \{f = 0\} \cap Y^* \subset Y^*$$

is the zero locus of $f := f_3 \in H^0(U \times \mathbb{P}^1, \mathcal{O}(1))$.

Next we define the polar curve. Let $L : Y^* \subset \mathbb{C}^{n+2} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to $\mathbb{P}^1$ and $z \in \mathbb{P}^1$ a regular value of L on $Y_0 \setminus V$. We may, after a change of coordinates, which corresponds to a canonical column operation on A, assume that $z = (0:1) = \infty$. In the chart $\{s_1 \neq 0\}$ we can do the same as in Lemma 4.2.3 in the whole chart at once to obtain a bending of L, which is sufficiently general for our needs. Any chosen bending in this chart will not alter the fiber of L over ∞.

Observe that on the overlap $\{s_1 \neq 0\} \cap \{s_2 \neq 0\}$ the polar loci with respect to L of the functions f/s_1 and $f/s_2 = f/s_1 \cdot s_1/s_2$ coincide. We can express L as s_2/s_1. Then, because

$$\mathrm{d}\frac{f}{s_1} = \mathrm{d}\left(\frac{f}{s_2} \cdot \frac{s_2}{s_1}\right) = \frac{s_2}{s_1} \cdot \mathrm{d}\frac{f}{s_2} + \frac{f}{s_2} \cdot \mathrm{d}\frac{s_2}{s_1}$$

clearly

$$\begin{aligned} \Gamma &= \overline{\left\{x \in Y^* \setminus Y_0 : \mathrm{d}\frac{f}{s_2}(x) \text{ and } \mathrm{d}\frac{s_2}{s_1}(x) \text{ are linearly dependent in } \Omega^1_{Y^*}\right\}} \\ &= \overline{\left\{x \in Y^* \setminus Y_0 : \mathrm{d}\frac{f}{s_1}(x) \text{ and } \mathrm{d}\frac{s_2}{s_1}(x) \text{ are linearly dependend in } \Omega^1_{Y^*}\right\}}. \end{aligned}$$

After possibly repeating the bending process of L on the other chart, we have a well defined global polar curve $\Gamma \subset Y^*$, which is smooth outside Y_0 and meets Y_0 only at finitely many points along V. We add those points to the zero-dimensional stratum of the Whitney stratification of Y_0.

4.3.1 The Generic Rank 1 Perturbation and the Axis

Since $f = a_{1,3} \cdot s_1 + a_{2,3} \cdot s_2$ is a section of $\mathcal{O}(1)$ and not a function on Y^*, we can not globally perturb by a constant, but we have to choose another section $b = b_1 \cdot s_1 + b_2 \cdot s_2 \in H^0(\mathbb{C}^{n+2} \times \mathbb{P}^1, \mathcal{O}(1))$ and consider

$$f - \delta \cdot b = 0$$

in $\mathbb{C}^{n+2} \times \mathbb{P}^1 \times \mathbb{C}$. Thus there will always be one point in V, the zero locus of b, at which we will perturb the local equation of f by zero. This point is called the *axis point* of the deformation. It is unavoidable, but we can choose its position by the parameters $(b_1 : b_2)$.

Let us assume that after a change of coordinates the point $(0,\infty) := (0,(0:1)) \in V$ is not in the stratum $\{p_1,\dots,p_N\}$ of Y_0 and consider the deformation, which has $(0,\infty)$ as the axis point. For the original ICMC2 singularity $(X_0,0)$ this means we consider the deformation given by the perturbation

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} - \delta \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.40}$$

This gives the equations for the total space $Y \subset \mathbb{C}^{n+2} \times \mathbb{P}^1 \times \mathbb{C}$ of the deformation of Y_0 in the obvious way.

Note that, due to the generality assumptions in the choices of Y^* and the axis point, this is a *generic rank 1 perturbation*. Every perturbation of A by a constant matrix B of rank 1 can be brought to this form using row and column operations on $A - \delta \cdot B$.

In the chart $\{s_1 \neq 0\}$ we now have a deformation of Y_0 given by the perturbation of

$$\frac{f}{s_1} : Y^* \setminus \{s_1 \neq 0\} \to \mathbb{C} \tag{4.41}$$

by δ. At the axis point $(0,\infty)$ on the other hand, we find

$$\frac{f}{s_2} : (Y^*, (0,\infty)) \to (\mathbb{C}, 0) \tag{4.42}$$

perturbed by $\delta \cdot s$ where $s = s_1/s_2$ is the local coordinate of $\mathbb{P}^1$ at ∞.

4.3.2 Y_δ at the Axis Point

By assumption the axis point $(0,\infty)$ of the generic rank 1 deformation was in general position along V. This means, if we let $g = \frac{f}{s_2}$ be the local equation (4.42) for Y_0 in Y^* at the axis point, we find ourselves in the setup of Theorem 4.2.5.

Let $s = s_1/s_2$ and $x_1,\dots,x_{n+2}$ be local coordinates in this chart such that the point $(0,\infty)$ is the origin and choose $\alpha, \beta > 0$ as in Theorem 4.2.5. Then for δ small enough the map

$$G := (g,s) : Y^* \cap \Delta_{\alpha\beta} \cap g^{-1}(D_\delta) \to D_\delta \times D_\beta$$

is a fiber bundle away from $Y_0 = G^{-1}(\{0\} \times D_\beta)$.

The Milnor fiber of g at 0 is the preimage of a line $\{\delta'\} \times D_\beta$ $0 < \delta' < \delta$, under this map. It inherits its product structure from the fibration by s. To obtain the deformed fiber Y_δ of the generic rank 1 deformation of Y_0 at $(0,\infty)$, we have to take a bent line

$$W = \{(\delta \cdot y, y) : y \in D_\beta\} \subset D_\delta \times D_\beta.$$

Now $Y_\delta \cap \Delta_{\alpha\beta} = G^{-1}(W)$. We deduce the following lemma:

Lemma 4.3.1. *Let $g : Y^* \to \mathbb{C}$ be the local equation for Y_0 at the axis point $(0,\infty) \in V$, s a local coordinate for V at $(0,\infty)$ with $s(0,\infty) = 0$ and $\Delta_{\alpha\beta}$ a chosen polydisc in the sense of Theorem 4.2.5. Then for $\delta > 0$ sufficiently small with respect to α and β the space*

$$Y_\delta \cap \Delta_{\alpha\beta} := Y^* \cap \Delta_{\alpha\beta} \cap \{g = s \cdot \delta\}$$

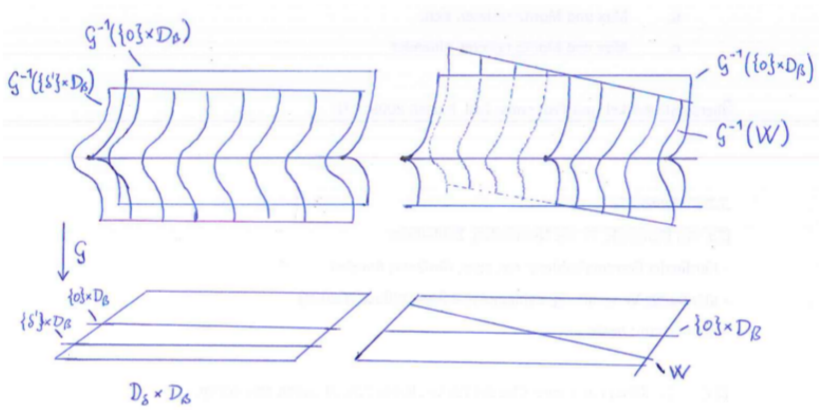

FIGURE 4.2: The fiber of the generic rank 1 perturbation at the axis point.

is the fiber over δ of the generic rank 1 perturbation (4.42) close to the axis point. The map

$$L : Y_\delta \cap \Delta_{\alpha\beta} \to \mathbb{P}^1$$

is a fibration over a punctured neighborhood $D_\beta^\times$ of $\infty \in \mathbb{P}^1$. The central fiber

$$Y_\infty^\pitchfork := Y_\delta \cap \Delta_{\alpha\beta} \cap \{L = \infty\} = Y_0 \cap \Delta_{\alpha\beta} \cap \{L = \infty\},$$

however, does not change as we pass from Y_0 to Y_δ. Consequently Y_δ may retain a singular point at $(0, \infty)$. If this happens, it is at most an ICIS.

Definition 4.3.2. The space $Y_\infty^\pitchfork$ is called the *axis of the deformation*.

Corollary 4.3.3. *The space $Y_\delta \cap \Delta_{\alpha\beta}$ as in Lemma 4.3.1 is contractible.*

Proof. The central fiber $Y_\infty^\pitchfork$ is a euclidean neighborhood retract of some open neighborhood U in $Y_\delta \cap \Delta_{\alpha\beta}$. Clearly the fiber bundle $(Y_\delta \cap \Delta_{\alpha\beta}) \setminus Y_\infty^\pitchfork$ can be retracted onto U and successively onto $Y_\infty^\pitchfork$. Being the representative of a germ of an isolated singularity in a Milnor ball, $Y_\infty^\pitchfork$ is contractible. Concatenation of these two contractions establishes the claim. □

4.3.3 The Global Picture in the Generic Rank 1 Perturbation

After we already have a description of what happens at the axis point $(0, \infty)$ in a generic rank 1 peturbation, let us now compute the topology of Y_δ in the other chart. To create a global setup, first choose Milnor balls B_i of radius ε for all special points $\{p_1, \dots, p_N\}$ of Y_0. Let B_i' be a Milnor ball of radius $\varepsilon/2$ around p_i and set

$$B = \bigcup_{i=1}^N B_i, \qquad B' = \bigcup_{i=1}^N B_i'.$$

We now choose $\alpha > 0$ sufficiently small such that

- all the local theory (Theorem 4.2.11 for threefolds or Theorem 4.2.13 in the surface case) works at the special points p_i,

- Theorem 4.2.5 holds along the set

$$V' := V \setminus (B' \cup \{(0, \infty)\}),$$

i.e. for $\delta > 0$ small enough the map

$$L : Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(V') \to V' \tag{4.43}$$

is a fiber bundle over V' with fiber $F^{\pitchfork}$.

Note that for the last requirement we can use Lemma 4.3.1 to achieve this behaviour in a neighborhood of the axis point. After that we're left with a compact subset of V, along which the existence of a global minimal $\alpha > 0$ can certainly be assured.

Now we choose $\delta > 0$ small enough with respect to all prior choices such that all the local theory developed above works at once along all points of the compact set V.

We can now piece together the topology of Y_δ from the topology of the several known patches. We regard the axis point $(0, \infty)$ as a further special point p_0 in the Whitney stratification of Y_0. Let Δ be the chosen polydisc around p_0 and set

$$U := Y_\delta \cap (B \cup \Delta), \qquad W := Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(V'). \tag{4.44}$$

Furthermore let $\partial_2 F_i$ be the second boundary of the local Milnor fiber of (Y_0, p_i) at p_i for $i > 0$. In case $i = 0$, i.e. at the axis point, we just set

$$\partial_2 F_0 := Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(\partial D_\beta)$$

where D_β is the chosen disc around $\infty \in \mathbb{P}^1$. We easily verify that the inclusion

$$\partial_2 F_i \hookrightarrow (U \cap W)_i$$

induces a homology equivalence

$$H_q(U \cap W) \cong \bigoplus_{i=0}^{N} H_q((U \cap W)_i) \cong \bigoplus_{i=0}^{N} H_q(\partial_2 F_i) \tag{4.45}$$

where $(U \cap W)_i$ is the component of $U \cap W$ close to p_i. For $q = 1$ and $i > 0$ let $[l_i]$ be the generator of $H_1(\partial_2 F_i)$ – respectively the generator of the vertical part in case $n = 2$ – represented by a section $l_i : S^1 \to \partial_2 F_i$ of $\arg L$ in (4.16), cf. Corollary 4.2.9 and 4.2.10.

The homology groups of W itself are determined by the structure of W as a fiber bundle over V' (4.43). Since V' has the homotopy type of a finite bouquet of circles around the points $p_1, \dots, p_N$, we can basically repeat the arguments leading to Corollary 4.2.9 and 4.2.10. In particular we can assume

$$H_0(W) = \mathbb{Z}, \quad H_1(W) = \begin{cases} \mathbb{Z}^N & \text{if } n = 3 \\ H_1' \oplus \mathbb{Z}^N & \text{if } n = 2 \end{cases}, \tag{4.46}$$

where H_1' is the quotient of $H_1(Y_\delta^{\pitchfork})$ by the monodromies around all loops in the base V'. In both cases $\mathbb{Z}^N$ is generated by the $[l_i]$. We can view the latter as sections of the generators of $H_1(V')$.

Theorem 4.3.4. *Let Y_δ be the fiber over $\delta \neq 0$ in the genereric rank 1 perturbation of the Tjurina transform $(Y_0, V) \subset (\mathbb{C}^{n+2} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ of an ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and type $(2,3,2)$. Let $L : Y_\delta \to \mathbb{P}^1$ be the projection to $\mathbb{P}^1$ and $G \subset H^\bullet(Y_\delta)$ the image of $L^* : H^\bullet(\mathbb{P}^1) \to H^\bullet(Y_\delta)$. Then Y_δ has a trivial first homology group and the total homology of Y_δ splits into*

$$H_\bullet(Y_\delta) \cong G^\perp \oplus \mathbb{Z}$$

where $G^\perp = \{[\sigma] \in H_\bullet(Y_\delta) : g \cap [\sigma] = 0 \quad \forall g \in G\}$ are the horizontal cycles of Y_δ. The cap product with $L^*(H^2(\mathbb{P}^1))$ gives a perfect pairing of the vertical cycles $H_\bullet(Y_\delta)/G^\perp = \mathbb{Z}$ with $H^2(\mathbb{P}^1)$. If $n = 3$, then $H_2(X_\varepsilon) \cong \mathbb{Z}$ consists of the vertical cycles only.

Proof. Consider the Mayer-Vietoris sequence for Y_δ for the choice (4.44) of the two patches U and W. First of all, the tail gives a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(U \cap W) & \longrightarrow & H_0(U) \oplus H_0(W) & \longrightarrow & H_0(Y_\delta) & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & \mathbb{Z}^{N+1} & \longrightarrow & \mathbb{Z}^{N+1} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

and Y_δ is clearly connected. The first homology group $H_1(Y_\delta)$ appears in the exact sequence

$$H_1(U \cap W) \xrightarrow{\iota_1} H_1(U) \oplus H_1(W) \longrightarrow H_1(Y_\delta) \longrightarrow 0 \tag{4.47}$$

We proceed with the proof for the case $n = 3$. From Theorem 4.2.11 we know that $H_1(U) = 0$. On the generators chosen above, the map ι_1 to the second summand is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{4.48}$$

and is, therefore, clearly surjective. Thus $H_1(Y_\delta) = 0$.

Proceeding along the Mayer-Vietoris sequence to the left, we see in (4.49) that $H_2(Y_\delta)$ must be nonzero because clearly the kernel of (4.48) is free of rank 1.

$$H_2(U) \oplus H_2(W) \xrightarrow{\kappa_2} H_2(Y_\delta) \xrightarrow{\partial_2} H_1(U \cap W) \xrightarrow{\iota_1} H_1(U) \oplus H_1(W)\,. \tag{4.49}$$

But κ_2 is, in fact, the zero map. To see this, observe the following. Every homology class $[\sigma] \in H_2(U)$ can be represented as a sum of 2-cycles in the boundaries

$$\sigma = \sum_{i=1}^{N} \sigma_i, \quad [\sigma_i] \in H_2(\partial_2 F_i)$$

as a consequence of Theorem 4.2.11. Corollary 4.2.9 then tells us that $[\sigma_i]$ even comes from a cycle in a transversal Milnor fiber $[\sigma_i] \in H_2(F_i^\pitchfork)$ close to

p_i. The same holds for any $[\sigma] \in H_2(W)$ and any other chosen transversal Milnor fiber over a point in V'.

Mapping any $[\sigma] \in H_2(U) \oplus H_2(W)$ into $H_2(Y_\delta)$, therefore, makes it homologous to a cycle in a transversal Milnor fiber arbitrary close to $Y_\infty^\pitchfork$, the fiber of L over the axis point. Here it collapses, because $Y_\delta \cap \Delta$ was contractible by Corollary 4.3.3.

Consequently $H_2(Y_\delta) = \ker \iota_1$. We construct a generator for $H_2(Y_\delta)$ as follows. Over $V' \cup \{\infty\}$ there exists a continous section

$$l : V' \to Y_\delta \cap L^{-1}(V')$$

of L because L gives $Y_\delta \cap L^{-1}(V')$ the structure of a fiber bundle with 1-connected fiber $Y_\delta^\pitchfork$ over a base, which is homotopic to a bouquet of 1-spheres. We can extend l over ∞ because we only glue in a contractible fiber. Let $D = \overline{\mathbb{P}^1 \setminus (V' \cup \{\infty\})}$ be the closure of the complement of the domain of definition of l. Then the fundamental class of the image of l defines a unique relative cycle

$$[l] \in H_2(Y_\delta, L^{-1}(D)).$$

Consider the following commutative diagram

$$\begin{tikzcd}
H_2(L^{-1}(D)) \arrow[r] & H_2(Y_\delta) \arrow[r, "\pi"] \arrow[d, "L_*"] & H_2(Y_\delta, L^{-1}(D)) \arrow[r] \arrow[d, "L_*"] & H_1(L^{-1}(D))\,. \\
 & H_2(\mathbb{P}^1) \arrow[r, "\cong"] & H_2(\mathbb{P}^1, D) &
\end{tikzcd} \tag{4.50}$$

The image of $[l]$ in $H_1(L^{-1}(D))$ is zero by Theorem 4.2.11: At each special point p_i the component of the boundary of $[l]$ in the local Milnor fiber is homologous to the generator of $H_1(\partial_2 F) \cong H_2(F, \partial_2 F)$. On the other hand the map on the left into $H_2(Y_\delta)$ is the zero map by the previous arguments: All 2-cycles of the local Milnor fibers become homologous to zero in Y_δ. A generator $[\sigma]$ of $H_2(Y_\delta)$ is therefore given as a preimage of $[l]$ under π.

The map L_* on the right is an isomorphism and, hence, on the left L_* maps $[\sigma]$ to the fundamental class of $\mathbb{P}^1$. This concludes the proof for the threefolds.

If $n = 2$, we need to modify the arguments above. First we show surjectivity of ι_1 in (4.47). Recall that we can split

$$H_1(U \cap W) \cong \bigoplus_{i=0}^{N} H_1(\partial_2 F_i) \cong \bigoplus_{i=0}^{N} \left(H_1'(\partial_2 F_i) \oplus \mathbb{Z} \right)$$

into its horizontal and vertical part, where $H_1'(\partial_2 F_i)$ is the cokernel of $H_1(F^\pitchfork)$ by the vertical monodromy at p_i.

We can restrict the first component of ι_1 mapping into $H_1(U) = \bigoplus_{i=1}^N H_1'(F_i)$ to the summand $\bigoplus_{i=1}^N H_1(\partial_2 F_i)$ and the second component of ι_1 mapping into $H_1(W)$ to $H_1'(\partial_2 F_0) \oplus \mathbb{Z}^{N+1}$. Both restrictions themselves are surjective by Theorem 4.2.13 and (4.46), hence, ι_1 is, as well. This makes sure that the first homology group of Y_δ vanishes.

On the vertical cycles $\mathbb{Z}^{N+1}$ of $H_1(U \cap W)$ the map ι_1 again takes the same form as in (4.48) and consequently we can choose a splitting

$$H_2(Y_\delta) = H_2'(Y_\delta) \oplus \mathbb{Z}$$

of the second homology group of Y_δ with the second summand mapping to the kernel of ι_1 on the vertical cycles. We can construct a generator $[\sigma]$ of the quotient $H_2(Y_\delta)/H_2'(Y_\delta) = \mathbb{Z}$ similar to the threefold case. Start with a continous section

$$l : V' \cup \{\infty\} \to Y_\delta$$

of $L : Y_\delta \to \mathbb{P}^1$. For surfaces the relative homology class $[l] \in H_2(Y_\delta, L^{-1}(D))$ is not unique, but depends on the choice of l. Nevertheless, any preimage $[\sigma] \in H_2(Y_\delta)$ under the map π in (4.50) generates the quotient $H_2(Y_\delta)/H_2'(Y_\delta) = \mathbb{Z}$ and the composite map $L_* \circ \pi$ is an isomorphism when restricted to the second summand of the splitting $H_2(Y_\delta) = H_2'(Y_\delta) \oplus \mathbb{Z}$.

Hence, $[\sigma]$ is mapped to the fundamental class of $\mathbb{P}^1$ again by L_*. All other cycles in $H_2(Y_\delta)$ can be represented as sitting in the preimage of discs or paths in $\mathbb{P}^1$ and are, therefore, mapped to zero by L_*. This concludes the proof for $n = 2$. $\square$

We can now prove the main theorem of this paper.

Proof. (of Theorem 4.1.3) Consider a deformation of $(X_0, 0)$ with two parameters (δ, ε) where the first one, δ, is for a generic rank 1 perturbation and the second one, ε, is for a smoothing. For the Tjurina transform $Y_{\delta,0}$ over $X_{\delta,0}$, the fiber over $(\delta, 0)$ for $\delta \neq 0$ small enough, the homology groups are described by Theorem 4.3.4. However, according to Lemma 4.3.1 there might still be an ICIS of Y_δ at the axis point.

In case $Y_{\delta,0}$ is smooth, its diffeomorphism type does not change as we pass to a smooth fiber $Y_{\delta,\varepsilon}$ for $\delta, \varepsilon \neq 0$. If it was not, its topology changes at most at the axis point $(0, \infty)$ where it is the smoothing of an ICIS.

This means that, in the notation above, the local Milnor fiber $Y_{\delta,\varepsilon} \cap \Delta$ of $(Y_{\delta,0}, (0, \infty))$ is $(n-1)$-connected. Hence, all $(n-1)$-cycles in $Y_\delta^{\pitchfork}$ appearing in the proof of Theorem 4.3.4 close to $Y_\infty^{\pitchfork}$ (i.e. representable by cycles in $Y_{\delta,\varepsilon} \cap \Delta$) still become homologous to zero in $Y_{\delta,\varepsilon}$ and we can literally repeat all the arguments. The theorem then follows from the isomorphism $Y_{\delta,\varepsilon} \cong X_{\delta,\varepsilon}$. $\square$

4.4 Concluding Remarks

The results presented in this thesis are merely a glimpse of what there might be to discover concerning the vanishing topology of determinantal singularities. Perhaps the most remarkable phenomenon is the existence of characteristic vanishing cycles, Definition 4.1.1, in the Milnor fiber.

For smoothable isolated determinantal singularities of type $(2, 3, 2)$ we saw that if there are vanishing cycles outside the middle degree, then they are indeed characteristic. It would be interesting to explore, whether this is always the case. Certainly, the methods of Chapter 4 are also applicable to singularities of type $(2, 2+k, 2)$ for all $k > 0$. For other shapes of the describing matrix, we saw in Example 3.2.4 and Theorem 3.2.5 that in case of isolated singularities in the Tjurina transform, we can also observe the characteristic cycles below the middle degree in the homology of the Milnor fiber. We give a further example with nonisolated singularities in the Tjurina transform for this matrix size.

Example 4.4.1. Consider the threefold singularity $(X_0, 0) \subset (\mathbb{C}^5, 0)$ defined by a generic embedding

$$A : \mathbb{C}^5 \hookrightarrow \mathrm{Mat}(4, 5; \mathbb{C})$$

of a 5-dimensional subspace into $\mathrm{Mat}(4, 5; \mathbb{C})$. The Tjurina transform now decomposes as

$$Y_0 = \overline{L(X_0 \setminus \{0\})} \cup (\{0\} \times \mathbb{P}^3) \subset \mathbb{C}^5 \times \mathbb{P}^3,$$

where $\overline{L(X_0 \setminus \{0\})}$ is the strict transform of X_0 and $\{0\} \times \mathbb{P}^3$ is an additional component. The locus

$$S = \overline{X_0} \cap (\{0\} \times \mathbb{P}^3),$$

where they meet, is a smooth projective hypersurface of degree 5, so we encounter "plane singularities" in the Tjurina transform in the sense that the singular locus itself has dimension two!

Nevertheless, the induced families in the Tjurina transform coming from determinantal deformations of $(X_0, 0)$ are flat. Experimental computations show that the fiber $\overline{Y}_\delta$ over $\delta \neq 0$ for a generic rank 2 perturbation is already smooth and hence diffeomorphic to the Milnor fiber $\overline{X}_\varepsilon$. The axis of such a deformation is a whole projective line $H \subset \mathbb{P}^3$ and the fiber $\overline{Y}_0 \cap L^{-1}(H) = \overline{Y}_\delta \cap L^{-1}(H)$ of L sits over it. This means that the fundamental class of $H \cong \mathbb{P}^1$ is also passed on to $\overline{X}_\varepsilon$ and then sitting over the corresponding cycle in $\mathbb{P}^3$. Yet, to develop a complete description of the topology of $\overline{X}_\varepsilon$ in the spirit of Chapter 4, we would need to deal with singular loci of dimension 2 and their interplay with the topology of S and the axis – a task which is far more evolved than what has been done in this chapter.

The characteristic cycles exhibit at most an indirect interplay with the infinitesimal deformations encoded in the $T^1_{X_0,0}$ or $\mathrm{Inf}(A)$ as can be seen from Table 3.1 and Remark 3.2.9. In the case of nonisolated singularities in the Tjurina transform, an analogue of Theorem 3.2.7 is not yet formulated. To address questions of the form μ vs. τ for general determinantal singularities beyond Cohen-Macaulay codimension 2, one would also need to develop a deeper understanding of the existence of semi-universal determinantal deformations. To this end, one could for example pick up the approach by M. Schaps in [62].

Also, it would be interesting to know, what happens for determinantal singularities defined by non-maximal minors concerning their vanishing topology and infinitesimal deformations. Are there examples of singularities, for which the spaces $\mathrm{Inf}(A)$ and $\mathrm{Inf}_t(A)$ do not agree? Do they also have characteristic vanishing cycles?

Another path that can be pursued is given by the non-smoothable EIDS as for example the ICMC2 fourfolds and EIDS with non-isolated singularities, cf. Example 2.2.2 ii). As remarked at the end of Chapter 2, it should be possible to create an algorithm for the computation of the vanishing Euler-characteristic of these singularities in terms of polar multiplicities. This could provide a further testing ground for hypotheses on their topological behaviour and reveal insights for new conjectures, just like the results by J. Damon and B. Pike did for the work done in this thesis.

Appendix A

Background and Notations

A.1 The Exterior Algebra

The exterior algebra of vector spaces, vector bundles and modules should be known to mathematicians. Nevertheless, we briefly recall definitions of the exterior algebra, exterior multiplication, dualities and orientations and induced maps in the exterior powers for a homomorphism of modules, to introduce and fix the notation for this thesis. For a more thorough treatment, the reader may consult any standard textbook on commutative algebra, e.g. [22].

A.1.1 Exterior Powers and Multiplication

Let M be a module over a commutative ring R. One can define the tensor algebra of M over R as

$$T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots = \bigoplus_{r=0}^{\infty} M^{\otimes r}. \quad \text{(A.1)}$$

This is a non-commutative graded algebra with multiplication given by the tensor product and grading by r. Consider the graded subalgebra K of $T(M)$ generated by the expressions

$$v \otimes w + w \otimes v, \quad v, w \in M.$$

We define the *exterior algebra* of M to be

$$\bigwedge M := T(M)/K, \quad \bigwedge^p M := M^{\otimes p}/K_p, \quad \text{(A.2)}$$

where K_p is the graded part of K in degree p of $T(M)$. We also say that $\bigwedge^p M$ is the p-th *exterior power* of M. Note that by this definition $\bigwedge^0 M = R$ for all modules M.

For the residue class of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_p \in M^{\otimes p}$ we write $a_1 \wedge a_2 \wedge \cdots \wedge a_p \in \bigwedge^p M$. The multiplication on $T(M)$ induces a graded multiplication on the exterior algebra, which we shall also denote by $\wedge : \bigwedge^p M \times \bigwedge^q M \to \bigwedge^{p+q} M$ given by

$$\wedge : (a_1 \wedge \cdots \wedge a_p, b_1 \wedge \cdots \wedge b_q) \mapsto a_1 \wedge \cdots \wedge a_p \wedge b_1 \wedge \cdots \wedge b_q. \quad \text{(A.3)}$$

This multiplication is skew-commutative, i.e. for $\omega \in \bigwedge^p M$ and $\eta \in \bigwedge^q M$ we have

$$\omega \wedge \eta = (-1)^{p \cdot q}\, \eta \wedge \omega. \quad \text{(A.4)}$$

A.1.2 Dualities of Exterior Algebras

If M is finitely generated over R by elements $e_1, \dots, e_r \in M$, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 0 < i_1 < \cdots < i_p \le r\}$$

generate $\bigwedge^p M$. In particular $\bigwedge^p M = 0$ for all $p > r$. To abbreviate the notation, we shall write

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$$

where $I = (i_1, \dots, i_p), \quad 0 < i_1 < \cdots < i_p \le r$ is an *ordered multiindex*. We shall write $I \subset \{1, \dots, r\}$ to indicate the range out of which the ordered elements i_k of I are chosen. By $\#I = p$ we indicate the order of the multiindex, i.e. the length of the sequence $0 < i_1 < \cdots < i_p \le r$ of the elements in I.

It is now easy to see that, if $M = R^r$ is a module freely generated by elements $e_1, \dots, e_r$, then for all $0 < p \le r$ the module $\bigwedge^p R^r$ is also free of rank $\binom{r}{k}$, and a set of free generators is given by $(e_I)_{I \subset \{1,\dots,r\}}$.

Moreover if we let $\varepsilon_1, \dots, \varepsilon_r \in (R^r)^\vee$ be the dual basis of $(R^r)^\vee = \mathrm{Hom}_R(R^r, R)$, then the natural pairing

$$(R^r)^\vee \otimes R^r \to R, \quad \varepsilon_i \otimes e_j \mapsto \varepsilon_i(e_j) = \delta_{i,j}$$

extends to a pairing of exterior powers given by

$$\bigwedge^p (R^r)^\vee \otimes \bigwedge R^r \to R, \quad \varepsilon_J \otimes e_I \mapsto \delta_{I,J}.$$

In this formula we generalize the Kronecker delta $\delta_{i,j}$ in the sense that for the ordered multiindices $\delta_{I,J} = 1$ if $I = J$ and $\delta_{I,J} = 0$ otherwise. Clearly, this pairing is non-degenerate, and thus we have canonical isomorphisms

$$(\bigwedge^p R^r)^\vee \cong \bigwedge^p (R^r)^\vee \cong \bigwedge^p (R^\vee)^r. \tag{A.5}$$

But there is another duality for exterior powers of free modules, which we shall exhibit now. Observe that, if R^r is freely generated by $e_1, \dots, e_r$, then

$$\bigwedge^r R^r = R \cdot e_1 \wedge \cdots \wedge e_r \cong R \tag{A.6}$$

is free of rank 1. A generator γ of $\bigwedge^r R^r$ is also called an *orientation* of the free module R^r. Now for any $0 \le p \le r$ also exterior multiplication

$$\wedge : \bigwedge^p R^r \otimes \bigwedge^{r-p} R^r \to \bigwedge^r R^r \tag{A.7}$$

induces a nondegenerate pairing with

$$e_I \wedge e_J = \begin{cases} \pm 1 & \text{if } i_k \ne j_l \quad \forall 0 < k \le p, 0 < l \le r-p \\ 0 & \text{otherwise} \end{cases}. \tag{A.8}$$

Thus (A.8) together with (A.5) and the choice of an orientation $\gamma \in \bigwedge^r R^r$ gives isomorphisms

$$(\bigwedge^p R^r)^\vee \cong \bigwedge^{r-p} R^r \tag{A.9}$$

for all $0 \le p \le r$.

A.1.3 Induced Maps on the Exterior Powers

Let us suppose the ring R is Noetherian so that for a finitely generated module M we can find a finite presentation

$$0 \longleftarrow M \longleftarrow R^r \xleftarrow{A} R^s$$

for some matrix $A \in \mathrm{Mat}(r, s; R)$. Then we have isomorphisms

$$\bigwedge^p M \cong \bigwedge^p R^r / \langle A \wedge \bigwedge^{p-1} R^r \rangle, \tag{A.10}$$

where $\langle A \wedge \bigwedge^{p-1} R^r \rangle$ is the submodule generated by the products of the columns of A with elements of $\bigwedge^{p-1} R^r$.

Suppose $\varphi : M \to N$ is a map of finitely generated R-modules. Then there is a map

$$\varphi^{\wedge p} : \bigwedge^p M \to \bigwedge^p N,$$

which is uniquely determined by requiring

$$\varphi^{\wedge p}(v_1 \wedge \cdots \wedge v_p) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_p).$$

Choosing generators for M and N is equivalent to a choice of presentations

$$\begin{tikzcd}
0 \arrow[r] & K_M \arrow[r] & R^r \arrow[r] \arrow[d, dotted, "\phi"] & M \arrow[r] \arrow[d, "\varphi"] & 0 . \\
0 \arrow[r] & K_N \arrow[r] & R^s \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The homomorphism $\phi : R^r \to R^s$ is a lift of φ. Since the underlying modules are free, we can write down a representing matrix $A \in \mathrm{Mat}(s, r; R)$ for ϕ.

Just like ϕ completely determines φ, also the morphism $\varphi^{\wedge p}$ is determined by $\phi^{\wedge p} : \bigwedge^p R^r \to \bigwedge^p R^s$. It is a well known fact that the module $\bigwedge^p R^r$ is *freely* generated by the e_I for $I \subset \{1, \ldots, r\}$ varying over all ordered multiindices as above. Hence, we find a representing matrix $A^{\wedge p}$ of $\phi^{\wedge p}$. Direct computation shows that the entries of this matrix are

$$A^{\wedge p}_{I,J} = \det A_{I,J},$$

where $A_{I,J}$ is the submatrix obtained from A by taking only the rows in I and columns in J. From this we may directly deduce the following lemma.

Lemma A.1.1. *Suppose we are given two successive maps of R-modules*

$$R^t \xleftarrow{A} R^s \xleftarrow{B} R^r$$

determined by matrices A and B. Then for the matrices representing the induced maps on the exterior powers we find

$$(A \cdot B)^{\wedge p} = A^{\wedge p} \cdot B^{\wedge p}, \tag{A.11}$$

or in other words

$$(A \cdot B)^{\wedge p}_{I,K} = \sum_{J \subset (1,\dots,s)} A^{\wedge p}_{I,J} \cdot B^{\wedge p}_{J,K}. \tag{A.12}$$

In particular we see that, if $p > \min\{r, s, t\}$, then $(A \cdot B)^{\wedge p} = 0$. The following corollary is immediate.

Corollary A.1.2. *Let $S \in \mathrm{GL}(m; R)$ be an invertible matrix with inverse S^{-1}. Then for all $0 < t \leq m$ also $S^{\wedge t} : \bigwedge^t R \to \bigwedge^t R$ is invertible with inverse $(S^{-1})^{\wedge t}$.*

A little more subtle is another map $\varphi^{\wedge d}_{\lrcorner}$ derived from $\varphi^{\wedge d}$ for a homomorphism $\varphi : R^r \to R^s$, which we shall call *contraction by* $\varphi^{\wedge d}$:

$$\varphi^{\wedge d}_{\lrcorner} : \bigwedge^p R^r \to \bigwedge^{p-d} R^r \otimes \bigwedge^d R^s. \tag{A.13}$$

If $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ generate $\bigwedge^p R^r$ and $f_J = f_{j_1} \wedge \cdots \wedge f_{j_d}$ generate $\bigwedge^d R^s$ as usual, it can be defined via

$$\varphi^{\wedge d}_{\lrcorner}(e_I) := \sum_{J \subset I, \#J = d} \quad \sum_{K \subset \{1,\dots,s\}, \#K = d} (-1)^{J \subset I} \varphi^{\wedge d}_{K,J} f_K \otimes e_J.$$

The sign $(-1)^{J \subset I}$ for two ordered multiindices $I, J \subset \{1, \dots, r\}$ is defined as

$$(-1)^{J \subset I} := \prod_{k=1}^{d} (-1)^{\min\{l \in \mathbb{N}_0 : j_k = i_l\} - k}.$$

This of course induces a corresponding map on finitely generated modules.

A.2 Grassmannians and Generalized Nash-Blowups

Just like the exterior algebra, Grassmannians are standard objects. But since notations and viewpoints differ throughout the literature, we include an account on them. Afterwards we define generalized Nash-blowups for coherent sheaves. Also this idea is not new, but needs to be carried out explicitely.

A.2.1 Grassmannians

Definition A.2.1. For two positive integes $0 < r \leq s$ the Grassmannian $\mathrm{Grass}(r, s)$ is the set

$$\mathrm{Grass}(r, s) = \{V \subset \mathbb{C}^s \text{ linear subspace } : \dim V = r\}.$$

A standard example for a Grassmannian is projective space $\mathbb{P}^n \cong \mathrm{Grass}(1, n+1)$. There is a way to give all Grassmannians the structure of a projective complex manifold similar to the standard affine charts of $\mathbb{P}^n$.

Any r-dimensional subspace $V \subset \mathbb{C}^s$ can be represented by an $r \times s$-matrix A of rank r, whose columns span V. Right-multiplication by invertible matrices $S \in \mathrm{GL}(s; \mathbb{C})$ do not change the span:

$$\operatorname{span} A = \operatorname{span}(A \cdot S) = V.$$

On the other hand any other matrix A', whose columns span a given subspace V can be obtained from A by right-multiplication with some S. Thus we can identify

$$\mathrm{Grass}(r,s) \cong (\mathrm{Mat}(s,r;\mathbb{C}) \setminus M^r_{s,r}) / \mathrm{GL}(r,\mathbb{C}),$$

where as usual $M^r_{s,r} = \{A \in \mathrm{Mat}(s,r;\mathbb{C}) : \operatorname{rank} A < r\}$.

For a given matrix A being of rank r means that at least one of the maximal minors of A does not vanish. This will give the conditions determining the charts of the standard cover: Let $I \subset \{1,\dots,s\}$ be an ordered multiindex with $\#I = r$ and $J \subset \{1,\dots,s\}$ the ordered multiindex complementary to I, i.e. J contains exactly those elements of $\{1,\dots,s\}$, which are not contained in I. We also set $K = (1,\dots,r)$, the only ordered multiindex of order $\#K = r$ in r elements.

On the open set $U_I = \{A \in \mathrm{Mat}(s,r;\mathbb{C}) : A^{\wedge r}_{I,K} \neq 0\}$ we can bring all matrices to a unique normal form, by multiplying from the right with $A^{-1}_{I,K}$. In case $I = (1,\dots,r)$, the result would be

$$\Xi^I := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ z^I_{r+1,1} & z^I_{r+1,2} & \cdots & z^I_{r+1,r} \\ z^I_{r+2,1} & z^I_{r+2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & z^I_{s-1,r} \\ z^I_{s,1} & \cdots & z^I_{s,r-1} & z^I_{s,r} \end{pmatrix}. \tag{A.14}$$

In general, the entries of the submatrix $Z^I := \Xi_{J,K} = (z^I_{j,k})_{j\in J,\, k\in K}$ give the *standard coordinates* on U_I. In particular

$$\dim \mathrm{Grass}(r,s) = r \cdot (s-r). \tag{A.15}$$

For any ordered multiindex $I \subset \{1,\dots,s\}$ and any chart U_I of $\mathrm{Grass}(r,s)$ we call Ξ^I as in (A.14) the *standard representative matrix* for the chart U_I.

The map

$$\mathrm{Mat}(r,s;\mathbb{C}) \setminus M^r_{r,s} \to \mathrm{Mat}\left(\binom{s}{r}, 1; \mathbb{C}\right) \setminus \{0\}, \quad A \mapsto A^{\wedge r} \tag{A.16}$$

induces a well defined map

$$\mathrm{Grass}(r,s) \hookrightarrow \mathbb{P}^{\binom{s}{r}-1},$$

the so called *Plücker embedding*.

Definition A.2.2. The *tautological bundle* over $\mathrm{Grass}(r,s)$ is

$$T = \{(v,V) \in \mathbb{C}^s \times \mathrm{Grass}(r,s) : v \in V\}.$$

Let $F = \mathbb{C}^s \times \mathrm{Grass}(r, s)$ be the trivial bundle. There is a short exact sequence of vector bundles

$$0 \longrightarrow T \longrightarrow F \longrightarrow Q \longrightarrow 0 \tag{A.17}$$

and Q is the *tautological quotient bundle*. The whole sequence (A.17) is referred to as the *tautological sequence* over $\mathrm{Grass}(r, s)$.

Note that Q is always globally generated: If we let

$$e_1, \ldots, e_s \in H^0(\mathrm{Grass}(r, s), F)$$

be the global sections coming from the standard basis of $\mathbb{C}^s$, then by construction the residue classes $[e_i] \in H^0(\mathrm{Grass}(r, s), Q)$ span all fibers of Q.

If we dualize (A.17), we obtain a short exact sequence

$$0 \longleftarrow T^\vee \longleftarrow F^\vee \longleftarrow Q^\vee \longleftarrow 0\ .$$

Now $F^\vee \cong F$ is self-dual in a canonical way and we obtain a map

$$D : \mathrm{Grass}(r, s) \to \mathrm{Grass}(s - r, s), \quad V \mapsto V^\perp = \ker((\mathbb{C}^s)^\vee \to V^\vee).$$

It is now not difficult to see the following.

Lemma A.2.3. *The map D is an isomorphism of complex manifolds. In particular the tautological sequence over* $\mathrm{Grass}(s - r, s)$ *is isomorphic to the dual of the tautological sequence over* $\mathrm{Grass}(r, s)$.

In this sense we will also say that $\mathrm{Grass}(s - r, s)$ is the *dual Grassmannian* of $\mathrm{Grass}(r, s)$ and write $\mathrm{Grass}(s-r, r) = \mathrm{Grass}(r, s)^\vee$. Since duality is symmetric, the same holds the other way around. From Lemma (A.2.3) it is clear that the standard coordinates of $\mathrm{Grass}(r, s)$ should induce another collection of standard coordinates on $\mathrm{Grass}(r, s)^\vee = \mathrm{Grass}(s - r, s)$. We will describe these coordinates now and refer to them as the *standard coordinates of the dual Grassmannian* $\mathrm{Grass}(r, s)^\vee$.

If we consider $\mathrm{Grass}(s - r, s)$ as $\mathrm{Grass}(r, s)^\vee$, then we mean the the set of $(s - r)$-planes in $(\mathbb{C}^s)^\vee$, we can represent every $W \in \mathrm{Grass}(s - r, s)$ by a matrix $B \in \mathrm{Mat}(s - r, s)$, the *rows* of which span W, with two matrices B and B' being equivalent, if there is an invertible matrix $S \in \mathrm{GL}(s - r; \mathbb{C})$ such that $B' = S \cdot B$.

Given two spaces $V \subset \mathbb{C}^s$ of dimension r and $W \subset (\mathbb{C}^s)^\vee$ of dimension $s - r$ represented by matrices $A \in \mathrm{Mat}(s, r; \mathbb{C})$ and $B \in \mathrm{Mat}(s - r, s; \mathbb{C})$ respectively, we have

$$D(V) = W \quad \Leftrightarrow \quad B \cdot A = 0. \tag{A.18}$$

For an ordered multiindex $I \subset \{1, \ldots, s\}$ with $\#I = r$ let V_I be the open set of $\mathrm{Mat}(s, s - r; \mathbb{C})$, on which the maximal minor with column indices *not* in I does not vanish. If $I = (1, \ldots, r)$, then we can normalize to the following form:

$$\Theta^I := \begin{pmatrix} x^I_{1,1} & x^I_{1,2} & \cdots & x^I_{1,r} & -1 & 0 & \cdots & 0 \\ x^I_{2,1} & x^I_{2,2} & \ddots & \vdots & 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x^I_{s-1,r} & \vdots & \ddots & \ddots & 0 \\ x^I_{s-r,1} & \cdots & x^I_{s-r,2} & x^I_{s-r,r} & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{A.19}$$

For arbitrary $I \subset \{1, \dots, s\}$ with $\#I = r$ let again $J = \{1, \dots, s\} \setminus I$ be the ordered multiindex complementary to I and $K = (1, \dots, s-r)$. Then the submatrix $\Theta^I_{K,J}$ of Θ^I is equal to $-\mathbf{1}_{s-r}$ and the other submatrix $X^I := \Theta^I_{K,I}$ gives the coordinates $(x^I_{k,i})_{k \in K,\, i \in I}$.

Now consider the charts U_I of $\mathrm{Grass}(r,s)$ and V_I of $\mathrm{Grass}(r,s)^\vee$. Let $J \subset \{1, \dots, s\}$ be the ordered multiindex complementary to I as above. From (A.18) we have

$$\Theta^I \cdot \Xi^I = \Theta^I_{K,I} - \Xi_{J,K} = X^I - Z^I = 0 \in \mathrm{Mat}(s-r, r; \mathbb{C}).$$

In other words coordinates in the block matrices

$$X^I = \Theta^I_{K,I} = \Xi^I_{J,K} = Z^I \tag{A.20}$$

are equal. From this we see that the standard coordinates $(x^I_{k,i})$ on the chart U_I of $\mathrm{Grass}(r,s)$ can be identified with the standard coordinates $(z^I_{j,k})$ on the chart V_I of $\mathrm{Grass}(r,s)^\vee$ via In the (k,l)-th entry of this resulting matrix we find

$$x^I_{k,i_k} - z^I_{j_l,l} = 0 \quad \Leftrightarrow x^I_{k,i_k} = z^I_{j_l,l}, \tag{A.21}$$

where i_k denotes the k-th entry of the ordered multiindex I and j_l the l-th entry of its complementary index J.

We extract the consequences of these observations for the tautological bundle in a lemma.

Lemma A.2.4. *Let $(U_I)_{I \subset \{1,\dots,s\},\, \#I=r}$ be the standard cover of $\mathrm{Grass}(r,s)$, Ξ^I as in (A.14) and Θ^I as in (A.18) with the canonical identification of the coordinate functions (A.21).*

For any I the tautological bundle T over U_I takes the form

$$T|_{U_I} = \{(p,v) \in U_I \times \mathbb{C}^s : \Theta^I(p) \cdot v = 0\}. \tag{A.22}$$

If we let J be the ordered multiindex complementary to I we can decompose v into v_I and v_J. Expanding the defining equations for $T|_{U_I}$ we obtain

$$v_J = X^I(p) \cdot v_I$$

and hence the components of v_I give a local trivialization of T on U_I.

A.2.2 Generalized Nash-blowups

Let (X,p) be the germ of a complex space, X a representative of (X,p) and $\mathcal{G}$ a coherent sheaf on X. By definition we can find a presentation

$$0 \longleftarrow \mathcal{G}_p \longleftarrow \mathcal{O}^s_{X,p} \xleftarrow{A} \mathcal{O}^t_{X,p} \tag{A.23}$$

of the stalk $\mathcal{G}_p$ of $\mathcal{G}$ at p with some matrix $A \in \mathrm{Mat}(s,t;\mathcal{O}_{X,p})$. Let

$$r = \min\{t \in \mathbb{N} : \langle A^{\wedge t}\rangle \neq \langle 0\rangle\} - 1$$

be the maximal rank of A at p. and $U = X \setminus V(\langle A^{\wedge r}\rangle)$ the set of points $x \in X$, where $\operatorname{rank} A(x) = r$. Clearly U is analytic, open and nonempty. On U we can define the map

$$\Psi_{\mathcal{G}} : U \to \mathrm{Grass}(r,s), \quad x \mapsto \operatorname{span} A(x). \tag{A.24}$$

Let $\Gamma(U, \Psi_{\mathcal{G}}) \subset X \times \mathrm{Grass}(r,s)$ be the graph of $\Psi_{\mathcal{G}}$ and $\pi : X \times \mathrm{Grass}(r,s) \to X$ the projection to X.

Definition A.2.5. The generalized Nash-blowup of X along $\mathcal{G}$ at p is defined as

$$Y = \overline{\Gamma(U, \Psi_{\mathcal{G}})} \subset X \times \mathrm{Grass}(r,s).$$

The set $E := \pi^{-1}(X \setminus U)$ is called the *exceptional set* of the blowup. By $(Y, \pi^{-1}(\{p\})$ and $(E, \pi^{-1}(\{p\}))$ we denote the germs of Y and E along the compact set $\pi^{-1}(\{p\}) \subset \{p\} \times \mathrm{Grass}(r,s)$.

Clearly, the germs $(Y, \pi^{-1}(\{p\})$ and $(E, \pi^{-1}(\{p\}))$ are independent of the representative X of (X,p).

In the coordinates of the Grassmannian introduced above we can give equations for $Y \subset (X,p) \times \mathrm{Grass}(r,s)$ in $\mathcal{O}_{X,p}\hat{\otimes}_{\mathbb{C}}\mathcal{O}_{\mathrm{Grass}(r,s)}$. For an ordered multiindex $I \subset \{1,\dots,s\}$, $\#I = r$ and the chart $(X,p) \times U_I$ of $(X,p) \times \mathrm{Grass}(r,s)$ we could built the composite matrix $(\Xi^I|A)$. Now a point $(x,W) \in X \times \mathrm{Grass}(r,s)$ is in Y if and only if $\operatorname{span} A \subset W$, so the equations

$$\left(\Xi^I|A\right)^{\wedge r+1} = 0 \tag{A.25}$$

have to hold along Y. On the other hand using the duality in Lemma A.2.3 and the explicit description of the tautological bundle, we can also require

$$\Theta^I \cdot A = 0. \tag{A.26}$$

It should be pointed out that these equations do not necessarily generate the ideal of Y in the respective chart. To obtain it in general, one needs to saturate with respect to $A^{\wedge r}_{I,J}$:

$$\langle \Theta^I \cdot A\rangle : (A^{\wedge r}_{I,J})^{\infty} \subset \mathcal{O}_{X,p}[Z^I]. \tag{A.27}$$

The same holds for the other choice of the equations. However, in both cases the equations (A.25) and (A.26) are contained in the respective ideals defining Y.

Let Q be the tautological quotient bundle from $\mathrm{Grass}(r,s)$ over $(Y, \pi^{-1}(\{p\}))$. It is a locally free coherent sheaf on Y, which is globally generated by $[e_1],\dots,[e_s]$ as above. Denote by π_*Q the pushforward to X. Since π is proper, π_*Q is a coherent sheaf of $\mathcal{O}_X$-modules. In particular the stalk $(\pi_*Q)_p$ is a finitely generated $\mathcal{O}_{X,p}$-module. We claim that there is a natural map

$$\overline{\pi}_* : \mathcal{G}_p \to (\pi_*Q)_p \tag{A.28}$$

on the stalks of $\mathcal{G}$ and $\pi_* Q$ induced from the following map of free modules:

$$\varphi : \mathcal{O}^s_{X,p} \to (\pi_* Q)_p, \quad (v_1, \ldots, v_s) \mapsto \sum_{i=1}^{s} v_i \cdot [e_i].$$

To see this, assume $(a_1, \ldots, a_s)^T$ is a column of the matrix A. Let U_I be one of the standard charts of $(X, p) \times \mathrm{Grass}(r, s)$. We may assume $I = (1, \ldots, r)$. Let Ξ^I be as in (A.14). Since the equations (A.25) vanish on $Y \cap U_I$, the element $\varphi(a_1, \ldots, a_s)^T \in \mathcal{O}^s_Y$ is in the span of Ξ^I, i.e. $\varphi(a_1, \ldots, a_s)^T \in H^0((X, p) \times U_I, T)$ is a section in the tautological bundle. Hence by the definition of the tautological quotient bundle, $[\varphi(a_1, \ldots, a_s)^T] = [0] \in Q$ in all charts. Thus $\overline{\pi}_*$ in (A.28) is well defined.

Definition A.2.6. The bundle Q over the Nash-blowup $(Y, \pi^{-1}(\{p\})$ of (X, p) along $\mathcal{G}$ is the *Nash bundle* of the blowup. We call $\overline{\pi}_*$ in (A.28) the *Nash homomorphism*.

From the above said and the explicit coordinates given, it is now not difficult to deduce the following lemma.

Lemma A.2.7. *Let $(Y, \pi^{-1}(\{p\})) \subset (X, p) \times \mathrm{Grass}(r, s)$ be the generalized Nash-blowup of (X, p) along $\mathcal{G}$ and Q the Nash bundle over $(Y, \pi^{-1}(\{p\})$. Over $U = \{\mathrm{rank}\, A = r\} \subset X$ the projection $\pi : Y \to X$ is an isomorphism and so is $\overline{\pi}_* : \mathcal{G}_q \to (\pi_* Q)_q$ for all $q \in U$.*

Remark A.2.8. The main application of the generalized Nash-blowup is to replace a coherent sheaf by a vector bundle. In case $A = (a_1, \ldots, a_r)^T$ is a $r \times 1$-matrix, we obtain the classical blowup of the ideal $\langle a_1, \ldots, a_r \rangle$.

A.3 Whitney Stratifications and Morse Theory

In this section we will give the definitions and main results concerning Whitney stratifications and Stratified Morse Theory as it can be found in [31]. Not only the results but also the exhibition of the subject can be found there. We include it in order to provide enough background for the reader to follow the outlines in this thesis, especially Chapter 2. In the end we proof a Corollary concerning Morse functions on manifolds with boundary.

The reader is assumed to be familiar with classical Morse theory (cf. [52]). That is, if we for example write "for some small $\varepsilon > 0$" at some point, we assume the reader to recognize the precise conditions for "small" from the context. Having done this, we apply the theory to manifolds with boundary.

A.3.1 Whitey stratified Sets and Thom's First Isotopy Lemma

We give the definition of a Whitney stratification from [31].

Definition A.3.1 ([31])**.** Let X be a closed subset of a smooth manifold M and suppose that

$$X = \bigcup_{i \in I} S_i$$

is a locally finite decomposition of Z into pairwise disjoint subsets S_i, called strata, such that each S_i is a locally closed submanifold of M and the boundary

$$\partial S_i = \overline{S_i} \setminus S_i \subset M$$

is again a union of strata of lower dimension.

The stratification is said to satisfy Whitney's conditions A and B, if the following holds. Suppose y_n is a sequence in S_i converging to a point $p \in S_j \subset \partial S_i$ and z_n a sequence in S_j converging to the same point. Fix a local coordinate system of M around p and let l_n be the secant line from z_n to y_n. Suppose l_n converges to a limit line $l \in \mathrm{Grass}_{\mathbb{R}}(1, \dim M)$ in the real Grassmannian and the sequence of tangent spaces $T_{y_n} S_i \subset T_{y_n} M$ converges to a limit $T \in \mathrm{Grass}_{\mathbb{R}}(\dim S_i, \dim M)$. Then the Whitney conditions are:

A The tangent space of S_j at the limit point $T_p S_j$ is contained in T.

B Also the limit line l is contained in T.

By definition a function $f : X \to \mathbb{R}$ on a Whitney stratified space X is smooth at a point $p \in X$, if for some embedding of a neighborhood $U \subset X$ as a Whitney stratified subspace of $\mathbb{R}^N$, f can be given as the restriction of a smooth function on $\mathbb{R}^N$ to U.

We now describe Thom's First Isotopy Lemma as stated in [31]. Let $X \subset M$ be a Whitney stratified set. A smooth function $f : M \to \mathbb{R}^n$ is called a *stratified submersion* if the restrition of f to all strata S_i of X is a submersion.

Theorem A.3.2 (Thom's First Isotopy Lemma, [31]). *Let $f : X \to \mathbb{R}^n$ be a proper stratified submersion on a Whitney stratified set X. Then there is a stratum preserving homeomorphism,*

$$h : X \to \mathbb{R}^n \times f^{-1}(\{0\}) \cap X$$

which is smooth on each stratum and commutes with the projection to $\mathbb{R}^n$. In particular the fibers of $f|_X$ are homeomorphic by a stratum preserving homeomorphism.

A.3.2 Stratified Morse Theory

Definition A.3.3 ([31]). Let X be a Whitney stratified space and $f : X \to \mathbb{R}$ a proper smooth function. We say that f is a Morse function on X, if the following holds:

i) All critical values of the restriction of f to a stratum $\Sigma^{(i)}$ of X are distinct.

ii) At each critical point $p \in \Sigma^{(i)} \subset X$ of f, the Hessian of f on $\Sigma^{(i)}$ is nondegenerate at p.

iii) The differential $\mathrm{d}f(p)$ does not annihilate any limit of tangent spaces of strata $\Sigma^{(j)}$ with $p \in \overline{\Sigma}^{(j)}$.

The *Morse data* of a Morse function f on X at a critical point $p \in \Sigma^{(i)} \subset X$ is defined as follows. For $c \in \mathbb{R}$ let $X_{\leq c} = \{x \in X : f(x) \leq c\}$. Choose some local embedding $(X, p) \hookrightarrow (\mathbb{R}^N, 0)$ as a Whitney stratified subspace and let D be a small ball around p in $\mathbb{R}^N$. If $p \in X$ is a critical point of f and $v = f(p)$ its critical value, then for some small $\varepsilon > 0$, the space

$$A = X_{\leq v+\varepsilon} \cap D \setminus X_{<v-\varepsilon} \cap D$$

is attached to $X_{\leq v-\varepsilon}$ as we cross the value v. The glueing of A happens along the locus

$$B = X_{v-\varepsilon} \cap D = \{x \in X : f(x) = v - \varepsilon\} \cap D.$$

Now the Morse data is given by the pair of spaces (A, B) up to homotopy equivalence. It measures the change in topology of $X_{\leq c}$ as c crosses v.

We have the following two theorems from [31].

Theorem A.3.4 (Stratified Morse Theory Part A,[31])**.** *As c varies within the open interval between two adjacent critical values, the topological type of $X_{\leq c}$ remains constant.*

This is, of course, merely a consequence of Thom's First Isotopy Lemma A.3.2. The interesting part for Morse theory is the following.

Theorem A.3.5 (Stratified Morse Theory Part B,[31])**.** *Let f be a Morse function on a Whitney stratified space X. Then, Morse data measuring the change in the topological type of $X_{\leq c}$ as c crosses the critical value v of the critical point p, is the product of the normal Morse data at p and the tangential Morse data at p.*

The normal and tangential Morse data mentioned in Theorem A.3.5 are defined as follows. For the tangential Morse data we just consider the Morse data of f restricted to the stratum $\Sigma^{(i)}$, in which the critical point p lives.

For the normal Morse data we choose some embedding $(X, p) \hookrightarrow (\mathbb{R}^N, 0)$ and take a hyperplane slice $N(p)$ through p transversal and of complementary dimension to $\Sigma^{(i)}$. Let $D \subset N(p)$ be a small disc centered around p. The normal Morse data of f at p is defined as the pair of spaces (A, B), where

$$A = D \cap \{x \in X : v - \varepsilon \leq f(x) \leq v + \varepsilon\}$$

and

$$B = D \cap \{x \in X : f(x) = v - \varepsilon\}$$

for some small $\varepsilon > 0$. Again, (A, B) is only considered up to homotopy equivalence.

We would like to apply Stratified Morse Theory to manifolds with boundary. Let M be a differentiable manifold of real dimension m with boundary ∂M. We will always assume that there exists a small extension M' of M beyond the boundary ∂M so that every point $p \in \partial M$ has a coordinate neighborhood in M' with coordinates $x_1, \ldots, x_m$ such that $x_1(p) = \cdots = x_m(p) = 0$, ∂M is given by $\{x_m = 0\}$ and $M = \{x_m \leq 0\}$. We will consider M as a stratified space with strata ∂M as one stratum and the interior $M \setminus \partial M$ as the other.

Let $p \in \partial M$ be a critical point of a Morse function f on the Whitney stratified space M. By Definition A.3.3 p cannot be a critical point on some extension M' of M at p since this would violate iii). So p is a critical point of f restricted to ∂M.

In a coordinate system as above on M' we use the classical Morse Lemma to change the coordinates $x_1, \dots, x_{m-1}$ in a way that

$$f|_{\partial M} : x \mapsto -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m-1} x_i^2.$$

Here, λ is the *Morse index* of $f|_{\partial M}$ at p. Outside ∂M we may have a difference

$$g = f - \left(-\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m-1} x_i^2 \right).$$

Now clearly

$$\frac{\partial g}{\partial x_m}(p) = \frac{\partial f}{\partial x_m}(p) =: a \neq 0$$

since f does not have a critical point at p on M'. We say that "the gradient of f points outwards" if a is positive. Otherwise we say that "the gradient of f points inwards". We again change coordinates on M' replacing x_m by g if $a > 0$, or, in case $a < 0$, by $-g$. Note that with this choice we still have $\partial M = \{x_m = 0\}$ and $M = \{x_m \leq 0\}$. In this coordinate system the function f finally takes the form

$$f = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m-1} x_i^2 \pm x_m.$$

Theorem A.3.6. *Let $p \in \partial M$ be a critical point with Morse index λ of the Morse function f on the manifold with boundary M. If the gradient of f is pointing outwards, then the Morse data of f at p is given by its tangential and normal parts*

$$\begin{aligned}
(A, B) &\cong (D^\lambda, S^{\lambda-1}) \times ([-1, 0], \{-1\}) \\
&= (D^\lambda \times [0, 1], D^\lambda \times \{-1\} \cup S^{\lambda-1} \times [-1, 0]) \\
&\cong (\{\mathrm{pt}\}, \{\mathrm{pt}\}).
\end{aligned}$$

If the gradient of f is pointing inwards, the Morse data is

$$\begin{aligned}
(A, B) &\cong (D^\lambda, S^{\lambda-1}) \times ([-1, 0], \emptyset) \\
&= (D^\lambda \times [-1, 0], S^{\lambda-1} \times [-1, 0]) \cong (D^\lambda, S^{\lambda-1}).
\end{aligned}$$

Consequently, as c crosses the critical value v, the topological type of $M_{\leq c}$ does not change in the first case and in the second a cell of real dimension λ is attached.

Here, D^λ denotes the closed ball of radius 1 in $\mathbb{R}^\lambda$ and $S^{\lambda-1}$ the sphere of dimension $\lambda - 1$.

Proof. In the coordinate system introduced above the tangential Morse data comes from the classical Morse data of the function $f|_{\partial M}$. The results can be found in [52]. For the computation of the normal Morse data we may restrict f to the set $\{x_1 = \cdots = x_{m-1} = 0\}$. The result follows from Theorem A.3.5. □

A.4 Complex Ordinary Differential Equations

While the basic theory of ordinary differential equations is contained in any undergraduate textbook on analysis, there seems to be no good reference for holomorphic or *complex* differential equations and the holomorphicity of their solutions. Hence we include them in this appendix. A good source for the non-holomorphic theory of the differential equations considered here is [56] from which we also adapt the exposition and formulation of the problems.

An ordinary differential equation is of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = F(u(t), t) & |t| \leq \varepsilon \\ u(0) = u_0 \end{cases} \tag{A.29}$$

where the function $u : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to V$ takes values in a Banach space V. The Picard Lindelöf Theorem assures existence and uniqueness of the solution under the assumption that F is Lipschitz-continuous in the first argument.

We are often confronted with the situation that we have an equation of this form, only that t is a complex parameter, u takes values in a Hilbert space H, and F is holomorphic function in both variables. This we call a *complex ordinary differential equation* (cODE).

In the real differentiable case it is a subtle amendment that the solution u of (A.29) depends smoothly on the initial condition v. We are interested more generally in the case where the right hand sides of (A.29) depend holomorphically on finitely many complex parameters x.

Theorem A.4.1. *Suppose $W \subset \mathbb{C}^k$, $B \subset \mathbb{C}^N$ and $D \subset \mathbb{C}$ are open sets and*

$$F : W \times B \times D \to \mathbb{C}^N, \quad v : W \to B \subset \mathbb{C}^N$$

are holomorphic functions. Consider the differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(x,t) & = F(x, u(t), t) \\ u(x, 0) & = v(x) \end{cases} \tag{A.30}$$

for $u : W \times D \to \mathbb{C}^N$. Then, for each $x_0 \in W$ and $t_0 \in D$ there are open neighborhoods W' and D' respectively and a unique solution

$$u : W' \times D' \to \mathbb{C}^N$$

of (A.30) depending holomorphically on both variables.

Corollary A.4.2. *In case $F : W \subset \mathbb{C}^N \to \mathbb{C}^N$ is a holomorphic vector field on $\mathbb{C}^N$, the flow Φ associated to F is holomorphic.*

Proof. Just let $v(x) = x$ be the dependence on the initial condition. □

For the proof of Theorem (A.4.1) we merely have to go through the standard proof of the Picard-Lindelöf Theorem as for example in [56] and make sure that it translates well to the realm of holomorphic functions. In fact the holomorphic dependence of the solutions on parameters of the equation becomes even easier compared to the smooth case. This is due to the following well known theorem.

Theorem A.4.3. *Let $f_n : U \subset \mathbb{C}^N \to \mathbb{C}$ be a sequence of holomorphic functions which converge uniformly to a function $f : U \to \mathbb{C}$ on every compact subset $K \subset U$. Then the limit f is also holomorphic.*

For the sake of completeness, we include a proof.

Proof. We use the Cauchy Integral Formula in several variables. Let $z = (z_1, \dots, z_N)$ be a point in U and Δ_ε a polydisc of polyradius $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ around z in U. Then, since all the f_n are holomorphic we have

$$f_n(z) = \int_{|\xi_1 - z_1| = \varepsilon_1} \cdots \int_{|\xi_N - z_N| = \varepsilon_N} \frac{f_n(\xi_1, \dots, \xi_N)}{(\xi_1 - z_1) \cdots (\xi_N - z_N)} \, \mathrm{d}\xi_N \cdots \mathrm{d}\xi_1.$$

If we let $n \to \infty$, we can, because of the compact domain of integration and uniform convergence of the f_n, take the limit below the integral on the right hand side. Thus

$$f(z) = \int_{|\xi_1 - z_1| = \varepsilon_1} \cdots \int_{|\xi_N - z_N| = \varepsilon_N} \frac{f(\xi_1, \dots, \xi_N)}{(\xi_1 - z_1) \cdots (\xi_N - z_N)} \, \mathrm{d}\xi_N \cdots \mathrm{d}\xi_1$$

and the right hand side depends holomorphically on all the z_i. □

Proof. (of Theorem A.4.1). Given $x_0 \in W$ and $t_0 \in D$, we construct a zeroth and a first approximation to the solution by setting

$$u^{(0)}(x,t) = v(x), \quad u^{(1)}(x,t) = v(x) + \int_{t_0}^{t} F(x, v(x), \tau) \, \mathrm{d}\tau$$

for $x \in W' \subset W$ and $t \in D' \subset D$ some neighborhoods of x_0 and t_0. Observe that the integral is defined via the choice of a path γ in D' from t_0 to t. Since the integrand depends holomorphically on τ, the result only depends on the homotopy class of the path γ. In order to have a well defined integral we therefore assume D' to be star-shaped around t_0 in what follows.

The function $u^{(1)} : W \times D \to \mathbb{C}^N$ takes values in $\mathbb{C}^N$. We would like to iterate the process and define

$$u^{(n+1)}(x,t) = v(x) + \int_{t_0}^{t} F(x, u^{(n)}(x,\tau), \tau) \, \mathrm{d}\tau.$$

But for this to work, $u^{(n)}$ has to actually take values in B, where F is defined.

To this end choose a neighborhood $B' \subset B$ of $v(x_0$ with compact closure $\overline{B}'$. Due to the continuity of v we may then choose a neighborhood $W' \subset W$ of x_0 such that $\overline{W}' \subset v^{-1}(B')$ and also $\overline{W}'$ compact. Furthermore we take a disc $D' \subset D$ around t_0 and define

$$\begin{aligned} C &:= \max\left\{\left\|\frac{\partial F}{\partial u}(x,u,t)\right\| : (x,u,t) \in \overline{W}' \times \overline{B}' \times \overline{D}'\right\} \\ K &:= \max\{\|F(x, v(x), t\| : (x,t) \in \overline{W}' \times \overline{D}'\} \\ \delta &:= \inf\{\|u - v(x)\| : u \in \partial\overline{B}', x \in W'\}. \end{aligned}$$

Since $v(\overline{W}')$ is properly contained in B', we clearly have $\delta > 0$.

We now want to show that we can choose the radius $\varepsilon > 0$ of the disc D' so small that the iteration process for $u^{(n+1)}(x,t)$ works for all $(x,t) \in W' \times D'$. The definition of $u^{(0)}$ and $u^{(1)}$ immediately implies

$$\|u^{(1)}(x,t) - u^{(0)}(x,t)\| \leq K \cdot \varepsilon \quad (x,t) \in W' \times D'.$$

Hence, if we choose $\varepsilon < \frac{\delta}{K}$, then certainly $u^{(1)}$ takes values only in B' when restricted to $W' \times D'$.

Now suppose $u^{(k)}(x,t) \in B'$ for all $k \leq n$ and $(x,t) \in W' \times D'$.

$$\|u^{(n+1)}(x,t) - u^{(n)}(x,t)\| \leq \varepsilon \cdot C \cdot \|u^{(n)}(x,t) - u^{(n-1)}(x,t)\| \tag{A.31}$$

and, hence, iteratively

$$\|u^{(n+1)}(x,t) - v(x)\| \leq \frac{1}{1 - \varepsilon \cdot C} \cdot K \cdot \varepsilon.$$

If we choose ε so small that the right hand side is smaller than δ (which certainly implies the previous assumption $K \cdot \varepsilon < \delta$, as well), then all the $u^{(n)}$ will take values in B' only.

Several things have shown up at this point. First of all, the iteration gives a contraction from the set

$$\{u : W' \times D' \to B'\}$$

to itself, which due to the Banach Fixed Point Theorem has a unique fixed point u solving the integral equation

$$u(x,t) = v(x) + \int_{t_0}^{t} F(x, u(x,\tau), \tau)\, \mathrm{d}\tau.$$

Second, the sequence of functions $\left(u^{(n)}(x,t)\right)_{n\in\mathbb{N}}$ converge uniformly on $W' \times D'$ because of the estimate

$$\|u^{(n)}(x,t) - u^{(m)}(x,t)\| \leq \frac{(\varepsilon \cdot C)^m}{1 - \varepsilon \cdot C} \cdot K \cdot \varepsilon$$

for any $n > m$ and all the $u^{(k)}$ are holomorphic in x and t. From Theorem A.4.3 we deduce that also the limit $u : W' \times D' \to B'$ is holomorphic. This finishes the proof.

□

We now formulate some consequences for linear cODEs which we will need.

A linear complex differential equation depending on a parameter $x \in \mathbb{C}^k$ is of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(x,t) &= F(x,t) \cdot u(x,t) \\ u(x,0) &= u_0(x) \end{cases}. \tag{A.32}$$

where u takes values in $\mathbb{C}^m$ and $A : W \times D \subset \mathbb{C}^k \times \mathbb{C} \to \mathrm{Mat}(m, m; \mathbb{C})$ and $u_0 : W \to \mathbb{C}^m$ are holomorphic functions.

Theorem A.4.4. *A linear cODE as above admits a unique solution operator*

$$U : W' \times D' \to \mathrm{GL}(m, \mathbb{C})$$

such that for any given u_0 the solution is of the form

$$u(x,t) = U(x,t) \cdot u_0(x).$$

Moreover U is holomorphic in x and t and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x,t) = F(x,t)U(x,t).$$

Proof. This follows directly from Theorem A.4.1 and the superposition principle for linear differential equations. We construct the i-th column of $U(x,t)$ as the solutions of the cODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(x,t) &= F(x,t) \cdot u(x,t) \\ u(x,0) &= e_i \end{cases},$$

where e_i is the i-th vector of the standard basis of $\mathbb{C}^N$. □

A special variant of linear cODEs is the following one for matrices:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}A(x,t) &= F(x,t) \cdot A(x,t) + A(x,t) \cdot G(x,t) \\ A(x,0) &= A_0(x) \end{cases} \tag{A.33}$$

where the function A takes values in the space $\mathrm{Mat}(m, n; \mathbb{C})$ and $F(x,t) \in \mathrm{Mat}(m; \mathbb{C})$, $G(x,t) \in \mathrm{Mat}(n; \mathbb{C})$ depend holomorphically on x and t.

Corollary A.4.5. *For a matrix cODE as above there exist unique solution operators $U(x,t) \in \mathrm{GL}(m; \mathbb{C})$ and $V(x,t) \in \mathrm{GL}(n; \mathbb{C})$ such that for any $A_0(x)$ the solution is given by*

$$A(x,t) = U(x,t) \cdot A_0(x) \cdot V(x,t).$$

Proof. We obtain $U(x,t)$ as the solution operator of the linear cODE given by $F(x,t)$ as in (A.32). For the operator $V(x,t)$ we take the transpose of the solution operator of equation (A.32) with $G^T(x,t)$ in place of $F(x,t)$.

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} U(x,t) \cdot A_0(x) \cdot V(x,t) \\
= \quad & \left(\frac{\mathrm{d}}{\mathrm{d}t} U(x,t) \right) \cdot A_0(x) \cdot V(x,t) + U(x,t) \cdot A_0(x) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} V(x,t) \right) \\
= \quad & F(x,t) \cdot U(x,t) \cdot A_0(x) \cdot V(x,t) + U(x,t) \cdot A_0(x) \cdot \left(G^T(x,t) \cdot V^T(x,t) \right)^T \\
= \quad & F(x,t) \cdot A(x,t) + A(x,t) \cdot G(x,t)
\end{aligned}$$

This solution is unique because (A.33) is in particular a linear cODE if we consider A as a vector. □

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