Fundamental energy cost of quantum measurements

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Abstract

Quantum measurements are an essential ingredient in numerous protocols and quantum technologies. As a characteristic trait, quantum measurements are typically employed a vast number of times to allow for higher accuracy, better control or just increased performance of quantum devices. In this thesis, we pursue a resource-theoretic approach to quantum measurements, aiming at the quantification of their energetic cost. While there exists a plethora of literature on this topic already, we develop a general framework that allows us to unify several of these previous results and to improve them systematically. We highlight an issue common to most of these previous works, where a notion of quantum measurement is employed which is not detailed enough to be applicable to several realistic and important applications for quantum measurement. More concretely, previous work typically involves a state transformation concept, similar to thermodynamic considerations, where only a specific initial and final state of the measured system are specified to characterise the process. For real applications however, quantum measurement devices are typically required to perform the measurement for *all* possible initial states of the measured system.

In this thesis we analyse this requirement and obtain as a consequence much stronger energy constraints on quantum measurements than previously known. For particular forms of measurement, namely dephasing channels, projective quantum instruments, and projection valued measures, we even determine the *exact* energy costs in contrast to the common lower bound statements in previous literature. Remarkably, the exact cost of projective quantum instruments constitutes the strongest energy constraint on quantum measurement known so far. We demonstrate the implications of these results in two important applications, where quantum measurements play a significant role, namely quantum Zeno stabilisation and quantum error correction. The exact energetic requirements we compute for these two protocols show that any restriction on the amount of available energy immediately places fundamental constraints on the performance of these protocols.

KEYWORDS: energy cost, quantum measurement, quantum thermodynamics

Kurzzusammenfassung

Quantenmessungen sind ein essentieller Bestandteil zahlreicher Protokolle und Quantentechnologien. Charakteristisch ist die typischerweise sehr große Anzahl von Messungen, die es erlaubt höhere Genauigkeiten, bessere Kontrolle oder eine erhöhte Leistungsfähigkeit von Quantengeräten zu erzielen. In dieser Dissertation verfolgen wir einen ressourcentheoretischen Ansatz mit dem Ziel die Energiekosten von Quantenmessungen zu quantifizieren. Während schon eine Fülle von Literatur zu diesem Thema existiert, entwickeln wir ein allgemeines mathematisches Gerüst, das uns erlaubt, viele dieser bisherigen Resultate zu vereinigen und systematisch zu verbessern. Wir stellen ein Problem von vielen vorigen Arbeiten zu diesem Thema heraus, welche einen Messbegriff nutzen, der nicht detailliert genug ist um ihn für viele realistische and wichtige Anwendungen verwenden zu können. Konkret behandeln diese vorigen Arbeiten ausschließlich das Konzept der Zustandstransformation, ähnlich zu thermodynamischen Herangehensweisen, bei der nur ein bestimmter Anfangs- und Endzustand des gemessenen Systems spezifiziert wird um den Messprozess zu beschreiben. Für realistische Anwendungen jedoch muss ein Messgerät die entsprechende Messung typischerweise für *alle* Anfangszustände durchführen.

In dieser Dissertation analysieren wir diese Bedingung und erhalten dadurch deutlich stärkere Restriktionen an die Energiekosten von Quantenmessungen als vorher bekannt. Für bestimmte Messungen (dephasierende Kanäle, projektive Quanteninstrumente und projektionswertige Maße) ermitteln wir sogar die *exakten* Energiekosten im Gegensatz zu den unteren Schranken, die man in bisheriger Literatur hauptsächlich findet. Insbesondere stellen die exakten Kosten für projektive Quanteninstrumente die stärksten Bedingungen an die Energiekosten von Quantenmessungen dar, die jemals berechnet wurden. Wir demonstrieren die Auswirkungen dieser Resultate in zwei wichtigen Anwendungen, in denen Quantenmessungen eine signifikante Rolle spielen, nämlich Quanten-Zeno-Stabilisation und Quantenfehlerkorrektur. Die exakten Energiekosten, die wir für die beiden Protokolle ermitteln, zeigen, dass jedwede Beschränkung an die Menge der zur Verfügung stehenden Energie sofort zu fundamentalen Einschränkungen an die Leistungsfähigkeit dieser Protokolle führt.

SCHLAGWORTE: Energiekosten, Quantenmessung, Quantenthermodynamik

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Contents

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CHAPTER 1

Introduction

Quantum measurements are a fundamental building block of quantum theory. Any experiment that aims at testing or employing quantum mechanical systems will ultimately invoke quantum measurements in order to link the “quantum world” with the “classical world” we live in [1]. The spectrum of applications of quantum measurements is hence enormous: It ranges from quantum-cryptographic protocols, where uncertainty relations are employed to prove unprecedented levels of security [2,3], to high-precision clocks [4,5], and to entire computing devices completely based on measurements as the only way of manipulating information [6–8]. Consequently, understanding the underlying process of quantum measurement is important from both a theoretical as well as a practical viewpoint.

A feature common to most applications of quantum measurements is the large number of measurements employed. Protocols such as measurement-based quantum computation [6–8], quantum error-correcting codes [9–11], quantum control (see [12, 13] and references therein) and quantum metrology [14, 15] schemes all involve a particularly large number of measurements. This naturally raises the following question:

How many resources do we need to expend
in order to conduct a quantum measurement?

The aim of this thesis is to provide answers to this question.

To begin with we need to make the question more precise: What exactly do we mean by “quantum measurement”? And what are relevant “resources” that we have to account for? Specifying these notions is important since the definitions of both quantum measurement and resources may depend strongly on the physical situation. For example, quantum measurements may be performed only to acquire classical information about the state of the quantum system via the measurement outcomes. Typical examples are quantum state tomography [16, 17] or quantum key distribution [18, 19]. Conversely, quantum measurements are sometimes applied for the sole reason of realising a desired state change on the “measured” physical system, e.g. as in quantum control theory [20]. All these different scenarios are abstractly described by different *formalisms* of quantum measurement, but all are referred to as “quantum measurement” in the literature. In this thesis, we will consider the four most prominent formalisms, namely quantum instruments, quantum channels, unselective measurements and POVMs.

Resources correspond, roughly speaking, to those physical properties of the initial setup which are essential to be able to conduct the desired process and which are valuable in some (not yet clarified) sense. Such resources can be as general as the time duration needed to conduct the process or as specific as the required Hilbert space dimension to realise the process. It is this type of valuable resource that we aim to account for in the context of measurement. In contrast, *free resources* correspond to those physical properties which are essential, but “easy to get”. Typical examples of such free resources are e.g. product states (in the resource theory of entanglement [21,22]), or thermal states (in the resource theory of athermality [22–24].

In this thesis, the resource we care about is *energy*. The intuition behind this resource is depicted in Figure 1.1: Consider an external agent who wants to perform a quantum measurement using a particular measurement device. Let us further impose the natural assumption that this external agent has only a finite energy supply available, which is symbolised by the battery in Figure 1.1. If the act of performing a quantum measurement consumes energy, then this amount of energy must be invested by the energy supply of the agent due to energy conservation. The finite energy supply of the agent therefore constrains the number of measurements the agent can perform. This is in parallel with similar considerations made in refs. [25, 26].

To see an example, where such a constraint is important, let us consider a typical situation where measurements are employed, namely measurement-based feedback protocols [12]. These are protocols where subsequent actions depend on previously obtained measurement outcomes, e.g. adaptive measurement schemes [27, 28], quantum error correction [9–11], or measurement-based quantum computation [6–8]. Such protocols share the property that their performance typically improves with the number of measurements that are conducted. For ex-

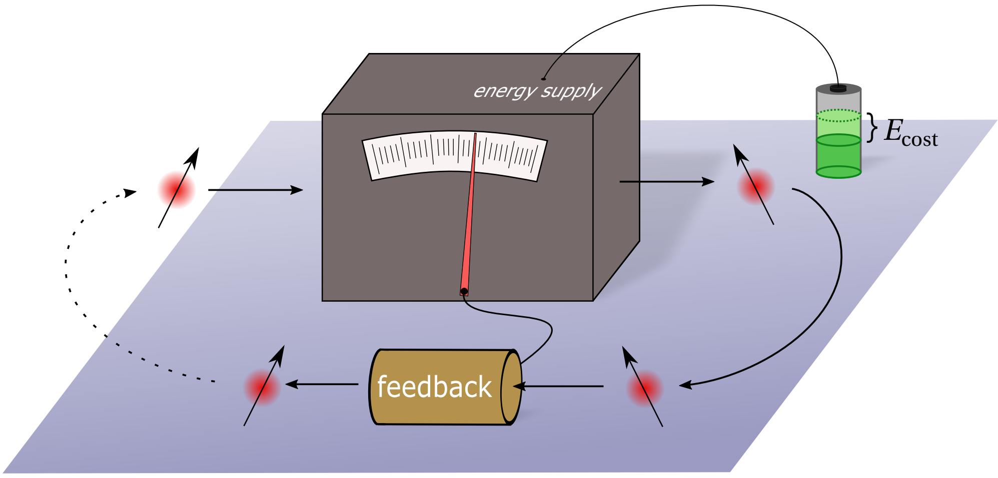

Figure 1.1: Applying a quantum measurement (grey box) on a quantum system (red circle) consumes an amount E_{cost} of energy (green battery). Such measurements are important to acquire classical information about the quantum system and/or to induce a state change on the measured system [29].

ample, quantum error correction codes usually yield better protection of quantum information against external noise when they are being concatenated [11], which requires an exponentially increasing amount of measurements to be performed. Following our reasoning from above, we therefore find that (even a tiny amount of) energy consumption for quantum measurements places fundamental and relevant restrictions on the performance of quantum error correction. Clearly, such limitations will arise for any protocol that benefits from a larger number of performable measurements.

Our interest in the energy cost of quantum measurements requires us to keep track of *all* energy expenses that arise in the process. This cannot be achieved if our description of the process is restricted to the abstract formalisms for measurement mentioned above, because these formalisms do not incorporate all physical systems that necessarily need to be involved in a quantum measurement.

In the first part of this thesis, we therefore develop a general framework for quantum measurement which properly includes *all* energy changes. The framework is centered around a purely quantum-mechanical, *microscopic* description of the abstract formalisms and specifies all physically relevant details of the measurement process. We refer to such a microscopic description as an *implementation* of the quantum measurement. In general, our framework admits many implementations of the same quantum measurement; this is in parallel with the existence of many physical measurement devices which perform the same quantum measurement.

Each implementation has its own specific energy expense. The energy cost of a quantum measurement therefore depends on the particular implementation employed. In this thesis however, we are interested in the *fundamental* energy requirements for quantum measurements, i.e. we compute the *minimal* energy expense over all possible implementations of a quantum measurement.

Our microscopic description of the measurement process also allows us to determine *thermodynamic* properties of quantum measurements, such as the resulting heat dissipation. This is not totally surprising since various aspects of thermodynamics can be understood as the resource theory of energy. In the case of state transformations (one of the four formalisms mentioned above) for example, our framework admits a reformulation of Landauer's famous principle [30–38], which posits a minimal amount of heat to be dissipated into an environment at temperature T whenever a quantum system experiences an entropy decrease,

$$\Delta Q \geq k_B T\big(S(\rho_S) - S(\rho'_S)\big)\,,$$

where ΔQ is the dissipated heat, $S(\rho_S)$ is the (von Neumann) entropy of the initial state ρ_S of the quantum system S, and $S(\rho'_S)$ the entropy of the final state ρ'_S of S. This principle is of fundamental importance: it provides a link between thermodynamics and information theory, it was employed to exorcise Maxwell's demon [32, 39, 40], and it is even believed to put severe limitations on computing technology within a few decades [41,42]. On the basis of ref. [38] we will show how Landauer's principle can be formulated and proven within our general framework. However, we also obtain drastic improvements of Landauer's principle for the other three, more detailed, formalisms of quantum measurement.

Indeed, several improvements or critical assessments of Landauer's principle have been formulated already (see e.g. [32, 34, 35, 38] and references therein). Most of these works are however only centered around the "state transformation idea", which is common to thermodynamic considerations. The state transformation formalism of measurement is however generally not detailed enough for important applications such as quantum state tomography or quantum error correction. Instead, these measurements must be properly described using the other three formalisms (e.g. POVMs for tomography, and quantum instruments for error correction [43]). These formalisms of measurement, in contrast to state transformations, determine the measurement process for *all* possible initial states of the measured system. To include this much stronger requirement into our description of quantum measurement is one of the main novelties of this work [44,45], and the strength of our results is largely based on exactly this requirement.

Our resource-theoretic approach is related to the field of quantum thermodynamics [46, 47] in various other aspects: For example, our framework will treat thermal states as free resources, e.g. just as in the theory of thermal operations (see e.g. [23, 48, 49]). The underlying assumption here is that thermal states are easily obtained if needed for the measurement process. This assumption is commonly made, but the question of whether, how and how fast systems actually thermalise is an ongoing field of research (see e.g. [46, 50–53]). Another example is that the energy cost we determine is directly related to the minimal *work cost* of quantum measurements as we explain in this thesis.

In summary, this thesis establishes a framework which allows us to determine the fundamental energy (or work) cost and heat dissipation of four different formalisms of quantum measurement. The results we obtain are significantly stronger than all previous predictions within the current literature and put severe limitations on the performance of, e.g., measurement-based feedback protocols. We exemplify this in detail for two relevant applications, namely quantum Zeno stabilisation and quantum error correction.

In the following chapter, we briefly review the basic mathematical notions needed to understand our framework and the results of this thesis.

In Chapter 3 we first review the four pre-eminent formalisms of quantum measurement mentioned above. We then introduce and explain in detail our framework which we use to develop our results. Additionally, we discuss the connection of our framework to thermodynamic notions.

Chapter 4 presents the strongest statements about the energy cost of quantum measurements currently known. This includes previous work, but most of these statements are novel contributions of this thesis. We clearly indicate when a particular statement has been shown in previous work. In particular, some of the results of this thesis were already published on the arXiv pre-print server [29]. Furthermore, we conclude Chapter 4 with a summary of the main works of this field of research.

In Chapter 5 we provide an analysis of the consequences of our theoretical results for two applications, namely quantum Zeno stabilisation and quantum error correction.

Finally, in Chapter 6 we present a summary of this thesis as well as a set of open questions.

CHAPTER 2

Mathematical preliminaries

In this chapter we collect and review some basic mathematical definitions and statements needed to understand the content of this thesis. Moreover, this chapter serves the purpose to fix some of the notation we will use extensively in the remainder of this thesis.

Most statements and definitions are standard textbook material. The presentation here is largely based on two such textbooks [43, 54] and lecture notes [55]. This is also where proofs can be found.

Operators on Hilbert spaces. A (finite dimensional) *Hilbert space*[1] $\mathcal{H}$ is a complex inner product space, i.e. a vector space over the complex field $\mathbb{C}$ equipped with an inner product, which is a map $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is linear in the second argument, is positive-definite, i.e. $\langle\psi|\psi\rangle > 0$ for all $0 \neq \psi \in \mathcal{H}$, and obeys conjugation symmetry, i.e. $\overline{\langle\phi|\psi\rangle} = \langle\psi|\phi\rangle$ for all $\psi, \phi \in \mathcal{H}$.

In this thesis we will employ the Dirac notation: This means that we assign to any vector $\psi \in \mathcal{H}$ a linear map $|\psi\rangle : \mathbb{C} \to \mathcal{H}$ that maps a complex number z to the vector $z\psi \in \mathcal{H}$. Additionally, we define for each vector $\psi \in \mathcal{H}$ the linear map $\langle\psi| : \mathcal{H} \to \mathbb{C}$ through the assignment $\phi \mapsto \langle\psi|\phi\rangle$ for all $\phi \in \mathcal{H}$, i.e. it maps each vector ϕ to its inner product with the vector ψ.

An *operator* between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is a linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. An operator *on* a Hilbert space $\mathcal{H}$ is a linear map from $\mathcal{H}$ to $\mathcal{H}$. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

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[1] Throughout this thesis all Hilbert spaces are finite dimensional such that we will not have to care about the different notions of limits and convergence essential in the treatment of infinite dimensional Hilbert spaces.

the set of all operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ and by $\mathcal{B}(\mathcal{H})$ the set of all operators on $\mathcal{H}$. Moreover, we denote by $\mathbb{1}_{\mathcal{H}}$ the *identity* operator on $\mathcal{H}$, which maps each vector on $\mathcal{H}$ to itself, i.e. $\mathbb{1}_{\mathcal{H}}\psi = \psi$ for all $\psi \in \mathcal{H}$. For a given operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we define the *adjoint* operator $A^\dagger \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ via

$$\langle \phi | A^\dagger \psi \rangle = \langle A\phi | \psi \rangle \qquad \forall \phi \in \mathcal{H}_1, \psi \in \mathcal{H}_2 \,.$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is called *normal* if $A^\dagger A = AA^\dagger$ and *hermitian* if $A = A^\dagger$. An operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *partial isometry* if $A = AA^\dagger A$ and an *isometry* if $A^\dagger A = \mathbb{1}_{\mathcal{H}_1}$. An isometry A is called *unitary* if additionally $AA^\dagger = \mathbb{1}_{\mathcal{H}_2}$. In this thesis we will typically denote unitary operators by the letter U. Since unitary operators are invertible ($U^{-1} = U^\dagger$), the operators A and $UAU^\dagger$ have the same spectrum.

An operator $A \in \mathcal{B}(\mathcal{H})$ is called *positive*, written as $A \geq 0$, if it satisfies one of the following equivalent properties: (1) $\langle \psi | A\psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$, (2) $A = B^\dagger B$ for some operator $B \in \mathcal{B}(\mathcal{H})$, or (3) A is hermitian and has non-negative spectrum. If a positive operator $A = B^\dagger B$ is the null operator $A = 0$, then also $B = 0$, since the singular values of B are the square roots of the eigenvalues of A, which are all zero. We say that an operator $A \in \mathcal{B}(\mathcal{H})$ is greater than or equal to an operator $B \in \mathcal{B}(\mathcal{H})$, written as "$A \geq B$", if $A - B$ is positive.

The *adjoint* T^* of a linear map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ between operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined by the linear map $T^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ that satisfies $\mathrm{tr}[T(A)B^\dagger] = \mathrm{tr}[A(T^*(B))^\dagger]$ for all $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Here, "$\mathrm{tr}[A]$" denotes the *trace* of a linear operator $A \in \mathcal{B}(\mathcal{H})$, which can be computed by adding up the diagonal entries of a matrix representing A.

Note that ψ and $|\psi\rangle$ are equivalent (up to a complex factor). In order to avoid confusion, we will from now on always write $|\psi\rangle$ to specify an element of a Hilbert space (the letter ψ will denote pure states in the following, see below).

Dynamics of quantum systems. We will later see that quantum measurements (defined in Chapter 3) will involve several *physical systems*. We label such systems by capital letters just as operators, but it will always be clear from the context whether we refer to a physical system or to an operator. To each physical system we associate a Hilbert space, where an additional subscript indicates the system, i.e. for example $\mathcal{H}_S$ for system S. The Hilbert space corresponds to the *state space* of the physical system. A *quantum state* of a system is an operator ρ on $\mathcal{H}$, which is positive, $\rho \geq 0$, and has unit trace, $\mathrm{tr}[\rho] = 1$. A state $\rho \in \mathcal{B}(\mathcal{H})$ is called *pure* if it can be written as $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$. We will often denote the pure state ρ by ψ.

The state space of two physical systems S_1 and S_2 with Hilbert spaces $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$, respectively, is given by $\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$. States of such a *bipartite* system are then again positive operators on $\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$ with unit trace. The *partial trace over system* S_1 of an operator $A \in \mathcal{B}(\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2})$, written as $\mathrm{tr}_{S_1}[A]$, is an operator on $\mathcal{H}_{S_2}$ defined uniquely through the requirement

$$\mathrm{tr}[A\,(\mathbb{1}_{S_1} \otimes B)] = \mathrm{tr}\big[\mathrm{tr}_{S_1}[A]\; B\big]$$

for all $B \in \mathcal{B}(\mathcal{H}_{S_2})$.

The *dynamics* of a *closed* quantum system with Hilbert space $\mathcal{H}$ is described by a unitary $U \in \mathcal{B}(\mathcal{H})$, i.e. the final system state ρ' is given by $\rho' = U\rho U^\dagger$, where ρ denotes the initial state of the system. The hermitian operator $H = i \log U \in \mathcal{B}(\mathcal{H})$ is called the *Hamiltonian* of the quantum system. The Hamiltonian H of a quantum system corresponds to an energy measurement of the quantum system in a sense which we explain in the next chapter.

The dynamics of an *open* quantum system is described by a *quantum channel*. Since quantum channels will play a major role in this thesis, we highlight the definition.

Definition 2.1. Quantum channel

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two Hilbert spaces. A *quantum channel* $\mathcal{F}$ from $\mathcal{B}(\mathcal{H}_1)$ to $\mathcal{B}(\mathcal{H}_2)$ is a linear mapping that is *trace preserving*, i.e.

$$\mathrm{tr}[\mathcal{F}(A)] = \mathrm{tr}[A] \qquad \forall A \in \mathcal{B}(\mathcal{H}_1)\,,$$

and *completely positive*, i.e., the map $\mathcal{F} \otimes \mathrm{id}_n : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathbb{C}^n)$ is positive for all $n \in \mathbb{N}$, where id_n denotes the identity map on $\mathcal{B}(\mathbb{C}^n)$, i.e. $\mathrm{id}_n(B) = B$ for all $B \in \mathcal{B}(\mathbb{C}^n)$.

The trace preserving and positivity requirements guarantee that a quantum channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ maps quantum states on S again to quantum states on S. Complete positivity ensures that this is still true if we include additional “bystanders” into our description of the physical system, i.e. systems with Hilbert space $\mathbb{C}^n$ on which the joint evolution $\mathcal{F} \otimes \mathrm{id}_n$ acts trivially.

The *adjoint* of a quantum channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is the linear, completely positive map $\mathcal{F}^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ satisfying

$$\mathrm{tr}[\mathcal{F}(A)B] = \mathrm{tr}[A\mathcal{F}^*(B)]$$

for all $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. The trace preserving requirement on $\mathcal{F}$ translates to *unitality* of the adjoint $\mathcal{F}^*$, i.e. $\mathcal{F}^*(\mathbb{1}_{\mathcal{H}_2}) = \mathbb{1}_{\mathcal{H}_1}$.

A *quantum operation* $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is a linear map, which is completely positive and satisfies $\mathrm{tr}[T(A)] \leq \mathrm{tr}[A]$ for all $0 \leq A \in \mathcal{B}(\mathcal{H}_1)$. Hence, any quantum channel is a quantum operation, but not vice versa.

The following proposition will prove very useful within this thesis.

Proposition 2.2. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two Hilbert spaces. Then* $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ *is a quantum operation if and only if there exist operators* $M_i : \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $\sum_i M_i^\dagger M_i \leq \mathbb{1}_{\mathcal{H}_1}$ *and*

$$T(A) = \sum_i M_i A M_i^\dagger \, .$$

If T *is a quantum channel, then the operators* M_i *satisfy* $\sum_i M_i^\dagger M_i = \mathbb{1}_{\mathcal{H}_1}$.

We call a decomposition as in Proposition 2.2 a *Kraus decomposition* of a quantum operation T, and the operators M_i *Kraus operators.* Moreover, there is a minimum number of Kraus operators needed for a Kraus decomposition of T; this number is called the *Kraus rank* of T.

Note that the Kraus operators M_i of a given linear and completely positive map T are not unique in general. Indeed, two sets $\{M_i\}$ and $\{N_l\}$ of operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ are sets of Kraus operators of T if (and only if) there is an isometry V such that $N_l = \sum_i V_{li} M_i$ [54, 55].

To any quantum channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ one can associate a linear operator τ defined via

$$\tau = (\mathcal{F} \otimes \mathrm{id}_{\mathcal{H}_1})(\Omega) \, ,$$

where

$$\Omega = \frac{1}{d} \sum_{j,k=1}^{d} |jj\rangle\langle kk|$$

is a maximally entangled state on $\mathcal{H}_1 \otimes \mathcal{H}_1$ and $d = \dim(\mathcal{H}_1)$ denotes the dimension of $\mathcal{H}_1$ [56]. Here, $\{|j\rangle\}$ denotes some orthonormal basis of $\mathcal{H}_1$ and we employ the notation $|jj\rangle \equiv |j\rangle \otimes |j\rangle$. The operator τ is called the *Choi matrix of* $\mathcal{F}$. The Choi matrix encodes all information about the quantum channel $\mathcal{F}$. It can for example be used to determine the Kraus rank of $\mathcal{F}$, which is, as one can show, identical to the matrix rank of τ [55].

The following proposition provides a useful representation of a quantum channel. It corresponds to a finite dimensional Schrödinger picture version of the so-called Stinespring dilation theorem [57].

Proposition 2.3. *Let* $\mathcal{H}_A$, $\mathcal{H}_B$ *be Hilbert spaces. Let* $\mathcal{F} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ *be a quantum channel and let* τ *denote the Choi matrix of* $\mathcal{F}$. *Then for any* $r \geq \mathrm{rank}(\tau)$, *there exists a Hilbert space* $\mathcal{H}_R$ *with* $\dim(\mathcal{H}_R) = r$ *and an isometry* $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_R$ *such that*

$$\mathcal{F}(\rho) = \mathrm{tr}_R[V \rho V^\dagger] \tag{2.1}$$

for all $\rho \in \mathcal{B}(\mathcal{H}_A)$.

We call a representation of a quantum channel $\mathcal{F}$ as in Proposition 2.3 a *Stinespring dilation* of $\mathcal{F}$, and the isometry V in such a representation a *Stinespring isometry*. If the representation is such that the Hilbert space dimension of $\mathcal{H}_R$ is identical to the Kraus rank of $\mathcal{F}$, i.e. $\dim(\mathcal{H}_R) = \mathrm{rank}(\tau)$, we call the Stinespring dilation *minimal*. A minimal Stinespring dilation is unique up to unitary rotations U_R on $\mathcal{R}$, i.e. if V and $\tilde{V}$ are to two minimal Stinespring isometries of the same channel, then $\tilde{V} = (\mathbb{1}_B \otimes U_R)V$ for some unitary $U_R \in \mathcal{B}(\mathcal{H}_R)$. All other Stinespring isometries $\hat{V} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{\hat{R}}$ are related to a minimal Stinespring isometry through an isometry $W : \mathcal{H}_R \to \mathcal{H}_{\hat{R}}$. As a consequence, any two Stinespring isometries of the same channel are related through a partial isometry. The basic intuition here is that the minimal Stinespring isometry provides the full information about the channel. We can of course always add unnecessary dimensions to the dilation space $\mathcal{H}_R$, but the corresponding Stinespring isometry will not map into these extra dimensions and act like the minimal isometry on the “necessary” dimensions.

Given a Stinespring dilation of a channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, we obtain another channel if we trace over the target space $\mathcal{B}(\mathcal{H}_B)$ in Eq. (2.1) instead of $\mathcal{H}_R$. This channel is called a *complementary channel* of $\mathcal{F}$.

Definition 2.4. Let $\mathcal{H}_A$, $\mathcal{H}_B$ be Hilbert spaces. Let $\mathcal{F} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel and let $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_R$ by an isometry of $\mathcal{F}$ such that

$$\mathcal{F}(\rho) = \mathrm{tr}_R[V\rho V^\dagger]$$

for all $\rho \in \mathcal{B}(\mathcal{H}_A)$.

Then the corresponding *complementary channel of* $\mathcal{F}$ is the channel $\mathcal{F}^c : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_R)$ defined through

$$\mathcal{F}^c(\rho) = \mathrm{tr}_B[V\rho V^\dagger]$$

for all $\rho \in \mathcal{B}(\mathcal{H}_A)$.

Just as the Stinespring isometries of a channel $\mathcal{F}$, the corresponding complementary channels are not unique. However, for any two complementary channels $\mathcal{F}^c : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_R)$ and $\mathcal{F}^{\tilde{c}} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{R'})$ of the same channel $\mathcal{F}$ there exists a partial isometry $W : \mathcal{H}_R \to \mathcal{H}_{R'}$ such that

$$\mathcal{F}^{\tilde{c}}(\rho) = W\mathcal{F}^c(\rho)W^\dagger \tag{2.2}$$

for all $\rho \in \mathcal{B}(\mathcal{H}_A)$ [58].

Importantly, we can always describe the dynamics of an open quantum system via the unitary dynamics of a closed quantum system in the following sense: Let

$\mathcal{F} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ be a quantum channel describing the dynamics of an open quantum system A. Note that we chose identical input and output spaces for simplicity. A Stinespring isometry V of this channel then maps from $\mathcal{H}_A$ to $\mathcal{H}_A \otimes \mathcal{H}_R$ and can therefore always be embedded into a unitary $U \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$ such that $V = U(\mathbb{1}_A \otimes |\phi\rangle)$ with some $|\phi\rangle \in \mathcal{H}_R$, and hence by Eq. (2.1):

$$\mathcal{F}(\rho) = \mathrm{tr}_R[U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger] \qquad \forall \rho \in \mathcal{B}(\mathcal{H}_A)\,.$$

Since such unitary representations of a quantum channel $\mathcal{F}$ will play a major role in this thesis, we formulate this observation as the next corollary.

> **Corollary 2.5.** *Let $\mathcal{H}_A$ be a Hilbert space. Let further $\mathcal{F} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ be a quantum channel and let τ denote the Choi matrix of $\mathcal{F}$. Then for any $r \geq \mathrm{rank}(\tau)$, there exists a Hilbert space $\mathcal{H}_R$ with $\dim(\mathcal{H}_R) = r$, a unitary $U \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$ and a vector $|\phi\rangle \in \mathcal{H}_R$ such that*
>
> $$\mathcal{F}(\rho) = \mathrm{tr}_R[U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger]$$
>
> *for all $\rho \in \mathcal{B}(\mathcal{H}_A)$.*

Entropies. A set $\{p_k\}$ is called a *probability distribution* if $p_k \geq 0$ for all k and $\sum_k p_k = 1$. A probability distribution $\{p_k\}$ is called *peaked* if one element of this set is equal to 1, which implies that all other elements are zero. If the number of elements of the probability distribution is K, a probability distribution is called *uniform* if all elements are the same and hence equal to $1/K$.

The *Shannon entropy* $H(\{p_k\})$ of a probability distribution $\{p_k\}$ is defined as [59]

$$H(\{p_k\}) = -\sum_k p_k \log p_k\,, \tag{2.3}$$

where we employ the standard convention to set $0 \log 0 := 0$. The Shannon entropy can take values in the interval $[0, \log K]$. It attains its minimal value, i.e. $H(\{p_k\}) = 0$, if and only if the probability distribution is peaked, and is maximal, i.e. $H(\{p_k\} = \log K$, if and only if the probability distribution is uniform.

The *von Neumann entropy* $S(\rho)$ of a quantum state $\rho \in \mathcal{B}(\mathcal{H})$ is defined as [60]

$$S(\rho) = -\mathrm{tr}[\rho \log \rho]\,. \tag{2.4}$$

Denote by $\{\lambda_k\}$ the set of eigenvalues of ρ. Note that $\{\lambda_j\}$ is then also a probability distribution. The von Neumann entropy $S(\rho)$ can then be expressed as the Shannon entropy of the set of eigenvalues, i.e. $S(\rho) = H(\{\lambda_k\})$. Hence, the von Neumann entropy can take values in the interval $[0, \log d]$, where d denotes the number of eigenvalues of ρ. It also follows that the von Neumann entropy is zero if and only if ρ is a pure state, and maximal if and only if ρ is the completely mixed state, i.e. $\rho = \mathbb{1}/d$. The von Neumann entropy is additive on tensor products, i.e. we have

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$

for any two states ρ and σ.

Let $\rho = \sum_k p_k \rho_k$, where $\{p_k\}$ is a probability distribution, be a convex combination of quantum states $\rho_k \in \mathcal{B}(\mathcal{H})$. Then the von Neumann entropy satisfies the following useful identity [43]

$$\sum_k p_k S(\rho_k) \leq S(\rho) = S\Big(\sum_k p_k \rho_k\Big) \leq H(\{p_k\}) + \sum_k p_k S(\rho_k)\,. \tag{2.5}$$

The first inequality is equivalent to saying that the von Neumann entropy is concave. Equality holds if and only if, all ρ_k, for which $p_k > 0$, are identical, i.e. $\rho_k = \rho$. The second inequality is saturated if and only if the states ρ_k are *mutually orthogonal*, i.e. they have support on orthogonal subspaces.

Let $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a bipartite quantum state, i.e. it is a quantum state on the tensor product of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The marginal states on the individual spaces are obtained by the partial trace introduced above, i.e. for example, $\rho_A = \mathrm{tr}_B[\rho_{AB}]$. For such a bipartite state, the *mutual information* $I(A:B)_{\rho_{AB}}$ between systems A and B is defined as

$$I(A:B)_{\rho_{AB}} = S(\rho_A) + S(\rho_B) - S(\rho_{AB})\,. \tag{2.6}$$

This quantity is non-negative, i.e.

$$I(A:B)_{\rho_{AB}} \geq 0 \tag{2.7}$$

for all quantum states $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, which implies that the entropy $S(\rho_{AB})$ of the joint state is never less than the sum $S(\rho_A) + S(\rho_B)$ of the entropies of the marginal states. The mutual information can be understood as a measure of the *correlations* between the systems A and B when they are in state ρ_{AB}. To provide some intuition, we consider two very simple examples: If $\rho_{AB} = \rho_A \otimes \rho_B$ is a product state, we expect the systems to be uncorrelated and indeed we have $I(A:B)_{\rho_{AB}} = 0$. Conversely, the strongest correlations are expected, if ρ_{AB} is a maximally entangled state, i.e. ρ_{AB} is pure and ρ_A and ρ_B are states on Hilbert spaces of equal dimension, say d, and are maximally mixed. We then have that $I(A:B)_{\rho_{AB}} = 2\log d$, which is also the maximal possible value that the mutual information can attain.

Given two states $\rho \in \mathcal{B}(\mathcal{H})$ and $\sigma \in \mathcal{B}(\mathcal{H})$, the *relative entropy* $D(\rho||\sigma)$ is defined as

$$D(\rho||\sigma) = \mathrm{tr}[\rho \ln \rho] - \mathrm{tr}[\rho \ln \sigma]\,, \tag{2.8}$$

with the convention to set $D(\rho||\sigma) := \infty$ if $\mathrm{supp}[\rho] \nsubseteq \mathrm{supp}[\sigma]$. Similar to all other entropic quantities introduced so far, the relative entropy is always non-negative,

$$D(\rho||\sigma) \geq 0$$

for any two states ρ and σ. Moreover, the relative entropy $D(\rho||\sigma)$ is zero if and only if $\rho = \sigma$. The relative entropy is hence similar to a metric, however it does not satisfy the triangle inequality and is not symmetric in its arguments.

Thermal states and free energy. A quantum state $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ of a system A with Hilbert space $\mathcal{H}_A$ and Hamiltonian H_A is called a *thermal state* (or *Gibbs state*) with respect to H_A at temperature $T \in \mathbb{R}_+$ if it can be written as (see e.g. [46, 47, 61])

$$\rho_A = \frac{1}{Z_A} \mathrm{e}^{-\frac{H_A}{k_B T}} ,$$

where k_B is the Boltzmann constant and $Z_A = \mathrm{tr}[\mathrm{e}^{-\frac{H_A}{k_B T}}]$ is the *so-called partition function.*

The *free energy* $F(\rho_A)$ of a quantum state $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ of a system A with Hilbert space $\mathcal{H}_A$ and Hamiltonian H_A at temperature $T \in \mathbb{R}_+$ is defined as [46, 62]

$$F(\rho_A) = \mathrm{tr}[H_A \rho_A] - k_B T S(\rho_A) . \tag{2.9}$$

The free energy $F(\rho_A)$ of a quantum state ρ_A determines the work value of ρ_A, i.e. the amount of energy that can be extracted from ρ_A by unitary dynamics on A alone.

There is a very useful alternative way to compute the free energy, which is based on the following identity

$$D(\rho_A||\rho_{\mathrm{th}}) = \mathrm{tr}[H_A \rho_A] - k_B T S(\rho_A) + \log Z_A ,$$

where $\rho_{\mathrm{th}} \in \mathcal{B}(\mathcal{H}_A)$ is the thermal state with respect to H_A at temperature T. This identity directly follows from the definitions for the relative entropy and for thermal states and its derivation will be explained in more detail in Eq. (4.2) from Chapter 4.

From this identity we immediately obtain a set of widely known, but still useful properties of thermal states: Firstly, thermal states have the largest entropy among all states with fixed average energy (recall that $D(\rho_A||\rho_{\mathrm{th}})$ attains its minimal value if and only if $\rho_A = \rho_{\mathrm{th}}$). Secondly, thermal states have the smallest energy among all states with constant entropy. And lastly, since

$$F(\rho_A) = -\log Z_A + D(\rho_A||\rho_{\mathrm{th}}) ,$$

we know that thermal states minimise the free energy, i.e. $F(\rho_A) \geq F(\rho_{\mathrm{th}})$ for all states $\rho_A \in \mathcal{B}(\mathcal{H}_A)$.

CHAPTER 3

Models of quantum measurement

This chapter is devoted to the framework that we use to determine the energy cost of quantum measurements. Importantly, the appropriate definition of what a "quantum measurement" is depends on the physical context [12, 43]. Hence, this chapter starts with a general introduction to the different notions of quantum measurement. In particular, we focus on four different formalisms for quantum measurement: Unselective measurements, quantum channels, quantum instruments, and POVMs. Note that we only describe the formalisms for measurements on finite dimensional quantum systems and with finite outcome sets.

Intuitively, it is clear that a physical implementation of a quantum measurement involves several quantum systems, i.e. not only the measured system, which we refer to as S throughout this thesis, but also e.g. a pointer system, which stores the measurement outcome and which we introduce more thoroughly below. The formalisms above however describe the quantum measurement solely on the basis of the measured system itself— the physical dynamics on the other involved quantum systems are ignored. This is because a detailed description of all the interactions and state changes is typically too complicated. For example, it is a rather daunting task to try to determine and describe the quantum state of the full measurement device used to perform the measurement. Moreover, for many applications such a detailed description of the measurement process is unimportant: All that matters is the way the state of the measured system S changes and/or which measurement outcome was obtained with which probability. The four formalisms above focus on exactly these properties.

However, the goal of this thesis is to derive the fundamental energy requirements for quantum measurement. This requires us to keep track of all (necessary) energy changes that occur in this process, that is, also those in the other involved systems such as the measurement device itself. The second part of this section therefore provides a general description of all the involved systems and the dynamics in a quantum measurement. Of course, there are typically various ways to perform the same quantum measurement. We refer to any of these ways as a particular *implementation* of the quantum measurement. Our general description includes all such implementations, which allows us to define the energy cost of a quantum measurement in the third part of this chapter.

We conclude this chapter by relating our framework to existing thermodynamic treatments of the quantum measurement process. Indeed, the energy costs we define can be understood as the amount of *thermodynamic work* needed to conduct the quantum measurement. Moreover, part of this thermodynamic work corresponds to the *heat* dissipated during the measurement. Our framework therefore allows us to characterise the thermodynamics of quantum measurement as well.

3.1 Abstract formalisms for quantum measurement

There exist various different notions for quantum measurement in the literature, each suitable for different physical situations. In the following we will briefly review some of these notions. Note that we will always only consider measurement on finite-dimensional quantum systems S and finite outcome sets.

The following list provides a general overview over the formalisms of measurement we discuss in this thesis. The precise definitions of these formalisms are given below in this section.

- **Unselective measurement.** An unselective measurement is a state transformation that determines the average post-measurement state ρ_S' for a *fixed* initial state ρ_S.

- **Quantum channel.** A quantum channel determines the average post-measurement state ρ_S' for *any* initial state ρ_S.

- **Quantum instrument.** This is the most detailed formalism of a quantum measurement. A quantum instrument determines the probabilities p_k to obtain measurement outcome k as well as the corresponding post-measurement state $\rho_{S,k}'$ for *any* initial state ρ_S. The average post-measurement state is then given by $\rho_S' = \sum_k p_k \rho_{S,k}'$.

- **POVM.** A POVM determines the probabilities p_k to obtain measurement outcome k for *any* possible initial state ρ_S.

This overview shows that two of these formalisms, unselective measurements and quantum channels, describe the dynamics of the system S. In contrast, POVMs describe the acquisition of classical information from quantum states but do not provide a notion of a post-measurement state. Only the formalism of quantum instruments describes both the dynamics as well as the information acquisition. The short descriptions above suggest an ordering of the formalism that is depicted in Figure 3.1. This ordering is made more precise shortly. But before that, let us first give more detailed definitions of the different formalisms for quantum measurements, starting with quantum instruments.

Definition 3.1. Quantum instrument

A *quantum instrument* on a system S with Hilbert space $\mathcal{H}_S$ is a set of completely positive linear maps $\{T_k\}_{k=1,\ldots,K}$ on $\mathcal{B}(\mathcal{H}_S)$ satisfying

$$\sum_k T_k^*(\mathbb{1}_S) = \mathbb{1}_S \,. \tag{3.1}$$

Here, the index k corresponds to the *measurement outcome* and ranges from 1 to the number of outcomes K. Moreover, T_k^* denotes the adjoint of the map T_k. If the initial state on S is $\rho_S \in \mathcal{B}(\mathcal{H}_S)$, a quantum instrument characterises both the *outcome probability*

$$p_k = \mathrm{tr}\Big[T_k(\rho_S)\Big]$$

to obtain outcome k as well as the corresponding *post-measurement state*[1]

$$\rho'_{S,k} = T_k(\rho_S)/p_k \,.$$

The *average post-measurement* state is then

$$\rho'_S = \sum_k p_k \rho'_{S,k} = \sum_k T_k(\rho_S)\,.$$

It is useful to think of quantum instruments as a tool to describe the measurement process in a black-box manner (see Figure 3.1): Given the initial state ρ_S of the quantum system S, the measurement is conducted via a black box, the abstract “measurement device”, which stores with probability p_k the outcome k and outputs the corresponding post-measurement state $\rho'_{S,k}$. The details of *how* the measurement device interacts with the quantum system are not specified in the quantum instrument formalism. It is the virtue of this formalism that the measurement process is independent of the microscopic properties of the measurement process and device. Instead, all we specify about the black box is the set $\{T_k\}$ of abstract maps on S.

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[1] If $p_k = 0$, we may simply choose $\rho'_{S,k}$ to be any desired quantum state on S.

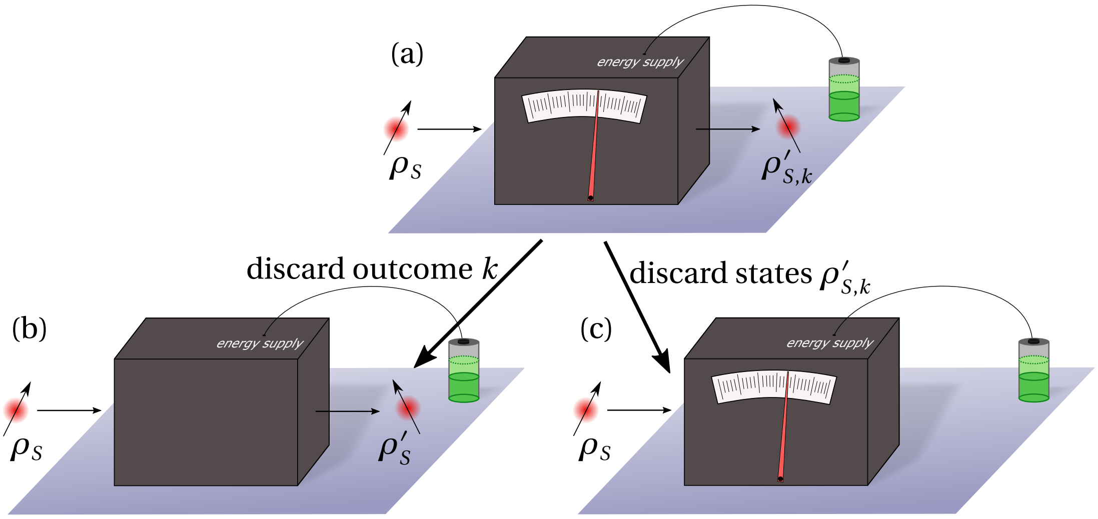

Figure 3.1: Hierarchy between the formalisms of quantum measurement. Quantum instruments (a) correspond to the parent quantity: All other formalism can be obtained from quantum instruments by requiring less "structure". For all initial states ρ_S of the measured system S, the quantum instrument $\{M_{ki}\}$ determines both the probability p_k to obtain outcome k as well as the post-measurement states $\rho'_{S,k}$ and outputs the state $\rho'_S = \sum_k p_k \rho'_{S,k}$ on average. Quantum channels (b) only describe this average post-measurement state $\rho'_S = \mathcal{F}(\rho_S)$. Unselective measurements (also (b)) determine this state only for a *fixed* input state ρ_S. POVMs (c) only characterise the probabilities p_k to obtain outcome k and hence the statistics of measurement outcomes for all initial states ρ_S.

The action of the map T_k on a state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ can always be written in terms of Kraus operators M_{ki} via $T_k(\rho_S) = \sum_{i=1}^{I(k)} M_{ki}\rho_S M_{ki}^\dagger$ (see Proposition 2.2). Here $I(k)$ denotes the Kraus rank of T_k, i.e. the minimal number of Kraus operators needed to decompose the channel T_k. Later on, we will find that the quantities $I(k)$ are an important physical property of the measurement as they are related to what is typically referred to as the *inefficiency* of the measurement [12,63,64], a notion that we define precisely later in this section. The inefficiency determines the "mixing" properties of the measurement, thereby playing an important role for the energy cost as we will see in Chapter 4. We call the index i the *inefficiency index*. If $I(k) = 1$ for all k, we simply drop the inefficiency index.

Instead of specifying the set $\{T_k\}$ of maps, we can equivalently provide the set of Kraus operators $\{M_{ki}\}$ to specify a quantum instrument.

Definition 3.2. Measurement operators of a quantum instrument

A quantum instrument $\{T_k\}$ on a system S with Hilbert space $\mathcal{H}_S$ can equivalently be described by the set $\{M_{ki}\}_{k=1,\dots,K,i=1,\dots,I(k)}$ of operators

$M_{ki} \in \mathcal{B}(\mathcal{H}_S)$ satisfying

$$T_k(\rho_S) = \sum_{i=1}^{I(k)} M_{ki} \rho_S M_{ki}^\dagger$$

for all states $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ and all measurement outcomes k, where $i = 1, ..., I(k)$ denotes the *inefficiency index* and $I(k)$ is the Kraus rank of T_k. We call the operators M_{ki} the *measurement operators* of the instrument $\{M_{ki}\}$. Because of Eq. (3.1), the measurement operators must satisfy

$$\sum_{ki} M_{ki}^\dagger M_{ki} = \mathbb{1}_S \, . \tag{3.2}$$

If the initial state on S is $\rho_S \in \mathcal{B}(\mathcal{H}_S)$, the quantum instrument $\{M_{ki}\}$ characterises both the probability

$$p_k = \mathrm{tr}\Big[\sum_i M_{ki} \rho_S M_{ki}^\dagger\Big]$$

to obtain outcome k as well as the corresponding post-measurement state

$$\rho'_{S,k} = \sum_i M_{ki} \rho_S M_{ki}^\dagger / p_k \, . \tag{3.3}$$

The average post-measurement state is then

$$\rho'_S = \sum_k p_k \rho'_{S,k} = \sum_{k,i} M_{ki} \rho_S M_{ki}^\dagger \, .$$

We will employ these two definitions for quantum instruments interchangeably in this thesis. It will always be clear from the context, whether we specify the maps T_k or the measurement operators M_{ki} to determine the quantum instrument.[2]

Many realistic scenarios require such a detailed specification of the measurement as provided by the quantum instrument formalism. This is for example the case for measurement-based feedback protocols, where a quantum system in state ρ_S is measured such that, based on the respective outcome k occuring with probability p_k, further actions on the post-measurement states $\rho'_{S,k}$ of the quantum system have to be performed. Clearly, for such a feedback protocol, outcomes k *and* post-measurement states $\rho'_{S,k}$ must be specified. In Section 5.2 we will consider an important application of our results on such a feedback protocol, namely quantum error correction.

Before continuing with the other three formalisms for quantum measurement, let us briefly comment on the "traditional" formalism of quantum measurement,

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[2] Note the Kraus operators of a given map T_k are not unique, see after Proposition 2.2.

namely the specification of a hermitian operator [12, 43, 54]. Indeed, most textbook examples of quantum measurement follow this formalism, e.g. when using the (hermitian) angular momentum operators to describe a spin measurement, or when providing the Hamiltonian of a quantum system to describe an energy measurement. In fact, this approach can be thought of as a special case of the quantum instrument formalism: Given a hermitian operator $A = A^\dagger \in \mathcal{B}(\mathcal{H}_S)$ on a quantum system S initially in state ρ_S, the traditional formalism postulates that the associated measurement outputs, with probability $p_k = \mathrm{tr}[P_k \rho_S]$, the outcome λ_k together with the post-measurement state $\rho'_{S,k} = P_k \rho_S P_k / p_k$, where P_k are the spectral projections of A, i.e. $A = \sum_k \lambda_k P_k$ [60, 65]. Note the identification of the eigenvalues λ_k of the operator A with the possible outcomes of the associated measurement, which implies that the outcomes of this measurement are real numbers. In practise, this restriction is unnatural: Measurement outcomes are often abstract quantities, which can be as general as "the red light is blinking". Of course, we can always assign real numbers to the outcomes, but this assignment is often completely arbitrary and therefore pointless. Instead, the associated measurement of A is better described by the quantum instrument $\{P_k\}$ where the measurement operators P_k correspond to the eigenprojections of A. From Definition 3.2 we directly see that this instrument yields the same probabilities and post-measurement states as the associated measurement of A. Since such *projective* quantum instruments are so common in the literature of quantum measurements, we briefly state the definition for further use.

Definition 3.3. Projective quantum instruments

A set $\{P_k\}$ is called a *projective quantum instrument* if it is a quantum instrument with measurement operators P_k satisfying $P_k = P_k^\dagger = P_k^2$ for all k. This projection requirement together with Eq. (3.2) implies that $\{P_k\}$ is a set of mutually orthogonal projections.

Let us now continue with the other formalisms of measurement, namely quantum channels, unselective measurements and POVMs.

There are situations in which measurements are performed although one does not wish to apply further feedback operations. Instead, such measurements are applied for the sole reason to induce the average state change on S from ρ_S to ρ'_S. This is for example the case in the stabilisation procotol by Zeno measurements, which we illustrate in Section 5.1. Specifying the output state ρ'_S for all initial states ρ_S is equivalent to fixing a quantum channel $\mathcal{F}$ on $\mathcal{B}(\mathcal{H}_S)$. This form of measurement therefore simply corresponds to *applying a quantum channel* and the "average" post-measurement state is given by

$$\rho'_S = \mathcal{F}(\rho_S)\,.$$

The definition of a quantum channel is provided in Definition 2.1 along with a set of useful properties.

The black-box picture of such a measurement is then the following (see Figure 3.1): The quantum system S in state ρ_S is input into a measurement device which only outputs the average post-measurement state $\rho'_S = \mathcal{F}(\rho_S)$ and which does not store any measurement outcome k.

How are these two formalisms, quantum instrument and quantum channel, related? Let us first fix a quantum instrument $\{M_{ki}\}$. The associated measurement channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is then given by

$$\mathcal{F}(\rho_S) = \sum_{ki} M_{ki} \rho_S M_{ki}^\dagger \qquad \forall \rho_S \ .$$

Clearly, this is indeed a quantum channel because of Proposition 2.2. For any input state ρ_S, this channel outputs the same average post-measurement state $\rho'_S = \mathcal{F}(\rho_S)$ as the quantum instrument $\{M_{ki}\}$ (which is also the reason why to call $\mathcal{F}(\rho_S)$ an average post-measurement state in the first place). It is in this sense that we say that a quantum instrument naturally induces a unique quantum channel. The opposite direction of this statement is in general not true. To see this, let us fix now a quantum channel $\mathcal{F}$. Recall from Proposition 2.2 that any quantum channel $\mathcal{F}$ has a Kraus decomposition $\mathcal{F}(\rho_S) = \sum_j M_j \rho_S M_j^\dagger$ with Kraus operators $M_j \in \mathcal{B}(\mathcal{H}_S)$. The set $\{M_j\}$ of Kraus operators can therefore be identified with the measurement operators of a quantum instrument, which will then output the state $\rho'_S = \mathcal{F}(\rho_S)$ on average as desired. Note however that the Kraus decomposition is not unique (see Proposition 2.2). Additionally, there is some further ambiguity in the identification: We can for example identify the index j with the measurement outcomes k. The quantum instrument $\{M_k\}$ would then correspond to an efficient measurement (see below) and the post-measurement states $\rho'_{S,k}$ are given as in Eq. (3.3). The other extreme would be to identify the index j with the inefficiency index i such that the measurement basically does not output any measurement outcome. Of course, many intermediate identifications are possible as well and each of them leads to different post-measurement states $\rho'_{S,k}$.

Quantum channels therefore do not naturally induce a specific quantum instrument. However, this observation shows that we can always, i.e. for any quantum channel $\mathcal{F}$, find a quantum instrument that generates the same dynamics as $\mathcal{F}$.

The third measurement formalism puts even less requirements on the measurement process by only specifying the average output dynamics for the *fixed* initial state ρ_S. We call this formalism an unselective measurement [66] or state transformation.

Definition 3.4. Unselective measurement / state transformation

An *unselective measurement* (or *state transformation*) on a system S with Hilbert space $\mathcal{H}_S$ is a mapping on $\mathcal{B}(\mathcal{H}_S)$ with the property that it maps a given, fixed initial state ρ_S to a given, fixed average post-measurement state ρ_S'. We denote such a state transformation by $\rho_S \mapsto \rho_S'$.

The black-box picture of this formalism corresponds to a measurement device that outputs the state ρ_S' specifically if the input state is ρ_S (see Figure 3.1). If any other state $\tilde{\rho}_S$ is measured with this device, the action of the device and hence the average post-measurement is not specified. As for quantum channels, there does not exist the notion of a measurement outcome within this formalism.

Note that we do not fix any mathematical properties, e.g. linearity, of this mapping as its action on the state space $\mathcal{B}(\mathcal{H}_S)$ is only determined for the given input state ρ_S. The fact that the dynamics are specified for the fixed input state ρ_S only, makes this formalism more general than quantum channels and quantum instruments. Indeed, any quantum channel $\mathcal{F}$ (and hence any quantum instrument) on an input state ρ_S naturally induces the unselective measurement $\rho_S \mapsto \mathcal{F}(\rho_S)$, but conversely there exist in general various quantum channels that lead to the same dynamics as specified by a given unselective measurement.

We introduced quantum channels and state transformations by ignoring the measurement statistics obtained through a quantum instrument. Of course, we can also think the other way around and search for a formalism that describes the measurement statistics, but ignores the dynamics of the measured system. Our last formalism of measurement describes exactly this setting.

Definition 3.5. POVMs

A POVM (positive operator valued measure) is a set $\{E_k\}_{k=1,\dots,K}$ of positive operators $E_k \in \mathcal{B}(\mathcal{H}_S)$ satisfying $\sum_k E_k = \mathbb{1}_S$. If the initial state of S is ρ_S, the probability to obtain outcome k in a measurement described by this POVM is given by

$$p_k = \mathrm{tr}[E_k \rho_S] \, .$$

A POVM is solely focused on the acquisition of classical information from the measured quantum state. In the black-box model, the measurement device only stores the measurement outcome k with probability p_k, but does not output a post-measurement state (see Figure 3.1). A reason to use POVMs over quantum instruments is that in some situations the post-measurement state is irrelevant or not even defined. A standard example of such situations is the detection of photons which are "destroyed" after measurement.

We already mentioned an ordering between the measurement formalisms (see Fig. 3.1) and provided some details of this ordering for quantum instruments, quantum channels and state transformations. How are POVMs related to these formalisms? For this, let us fix a quantum instrument $\{M_{ki}\}$ on S. We can always define a POVM $\{E_k\}$ with elements $E_k = \sum_i M_{ki}^\dagger M_{ki} \geq 0$. This POVM outputs outcome k with the same probability $p_k = \mathrm{tr}[E_k \rho_S]$ as the quantum instrument $\{M_{ki}\}$. A quantum instrument therefore naturally induces a unique POVM on S, the opposite statement is however not true: For any fixed POVM $\{E_k\}$ we can always find several quantum instruments that yield the same measurement statistics. For example, if $\{M_{ki}\}$ is one such quantum instrument, we find another such quantum instrument $\{\tilde{M}_{ki}\}$ by defining the measurement operators $\tilde{M}_{ki} = U_{ki} M_{ki}$, where $\{U_{ki}\}$ is any set of unitaries U_{ki} on $\mathcal{H}_S$. These different quantum instruments provide the same measurement statistics, but output different post-measurement states, showing that the post-measurement state is irrelevant and hence not uniquely specified when the formalism of POVMs is applied.

Efficient vs. inefficient quantum instruments

Let us consider again a quantum instrument $\{M_{ki}\}$. The measurement operators M_{ki} have two indices and we already know that, by definition, the index k corresponds to the measurement outcome obtained in the measurement. What is the physical interpretation of the inefficiency index i? And why can't we just describe such a measurement with a single index k only? To answer these question let us first fix some terminology which is also used in ref. [12, 63].

Definition 3.6. Inefficiency of a quantum instrument

Let $\{M_{ki}\}_{k=1,\ldots,K, i=1,\ldots,I(k)}$ be a quantum instrument on a quantum system S with Hilbert space $\mathcal{H}_S$ as in Definition 3.2. The *inefficiency* I of the quantum instrument $\{M_{ki}\}$ is then defined as the maximal $I(k)$ for all k, i.e.

$$I = \max_k I(k)\,.$$

We call a quantum instrument *efficient* if $I = 1$, implying that there is a one-to-one correspondence between measurement operator M_{ki} and measurement outcome k. Any quantum instrument that is not efficient is called *inefficient.*

For now, this separation between efficient and inefficient measurements may seem rather artificial. However, we will find that these two kinds of measurement can be quite different from an energetic perspective.

Let us now come back to our question about the physical interpretation of the inefficiency index i. For this, consider the following scenario, which aims at showing that some realistic and physically well motivated instances of quantum measurements cannot be properly described by efficient instruments only. This scenario is also discussed in ref. [63]. Suppose we wish to perform an efficient quantum instrument $\{N_i\}$, where $\sum_i N_i^\dagger N_i = \mathbb{1}_S$, on a quantum system in state ρ_S[3]. According to Definition 3.2 this measurement outputs outcome i with probability $p_i = \mathrm{tr}[N_i \rho_S N_i^\dagger]$ together with the state $\rho'_{S,i} = \sum_i N_i \rho_S N_i^\dagger / p_i$. Now imagine we have only imperfect ability to read off the actual outcome i. Instead we simply guess according to some random variable what the outcome is. Let us denote by p_{ki} the joint probability for the event where the outcome was i and this random variable outputs k. Furthermore, we denote by $p(k|i) = p_{ki}/p_i$ the conditional probability with which the random variable outputs some value k if the actual outcome was i. The total probability p_k that we guess that the outcome was k is then given by

$$p_k = \sum_i p(k|i) p_i \ ,$$

while the post-measurement state $\rho'_{S,k}$ we obtain is given by

$$\rho'_{S,k} = \sum_i p(i|k) \rho'_{S,i} \qquad \forall \rho_S \ . \tag{3.4}$$

Our claim is now that this situation cannot be described by efficient quantum instruments alone in the sense that no efficient measurement can yield the correct probabilities p_k and post-measurement states $\rho'_{S,k}$ for all possible initial states ρ_S. More concretely, the claim is that there do not exist operators $\tilde{N}_k$ with $\sum_k \tilde{N}_k^\dagger \tilde{N}_k = \mathbb{1}_S$ such that Eq. (3.4) is satisfied, i.e.

$$\tilde{N}_k \rho_S \tilde{N}_k^\dagger / p_k = \sum_i p(i|k) \frac{N_i \rho_S N_i^\dagger}{p_i} \qquad \forall \rho_S \ . \tag{3.5}$$

To see this, we use the identity

$$\frac{p(i|k)}{p_i} = \frac{p(k|i)}{p_k} \qquad \forall i, k$$

to find that Eq. (3.5) is equivalent to requiring

$$\tilde{N}_k \rho_S \tilde{N}_k^\dagger = \sum_i p(k|i) N_i \rho_S N_i^\dagger \qquad \forall \rho_S \ .$$

---

[3] The index i appearing in the instrument $\{N_i\}$ is *not* the inefficiency index, but the outcome index. We deliberately use the index i here, since this index will turn out to be the inefficiency index of *another* quantum instrument.

The left-hand side corresponds to a quantum channel with unit Kraus rank, while the right hand side is, in general, a quantum channel with Kraus rank larger than one depending on the conditional probabilities $p(k|i)$ and the measurement operators N_i. But this is a contraction: If the equality would really hold, the same channel would have different Kraus ranks. But this is in contradiction to the definition of the Kraus rank as given in Chapter 2. To conclude, efficient measurements cannot properly describe the physical setting above.

Instead, we need to employ the more general notion of inefficient measurements $\{M_{ki}\}$. Indeed, the measurement operators

$$M_{ki} = \alpha_{ki} N_i$$

with $\alpha_{ki} \in \mathbb{C}$ satisfying $|\alpha_{ki}|^2 = p(k|i)$ yield the correct probabilities p_k as well as the correct post-measurement states $\rho'_{S,k}$. To see this, we compute

$$p_k = \mathrm{tr}[\sum_i M_{ki} \rho_S M_{ki}^\dagger] = \sum_i p(k|i) \mathrm{tr}[N_i^\dagger N_i \rho_S] = \sum_i p(k|i) p_i$$

and

$$\rho'_{S,k} = \sum_i M_{ki} \rho_S M_{ki}^\dagger / p_k = \sum_i \frac{p(i|k) p_k}{p_k p_i} N_i \rho_S N_i^\dagger = \sum_i p(i|k) \rho'_{S,i} \, .$$

Inefficient quantum instruments are hence not only relevant for mathematical generality, but also essential to describe realistic measurement settings.

3.2 The detailed framework for quantum measurements

Given a quantum measurement, how much energy do we have to expend? As explained in Section 3.1, we need to specify our notion of measurement to make this question more precise. Indeed, we will find that the different formalisms of measurement require different energy consumption. In this section we characterise the framework that we use to specify how to account for the energetic cost of such processes.

Our framework is based on three basic assumptions. These assumptions are used to provide a detailed mathematical description of what we referred to as a measurement device in Figure 3.1. In contrast to the abstract *formalisms* for quantum measurement discussed above which treated the device as a black blox, our *framework* describes the microscopic details of the device and all other quantum systems that are involved in the measurement process. This will enable us to include all energy changes into our calculations which necessarily occur in a quantum measurement.

The assumptions are:

Assumption 1. Involved systems
Any quantum measurement on a system S is a process that involves S and other quantum systems, namely the measurement device M itself and a thermal environment E. These quantum systems serve different purposes and hence have different properties which may depend on the formalism of quantum measurement. The measurement device M has a Hilbert space $\mathcal{H}_M$ and stores the measurement outcomes k if necessary. The thermal environment E is a quantum system with Hilbert space $\mathcal{H}_E$ and has the property that it always starts in a thermal state at the beginning of a quantum measurement (see below)[4].

Each of the involved systems comes with its own free Hamiltonian, H_S, H_M, and H_E. The total Hamiltonian of all involved systems before and after the measurement is given as

$$H_{SME} = H_S \otimes \mathbb{1}_{ME} + \mathbb{1}_S \otimes H_M \otimes \mathbb{1}_E + \mathbb{1}_{SM} \otimes H_E \ . \tag{3.6}$$

This corresponds to saying that the involved systems do not interact with one another before and after the measurement. During the measurement however we, of course, allow for interactions.

Assumption 2. No initial correlations
If a quantum measurement is performed on a quantum system S in state ρ_S, we assume that the initial state of all involved systems is a product state and hence does not contain any correlations, i.e.

$$\rho_{SME} = \rho_S \otimes \rho_M \otimes \rho_E \ ,$$

where ρ_M and ρ_E denote the initial states on M and E, respectively (see below for further properties of the initial states). This corresponds to the intuition that a measured system did not interact with the measurement device before the measurement is performed and that the thermal bath is independent[5].

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[4] For some frameworks, it is important to include so-called *catalysts*, which are quantum systems with the property that the initial and final state after the process are identical (see e.g. refs. [46, 67, 68]). Our framework already naturally includes such catalysts, since they can always be included into the memory M.

[5] It can be shown that any form of correlations in the initial state allows us to extract energy [69–71]. However, to establish these correlations requires at least this amount of energy. Hence, we simply restrict our study to uncorrelated input states and note that our results can easily be extended to the correlated case

Assumption 3. Unitary dynamics

Whereas the dynamics on the measured system S itself may not be unitary, the overall dynamics of any quantum measurement on a Hilbert space $\mathcal{H}_S$ is governed by a unitary U_{SME} on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_M \otimes \mathcal{H}_E$ of all involved systems. This unitary must satisfy certain properties that depend on the measurement to be performed and that we discuss in Section 3.3. Such unitary dynamics allow for a "complete" energy cost definition: Since unitaries describe the dynamics of closed quantum systems, we can be sure that we incorporated all relevant quantum systems into our description and that we do not "cheat" by forgetting additional energy contributions.

Let us specify the involved systems from Assumption 1 more precisely.

- **Measurement device/memory.** The primary role of the measurement device M is to store the measurement outcomes k. This is why we will also refer to the measurement device as a *memory*. Throughout this thesis we will use these two terms interchangeably. Although being quantum, the memory M is required to store only classical information in the following sense: The memory is modelled by a Hilbert space $\mathcal{H}_M = \bigoplus_k \mathcal{H}_{M,k}$ that is composed of several mutually orthogonal (not necessarily one-dimensional) Hilbert spaces $\mathcal{H}_{M,k}$, one for each measurement outcome k [62, 64]. One may wonder why it does not suffice to just consider one-dimensional Hilbert spaces $\mathcal{H}_{M,k}$. Indeed, we could assign to each outcome k a basis element $|k\rangle \in \mathcal{H}_M$ to achieve that the outcome is properly stored. The subspaces $\mathcal{H}_{M,k}$ spanned by this one basis element $|k\rangle$ would indeed be one-dimensional and we call a system M with a Hilbert space $\mathcal{H}_M = \bigoplus_k \mathcal{H}_{M,k}$ consisting of such one-dimensional subspaces a *pointer* system and denote the subspaces by $\mathcal{H}_{\text{pointer},k}$ (see e.g. [12, 72]). There are however two reasons why we do not restrict the memory to be a pointer system: The first reason is mathematical generality, that is, our results can easily be phrased within this less restrictive framework, where the dimension of the subspaces $\mathcal{H}_{M,k}$ may be greater than one. The second reason is more physically motivated: In our framework the memory M is to be understood as the subsumed description of all relevant systems in the measurement process, that are neither the measured system nor the thermal bath. This description typically includes the pointer system, but may also include *other* systems. For example the measurement device could work by coupling a photon to the measured system S, which then is being detected, thereby changing the state of the pointer system. The total Hilbert space $\mathcal{H}_M = \mathcal{H}_{\text{photon}} \otimes \mathcal{H}_{\text{pointer}}$ of M would then be composed of the Hilbert space $\mathcal{H}_{\text{photon}}$ of the photon and the Hilbert space $\mathcal{H}_{\text{pointer}}$ of the pointer. But this implies that the corresponding subspaces $\mathcal{H}_{M,k} = \mathcal{H}_{\text{photon}} \otimes \mathcal{H}_{\text{pointer},k}$ of the memory are also orthogonal with dimension larger than one if the Hilbert space of the photon is not zero dimensional.

To each subspace $\mathcal{H}_{M,k}$ of the memory M belongs a projection operator $Q_{M,k}$ on M such that $Q_{M,k}|\psi\rangle_M = |\psi\rangle_M$ for all vectors $|\psi\rangle_M$ within $\mathcal{H}_{M,k}$ and $\sum_k Q_{M,k} = \mathbb{1}_M$. Any state on M can then be written as $\rho_M = \sum_{k,k'} Q_{M,k}\rho_M Q_{M,k'}$. We call a state on M *classical* if it is of the particular form

$$\rho_M = \sum_k p_k \rho_{M,k} \tag{3.7}$$

with $\{p_k\}$ a probability distribution and $\rho_{M,k}$ a state with support on $\mathcal{H}_{M,k}$ only. This is equivalent to the condition that ρ_M commutes with all projections $\{Q_{M,k}\}$, i.e. $[\rho_M, Q_{M,k}] = 0$ for all k. We say that *each of the states $\rho_{M,k}$ on M stores a specific outcome k* since they are, by construction, mutually orthogonal for different k and hence perfectly distinguishable by the projections $\{Q_{M,k}\}$. These projections can therefore be used to read off the outcome k that is currently stored in M. Moreover, the information about the measurement outcome k is *classical* in the sense that we access M by the projections $\{Q_{M,k}\}$ *only* [73]. The state ρ_M as in Eq. (3.7) can therefore be interpreted as the state of a classical register that encodes the probabilities p_k that outcome k was outputted in the measurement.

In contrast to other frameworks (see e.g. [64] and Section 4.7), we need not place any (e.g. thermality) restriction on the initial state $\rho_M \in \mathcal{B}(\mathcal{H}_M)$ of the measurement device.

- **Thermal environment.** All systems that are neither the measured system S, nor the measurement device M, are assumed to be part of a thermal bath E that basically models the environment of the measurement device. Its defining property is that, whenever a measurement is performed, the environment is initially in a *thermal state* [46,74] at temperature T with respect to its Hamiltonian $H_E \in \mathcal{B}(\mathcal{H}_E)$, i.e.

  $$\rho_E = \frac{e^{-\frac{H_E}{k_B T}}}{Z_E},$$

  where k_B is the Boltzmann constant and $Z_E = \operatorname{tr}[e^{-\frac{H_E}{k_B T}}]$ is a normalising factor which is typically called the *partition function*.

  Later on, we will find that a resetting procedure is needed to include all energetic costs from a quantum measurement (the exact details of this procedure are explained in Section 3.3). This resetting procedure also has to make use of a thermal environment, which we will call B. It satisfies the same properties as the thermal bath E stated above. That is, its initial state ρ_B must be thermal at temperature T with respect to some Hamiltonian H_B, which may, but need not be identical to H_E. It is important that both environments E and B have the same temperature. Otherwise it is a well-known thermodynamic statement that an infinite amount of energy can be extracted from these two baths (see e.g. [39]). Without this assumption we may therefore not hope to establish any sensible bounds on the energy consumption of quantum measurement.

We assume *thermal* states on the environment for similar reasons: If ρ_E were just an arbitrary state, we could in general extract energy by unitary dynamics on E alone and hence without changing S at all. States ρ_E, which do not allow such energy extraction, i.e. $\mathrm{tr}[H_E(U_E\rho_E U_E^\dagger - \rho_E)] \leq 0$ for all unitaries U_E on E, are called *passive* [46,75]. Thermal states are even *completely passive*, because also the tensor product of arbitrarily many independent copies of these states is still passive, i.e. $\mathrm{tr}[H_E(U_E\rho_E^{\otimes n} U_E^\dagger - \rho_E^{\otimes n})] \leq 0$ for all unitaries $U_E \in \mathcal{B}(\mathcal{H}^{\otimes n})$ and all $n \in \mathbb{N}$ [74]. Moreover, thermal states are the *only* completely passive states. Hence, thermality of the environment is a natural assumption to prevent us from extracting energy directly from the environment. In addition, this assumption is supported by experimental data [50, 51].

Our description also includes (but is not restricted to) the intuitive model where the environment consists of many quantum systems, each with its own Hamiltonian, but all initially in a thermal state. The only fixed property of the environment is hence its temperature T, which can be considered to be a macroscopically accessible property. If a measurement is performed, only a certain part of the environment interacts with the measured system S and the measurement device M. We do not know which part of the thermal bath it is exactly, hence we also do not know its Hamiltonian. However, we know that this part will be in some thermal state ρ_E at temperature T with respect to some unknown, but fixed Hamiltonian H_E. After the measurement we cannot access this part of the environment any more. Instead, future measurements will affect *another* part of the environment, which is again in a thermal state.

3.3 The energy cost of quantum measurements

Based on the Assumptions 1 to 3, we now develop a definition for the energy cost of quantum measurements. For this, we introduce the notion of an *implementation* of a quantum measurement which corresponds to a specific realisation of the measurement, which specifies all the relevant microscopic details of the process. Moreover, we motivate the need of an additional *resetting step* which follows the measurement, in order to allow for many uses of the same implementation of the quantum measurement. Just like the quantum measurement itself, this resetting process can also be realised in various ways, i.e. there are various *implementations of the resetting step*. Given these two implementations, i.e. the microscopic realisations of measurement and resetting, and the initial configuration of the measured system S, i.e. initial state ρ_S and Hamiltonian H_S, we can then compute the average energy consumption of these processes. This energy cost will depend on the chosen implementations. This seems problematic since, in practical situations we generally do not know the microscopic details of all involved systems; in contrast, we are only given the initial configuration of the measured system S and the specification of the abstract quantum measurement, i.e. for example the measurement operators $\{M_{ki}\}$ of a quantum instrument. We therefore define the energy cost of a quantum measurement as the *minimal* energy consumption of all possible implementations of the measurement and resetting.

Let us now put precise meanings to the notions just introduced.

According to the Assumptions 1 to 3, performing a quantum measurement on system S corresponds to applying a unitary U_{SME} on S, memory M, and thermal environment E, which are initially in the product state ρ_{SME}. We will later see that this unitary must satisfy certain properties which depend on the quantum measurement and which we have not specified yet. Let us nevertheless assume that we have found such a unitary for a given quantum measurement. Our microscopic description of the measurement is then completely determined by the state spaces $\mathcal{H}_M$ (along with the set of projections $\{Q_{M,k}\}$) and $\mathcal{H}_E$, the initial states ρ_M and ρ_E, the Hamiltonians H_M and H_E, and the unitary U_{SME}. We call the collection of all these quantities an *implementation of the quantum measurement*. In order to simplify the notation, we will denote an implementation by the tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$, whenever the other quantities are clear from the context or irrelevant. A precise definition of implementations is provided for each formalism of quantum measurement in the following sections.

The set of all implementations of a given quantum measurement is denoted by $\mathcal{U}_{\mathcal{M}}$, where the index $\mathcal{M}$ specifies the quantum measurement. For example, the set of all implementations of the state transformation $\rho_S \mapsto \rho'_S$ is denoted by $\mathcal{U}_{\rho_S \mapsto \rho'_S}$. Typically, there exist many implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\mathcal{M}}$ for a given quantum measurement. This is in parallel with the intuition that there exist many measurement devices that can be used to perform the same measurement.

Assume now that the tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is an implementation of a given quantum measurement and suppose we perform this measurement via this implementation on a system S in state ρ_S. After the measurement, the involved systems are in the state $\rho'_{SME} = U_{SME}\rho_{SME}U^{\dagger}_{SME}$ and all these systems have experienced state changes during the measurement in general. Clearly, this is to be expected for the measured system S. Also the state of the memory needs to change from ρ_M to the classical state ρ'_M in order to store the measurement outcomes properly (if we employ a formalism capable of recording outcomes). Suppose now that we want to perform a second measurement using the *same* measurement device and the *same* environment E. Note however that whereas the state of the thermal environment is again ρ_E as for the first measurement (see Assumption 1 and the description of the thermal environment), the state of the measurement device is now ρ'_M. Applying the same unitary U_{SME} on $\rho_S \otimes \rho'_M \otimes \rho_E$ will in general not induce the desired dynamics since the tuple $(\rho'_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ will not be an implementation of the quantum measurement. It is easy to make this observation more precise once we have provided the proper mathematical definitions for implementations. For now, however, it suffices to think of this argument as saying that a measurement device, which is not initialised in its proper initial state, cannot in general be used for further measurements.

The key issue is hence that the initial state of the memory is not restored before the next measurement is performed. We therefore include a *resetting step* into our description of a quantum measurement, whose sole purpose is to restore the initial state of the memory.

Following Assumption 3 the overall dynamics of the resetting step shall be unitary to ensure that we include all relevant energy contributions. The resetting step can however not be achieved by unitary dynamics on M alone, because the spectrum of the state on M will typically change during the measurement. The unitary must therefore act on other quantum systems. Following Assumption 1, we could either use the (already measured) system S or a thermal environment B[6]. However, the resetting step should be independent of the measured system S, because the state of this system may already have changed before the resetting step acts: Consider for example a feedback protocol, where we want to perform further actions on the measured system S based on the measurement outcome k. Since these actions depend the outcome, we must not reset the memory before conducting the feedback protocol. But after the feedback protocol the state of S will have changed, such that the resetting should be independent of S.

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[6] We could, of course, also use the thermal environment E instead of B. The only important physical property here is that the environment starts in a thermal state as usual. The use of two environments E and B is only for notational reasons: Whenever an environment E is used, the considered process is the measurement step, whereas environments B only appear in the corresponding resetting step.

We therefore model the resetting step by including a thermal bath B, which leads to the following definition.

Definition 3.7. The resetting step.

Consider the state $\rho'_M = \mathrm{tr}_{SE}[U_{SME}\rho_{SME}U^\dagger_{SME}]$ of a memory M after an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_\mathcal{M}$ of some given quantum measurement $\mathcal{M}$ has been performed on a system S in state ρ_S. Then an *implementation of the resetting step* is a tuple (ρ_B, U_{MB}) such that $\rho_B = \mathrm{e}^{-\frac{H_B}{k_B T}}/Z_B$ is the initial state of a thermal environment B as in Section 3.2 and U_{MB} is a unitary satisfying

$$\mathrm{tr}_B[U_{MB}(\rho'_M \otimes \rho_B)U^\dagger_{MB}] = \rho_M$$

for all ρ_S that the quantum measurement may be performed on.

Note that we require in this definition that an implementation of the resetting step yields the proper initial state ρ_M on M no matter which state ρ_S was being measured in the first place. For example, if the measurement was a quantum channel $\mathcal{F}$, then the resetting step must reset the state ρ'_M to ρ_M *for all* states $\rho_S \in \mathcal{B}(\mathcal{H}_S)$, since we could have applied the channel $\mathcal{F}$ to any of these states. In contrast, if the measurement is just the unselective measurement $\rho_S \mapsto \rho'_S$, then the resetting step must only reset M for this particular input state ρ_S.

Let us now define the energy consumption of such implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ and (ρ_B, U_{MB}) of the quantum measurement and the resetting step, respectively. Note that we have not yet specified what requirements we place on implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of the measurement. We will therefore provide the formal definition of their energy cost later in this section, but discuss here already the general scheme of this definition. For the resetting step however, we already defined when a tuple (ρ_B, U_{MB}) is an implementation, so we can directly provide the formal definition of the energy cost of the resetting step.

Definition 3.8. The energy cost of the resetting step.

Let (ρ_B, U_{MB}) be an implementation of the resetting step of the memory M in state ρ'_M after some implementation of a quantum measurement $\mathcal{M}$ as in Definition 3.7. The *energy cost of this implementation* is then defined as the average energy change on M and B

$$E_{\mathrm{res}}^{(\rho_B, U_{MB})} = \mathrm{tr}[H_{MB}(U_{MB}(\rho'_M \otimes \rho_B)U^\dagger_{MB} - \rho'_M \otimes \rho_B)] \,.$$

The *energy cost of the resetting step* is defined as the minimal energy cost of all implementations of the resetting step

$$E_{\mathrm{res}}^{\mathcal{M}} = \inf_{(\rho_B, U_{MB})} E_{\mathrm{res}}^{(\rho_B, U_{MB})} \,.$$

The energy cost of the resetting step therefore corresponds to the minimal average energy change of the two systems which are involved in the process, namely memory M and environment B. Note that the unitarity of the process ensures that there are no other systems whose energy is changed in the resetting step. The quantity $E_{\text{res}}^{\mathcal{M}}$ therefore precisely captures the overall energy consumption of the resetting step.

We define the energy cost of a particular implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of a quantum measurement in a similar fashion: Since we want to account for *all* energy changes that occur in the involved systems in the process, we first consider the quantity

$$E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \operatorname{tr}[H_{SME}(\rho'_{SME} - \rho_{SME})] \,,$$

where $\rho'_{SME} = U_{SME}\rho_{SME}U_{SME}^{\dagger}$ is the state of all involved systems after the unitary interaction U_{SME} specified by the implementation. The quantity $E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ therefore corresponds to the average energy change of all involved systems in the quantum measurement (without resetting).

Since we always need to include the energy cost $E_{\text{res}}^{\mathcal{M}}$ of the resetting step for a correct overall energy balance, we define the energy cost $E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ of a particular implementation as in the following definition. Based on this quantity, we can then also define the energy cost of the quantum measurement $\mathcal{M}$ as the minimal average energy change for all implementations.

Definition 3.9. The energy cost of a quantum measurement.

Let $\mathcal{M}$ be a quantum measurement, i.e. either a state transformation, a quantum channel, a quantum instrument or a POVM, on a quantum system S with Hamiltonian H_S in state ρ_S. Denote by $\mathcal{U}_{\mathcal{M}}$ the set of implementations of $\mathcal{M}$ and let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\mathcal{M}}$ be a particular implementation of $\mathcal{M}$. Then the *energy cost of the implementation* $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is defined as

$$E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} + \inf_{(\rho_B,U_{MB})} E_{\text{res}}^{(\rho_B,U_{MB})} \,,$$

which corresponds to the total average energy change of measurement and resetting if the measurement is implemented via $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ and the resetting is implemented *optimally.*

The *energy cost of the quantum measurement* $\mathcal{M}$ is then defined as

$$\begin{aligned} E_{\text{cost}}^{\mathcal{M}} &= \inf_{\mathcal{U}_{\mathcal{M}}} E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} \\ &= \inf_{(\rho_B,U_{MB})} \inf_{\mathcal{U}_{\mathcal{M}}} \Big(E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} + E_{\text{res}}^{(\rho_B,U_{MB})} \Big) , \end{aligned}$$

i.e. as the minimal average energy change on all involved systems for both measurement and resetting over all implementations.

Let us provide some intuition for this rather abstract definition by comparing it to the following naive definition: Consider for example a quantum instrument $\{M_{ki}\}$ on a system S with Hamiltonian H_S initially in state ρ_S. This formalism of measurement determines the average post-measurement state $\rho'_S = \sum_{ki} M_{ki} \rho_S M_{ki}^\dagger$ on S (see Definition 3.2). Naively, one may then think that the energy cost of this quantum instrument is simply given as the average energy change on S, i.e.

$$\Delta E_S = \text{tr}[H_S(\rho'_S - \rho_S)] .$$

But note that this description ignores all other systems needed to physically implement the measurement. Indeed, by Eq. (3.6) from Assumption 1 of our framework and by Definition 3.9, we find that the correct energy cost of the quantum instrument $\{M_{ki}\}$ is instead given by

$$E_{\text{cost}}^{\{M_{ki}\}} = \Delta E_S + \inf_{(\rho_B,U_{MB})} \inf_{\mathcal{U}_{\{M_{ki}\}}} \Big(\Delta E_E + \Delta E_B \Big) . \tag{3.8}$$

This shows that physically implementing the quantum instrument $\{M_{ki}\}$ demands the *extra* cost

$$\Delta Q_{\{M_{ki}\}} = \inf_{(\rho_B,U_{MB})} \inf_{\mathcal{U}_{\{M_{ki}\}}} \Big(\Delta E_E + \Delta E_B \Big) , \tag{3.9}$$

in addition to ΔE_S, where $\Delta E_E = \text{tr}[H_E(\rho'_E - \rho_E)]$ and ΔE_B analogously (note that $\Delta E_M = 0$, since initial and final states of M are identical due to the resetting). In Section 3.4 we explain why the additional cost $\Delta Q_{\{M_{ki}\}}$ corresponds to the *heat dissipation* caused by the quantum instrument.

Of course, this discussion also holds for all other formalisms of quantum measurement (except POVMs): Quantum channels and unselective measurements also determine the average post-measurement state and their energy consumption is also determined by ΔE_S and an additional heat dissipation term.

We still have not specified how exactly an implementation of a quantum measurement is defined. However, it was sufficient so far to first understand the general construction for our definition of energetic cost, before delving into the detailed definition of implementations. We now provide these definitions separately for each of the four formalism of quantum measurement.

Implementations of unselective measurements

An unselective measurement $\rho_S \mapsto \rho'_S$ on a quantum system S corresponds to a map that changes the fixed input ρ_S to the fixed output state ρ'_S (see Definition 3.4). According to Assumptions 1 to 3, the physical process behind these dynamics on S is governed by a unitary U_{SME} on S, memory M, and environment E. The minimal assumption on this unitary is that the final state on S after applying U_{SME} is identical to the desired fixed output state ρ'_S, i.e.

$$\mathrm{tr}_{ME}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U^\dagger_{SME}] = \rho'_S\,. \tag{3.10}$$

for the given intial state ρ_S. This suggests to define an implementation of the unselective measurement as a tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$, where ρ_M is any state on a memory M, $\{Q_{M,k}\}$ is any set of mutually orthogonal projections such that $\sum_k Q_{M,k} = \mathbb{1}_M$, ρ_E is a thermal state on E and U_{SME} is a unitary satisfying Eq. (3.10). Following Definition 3.9 and Eq. (3.8), the energy cost of an unselective measurement is then given as

$$E^{\rho_S \mapsto \rho'_S}_{\text{cost}} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\rho_S \mapsto \rho'_S}} \Big(\Delta E_E + \Delta E_B\Big)\,. \tag{3.11}$$

While this is basically how our final definition of the energy cost will look like, we can simplify this quantity: Note that the formalism of unselective measurements does not include the notion of measurement outcomes k, hence the memory does not have any particular purpose for this measurement formalism. This is reflected by the fact that our only requirement on the unitary is given by Eq. (3.10), which does not make any difference between the memory M and the environment E. Still however, there is a *physical* difference between the memory M and the environment E, since M need not be thermal and needs to be reset. We can thus simplify the description as follows: If we consider the overall process of measurement and resetting for given implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ and (ρ_B, U_{MB}), the state of all systems after this process can be written as

$$\tilde{\rho}_{SMEB} = (\mathbb{1}_S \otimes U_{MB} \otimes \mathbb{1}_E)(U_{SME} \otimes \mathbb{1}_B)(\rho_S \otimes \rho_M \otimes \rho_E \otimes \rho_B)(U_{SME} \otimes \mathbb{1}_B)^\dagger(\mathbb{1}_S \otimes U_{MB} \otimes \mathbb{1}_E)^\dagger\,,$$

which simply corresponds to first applying the measurement interaction U_{SME}, followed by the resetting interaction U_{MB}. Since ρ_E and ρ_B are thermal states, their product $\rho_{\tilde{E}} = \rho_E \otimes \rho_B$ is again thermal. We can therefore always rewrite $\tilde{\rho}_{SMEB}$ as

$$\tilde{\rho}_{SMEB} = U_{SM\tilde{E}}(\rho_S \otimes \rho_M \otimes \rho_{\tilde{E}})U^\dagger_{SM\tilde{E}}\,,$$

where the unitary $U_{SM\tilde{E}} = (\mathbb{1}_S \otimes U_{MB} \otimes \mathbb{1}_E)(U_{SME} \otimes \mathbb{1}_B)$ is just the product of measurement interaction and the resetting interaction. The state $\tilde{\rho}_{SMEB}$ after measurement and resetting can therefore be understood as the resulting state of a *single* measurement step *without* resetting, characterised by the “modified” measurement interaction $U_{SM\tilde{E}}$ applied to system S, environment $\tilde{E}$ and a system M, whose initial and final state now coincide. Hence, M serves as a mere *catalyst* in this measurement [46, 67, 68]. The energy cost $E_{\text{cost}}^{\rho_S \mapsto \rho'_S}$ of the unselective measurement from Eq. (3.11) is then given as the average energy change of all involved systems

$$E_{\text{cost}}^{\rho_S \mapsto \rho'_S} = \Delta E_S + \inf_{\tilde{\mathcal{U}}_{\rho_S \mapsto \rho'_S}} \Delta E_{\tilde{E}} \ ,$$

where $\tilde{U}_{\rho_S \mapsto \rho'_S}$ denotes the set of all the corresponding “modified” implementations $(\rho_M, \rho_{\tilde{E}}, U_{SM\tilde{E}})$ of $\rho_S \mapsto \rho'_S$.

Using results which we develop in Chapter 4 (see, in particular, the end of Section 4.2.2), it is easy to see that allowing for catalysts in the process does not decrease the energy cost of unselective measurements. We may therefore without loss of generality ignore the catalyst M, and state the following final definitions for the case of unselective measurements.

Definition 3.10. Implementation of an unselective measurement

Let $\rho_S \mapsto \rho'_S$ be an unselective measurement on a quantum system S in state ρ_S. An *implementation of the unselective measurement* is then a tuple (ρ_E, U_{SE}) such that ρ_E is the thermal initial state of the environment E and U_{SE} is a unitary satisfying

$$\mathrm{tr}_E[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger] = \rho'_S \ .$$

The set $\mathcal{U}_{\rho_S \mapsto \rho'_S}$ is defined as the set of all implementations of the given unselective measurement $\rho_S \mapsto \rho'_S$.

The energy cost of an unselective measurement is then defined as follows.

Definition 3.11. Energy cost and heat dissipation of an unselective measurement

Let $\rho_S \mapsto \rho'_S$ be an unselective measurement on a quantum system S with Hamiltonian H_S in state ρ_S and denote by $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ an implementation of $\rho_S \mapsto \rho'_S$.
The *energy cost of this implementation* is then defined as

$$E_{\text{cost}}^{(\rho_E, U_{SE})} = \Delta E_S + \Delta E_E \ ,$$

where $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy change on the measured system S and ΔE_E analogously. The *energy cost of the unselective* *measurement* $\rho_S \mapsto \rho'_S$ is defined as

$$E_{\text{cost}}^{\rho_S\mapsto\rho'_S} = \Delta E_S + \inf_{\mathcal{U}_{\rho_S\mapsto\rho'_S}} \Delta E_E \ .$$

Moreover, the *dissipated heat* during the unselective measurement is defined as

$$\Delta Q_{\rho_S\mapsto\rho'_S} = \inf_{\mathcal{U}_{\rho_S\mapsto\rho'_S}} \Delta E_E \ .$$

Implementations of quantum channels

Quantum channels are similar to unselective measurements in that both formalisms do not track the measurement outcomes k. Quantum channels however possess additional structure since they specify the average post-measurement state $\rho'_S = \mathcal{F}(\rho_S)$ for *all* states ρ_S on S. We need to adjust our definition for implementations accordingly.

Definition 3.12. Implementations of a quantum channel

Let $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ be a quantum channel on a quantum system S. An *implementation of the quantum channel* is then a tuple (ρ_E, U_{SE}) such that ρ_E is the thermal initial state of the environment E and U_{SE} is a unitary satisfying

$$\operatorname{tr}_E[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger] = \mathcal{F}(\rho_S) \qquad \forall \rho_S \in \mathcal{B}(\mathcal{H}_S)\,.$$

The set $\mathcal{U}_\mathcal{F}$ is defined as the set of all implementations of the quantum channel $\mathcal{F}$.

Note the additional quantifier "$\forall \rho_S$" in the requirement on the unitary U_{SE}, which ensures that the dynamics on S are given as prescribed by the quantum channel formalism.

Do such implementations exist for all quantum channels? Unfortunately, this is not the case. The key issue is here the requirement for ρ_E to be thermal and hence of full rank. To see why this can be problematic, consider the channel $\mathcal{F}(\rho_S) = \psi_S$ that maps any initial state ρ_S to a fixed pure state ψ_S. For such a channel there does not exist an implementation as defined above: Since any implementation of this quantum channel must output the pure state ψ_S for all initial states ρ_S by definition, we may choose ρ_S to be of full rank for our argument. Now, unitaries preserve the spectrum and hence the rank of the state. This implies that the state

$\rho'_{SE} = U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger$ is still of full rank. But then the state $\rho'_S = \operatorname{tr}_E \rho'_{SE}$ is not pure as required since its rank is greater than one (unless, of course, $\dim \mathcal{H}_S = 1$).[7]

While it is helpful to keep this mathematical issue in mind, the physical consequences are rather limited: We can always find a Stinespring dilation of the "problematic" $\mathcal{F}$, such that $\mathcal{F}(\rho_S) = \operatorname{tr}_A[U_{SA}(\rho_S \otimes \rho_A)U_{SA}^\dagger]$ for all ρ_S, where A is an ancilla system initially in a pure state ρ_A and U_{SA} is a unitary (see Corollary 2.5). Now consider the channel $\tilde{\mathcal{F}}(\rho_S) = \operatorname{tr}_A[U_{SA}(\rho_S \otimes \tilde{\rho}_A)U_{SA}^\dagger]$ for all ρ_S, where $\tilde{\rho}_A$ is a full-rank state. Note that $\tilde{\rho}_A$ can be chosen as close to the pure state ρ_A as desired. The dynamics of this modified channel $\tilde{\mathcal{F}}$ cannot be distinguished from the problematic channel $\mathcal{F}$ by any means (if the approximation is good enough) and furthermore admits a proper implementation as defined above, namely $(\tilde{\rho}_A, U_{SA})$.

In complete analogy to Definition 3.11 for unselective measurements, we define the energy cost for quantum channels as follows.

Definition 3.13. Energy cost and heat dissipation of a quantum channel

Let $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ be a quantum channel on a quantum system S with Hamiltonian H_S in state ρ_S and denote by $(\rho_E, U_{SE}) \in \mathcal{U}_\mathcal{F}$ an implementation of $\mathcal{F}$.

The *energy cost of this implementation* is then defined as

$$E_{\text{cost}}^{(\rho_E, U_{SE})} = \Delta E_S + \Delta E_E \ ,$$

where $\Delta E_S = \operatorname{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy change on the measured system S and ΔE_E analogously. The *energy cost of the quantum channel $\mathcal{F}$* is given as

$$E_{\text{cost}}^{\mathcal{F}} = \Delta E_S + \inf_{\mathcal{U}_\mathcal{F}} \Delta E_E \ .$$

Moreover, the *dissipated heat* during the quantum channel is defined as

$$\Delta Q_\mathcal{F} = \inf_{\mathcal{U}_\mathcal{F}} \Delta E_E \ .$$

Implementations of quantum instruments

In contrast to unselective measurements and quantum channels, the last two formalisms, quantum instruments and POVMs, keep track of the measurement outcomes k. The memory M in the corresponding implementations will therefore play

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[7] This issue can also not be resolved by explicitly including the memory into our definition for implementations and letting the memory M start in an initially pure state ρ_M. While this would allow to find physical implementations for the quantum channel, this line of thought only shifts the problem to the resetting step, where then again a non-pure quantum state ρ'_M needs to be mapped to a fixed pure state ρ_M with the help of a (full rank) thermal state ρ_B.

an important role and cannot any more be neglected without missing essential energetic contributions.

Recall from Assumption 1 that the memory is required to store the classical information about the outcomes after the measurement, i.e. the post-measurement state of the memory must be classical as in Eq. (3.7). The following definition for an implementation of a quantum instrument incorporates these requirements.

Definition 3.14. Implementations of a quantum instrument

Let $\{M_{ki}\}$ be a quantum instrument on a quantum system S. An *implementation of this quantum instrument* is then a tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ such that

Condition 0: ρ_M is a quantum state on a memory M, $\{Q_{M,k}\}$ are mutually orthogonal projections on M such that $\sum_k Q_{M,k} = \mathbb{1}_M$, ρ_E is a thermal state on environment E at temperature T and U_{SME} is a unitary on S, M, and E;

Condition 1: The probabilities and post-measurement states agree, i.e.

$$\sum_i M_{ki}\rho_S M_{ki}^\dagger = \mathrm{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME}\rho_{SME} U_{SME}^\dagger] \tag{3.12}$$

for all states $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ and all possible measurement outcomes k;

Condition 2: The memory M is classical after the measurement, i.e.

$$\left[(\mathbb{1}_S \otimes Q_{M,k}), \rho'_{SM}\right] = 0 \qquad \forall \rho_S \forall k\,, \tag{3.13}$$

where $\rho'_{SM} = \mathrm{tr}_E[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger]$ denotes the state on S and M after measurement.

The set $\mathcal{U}_{\{M_{ki}\}}$ is defined as the set of all implementations of the quantum instrument $\{M_{ki}\}$.

Let us provide some intuition for this definition. Condition 0 is simply a restatement of the requirements on the involved system in an implementation (see Assumption 1). Condition 1 is analogous to the requirements on the unitary U_{SE} in Definitions 3.10 and 3.12 for unselective measurements and quantum channels, respectively: It ensures that the dynamics on the measured system are as prescribed by the measurement operators M_{ki} of the quantum instrument. On the one hand it states that the probability that we read off outcome k after measurement is

$$p_k = \mathrm{tr}[Q_{M,k}\rho'_M] = \mathrm{tr}\left[\sum_i M_{ki}\rho_S M_{ki}^\dagger\right]$$

for all ρ_S, and is hence identical to the prescription in Definition 3.2 for quantum instruments. On the other hand, also the post-measurement states on S

$$\rho'_{S,k} = \mathrm{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME} \rho_{SME} U^\dagger_{SME}]/p_k = \sum_i M_{ki} \rho_S M^\dagger_{ki}/p_k$$

agree for any outcome k and any initial state ρ_S with the requirement of Definition 3.2.

Condition 2 ensures that the memory is in a classical state after the measurement, i.e. it is block-diagonal with respect to the projections $\{Q_{M,k}\}$ and hence does not contain coherences between subspaces $\mathcal{H}_{M,k}$ and $\mathcal{H}_{M,k'}$ corresponding to different outcomes $k \neq k'$. However, there should be more to this definition, since classicality of the memory would already be obtained if we would replace Condition 2 with the weaker requirement

$$\left[Q_{M,k}, \rho'_M\right] = 0 \qquad \forall \rho_S \forall k\,, \tag{3.14}$$

where $\rho'_M = \mathrm{tr}_S[\rho'_{SM}]$ is the state on M after the measurement (see explanation after Eq. (3.7)). Interestingly, this requirement is appropriate for the definition of implementations of POVMs (see Definition 3.19). Why then is this requirement inappropriate for quantum instruments? To see this, consider a feedback protocol where the *feedback unitary*

$$U_{\mathrm{fb}} = \sum_k V_{S,k} \otimes Q_{M,k}\,, \tag{3.15}$$

with $V_{S,k}$ being unitaries on the measured system, is applied on the state ρ'_{SM} on S and M after measurement. Intuitively, one would expect that the memory remains unchanged by this operation: The feedback unitary just reads out the classical information k from the classical memory M and correspondingly applies the unitary $V_{S,k}$ on S. However, this is generally not true if we only demand the weaker requirement Eq. (3.14). That is, demanding $\left[Q_{M,k}, \rho'_M\right] = 0$ for all ρ_S and k does not generally imply that

$$\left[Q_{M,k}, \mathrm{tr}_S[U_{\mathrm{fb}} \rho'_{SM} U^\dagger_{\mathrm{fb}}]\right] = 0 \qquad \forall \rho_S \forall k\,,$$

which means that the state $\mathrm{tr}_S[U_{\mathrm{fb}} \rho'_{SM} U^\dagger_{\mathrm{fb}}]$ on M after feedback is not classical any more.

In contrast, when using the stronger requirement Eq. (3.13) from Definition 3.14 the memory remains classical even after feedback. More concretely, if $\left[(\mathbb{1}_S \otimes Q_{M,k}), \rho'_{SM}\right] = 0$ holds for all ρ_S and k, then we also have that $\left[(\mathbb{1}_S \otimes Q_{M,k}), U_{\mathrm{fb}} \rho'_{SM} U^\dagger_{\mathrm{fb}}\right] = 0$ for all ρ_S, all k and all feedback unitaries U_{fb} of the form Eq. (3.15). This is simply because, by construction, the feedback unitary U_{fb} commutes with the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$ and implies that

$$\big[Q_{M,k}, \operatorname{tr}_S\big[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\big]\big] = 0$$

for all ρ_S, all k and all feedback unitaries U_{fb} of the form Eq. (3.15). This means that the memory state is classical with respect to the projections $\{Q_{M,k}\}$ even after feedback if ρ'_{SM} satisfies Condition 2 from Definition 3.14.

Indeed, we can even show this to be an "if and only if" statement: As soon as the state ρ'_{SM} does not commute with the projections $(\mathbb{1}_S \otimes Q_{M,k})$, i.e. $\big[(\mathbb{1}_S \otimes Q_{M,k}), \rho'_{SM}\big] \neq 0$ for some k, there are feedback unitaries U_{fb} such that even the marginal state $\operatorname{tr}_S\big[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\big]$ on M after this feedback unitary is not classical anymore with respect to the projections $\{Q_{M,k}\}$, i.e. $\big[Q_{M,k}, \operatorname{tr}_S\big[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\big]\big] \neq 0$ for some k. For clarity, we formulate this statement in the following lemma.

> **Lemma 3.15.** *Let $\{Q_{M,k}\}$ be mutually orthogonal projections on a quantum system M such that $\sum_k Q_{M,k} = \mathbb{1}_M$ and let ρ'_{SM} be any quantum state on quantum systems S and M such that $\big[(\mathbb{1}_S \otimes Q_{M,k}), \rho'_{SM}\big] \neq 0$ for some k. Then there exist unitaries $V_{S,k}$ on S such that*
>
> $$\big[Q_{M,k}, \operatorname{tr}_S\big[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\big]\big] \neq 0$$
>
> *for some k, where $U_{\text{fb}} = \sum_k V_{S,k} \otimes Q_{M,k}$.*

Proof. For any fixed k we choose an orthonormal basis $\{|ki\rangle\}_i$ of the subspace $Q_{M,k}\mathcal{H}_M$, where the range of the index i corresponds to the dimension of the subspace $Q_{M,k}\mathcal{H}_M$. The set $\{|ki\rangle\}_{k,i}$ of all these subspace bases is therefore an orthonormal basis of the full Hilbert space $\mathcal{H}_M$.

We can then always write ρ'_{SM} as

$$\rho'_{SM} = \sum_{k,k'} \sum_{i,j} X_{ij}^{(kk')} \otimes |ki\rangle_M \langle k'j| \,,$$

where $X_{ij}^{(kk')} \in \mathcal{B}(\mathcal{H}_S)$ for all k, k', i, j. In this representation, we can compute the state on M after applying a feedback unitary U_{fb} of the form $U_{\text{fb}} = \sum_k V_{S,k} \otimes Q_{M,k}$:

$$\operatorname{tr}_S\big[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\big] = \sum_{k,k'} \sum_{i,j} \operatorname{tr}\big[V_{S,k} X_{ij}^{(kk')} V_{S,k'}^\dagger\big] \cdot |ki\rangle_M \langle k'j| \,.$$

This state commutes with all projections $Q_{M,k}$ if and only if $\operatorname{tr}\big[V_{S,k} X_{ij}^{(kk')} V_{S,k'}^\dagger\big] = 0$ for all i, j and all $k \neq k'$. Conversely, we know that this state does *not* commute with these projections, if we have that

$$\operatorname{tr}\big[V_{S,k} X_{ij}^{(kk')} V_{S,k'}^\dagger\big] \neq 0$$

for some fixed i, j and some fixed $k \neq k'$.

Let now ρ'_{SM} be such that $\left[(\mathbb{1}_S \otimes Q_{M,k}), \rho'_{SM}\right] \neq 0$ for some k. This implies that there exist $\hat{k}$ and $\hat{k}'$ with $\hat{k} \neq \hat{k}'$ such that $X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')}$ is not zero for some $\hat{\imath}, \hat{\jmath}$. We can then consider the singular value decomposition of $X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')}$, i.e.

$$X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')} = U D \tilde{U} \ ,$$

where U and $\tilde{U}$ are unitary matrices and D is the diagonal matrix containing the non-negative singular values of $X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')}$. Furthermore, since $X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')} \neq 0$, we know that at least one singular value is strictly positive, which implies that $\mathrm{tr}[D] > 0$.

Let us now choose the feedback unitary U_{fb} to be such that $V_{S,\hat{k}} = U^\dagger$ and $V_{S,\hat{k}'} = \tilde{U}$ (and all other $V_{S,k}$ arbitrary). We then have that

$$\mathrm{tr}\left[V_{S,\hat{k}} X_{\hat{\imath}\hat{\jmath}}^{(\hat{k}\hat{k}')} V_{S,\hat{k}'}^\dagger\right] = \mathrm{tr}[D] > 0 \ .$$

As argued above, this implies that the state $\mathrm{tr}_S\left[U_{\text{fb}} \rho'_{SM} U_{\text{fb}}^\dagger\right]$ does not commute with all projections $Q_{M,k}$, which concludes the proof. □

This lemma shows that the stronger requirement Eq. (3.13) from Definition 3.14 is essential to ensure that the memory above remains classical even after feedback.

Do implementations as in Definition 3.14 exist for all quantum instruments? The following proposition shows that this is indeed the case.

> **Proposition 3.16.** *For any quantum instrument $\{M_{ki}\}$, there exists an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$. Conversely, for any tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ satisfying Condition 0 and Condition 2, there exists a quantum instrument $\{M_{ki}\}$ such that Condition 1 is satisfied, i.e the tuple is an implementation of this instrument.*

Proof. In order to prove the first part of the statement we construct an implementation directly from the quantum instrument. For this, let $\{M_{ki}\}$ denote the measurement operators of the quantum instrument. We take the memory M to be a quantum system with Hilbert space $\mathcal{H}_M = \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}$ where $\mathcal{H}_{M_2}$ is a Hilbert space with dimension $\dim \mathcal{H}_{M_2} \geq I$ with I the inefficiency of $\{M_{ki}\}$ and $\mathcal{H}_{M_1} = \bigoplus_k \mathcal{H}_{M_1,k}$ is the direct sum of one-dimensional Hilbert spaces $\mathcal{H}_{M_1,k}$ with projections $\tilde{Q}_{M_1,k} = |k\rangle_{M_1}\langle k|$ satisfying $\tilde{Q}_{M_1,k}\psi_k = \psi_k$ for all $\psi_k \in \mathcal{H}_{M_1,k}$. The memory is initialised in the pure product state $\rho_M = |0\rangle_{M_1}\langle 0| \otimes |0\rangle_{M_2}\langle 0|$, where $|0\rangle_{M_1} \in \mathcal{H}_{M_1}$ and $|0\rangle_{M_2} \in \mathcal{H}_{M_2}$, and the projection operators are chosen as $Q_{M,k} = \tilde{Q}_{M_1,k} \otimes \mathbb{1}_{M_2}$. The initial state ρ_E of the environmental thermal bath E may be arbitrary as long as it is thermal with respect to some Hamiltonian H_E at temperature T and as long as the dimension of E is large enough (this is specified in the following). The unitary interaction U_{SME} is chosen to be

$$U_{SME} = \sum_k (\mathbb{1}_S \otimes Q_{M,k} \otimes V_{E,k})(U_{SM} \otimes \mathbb{1}_E)\,, \tag{3.16}$$

where $V_{E,k}$ is a unitary satisfying

$$\operatorname{tr}[V_{E,k}\rho_E V_{E,k'}^\dagger] = \delta_{k,k'} \qquad \forall k,k' \tag{3.17}$$

and U_{SM} is any unitary completion of the operator $\sum_{kj} M_{kj} \otimes |k\rangle_{M_1}\langle 0| \otimes |j\rangle_{M_2}\langle 0|$ on S and M. Note that such a set $\{V_{E,k}\}$ of unitaries always exists if the dimension of E is large enough, at least if we choose $\rho_E = \mathbb{1}_E/d_E$ maximally mixed. For this state, the condition Eq. (3.17) translates to $\operatorname{tr}[V_{E,k}V_{E,k'}^\dagger] = d_E\delta_{k,k'}$, implying that the operators $\{V_{E,k}\}$ form a unitary operators basis with respect to the standard Hilbert-Schmidt scalar product. A specific construction are the so-called Heisenberg-Weyl operators [55, 76].

Let us check that $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is really an implementation of the quantum instrument $\{M_{ki}\}$. For this we need to check that Conditions 0, 1 and 2 of Definition 3.14 are satisfied. By construction, we know that Condition 0 is satisfied. Also, due to the form of U_{SME}, it is clear that the state

$$\begin{aligned}
\rho'_{SM} &= \operatorname{tr}_E[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger] \\
&= \sum_{k,k'} (\mathbb{1}_S \otimes Q_{M,k})U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}_S \otimes Q_{M,k'}) \cdot \operatorname{tr}[V_{E,k}\rho_E V_{E,k'}^\dagger] \\
&= \sum_k (\mathbb{1}_S \otimes Q_{M,k})U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}_S \otimes Q_{M,k})
\end{aligned}$$

after the unitary interaction is block diagonal with respect to the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$ as required by Condition 2. It therefore remains to verify Condition 1. Indeed, we have

$$\begin{aligned}
&\operatorname{tr}_{ME}\big[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E)U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger\big] \\
&= \sum_{k',k''} \operatorname{tr}_M\big[(\mathbb{1}_S \otimes Q_{M,k})(\mathbb{1}_S \otimes Q_{M,k'})U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}_S \otimes Q_{M,k''})\big] \cdot \operatorname{tr}[V_{E,k'}\rho_E V_{E,k''}^\dagger] \\
&= \operatorname{tr}_M[(\mathbb{1}_S \otimes Q_{M,k})U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}_S \otimes Q_{M,k})] \\
&= \operatorname{tr}_M\Big[(\mathbb{1}_S \otimes Q_{M,k})\Big(\sum_{k,j,k',j'} M_{kj}\rho_S M_{k'j'}^\dagger \otimes |k\rangle_{M_1}\langle k'| \otimes |j\rangle_{M_2}\langle j'|\Big)(\mathbb{1}_S \otimes Q_{M,k})\Big] \\
&= \operatorname{tr}_M\Big[\sum_{j,j'} M_{kj}\rho_S M_{kj'}^\dagger \otimes |k\rangle_{M_1}\langle k| \otimes |j\rangle_{M_2}\langle j'|\Big] \\
&= \sum_j M_{kj}\rho_S M_{kj}^\dagger\,.
\end{aligned}$$

To show the second part of the statement, let us consider any tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ satisfying Condition 0 and Condition 2. Our goal is to show that for any such tuple there exists a quantum instrument $\{T_k\}$ such that Condition 1 is satisfied. This is equivalent to saying that, for any such tuple, there exists a quantum instrument, such that $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is an implementation of this instrument. For this, let us define the completely positive maps $T_k : \mathcal{B}(\mathcal{H}_S) \rightarrow \mathcal{B}(\mathcal{H}_S)$ via

$$T_k(\rho_S) := \mathrm{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME} \rho_{SME} U_{SME}^\dagger (\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E)]$$

for all ρ_S on S. Note that $\sum_k T_k(\rho_S) = \mathrm{tr}_{ME}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger]$ is trace-preserving. Hence $\sum_k T_k^*(\mathbb{1}_S) = \mathbb{1}_S$, which already implies by Definition 3.1 that the set $\{T_k\}$ is a quantum instrument.

□

Having specified what we refer to as an implementation of a quantum instrument, we are now ready to define the energy cost of a quantum instrument.

Definition 3.17. Energy cost and heat dissipation of a quantum instrument

Let $\{M_{ki}\}$ with measurement operators $M_{ki} \in \mathcal{B}(\mathcal{H}_S)$ be a quantum instrument on a quantum system S with Hamiltonian H_S in state ρ_S and denote by $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{M_{ki}\}}$ an implementation of $\{M_{ki}\}$. The *energy cost of this implementation* is then defined as

$$E_{\mathrm{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \Delta E_E + \inf_{(\rho_B, U_{MB})} E_B \,, \tag{3.18}$$

where $\Delta E_S = \mathrm{tr}[H_S(\rho_S' - \rho_S)]$ is the average energy change on the measured system S and $\Delta E_E, \Delta E_B$ analogously. The *energy cost of the quantum instrument* $\{M_{ki}\}$ is defined as

$$E_{\mathrm{cost}}^{\{M_{ki}\}} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{M_{ki}\}}} \Big(\Delta E_E + \Delta E_B \Big) \,,$$

Moreover, the *dissipated heat* during the quantum instrument is defined as

$$\Delta Q_{\{M_{ki}\}} = \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{M_{ki}\}}} \Big(\Delta E_E + \Delta E_B \Big) \,.$$

Before continuing with the last formalism of quantum measurement, namely POVMs, let us give a name to the specific construction of implementations of a quantum instrument which we introduced in the proof of Proposition 3.16.

Definition 3.18. Standard implementation of a quantum instrument

The *standard implementation of a quantum instrument* $\{M_{ki}\}$ is an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{M_{ki}\}}$ chosen such that

$$\rho_M = |0\rangle_{M_1}\langle 0| \otimes |0\rangle_{M_2}\langle 0|$$

is an initially pure product state of the (bipartite) memory with Hilbert space $\mathcal{H}_M = \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}$ of "large enough dimension", where $|0\rangle_{M_1} \in \mathcal{H}_{M_1}$ and $|0\rangle_{M_2} \in \mathcal{H}_{M_2}$,

$$Q_{M,k} = |k\rangle_{M_1}\langle k| \otimes \mathbb{1}_{M_2}$$

are the "read-off" projections on M with $\{|k\rangle_{M_1}\}$ an orthonormal basis of $\mathcal{H}_{M_1}$,

$$\rho_E = \frac{\mathbb{1}_E}{d_E}$$

is the initial state of the thermal environment E with trivial Hamiltonian $H_E = 0$, and

$$U_{SME} = \sum_k (\mathbb{1}_S \otimes Q_{M,k} \otimes V_{E,k})(U_{SM} \otimes \mathbb{1}_E) \tag{3.19}$$

is the same unitary interaction as in Eq. (3.16) from the proof of Proposition 3.16.

Let us note that we still have some freedom within this definition; in particular, this specification does not fix the Hamiltonian H_M on M. It is however convenient to refer to *all* implementations of the form given in Definition 3.18 as "the" standard implementation.

From the proof of Proposition 3.16 we know that the standard implementation can be used to realise any quantum instrument (if we adapt the unitary U_{SME} correspondingly). Moreover, the standard implementation has a simple structure: Applying the unitary U_{SME} from Eq. (3.19) on the state $\rho_S \otimes \rho_M \otimes \rho_E$ effectively reduces to applying the unitary U_{SM} on $\rho_S \otimes \rho_M$ followed by the read-out projections $Q_{M,k}$, i.e. the average post-measurement state on S and M is given as (see Figure 3.2)

$$\rho'_{SM} = \sum_k (\mathbb{1}_S \otimes Q_{M,k}) U_{SM} (\rho_S \otimes \rho_M) U_{SM}^\dagger (\mathbb{1}_S \otimes Q_{M,k}) \,.$$

The standard implementation corresponds exactly to the implementation considered in ref. [64]. Due to its simple structure, it is relatively easy to compute its energy cost. Indeed, by Eq. (3.18) we find that the energy cost of the standard implementation is simply given by

$$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \Delta E_B \,, \tag{3.20}$$

i.e. by the average energy change ΔE_S on S in addition to the minimal resetting cost of the memory.

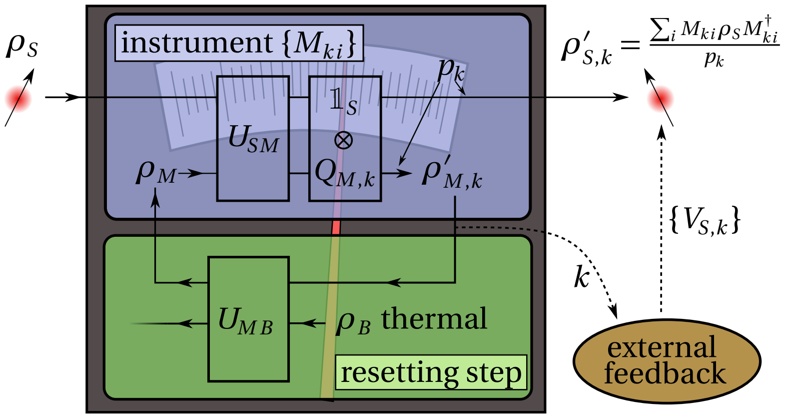

Figure 3.2: The standard implementation of a quantum instrument. To perform any general quantum instrument $\{M_{ki}\}$ on state ρ_S of system S, one can always employ the standard implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ from Definition 3.18, which first applies the unitary interaction U_{SM} on S and the memory M and then dephases the memory with projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$. As required for any implementation (see Definition 3.14), the measurement outcome k will be stored with probability p_k in the memory after the process (which allows for subsequent feedback on S via the unitary $U_{\text{fb}} = \sum_k V_{S,k} \otimes Q_{M,k}$ from Eq. (3.15)) and the post-measurement state $\rho'_{S,k}$ on S is given as prescribed by the measurement operators $\{M_{ki}\}$. To allow for several uses of the same implementation, the memory has to be eventually erased in a resetting step (Definition 3.7). The standard implementation fixes all these microscopic details (blue and green parts) of the measurement process [29].

Implementations of POVMs

The last formalism of quantum measurement, for which we need to define the notion of implementations, is the POVM formalism. POVMs do not determine the post-measurement states on S, but track the measurement outcomes. This is expressed by the following definition.

Definition 3.19. Implementations of a POVM

Let $\{E_k\}$ be a POVM on a quantum system S. An *implementation of this POVM* is then a tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ such that:

Condition 0: ρ_M is a quantum state on a memory M, $\{Q_{M,k}\}$ are orthogonal projections on M such that $\sum_k Q_{M,k} = \mathbb{1}_M$, ρ_E is a thermal state on environment E at temperature T and U_{SME} is a unitary on S, M, and E;

Condition 1: The probabilities agree, i.e.

$$\mathrm{tr}[E_k \rho_S] = \mathrm{tr}[Q_{M,k} \rho'_M]$$

for all states $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ and all possible measurement outcomes k, where $\rho'_M = \mathrm{tr}_{SE}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U^\dagger_{SME}]$ denotes the state on M after measurement;

Condition 2: The memory M after the measurement is classical with respect to the projections $\{Q_{M,k}\}$, i.e.

$$\left[Q_{M,k}, \rho'_M\right] = 0 \qquad \forall \rho_S \forall k \, . \tag{3.21}$$

The set $\mathcal{U}_{\{E_k\}}$ is defined as the set of all implementations of the POVM $\{E_k\}$.

We briefly compare this definition with Definition 3.14 for quantum instruments. Conditions 0 is the same, but Condition 1 is now weaker: Since the post-measurement states on S are not fixed by the POVM formalism, we merely have the requirement that the probability $\mathrm{tr}[Q_{M,k}\rho'_M]$ to read out outcome k from the state $\rho'_M = \mathrm{tr}_{SE}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U^\dagger_{SME}]$ of the memory M after measurement coincides, for all initial states ρ_S on S, with the probability $p_k = \mathrm{tr}[\sum_i M_{ki}\rho_S M^\dagger_{ki}]$ to obtain outcome k as prescribed by the measurement operators M_{ki} of the instrument. Condition 2 has also changed in that we only demand classicality of the memory after measurement with respect to the projections $\{Q_{M,k}\}$ instead classicality with respect to the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$ as for quantum instruments. The reason for this is again the fact that the post-measurement states on S are not fixed for a given POVM: In this formalism we do not have access to post-measurement states on S. Feedback protocols, which served as the reason to require the stronger classicality requirement for instruments (see Lemma 3.15), can therefore not act on the measured system S. Still, of course, it is possible to read out the memory to apply actions on *other* quantum systems that depend on the measurement outcome. But, for any such protocols, the memory will remain classical as in Eq. (3.21), i.e. with respect to the projections $\{Q_{M,k}\}$, such that we allow the weaker classicality requirement for POVMs for generality.

The average energy change $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ on S, which was constant for the other formalisms, does not take a prescribed value for POVMs because ρ'_S is not fixed. Since we are searching for the minimal energy requirements, we therefore need to include this energy change in the optimisation as well.

Definition 3.20. Energy cost and heat dissipation of a POVM

Let $\{E_k\}$ with positive operators $E_k \in \mathcal{B}(\mathcal{H}_S)$ be a POVM on a quantum system S with Hamiltonian H_S in state ρ_S and denote by $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{E_k\}}$ an implementation of $\{E_k\}$.
The *energy cost of this implementation* is then defined as

$$E_{\mathrm{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \Delta E_E + \inf_{(\rho_B, U_{MB})} \Delta E_B \, ,$$

where $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy change on the measured system S and ΔE_E, ΔE_B analogously. The *energy cost of the POVM* $\{E_k\}$ is defined as

$$E^{\{E_k\}}_{\mathrm{cost}} = \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{E_k\}}} \left(\Delta E_S + \Delta E_E + \Delta E_B\right) .$$

Moreover, the *dissipated heat* during the POVM is defined as

$$\Delta Q_{\{M_{ki}\}} = \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{E_k\}}} \left(\Delta E_E + \Delta E_B\right) .$$

3.4 The thermodynamics of quantum measurement

In the previous sections we provided various definitions for the *energy cost of* measurement, depending on the specific form of measurement. In this section, we show that all these definitions basically correspond to the *thermodynamic work cost* needed to conduct the respective measurement. Moreover, we relate our framework to resource-theoretic considerations, which provides useful intuition about the measurement process and will make it easier to understand the physics behind the results we develop in Chapter 4. Finally, we explain why we referred to the respective energetic quantities in the previous definitions as “heat dissipation”.

Thermodynamic work cost

There still exist many controversies regarding the way to properly define thermodynamic work in the quantum regime. Here, we show that the energy cost as defined in our framework directly coincides with the definition of work most frequently used in the current literature on quantum thermodynamics (see e.g. [46,70,75,77]). Let us state this definition of work.

Definition 3.21. Work cost of a unitary process

Consider a quantum system X with Hamiltonian H_X initially in state $\rho_X \in \mathcal{B}(\mathcal{H}_X)$. Let the dynamics of some process on X be given by a unitary U_X on X, i.e. the final state is given as

$$\rho'_X = U_X \rho_X U_X^\dagger .$$

Then the *work cost*, i.e. the amount of work needed to conduct this process, is defined as

$$\Delta W_{\mathrm{cost}} = \Delta E_X ,$$

i.e. as the average energy change $\Delta E_X = \mathrm{tr}[H_X(\rho'_X - \rho_X)]$ on X.

Note that this definition applies only to unitary processes. Whereas a rigorous work definition, valid for arbitrary quantum processes, is definitely desirable, we do not need such a general definition in this thesis, for reasons we explain later in this section.

In the following two paragraphs we briefly review two standard (heuristic) arguments why this definition properly generalises the thermodynamic notion of work to the quantum regime. Both arguments suggest that unitary dynamics does not generate thermodynamic heat. Denoting this heat by ΔQ and invoking the First Law of Thermodynamics, which states that the average energy change splits into the amount of work done on X and the heat dissipated in the process, i.e. $\Delta E_X = \Delta W_{\text{cost}} - \Delta Q$, then "implies" Definition 3.21. We emphasise that the following arguments do not correspond, by any means, to rigorous proofs, but should rather be understood as strong evidence that the above definition is well motivated.

The first argument is as follows: Unitary dynamics are special within quantum theory as they describe the dynamics of *closed* quantum systems (see Chapter 2). Hence, these dynamics do not involve interactions with other quantum systems, e.g. external thermal environments. As a consequence there should not be any heat dissipated to these external environments in such processes. This then implies $\Delta Q = 0$.

The second standard argument for $\Delta Q = 0$ for unitary processes is that such processes do not generate entropy. Indeed, since unitaries do not change the spectrum of a quantum state, we have $S(U_X \rho_X U_X^\dagger) = S(\rho_X)$, where S denotes the von Neumann entropy (see Chapter 2). According to the Second Law of Thermodynamics, heat is however always accompanied with an increase in (thermodynamic) entropy, which suggests that $\Delta Q = 0$ for unitary dynamics. Note that we need to identify thermodynamic entropy with von Neumann entropy $S(\rho_X)$ for this argumentation. This identification is commonly made and also critically assessed in the literature (see e.g. [49, 66, 78–80] and the references therein).

We have not yet established the connection between our energy cost definitions and Definition 3.21 of the work cost above. However, this is quite simple: We modelled the measurement (and resetting) as a unitary process on all involved systems. Moreover, we defined the energy cost as the minimal average energy change on all these systems. Identifying the involved systems with the quantum system X from Definition 3.21 then implies that the energy cost of quantum measurements as defined in our framework (see Section 3.3) is identical to the minimal amount of work needed to conduct the measurement.

Let us provide a small consistency check: We consider the following unitarily implemented quantum channel on a quantum system S with Hamiltonian H_S initially in state ρ_S,

$$\mathcal{F}(\sigma_S) = U_S \sigma_S U_S^\dagger \qquad \forall \sigma_S \, ,$$

where the unitary U_S on S is arbitrary, but fixed. We just argued that the energy cost $E^{\mathcal{F}}_{\text{cost}}$ as defined in Definition 3.13 is identical to the work cost associated to this channel, i.e. $E^{\mathcal{F}}_{\text{cost}} = \Delta W^{\mathcal{F}}_{\text{cost}}$. However, Definition 3.21 states that this work cost is given by the average energy change ΔE_S on S, i.e. $\Delta W^{\mathcal{F}}_{\text{cost}} = \operatorname{tr}[H_S(U_S \rho_S U_S^\dagger - \rho_S)]$. For these statements to be consistent, the energy cost $E^{\mathcal{F}}_{\text{cost}}$ of this quantum channel therefore better be equal to ΔE_S. We show that this is indeed the case in Corollary 3.23 by using the following lemma.

Lemma 3.22. *Let U_S be a unitary on a quantum system S and ρ_X be a thermal state with respect to a Hamiltonian H_X on a quantum system X. Moreover, let $\mathcal{G} : \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_X) \to \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_X)$ be a unital channel such that*

$$\operatorname{tr}_X[\mathcal{G}(\rho_S \otimes \rho_X)] = U_S \rho_S U_S^\dagger \qquad \forall \rho_S \, . \tag{3.22}$$

Then the energy change on X is non-negative, i.e.

$$\operatorname{tr}[H_X(\operatorname{tr}_S[\mathcal{G}(\rho_S \otimes \rho_X)] - \rho_X)] \geq 0. \tag{3.23}$$

Proof. This lemma is a direct consequence of Proposition 4.1, stated and proven in the next chapter, and the explanation below this proposition. Applied to the channel at hand, it states that the energy change on X due to $\mathcal{G}$ is lower bounded by the entropy decrease on S, i.e.

$$\operatorname{tr}[H_X(\operatorname{tr}_S[\mathcal{G}(\rho_S \otimes \rho_X)] - \rho_X)] \geq k_B T \left(S(\rho_S) - S(\operatorname{tr}_X[\mathcal{G}(\rho_S \otimes \rho_X)]) \right) \, .$$

But from Eq. (3.22) we know that

$$S(\operatorname{tr}_X[\mathcal{G}(\rho_S \otimes \rho_X)]) = S(\rho_S) \, ,$$

which already implies inequality Eq. (3.23).

□

Corollary 3.23. *Let $\mathcal{F}$ be a quantum channel on a quantum system S with Hilbert space $\mathcal{H}_S$ initially in state ρ_S, satisfying*

$$\mathcal{F}(\sigma_S) = U_S \sigma_S U_S^\dagger \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H}_S) \, ,$$

where U_S is a unitary on S. Then the energy cost $E^{\mathcal{F}}_{\text{cost}}$ of $\mathcal{F}$ is exactly given by the work cost $\Delta W_{\text{cost}} = \Delta E_S$ of the unitary U_S,

$$E^{\mathcal{F}}_{\text{cost}} = \Delta W_{\text{cost}} = \Delta E_S \, .$$

Proof. Recall from Definition 3.13 that the energy cost of $\mathcal{F}$ is defined as

$$E_{\text{cost}}^{\mathcal{F}} = \Delta E_S + \inf_{\mathcal{U}_{\mathcal{F}}} \Delta E_E \ . \tag{3.24}$$

From Lemma 3.22 we know that

$$\inf_{\mathcal{U}_{\mathcal{F}}} \Delta E_E \geq 0 \ ,$$

since any implementation $(\rho_X, U_{SX}) \in \mathcal{U}_{\mathcal{F}}$ of $\mathcal{F}$ induces the unital channel $\mathcal{G}$ defined as $\mathcal{G}(\sigma_{SX}) = U_{SX} \sigma_{SX} U_{SX}^{\dagger}$ for all σ_{SX}. This implies for the energy cost that

$$E_{\text{cost}}^{\mathcal{F}} \geq \Delta E_S \ . \tag{3.25}$$

Now consider the pair (ρ_X, U_{SX}), where ρ_X is the maximally mixed state on a system X with Hamiltonian $H_X = 0$ and $U_{SX} = U_S \otimes \mathbb{1}_X$ is a unitary on S and X. Clearly, this tuple is an implementation of $\mathcal{F}$. Moreover, the average energy change ΔE_X on X for this implementation is precisely zero, i.e. $\Delta E_E = 0$. This together with Eqs. (3.24) and (3.25) then implies the desired statement, $E_{\text{cost}}^{\mathcal{F}} = \Delta E_S$.

□

Resource considerations

Let us now focus on the *resource consumption* of a quantum measurement $\mathcal{M}$. Although the overall dynamics of the measurement process of all involved systems is governed by an (invertible) unitary (see Assumption 3 from Section 3.2), the measurement process as modelled in our framework (see Section 3.2) is typically *not* reversible in the sense that this process will in general consume *free resources* (in addition to the energy resource). More concretely, in our framework, there are two main free resources: firstly, the initial state ρ_S of the measured system S and, secondly, the thermal states ρ_E and ρ_B of the thermal environments E and B, respectively. What characterises these states as free resources is the fact that they change in the process and that we do not account for the cost to re-create them. The memory M, however, is *not* a free resource as its final state equals its initial state (due to the resetting step).

It is useful to note that the energy cost $E_{\text{cost}}^{\mathcal{M}}$ of a quantum measurement $\mathcal{M}$ (which we just identified with the respective work cost) stems *precisely* from the energy changes on the “free resource” systems (see Definition 3.9). Conversely, if we did not treat these states as free resources, we would need to include the energy cost of re-creating them into the overall energy balance. But this cost would exactly cancel $E_{\text{cost}}^{\mathcal{M}}$, yielding the trivial result that the overall energy is conserved.

Let us emphasise that it is natural to consider exactly the states ρ_S, ρ_E and ρ_B as free resources in the treatment of quantum measurements. If we like to perform a measurement, we typically assume the initial state ρ_S on the system to be measured as given. We do not care about how this state was created or whether energy had to be invested to obtain this state. Also, we do not need to re-create ρ_S after the end of the process, since this state is not needed to allow for further quantum measurements. This is in contrast to the thermal states ρ_E and ρ_B, which need to be reset after the process — otherwise the same implementation of the measurement cannot be used another time. However, in contrast to the resetting step for the memory, we assume that the resetting process of the thermal baths "just so happens in Nature". The assumption here is that the initial states of the baths are obtained by thermalisation processes that we do not need to control. If this assumption proves to be incorrect in certain situations, i.e. if we need to create these thermal states in a process that itself consumes valuable resources, then of course we would need to adapt our framework accordingly.

Summarising, what we account for in our definition of $E_{\text{cost}}^{\mathcal{M}}$ is the *consumption of free resources.* This is in parallel to one of the main motivation behind thermodynamic (or rather resource-theoretic) considerations in general [23,24,48,49,81]: Given a set of free resources, how much work does one need to invest in order to realise a desired state change? For processes which correspond to quantum measurements, the quantity $E_{\text{cost}}^{\mathcal{M}}$ provides the answer to this question.

Heat dissipation of quantum measurements

As promised in Section 3.3, we now explain why we defined the heat dissipation $\Delta Q_{\mathcal{M}}$ of a quantum measurement $\mathcal{M}$ as in Eq. (3.9). In light of the discussions we presented so far in this section, this definition may seem especially confusing for reasons, which we discuss and clarify in the remainder of this section.

The first question that may arise is the following: Since we just argued that the heat dissipation in unitary processes should be zero, doesn't this imply that $\Delta Q_{\mathcal{M}} = 0$ always? To answer this question, note the (somewhat subtle) difference in the following wording: According to our previous discussion, the quantity $E_{\text{cost}}^{\mathcal{M}}$ describes the amount of work done on *all involved systems* in order to realise the quantum measurement. In particular, this includes the work done on the thermal baths E and B. In this process, no heat is dissipated to an additional environment which "surrounds" the involved systems (recall again that unitaries describe the dynamics of *closed* quantum systems). This corresponds to an "outside" perspective, where we consider the dynamics of all involved systems. In contrast, we can take an "inside" perspective and consider the work ΔW_S done on *only the measured system S* in a quantum measurement. For this recall from the First Law of Thermodynamics, that the average energy change ΔE_S on S splits into ΔW_S and the heat dissipated into the environment. Note that the environment of S are precisely the thermal baths E and B (no other environments may participate due to the unitarity of the process). It is therefore natural to identify the dissipated heat with the average energy change $\Delta Q_{\mathcal{M}}$ on these thermal baths. This identification is plausible since energy changes on these systems cannot be recovered, as these systems immediately thermalise after the measurement (see Assumption 1 and the description of the thermal environment). This fits the common picture that thermodynamic heat corresponds to the energy that is inevitably lost in the microscopic degrees of freedom of a bath. Moreover, note that the dynamics on S is *not* unitary and hence the dissipated heat $\Delta Q_{\mathcal{M}}$ need not be zero, which answers our first question above. For the "inside" perspective, the First Law then reads

$$\Delta E_S = \Delta W_S - \Delta Q_{\mathcal{M}} ,$$

Hence, the work done on S just corresponds to the work done on all involved systems, i.e. $\Delta W_S = E^{\mathcal{M}}_{\text{cost}}$, implying that the "outside" and "inside" perspective are equivalent.

The second question about the heat dissipation $\Delta Q_{\mathcal{M}}$ of a quantum measurement $\mathcal{M}$ concerns the conversion of heat into work: Consider for example a measurement on a quantum system with trivial Hamiltonian $H_S = 0$. From its very definition, it is not at all obvious that $\Delta Q_{\mathcal{M}}$ needs to be positive. Indeed, we will find in Chapter 4 that some measurements on certain initial states of S yield strictly negative heat dissipation (see for example Proposition 4.18). But doesn't this imply that heat can be converted into work? Indeed, we then have $\Delta E_S = 0$ and the work cost $E^{\mathcal{M}}_{\text{cost}} = \Delta Q_{\mathcal{M}}$ is strictly negative, meaning that the agent who conducts the measurement gains work. At first glance this may look like a contradiction to the Second Law of Thermodynamics. However, note that such a heat-to-work is perfectly allowed and reasonable as long as there are other resources consumed in the process (which implies that the process is not cyclic). And indeed, we find in Chapter 4 that this conversion is only possible if the purity of the initial state ρ_S is consumed. All our results are therefore in perfect accordance with the Second Law of Thermodynamics.

CHAPTER 4

Energy cost for quantum measurements

This chapter is devoted to the derivation of (lower bounds on) the fundamental energy costs of the different formalisms of quantum measurement introduced in Chapter 3. Namely, we derive energy costs for simple state transformations, for quantum channels, for quantum instruments, and for POVMs.

For a short summary of the main results presented in this chapter we refer to Table 6.1 of Chapter 6. Some of these results have already been shown in previous literature. We still include them in this chapter when they can be formulated within the framework we use. This will make the (sometimes subtle) distinctions between our novel and these older results much clearer. In each section, we precisely indicate whether the statements were already obtained previously and give the corresponding reference. Additionally, we devote Section 4.7 to the discussion of previous literature related to the energy cost of measurements and discuss also those references whose results cannot be straightforwardly formulated within our framework.

All results in this chapter can be proven using our microscopic framework detailed in Chapter 3, in which the energy cost $E^{\mathcal{M}}_{\text{cost}}$ of an abstract quantum measurement $\mathcal{M}$ is defined by an optimisation over the set $\mathcal{U}_{\mathcal{M}}$ of all possible physical implementations (see Section 3.3). In general, this set $\mathcal{U}_{\mathcal{M}}$ is not easy to characterise and consequently computing the minimal energy cost over all implementations is difficult. But, instead of computing this cost *exactly*, we can enlarge the set $\mathcal{U}_{\mathcal{M}}$ in order to make the optimisation easier. For example, instead of optimising only over those unitaries that satisfy some restriction depending on the measurement $\mathcal{M}$, we may also just optimise over all unitaries. Such a strategy will always produce *lower bounds* on the energy cost and we will employ this method quite often within this chapter. For certain measurements, obtaining such lower bounds is the best we can do. In some special and nevertheless relevant cases, however, we are able to determine the *exact* costs, namely for so-called dephasing channels, for projective quantum instruments (see Definition 3.3) and for projection-valued measurements (PVM). We provide the definitions of dephasing channels and PVMs in the corresponding sections. These three measurements have in common that their implementations involve projections instead of general linear operators. Such projections provide a lot of structure, which enables us to compute the exact energy costs.

For clarity, let us emphasise that even in those cases where the energy cost of a quantum measurement $\mathcal{M}$ can be computed exactly, not *all* implementations of $\mathcal{M}$ necessarily demand this energy expense. As expected physically and as can be seen directly from the definitions in Section 3.3, different implementations come with different costs. Our results however are independent of the explicit implementation and just require knowledge about the initial state ρ_S and the Hamiltonian H_S of the measured system S and the mathematical specification of the measurement $\mathcal{M}$. This is because the energy cost of a quantum measurement is defined as the *minimal* energy expense over all possible implementations (see Section 3.3) and as such corresponds to a property of the quantum measurement itself, irrespective of its physical realisation. Note however that it may well be that an energetically optimal implementation of a given measurement may require us to be able to control all the degrees of freedom of the environment or to ”engineer” its Hamiltonian. Hence, such optimal implementations cannot always be realised in practise. In this thesis however, we allow such unfeasible implementations (in particular for our lower bound results) as we are interested in the *fundamental* energy requirements of quantum measurement.

In the first section of this chapter, we summarise the most important aspects of our framework presented in Chapter 3. The subsequent sections are then devoted to the derivation of the (lower bounds on the) energy cost of the four formalisms of measurements, namely unselective measurements (Section 4.2), quantum channels (Section 4.3), quantum instruments (Section 4.4), and POVMs (Section 4.6). The chapter concludes with a comparison section, where our results are related to existing results within current literature.

Note that some of the novel results we obtain in this chapter were already published on the arXiv pre-print server [29].

4.1 Notation and statement of the problem

In the following sections we derive lower bounds or exact results on the energy cost of quantum measurements, for each of the four formalisms presented in Chapter 3. To simplify the presentation of these results, we state here in general terms the mathematical problem behind the computation of the energy cost, thereby summarising the details of our framework from Chapter 3 and specifying the notation we use.

We consider a quantum measurement, denoted by $\mathcal{M}$, on a quantum system labelled S. This quantum measurement can be either an unselective measurement, a quantum channel, a quantum instrument, or a POVM. Following Chapter 3, such measurements involve further quantum systems: the memory M and/or thermal environments E and B. To any of these systems we assign a specific (finite dimensional) Hilbert space $\mathcal{H}$, a Hamiltonian H and an initial state ρ, where an additional subscript indicates the respective physical system; the initial state of the measured system S is e.g. labelled by ρ_S.

Our framework assigns *implementations* to a given quantum measurement $\mathcal{M}$. These implementations correspond to a microscopic description of the abstract mathematical prescription $\mathcal{M}$ of the measurement. More concretely, an implementation is a tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ specifying the initial properties of all involved systems (except S) and the interaction U_{SME} which depends on $\mathcal{M}$. Although not explicitly mentioned in the notation, an implementation also fixes the Hamiltonians (and, of course, state spaces) of all these systems. Since the notion of an implementation depends on the measurement formalism, we will not repeat the exact definitions here and instead refer to Section 3.3.

Given a specific implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of a measurement $\mathcal{M}$, the state of all involved systems before the measurement is denoted by

$$\rho_{SME} = \rho_S \otimes \rho_M \otimes \rho_E$$

and is of product form (see Assumption 2 of our framework). The state after the measurement is given by $\rho'_{SME} = U_{SME} \rho_{SME} U^\dagger_{SME}$ and we obtain its marginal states by taking the partial trace, i.e. for example $\rho'_S = \mathrm{tr}_{ME}[\rho'_{SME}]$ for the average post-measurement state on S.

The set of all implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of a given quantum measurement $\mathcal{M}$ is denoted by $\mathcal{U}_{\mathcal{M}}$.

Importantly, to allow for further uses of this implementation, the memory M must be erased in a resetting step $\mathcal{R}$[1]. This step is realised using implementa-

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[1] In Section 3.3 we have seen that a memory (and the subsequent resetting step) are only needed for those quantum measurements, where the outcomes are recorded, namely quantum instruments and POVMs.

tions (ρ_B, U_{MB}) which are defined in Definition 3.7. Hence the complete microscopic specification of a given quantum measurement $\mathcal{M}$ consists of the implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\mathcal{M}}$ for the measurement and the implementation (ρ_B, U_{MB}) for the resetting. We will later see that it is easy to characterise the energetically optimal implementation of the resetting step for any implementation of the measurement. Hence, we will often only specify the implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ to provide the complete microscopic specification of the measurement, thereby assuming that the resetting step is conducted optimally, i.e. in such a way that it consumes the least amount of energy.

We define the energy cost $E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ of any given implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\mathcal{M}}$ of the measurement $\mathcal{M}$ as the average energy change on all involved systems, i.e.

$$E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \Delta E_S + \Delta E_E + \inf_{(\rho_B,U_{MB})} \Delta E_B \ . \tag{4.1}$$

We take the infimum here, because this translates to the assumption of an energetically optimal resetting step, as just stated.

The energy cost of the quantum measurement $\mathcal{M}$ is then defined as the minimal average energy change on all involved systems over all implementations, i.e.

$$E_{\text{cost}}^{\mathcal{M}} = \inf_{\mathcal{U}_{\mathcal{M}}} E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \inf_{(\rho_B,U_{MB})} \inf_{\mathcal{U}_{\mathcal{M}}} (\Delta E_S + \Delta E_E + \Delta E_B) \ ,$$

where "$\inf_{\mathcal{U}_{\mathcal{M}}}$" denotes the infimum over all implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ in $\mathcal{U}_{\mathcal{M}}$. This directly corresponds to Definitions 3.17 and 3.20. For unselective measurements and quantum channels, we can simplify this definition, since the memory and hence the resetting step is obsolete (see Section 3.3). For these measurements, the energy cost of a specific implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\mathcal{M}}$ is simply given as

$$E_{\text{cost}}^{(\rho_E,U_{SE})} = \Delta E_S + \Delta E_E$$

and the energy cost of the measurement $\mathcal{M}$ as

$$E_{\text{cost}}^{\mathcal{M}} = \inf_{\mathcal{U}_{\mathcal{M}}} E_{\text{cost}}^{(\rho_E,U_{SE})} = \Delta E_S + \inf_{\mathcal{U}_{\mathcal{M}}} \Delta E_E \ ,$$

which agrees with Definitions 3.11 and 3.13.

The goal of the following sections is to compute these costs for the four different formalisms of measurement. Note that these costs depend on the initial state ρ_S and the Hamiltonian H_S of the measured system, as well as the mathematical specification of the measurement $\mathcal{M}$. Hence, our desired statements are of the following form: Given $\mathcal{M}$, ρ_S, and H_S, the energy cost $E_{\text{cost}}^{\mathcal{M}}$ is precisely given (or lower bounded) by a quantity, which can be easily computed from the given quantities.

4.2 Unselective measurements

As explained in Section 3.1, unselective measurements constitute the most general description of quantum dynamics since they simply correspond to state transformations with no further structure. Hence, quantifications of the energy cost of such unselective measurements can be directly applied to various situations (see for example the application of quantum Zeno stabilisation in Section 5.1) and has been the focus of much previous literature (see Section 4.7). In this section we will briefly review the main results of this literature. While this section does not contain any new results, it illustrates how our framework can be used to re-derive these results and allows us to compare them with the new results we obtain for the other formalisms of quantum measurement in the subsequent sections.

Our reformulation of existing results on unselective measurements also illustrates our basic strategy to obtain our results: As it is in general difficult to characterise the set of implementations $\mathcal{U}_{\rho_S \mapsto \rho'_S}$ of a given state transformation $\rho_S \mapsto \rho'_S$, we will first derive a lower bound on the energy cost $E_{\text{cost}}^{\rho_S \mapsto \rho'_S}$. In a second step, we provide an explicit construction of an implementation whose energy cost approaches this lower bound arbitrarily closely.

4.2.1 General lower bound

Let us consider an unselective measurement $\rho_S \mapsto \rho'_S$ that maps the fixed initial state ρ_S to the fixed final state ρ'_S. Our goal is to derive a lower bound on the energy cost $E_{\text{cost}}^{\rho_S \mapsto \rho'_S}$ of this measurement. Such lower bounds can be obtained by optimising over a larger set than only the set $\mathcal{U}_{\rho_S \mapsto \rho'_S}$ of implementations. Here, we do not need to employ this strategy. In contrast, we directly formulate a lower bound on the energy cost $E_{\text{cost}}^{(\rho_E, U_{SE})} = \text{tr}[H_{SE}(\rho'_{SE} - \rho_{SE})]$, valid *for all* implementations $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ of the unselective measurement. Since $E_{\text{cost}}^{\rho_S \mapsto \rho'_S} = \inf_{\mathcal{U}_{\rho_S \mapsto \rho'_S}} E_{\text{cost}}^{(\rho_E, U_{SE})}$, this lower bound will also be a lower bound on the energy cost $E_{\text{cost}}^{\rho_S \mapsto \rho'_S}$ of the unselective measurement.

The energy cost of an implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$

$$E_{\text{cost}}^{(\rho_E, U_{SE})} = \text{tr}[H_{SE}(\rho'_{SE} - \rho_{SE})] = \Delta E_S + \Delta E_E$$

splits into the average energy changes ΔE_S and ΔE_E on S and E, respectively. Whereas ΔE_S is fixed for all implementations of the measurement, the average energy change ΔE_E on the thermal environment E varies with the choice of implementation. The next proposition provides a general, yet important tool to obtain a lower bound on this energy change [38]. The importance of this proposition lies in the fact that it can be applied to bound the energy change of any initially thermal system, irrespective of the formalism of measurement.

Proposition 4.1. *Let X be a quantum system with Hamiltonian H_X and initially in a thermal state*

$$\rho_X = \frac{\mathrm{e}^{-\frac{H_X}{k_B T}}}{Z_X}$$

at some fixed temperature T. Let A be another quantum system, initially uncorrelated with X, i.e. $\rho_{AX} = \rho_A \otimes \rho_X$, where ρ_A denotes the initial state of A. Denote by $\rho'_{AX} = U_{AX} \rho_{AX} U_{AX}^\dagger$ the state after applying a unitary U_{AX} on A and X.
Then we have that

$$\mathrm{tr}[H_X(\rho'_X - \rho_X)] = k_B T\Big(S(\rho_A) - S(\rho'_A) + I(A:X)_{\rho'_{AX}} + D(\rho'_X \| \rho_X)\Big),$$

where $I(A:X)_{\rho'_{AX}}$ denotes the mutual information between A and X after the unitary (see Eq. (2.6)), $D(\rho'_X \| \rho_X)$ denotes the relative entropy between the final state $\rho'_X = \mathrm{tr}_A[\rho'_{XA}]$ and initial state on the system X (see Eq. (2.8)) and $S(\rho'_A)$ denotes the von Neumann entropy of the final state $\rho'_A = \mathrm{tr}_X[\rho_{XA}]$ on A (see Eq. (2.4)).

Proof. The presentation of the proof is largely based on [38].

The proof consists of two steps: First we relate the energy change $\mathrm{tr}[H_X(\rho'_X - \rho_X)]$ to the entropy increase $S(\rho'_X) - S(\rho_X)$ on X by using the relative entropy $D(\rho'_X \| \rho_X)$. The second step is to show that this entropy increase on X is never less than the entropy decrease on A, $S(\rho'_X) - S(\rho_X) \geq S(\rho_A) - S(\rho'_A)$, the gap term being exactly the mutual information $I(A:X)_{\rho'_{AX}}$ for unitary dynamics.

For the first step, we compute the relative entropy $D(\rho'_X \| \rho_X)$ using the fact that ρ_X is thermal,

$$\begin{aligned} D(\rho'_X \| \rho_X) &= -S(\rho'_X) - \mathrm{tr}[\rho'_X \log \rho_X] \\ &= -S(\rho'_X) - \mathrm{tr}[\rho'_X \log[\mathrm{e}^{-H_X/(k_B T)}/Z_X]] \\ &= -S(\rho'_X) + \frac{1}{k_B T}\mathrm{tr}[\rho'_X H_X] + \log[Z_X]\mathrm{tr}[\rho'_X] \\ &= -S(\rho'_X) + \frac{1}{k_B T}\mathrm{tr}[\rho'_X H_X] + \log[Z_X]\,. \end{aligned} \tag{4.2}$$

The same computation yields that the relative entropy between the same states ρ_X satisfies

$$0 = D(\rho_X \| \rho_X) = -S(\rho_X) + \frac{1}{k_B T}\mathrm{tr}[\rho_X H_X] + \log[Z_X]\,,$$

which implies that

$$\log[Z_X] = S(\rho_X) - \frac{1}{k_B T}\mathrm{tr}[\rho_X H_X]\,.$$

This together with Eq. (4.2) shows that the energy change $\mathrm{tr}[H_X(\rho'_X - \rho_X)]$ on X is given by

$$\mathrm{tr}[H_X(\rho'_X - \rho_X)] = k_B T(S(\rho'_X) - S(\rho_X) + D(\rho'_X \| \rho_X))\,. \tag{4.3}$$

This identity will prove useful at several places throughout this thesis as it provides the basic way to relate the average energy changes on a quantum system to its entropy change in any process where an initially thermal state ρ_X is mapped to some final state ρ'_X[2].

The second step employs the unitarity of the process. Since unitary transformations preserve the spectrum of a quantum state, such processes also preserve its entropy and we have on the one hand that

$$S(\rho'_{AX}) = S\big(U_{AX}(\rho_A \otimes \rho_X)U^\dagger_{AX}\big) = S(\rho_A \otimes \rho_X) = S(\rho_A) + S(\rho_X)\,. \tag{4.4}$$

On the other hand, we know from Eq. (2.6), that the entropy of the joint state ρ'_{AX} is related to the entropies of the marginal states ρ'_A and ρ'_X through the mutual information $I(A:X)_{\rho'_{AX}}$,

$$S(\rho'_{AX}) = S(\rho'_A) + S(\rho'_X) - I(A:X)_{\rho'_{AX}}\,. \tag{4.5}$$

Eqs. (4.4) and (4.5) together imply

$$S(\rho'_X) - S(\rho_X) = S(\rho_A) - S(\rho'_A) + I(A:X)_{\rho'_{AX}}\,. \tag{4.6}$$

This statement is the second important identity we develop in this proof as it relates the entropy change of one marginal system (X) to the entropy change of another marginal system (A). Inserting (4.6) into (4.3) yields the desired statement

$$\mathrm{tr}[H_X(\rho'_X - \rho_X)] = k_B T(S(\rho_A) - S(\rho'_A) + I(A:X)_{\rho'_{AX}} + D(\rho'_X \| \rho_X))$$

and concludes the proof. □

Note from Eq. (4.4) in this proof that unitarity of the process is only used to relate the von Neumann entropy of the final state ρ'_{AX} to the von Neumann entropy of the initial states ρ_A and ρ_X. Hence, we can generalise the statement of Proposition 4.1 to unital channels $\mathcal{G}: \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_X) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_X)$ where the final state of the process

---

[2] One can even drop the thermality requirement: If ρ_X is not thermal, we can still denote by $\rho_{X,\mathrm{th}}$ the thermal state with respect to Hamiltonian H_X at ambient temperature T such that the same computation as above yields $\mathrm{tr}[H_X(\rho'_X - \rho_X)] = k_B T(S(\rho'_X) - S(\rho_X) + D(\rho'_X \| \rho_{X,\mathrm{th}}) - D(\rho_X \| \rho_{X,\mathrm{th}}))$.

is given as $\rho'_{SA} = \mathcal{G}(\rho_{SA})$. This is because the von Neumann entropy never decreases under unital channels [82], i.e. $S(\mathcal{G}(\rho_{SA})) \geq S(\rho_{SA})$ for all $\rho_{SA} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_X)$, and it is easy to see that this inequality "goes into the right direction" to prove that

$$\mathrm{tr}[H_X(\rho'_X - \rho_X)] \geq k_B T\Big(S(\rho_A) - S(\rho'_A) + I(A:X)_{\rho'_{AX}} + D(\rho'_X \| \rho_X)\Big)\,,$$

where the equality in Proposition 4.1 is now replaced by an inequality. We used this statement in the proof of Lemma 3.22.

As a direct corollary, Proposition 4.1 admits the following lower bound on the energy cost for unselective measurements.

Corollary 4.2. *Let S be a quantum system with Hilbert space $\mathcal{H}_S$ and Hamiltonian H_S. Let ρ_S and ρ'_S be two quantum states on S. Then the energy cost $E_{\mathrm{cost}}^{\rho_S \mapsto \rho'_S}$ of the unselective measurement $\rho_S \mapsto \rho'_S$ is lower bounded as*

$$E_{\mathrm{cost}}^{\rho_S \mapsto \rho'_S} \geq \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S))\,,$$

where $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy increase and $S(\rho_S) - S(\rho'_S)$ is the entropy decrease on S.

Proof. From Definition 3.11 we know that the energy cost of the unselective measurement $\rho_S \mapsto \rho'_S$ is defined as

$$E_{\mathrm{cost}}^{\rho_S \mapsto \rho'_S} = \Delta E_S + \inf_{\mathcal{U}_{\rho_S \mapsto \rho'_S}} \Delta E_E\,, \tag{4.7}$$

i.e. we need to minimise the average energy change on the thermal environment E. Let now $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ be an implementation of $\rho_S \mapsto \rho'_S$. Clearly, identifying the system S with the system A from Proposition 4.1 and system E with X, we find that by this proposition the average energy change on E is given as

$$\Delta E_E = \mathrm{tr}[H_E(\rho'_E - \rho_E)] = k_B T(S(\rho_S) - S(\rho'_S) + I(S:E)_{\rho'_{SE}} + D(\rho'_E \| \rho_E))\,,$$

where $\rho'_{SE} = U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger$ as usual and the marginal states are obtained by taking the respective partial traces. Now, since both the mutual information and the relative entropy are non-negative, this implies that $\Delta E_E \geq k_B T(S(\rho_S) - S(\rho'_S))$. Importantly, this inequality holds for any given implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$. Hence, inserting this into Eq. (4.7) yields

$$E_{\mathrm{cost}}^{\rho_S \mapsto \rho'_S} \geq \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S))\,,$$

which is the desired statement.

□

4.2.2 Tightness of the lower bound

How good is the lower bound on the energy cost of general unselective measurements provided by Corollary 4.2? To answer this question let us first ask whether we can find, for a given unselective measurement $\rho_S \mapsto \rho'_S$, implementations $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ whose energy cost coincides *exactly* with the lower bound, i.e. $E_{\text{cost}}^{(\rho_E,U_{SE})} = \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S))$.

Interestingly, this is not the case, as was shown in [38]:

> **Lemma 4.3.** *The only unselective measurement $\rho_S \mapsto \rho'_S$ for which there is an implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ with cost*
>
> $$E_{\text{cost}}^{(\rho_E,U_{SE})} = \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S))$$
>
> *is the state transformation $\rho_S \mapsto \rho'_S$, where ρ'_S can be written as $\rho'_S = U_S \rho_S U_S^\dagger$ for some unitary U_S on S (implying $S(\rho_S) = S(\rho'_S)$).*

Proof. The proof is based on [38].

From the proof of Corollary 4.2 we know that the energy cost of any particular implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\rho_S \mapsto \rho'_S}$ of the unselective measurement $\rho_S \mapsto \rho'_S$ is exactly given by

$$\begin{aligned} E_{\text{cost}}^{(\rho_E,U_{SE})} &= \Delta E_S + \Delta E_E \\ &= \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S) + I(S:E)_{\rho'_{SE}} + D(\rho'_E \| \rho_E)) \,. \end{aligned}$$

An implementation with the desired cost $E_{\text{cost}}^{(\rho_E,U_{SE})} = \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S))$ must therefore satisfy that the mutual information term $I(S:E)_{\rho'_{SE}}$ as well as the relative entropy term $D(\rho'_E \| \rho_E)$ vanish.

Now, $I(S:E)_{\rho'_{SE}} = 0$ implies that there exist no correlations between S and E, i.e. $\rho'_{SE} = \rho'_S \otimes \rho'_E$. The relative entropy $D(\rho'_E \| \rho_E)$ vanishes only if $\rho'_E = \rho_E$. These two statements together imply that the state ρ'_{SE} of the involved systems after the unitary U_{SE} is of the form [38]

$$\rho'_{SE} = U_{SE}(\rho_S \otimes \rho_E) U_{SE}^\dagger = \rho'_S \otimes \rho_E \ .$$

But this implies that there is a unitary U_S on S such that $\rho'_S = U_S \rho_S U_S^\dagger$.

□

Although the lower bound in Corollary 4.2 can therefore not be attained precisely in general, there is no better lower bound for unselective measurements as was shown in ref. [38].

Theorem 4.4. ***Energy cost for unselective measurements***

Let S be a quantum system with Hamiltonian H_S. Let ρ_S and ρ'_S be two quantum states on S such that $\mathrm{rank}[\rho'_S] \geq \mathrm{rank}[\rho_S]$. *Then the energy cost $E_{\mathrm{cost}}^{\rho_S\mapsto\rho'_S}$ of the unselective measurement $\rho_S \mapsto \rho'_S$ is* exactly *given by*

$$E_{\mathrm{cost}}^{\rho_S\mapsto\rho'_S} = \Delta E_S + k_B T\big(S(\rho_S) - S(\rho'_S)\big)\,,$$

where $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy increase and $S(\rho_S) - S(\rho'_S)$ is the entropy decrease on S.

Before presenting the proof, let us quickly comment on this theorem, which is often referred to as the quantum-mechanical analogue of Landauer's principle [30–38, 83]. This principle states that a change in entropy of a physical system is always accompanied by a heat cost and is considered a fundamental link between information theory and thermodynamics. Theorem 4.4 can be understood as a proof of this principle in the quantum regime: Recalling Definition 3.11 and the discussion in Section 3.4, the theorem states that the dissipated heat of the state transformation $\rho_S \mapsto \rho'_S$ is precisely given by $\Delta Q_{\rho_S\to\rho'_S} = k_B T\big(S(\rho_S) - S(\rho'_S)\big)$.

We need the assumption that $\mathrm{rank}[\rho'_S] \geq \mathrm{rank}[\rho_S]$, because otherwise either $T = 0$ or $E_{\mathrm{cost}}^{\rho_S\mapsto\rho'_S} = \infty$ as was shown in ref. [38]. However, this is a rather technical requirement, since we can always approximate ρ'_S by a full rank state as closely as desired.

Due to the generality of unselective measurements, the statement of this theorem will also play a role in the remaining formalisms of quantum measurement.

Proof. The complete proof can be found in [38]. Here we just provide some intuition about the proof strategy.

As a first step, let us provide the construction of a simple (but not yet energetically optimal) implementation that works *for all* unselective measurements. For this, let $\rho_S \mapsto \rho'_S$ be an arbitrary, but fixed unselective measurement and recall that an implementation of $\rho_S \mapsto \rho'_S$ is by Definition 3.10 a tuple (ρ_E, U_{SE}) such that ρ_E is a thermal state with respect to a Hamiltonian H_E at temperature T and U_{SE} is a unitary satisfying

$$\mathrm{tr}_E[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger] = \rho'_S\,. \tag{4.8}$$

A simple implementation of $\rho_S \mapsto \rho'_S$ is hence the following: We choose the environment E to be a quantum system with the same Hilbert space dimension as S. Additionally, we choose the initial state ρ_E of E to be identical to the *final* state on S, i.e.

$$\rho_E = \rho'_S\,.$$

The Hamiltonian H_E of E is chosen to be $H_E = -k_B T \log \rho'_S$ such that ρ_E is thermal with respect to this Hamiltonian at temperature T. Finally, we choose the unitary interaction U_{SE} on S and E to be the swap operation, that exchanges the states on S and E. Clearly, the tuple (ρ_E, U_{SE}) is an implementation of $\rho_S \mapsto \rho'_S$, since Eq. (4.8) is satisfied. An important property of this construction is that it is independent of the initial state ρ_S on S. As a consequence, a similar construction can be used to implement other forms of measurement as long as the final state ρ'_S on S is fixed (such that this state can be chosen as the initial state on E). We will use this implementation at several other places in this thesis (not only for system S, but also for the memory M).

From Definition 3.11 we know that the energy cost of this implementation is given as

$$E_{\text{cost}}^{(\rho_E, U_{SE})} = \Delta E_S + \Delta E_E \ .$$

We can use Proposition 4.1 to compute the energy change on E,

$$\begin{aligned}\operatorname{tr}[H_{SE}(U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger - \rho_S \otimes \rho_E)] &= \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S) + D(\rho'_E \| \rho_E)) \\ &= \Delta E_S + k_B T(S(\rho_S) - S(\rho'_S) + D(\rho_S \| \rho'_S)) \,,\end{aligned}$$

where we used that $I(S:E)_{\rho'_{SE}} = 0$ since the final state ρ'_{SE} is a product state and hence does not contain correlations. As hinted at before, the cost of this implementation therefore exceeds the lower bound from Corollary 4.2, the gap term being precisely the cost $k_B T D(\rho_S \| \rho'_S)$.

To conclude the proof, we therefore need to adapt this construction in such a way, that this gap term vanishes. This is achieved by "interpolating" the transition from ρ_S to ρ'_S. What we mean by this is the following: Consider a thermal bath E that consists of two subsystems E_1 and E_2, each with the same Hilbert space dimension as S. We choose the initial state of E to be the product state $\rho_E = \rho_{E_1} \otimes \rho_{E_2}$, where $\rho_{E_2} = \rho'_S$ is identical to the final state on S, as in the construction above. The state ρ_{E_1} on E_1 however is chosen as an *interpolating* state, i.e.

$$\rho_{E_1} = \frac{\rho_S + \rho'_S}{2} \equiv \rho_{S,\text{int}} \ .$$

The unitary U_{SE} then works by first swapping S with E_1 and then swapping E_1 with E_2. Clearly, the final state on S is then just given by ρ'_S. The tuple (ρ_E, U_{SE}) is hence a valid implementation of $\rho_S \mapsto \rho'_S$. We use again Proposition 4.1 to compute the energy cost of this implementation:

$$\begin{aligned}&\operatorname{tr}[H_E(U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger - \rho_S \otimes \rho_E)] \\ &\quad= \operatorname{tr}[H_S(\rho_{S,\text{int}} - \rho_S) + k_B T\big(S(\rho_S) - S(\rho_{S,\text{int}}) + D(\rho_S \| \rho_{S,\text{int}})\big) \\ &\quad\quad + \operatorname{tr}[H_S(\rho'_S - \rho_{S,\text{int}}) + k_B T\big(S(\rho_{S,\text{int}}) - S(\rho'_S) + D(\rho_{S,\text{int}} \| \rho'_S)\big) \\ &\quad= \Delta E_S + k_B T\big(S(\rho_S) - S(\rho'_S) + D(\rho_S \| \rho_{S,\text{int}}) + D(\rho_{S,\text{int}} \| \rho'_S)\big) \,,\end{aligned}$$

i.e. the gap term for this implementation is given as $k_B T(D(\rho_S||\rho_{S,\text{int}})+D(\rho_{S,\text{int}}||\rho'_S))$. While it is not clear that this gap term is smaller than $k_B T D(\rho_S||\rho'_S)$, it was shown in e.g. ref. [38,77] that introducing more and more such interpolating states, will let the gap term converge to zero. Concretely, one uses a bath initially in the product state $\rho_E = \rho_{E,1} \otimes \rho_{E,2} \otimes ... \otimes \rho_{E,n}$ with

$$\rho_{E,i} = \left(1 - \frac{i}{n}\right)\rho_S + \frac{i}{n}\rho'_S$$

for $i = 1, ..., n$. Note that the previous construction corresponds to the case where $n = 2$ such that $\rho_{E,1} = \rho_{S,\text{int}}$ and $\rho_{E,2} = \rho'_S$. The unitary interaction in this new implementation now corresponds to a swap between the system S and the first subsystem of E, followed by a swap between system S and the second subsystem of E, and so on. The gap term for such an implementation can be computed analogously to the case $n = 2$ which yields $k_B T \sum_{i=0}^{n-1} D(\rho_i||\rho_{i+1})$, where we set $\rho_{E,0} \equiv \rho_S$. This gap term converges to zero in the limit of large n [38].

Intuitively, this process is analogous to what is called an adiabatic process in thermodynamics: The changes of the system state in each step are small, the process employs a bath which is basically infinite-dimensional (in the limit of large n) and, if we associate to each of the swap operations a constant and finite time duration, the process needs infinitely long to output the target state ρ'_S.

The exact details of this process are not important here. However, we note that the construction of this implementation is still independent of the initial state ρ_S and hence works for all processes where the final state ρ'_S on S is fixed.

□

One may wonder why we split the results in this chapter into first proving a lower bound and then showing that the lower bound is tight in the sense that we can replace the inequality by an equality. Indeed, we could have just stated the exact result in the first place. We did this for the purpose of clarity, that is to show the structure of the proofs in the following subsection: in order to prove our exact results we always first compute a lower bound valid for all implementations and then show the existence of a particular implementation that attains the bound.

Now that we computed the exact energy cost of unselective measurements, let us finally comment on the usefulness of catalysts in our framework, as promised in Section 3.3. Recall that the defining property of a catalyst C is that its initial and final state are identical. Assume now that we allowed for such a catalyst C within our Definition 3.10 of implementations of an unselective measurement $\rho_S \mapsto \rho'_S$, i.e. we consider tuples $(\rho_C, \rho_E, U_{SCE})$ where ρ_C is any state on a Hilbert space $\mathcal{H}_C$,

ρ_E is thermal as usual and U_{SCE} is a unitary such that

$$\mathrm{tr}_{CE}[U_{SCE}(\rho_S \otimes \rho_C \otimes \rho_E)U^\dagger_{SCE}] = \rho'_S$$

and

$$\mathrm{tr}_{SE}[U_{SCE}(\rho_S \otimes \rho_C \otimes \rho_E)U^\dagger_{SCE}] = \rho_C$$

for the fixed input ρ_S. The first equation just corresponds to the implementation requirement as in Definition 3.10, whereas the second equation ensures that C is indeed a catalyst.

Since initial and final states of the catalyst are identical, there is no average energy change on C, i.e. $\Delta E_C = 0$ irrespective of the Hamiltonian H_C. The energy cost of the unselective measurement $\rho_S \mapsto \rho'_S$ in this more general framework which includes catalysts would then be defined as

$$\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}} = \Delta E_S + \inf_{\tilde{\mathcal{U}}_{\rho_S \mapsto \rho'_S}} \Delta E_E \ ,$$

where $\tilde{\mathcal{U}}$ denotes the set of all implementation $(\rho_C, \rho_E, U_{SCE})$, in complete analogy to Definition 3.10. We now show that this cost $\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}}$ is *identical* to the energy cost $E^{\rho_S \mapsto \rho'_S}_{\text{cost}}$, we discussed in this section. This observation is the reason why we did not include catalysts within our framework in the first place.

To see that $\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}} = E^{\rho_S \mapsto \rho'_S}_{\text{cost}}$, we compute a lower bound on $\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}}$ via Proposition 4.1,

$$\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}} \geq \Delta E_S + k_B T \big(S(\rho_{SC} - S(\rho'_{SC}) \big) \, .$$

Due to the non-negativity of the mutual information $I(S:C)_{\rho'_{SC}}$ (see Eq. (2.7)) we have $S(\rho'_{SC}) \geq S(\rho'_S) - S(\rho_C)$, which implies that

$$\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}} \geq \Delta E_S + k_B T \big(S(\rho_S - S(\rho'_S) \big) \, .$$

But the right hand side of this inequality is just the (attainable) lower bound $E^{\rho_S \mapsto \rho'_S}_{\text{cost}}$ from our framework that does not include catalysts. Hence, the energy cost $\tilde{E}^{\rho_S \mapsto \rho'_S}_{\text{cost}}$ is exactly given by $E^{\rho_S \mapsto \rho'_S}_{\text{cost}}$ and catalysts provide no energetic advantage.

4.3 Quantum channels

Quantum channels are a basic tool to describe the dynamics of an open quantum system. As explained in Section 3.1, any channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ can be decomposed via $\mathcal{F}(\rho_S) = \sum_k M_k \rho_S M_k^\dagger$ and hence can be abstractly interpreted as a measurement of its Kraus operators $\{M_k\}$. The goal of this section is to derive fundamental lower bounds on the energy cost of implementing a given quantum channel.

In particular, we derive a lower bound for general quantum channels which is straightforwardly obtained from Theorem 4.4 for unselective measurements. This lower bound is in general not tight, as was shown only recently [45], and we provide reasons why this is to be expected. Unfortunately, up to now there is no tight lower bound known for general quantum channels.

For a specific type of quantum channels, the erasing channels, though, we show that the lower bound *is* tight, meaning that we can determine the energy cost of these channels exactly.

We are able to derive the *exact* energy costs for another subclass of channels as well, namely the so-called dephasing channels. As dephasing is important for various quantum information protocols, this result can be considered the main contribution of this section. Remarkably, the energy costs of a dephasing channel exceed the general lower bound from above. Our result can therefore be used to formulate an alternative proof to show that the general lower bound for quantum channels is not tight [45].

4.3.1 General lower bound

Let us consider a general quantum channel $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ on S. In contrast to unselective measurements, quantum channels determine the average post-measurement state $\rho'_S = \mathcal{F}(\rho_S)$ for *all* input states ρ_S. This formalism of measurement is therefore more stringent than unselective measurements and is for example suitable in the context of Zeno stabilisation protocols as was already hinted at in Section 3.1 and is explained in more detail in Section 5.1. Intuitively, the additional requirements should lead to higher energetic requirements compared to those in Theorem 4.4. Interestingly however, until recently [45] the best known bound was just the one obtained by applying the Theorem 4.4 directly for quantum channels. For completeness and clarity, we formulate this lower bound as the next corollary:

Corollary 4.5. ***Energy cost for general quantum channels.***

Let S be a quantum system with Hilbert space $\mathcal{H}_S$ and Hamiltonian H_S initially in state ρ_S and let $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ be a quantum channel on S. Then the energy cost of $\mathcal{F}$ is lower bounded as

$$E_{\text{cost}}^{\mathcal{F}} \geq \Delta E_S + k_B T \left(S(\rho_S) - S\big(\mathcal{F}(\rho_S)\big)\right),$$

where $\Delta E_S = \operatorname{tr}[H_S(\mathcal{F}(\rho_S) - \rho_S)]$ denotes the average energy change on S.

Proof. The quantum channel $\mathcal{F}$ applied to the initial state ρ_S of S induces the unselective measurement $\rho_S \mapsto \rho'_S = \mathcal{F}(\rho_S)$. We may therefore apply Corollary 4.2 which gives the desired statement. □

4.3.2 Energy cost of the erasing channel

The general lower bound on the energy cost of quantum channels in Corollary 4.5 is a direct consequence of Theorem 4.4. In this theorem we showed that the lower bound in Corollary 4.2 is tight by providing an explicit construction of an implementation whose cost approaches the lower bound arbitrarily closely. It is not at all clear, whether the bound in Eq. 4.5 is tight as well. In fact, we will soon see that this is not the case. Before coming to that, we now show that there is an important class of channels, for which the lower bound in Corollary 4.5 *is* tight: namely, the erasing channels.

Definition 4.6. Erasing channel

For a state $\rho'_S \in \mathcal{B}(\mathcal{H}_S)$, the *erasing channel* $\mathcal{E} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ *with target state* ρ'_S on a quantum system S is the channel satisfying

$$\mathcal{E}(\rho_S) = \rho'_S \qquad \forall \rho_S \in \mathcal{B}(\mathcal{H}_S)\,.$$

The erasing channel maps any initial state ρ_S to the same, fixed target state ρ'_S and therefore erases all the information contained in ρ_S. We already made use of a related channel within our framework: A quantum instrument needs to store the measurement outcome k in a physical device M. This device has to be reset to its initial state for further uses, i.e. the (classical) information stored in the state ρ'_M needs to be erased. While the resetting in a quantum instruments relies on slightly weaker requirements (see Definition 3.7 and Section 4.4.2), this process can always be realised by an erasing channel and the resetting cost will turn out to be directly related to the energy cost of the erasing channel.

Theorem 4.7. ***Energy cost for erasing channels.***

Let $\mathcal{E} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ *be an erasing channel with target state* ρ'_S *on a quantum system* S *with Hilbert space* $\mathcal{H}_S$ *and Hamiltonian* H_S*. Let further* ρ_S *be the initial state of* S[3]*. Then the energy cost of* $\mathcal{E}$ *is exactly given by*

$$E^{\mathcal{E}}_{\text{cost}} = \Delta E_S + k_B T\big(S(\rho_S) - S(\rho'_S)\big)\,,$$

where $\Delta E_S = \text{tr}[H_S(\rho'_S - \rho_S)]$*, i.e. the lower bound in Corollary 4.5 is tight for erasing channels.*

---

[3] Just as in Theorem 4.4 we formally need to require that rank$[\rho_S] \leq$ rank$[\rho'_S]$, otherwise $T = 0$ or $E^{\mathcal{E}}_{\text{cost}} = \infty$ [38]. We dropped this assumption, since we can always approximate the target state ρ'_S by a suitable full rank state.

Proof. From Corollary 4.5 we already know that

$$E_{\text{cost}}^{\mathcal{E}} \geq \Delta E_S + k_B T\big(S(\rho_S) - S(\rho_S')\big)\,.$$

We therefore need to show the tightness of this lower bound, i.e. the existence of an implementation $(\rho_E, U_{SE}) \in \mathcal{U}_{\mathcal{E}}$ of the erasing channel $\mathcal{E}$ with a cost

$$E_{\text{cost}}^{(\rho_E, U_{SE})} < \Delta E_S + S(\rho_S) - S(\rho_S') + \epsilon$$

for any $\epsilon \geq 0$. From Theorem 4.4 we know that there are implementations (ρ_E, U_{SE}) of the state transformation $\rho_S \mapsto \rho_S' = \mathcal{E}(\rho_S)$ with cost arbitrarily close to $\Delta E_S + S(\rho_S) - S(\rho_S')$. And from the proof of Theorem 4.4 we know that there exists a valid construction of these implementations (ρ_E, U_{SE}) which output the correct states ρ_S' for *all* initial states ρ_S. Hence these implementations of the unselective measurement $\rho_S \mapsto \mathcal{E}(\rho_S)$ are also implementations of the erasing channel $\mathcal{E}$, which concludes the proof. □

4.3.3 Tightness for general quantum channels

The lower bound on the energy cost in Corollary 4.5 is tight for erasing channels. Can we extend this statement to arbitrary channels? An indication why this is probably not the case is that our previous methods for proving tightness were severely limited: The basic idea of the implementations we used in the proofs of Theorem 4.4 and Theorem 4.7 was basically to use a thermal environment E whose initial state ρ_E is identical to the final state ρ_S' on the measured system S. The interaction then just consisted of a swapping unitary that exchanged S and E. Such implementations are legitimate for unselective measurements and for the erasing channel, because the final state on S for both measurement formalisms is fixed. For general quantum channels however, the implementation must produce the correct state $\rho_S' = \mathcal{F}(\rho_S)$ for all inputs ρ_S (see Definition 3.12). We may therefore not simply initialise the thermal bath in the target state ρ_S', since the environment (and likewise the whole implementation) must be independent of the initial state ρ_S of the measured system S.

Interestingly, the question of tightness of the general lower bound in Corollary 4.5 was not considered only until recently, when Bedingham and Maroney showed the following result [45] (also see Section 4.7).

Proposition 4.8. *Let S be a quantum system with Hilbert space $\mathcal{H}_S$ and Hamiltonian H_S. Then there exist quantum channels $\mathcal{F}:\mathcal{B}(\mathcal{H}_S)\rightarrow\mathcal{B}(\mathcal{H}_S)$ and states ρ_S on S such that*

$$E_{\text{cost}}^{\mathcal{F}} > \Delta E_S + k_B T\left(S(\rho_S) - S\big(\mathcal{F}(\rho_S)\big)\right)$$

i.e. the energy cost is strictly *larger than predicted by the lower bound in Corollary 4.5.*

Proof. The proof of this statement is not easily phrased within our framework. To get a rough idea about the basic strategy we therefore refer to Section 4.7. □

In the next section we show that this statement can be strengthened for the case of dephasing channels, for which we obtain the exact costs which are strictly stronger than predicted by the general lower bound given by Corollary 4.5. We thereby provide an alternative proof for Proposition 4.8.

4.3.4 Energy cost of dephasing channels

Another important class of channels are the dephasing channels as they provide a simple way to model decoherence in a quantum system. Let us first define dephasing channels.

Definition 4.9. A dephasing channel with respect to the projections $\{P_k\}_{k=1}^K$, where $P_k\in\mathcal{B}(\mathcal{H}_S)$ with $P_k = P_k^\dagger = P_k^2$ and $\sum_{k=1}^K P_k = \mathbb{1}_S$, is a channel $\mathcal{F}:\mathcal{B}(\mathcal{H}_S)\rightarrow\mathcal{B}(\mathcal{H}_S)$ with Kraus decomposition

$$\mathcal{F}(\rho_S) = \sum_{k=1}^{K} P_k \rho_S P_k \qquad \forall \rho_S\,. \tag{4.9}$$

The projections P_k therefore correspond to the Kraus operators of this channel and the Kraus rank is K.

In the following we derive the *exact* energy cost of such a dephasing channel, which is, for almost all initial states ρ_S, strictly larger than the lower bound for general channels in Corollary 4.5. Our result on dephasing channels is also important for quantum instruments: If we apply the dephasing channel with respect to projections $\{Q_{M,k}\}$ on the state $\tilde{\rho}_M$ of a quantum system M, the final state will be classical with respect to these projections. Dephasing channels are therefore a basic example to model the dynamics on the memory M of a quantum instrument, where classical information (i.e. measurement outcomes) is stored in M.

Before presenting the main theorem of this section on the energy cost of dephasing channels, let us prove the following two important statements, which together characterise the set of all possible implementations of dephasing channels: The first statement determines the structure of the environmental system E after the unitary interaction U_{SE} of any such implementation. The second statement shows that these unitary interactions must be of a very specific form.

The following lemma [29] specifies the final state ρ'_E on E of any implementation after applying the dephasing channel. The formulation of the lemma is such that it does not require the initial state of the environment to be of full rank (and hence not necessarily thermal), which is why we denote the environmental system by the letter X to avoid confusion.

> **Lemma 4.10.** *Let $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ be a dephasing channel with respect to projections $\{P_k\}$ on a quantum system S with Hilbert space $\mathcal{H}_S$. Let further ρ_X be an initial (not necessarily thermal) state on a quantum system X with Hilbert space $\mathcal{H}_X$ and U_{SX} be a unitary on S and X such that*
>
> $$\mathcal{F}(\rho_S) = \operatorname{tr}_X[U_{SX}(\rho_S \otimes \rho_X)U_{SX}^\dagger] \qquad \forall \rho_S\ . \tag{4.10}$$
>
> *Then there exist quantum states $\sigma_{X,k} \in \mathcal{B}(\mathcal{H}_X)$ with $S(\sigma_{X,k}) = S(\rho_X)$ for all k such that, for all quantum states ρ_S, the state of X after the unitary interaction U_{SX}, $\rho'_X = \operatorname{tr}_S\left[U_{SX}(\rho_S \otimes \rho_X)U_{SX}^\dagger\right]$, can be written as*
>
> $$\rho'_X = \sum_k \operatorname{tr}[P_k \rho_S]\sigma_{X,k}\ .$$

Let us compare this result to previous statements in the literature: If the state ρ_X were pure, then the map $\rho_S \mapsto \operatorname{tr}_S\left[U_{SX}(\rho_S \otimes \rho_X)U_{SX}^\dagger\right]$ for all ρ_S corresponds to a complementary channel of the dephasing channel $\mathcal{F}$ (see Chapter 2). Such complementary channels of the dephasing channel have already been computed in many previous works (see e.g. [58, 73, 84, 85])[4]. Here, we generalise these statements to arbitrary ancilla states ρ_X. Note that this is necessary to apply them to our framework where ρ_X will be the initial state of a thermal environment. Moreover, Lemma 4.10 is not only a statement about the general form of the output of such "generalised" complementary channels, but also determines its *entropy*, which will be crucial for our Theorem 4.12 on the energy cost of dephasing channels.

Proof. The proof is based on the dilation theory of channels. Let $\mathcal{F} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ be the dephasing channel as in Eq. (4.9). To make the notation more precise let us refer to this channel as $T_{S\to S} = \mathcal{F}$ highlighting the fact that the dephasing channel maps states on S to states on S.

---

[4] Dephasing channels are a special case of Hadamard channels which are the complementary channels of entanglement-breaking channels [58, 86–88].

Before diving directly into the proof, let us first state its basic idea: By assumption we are given two descriptions of the same channel, namely

$$T_{S\to S}(\rho_S) = \sum_{k=1}^{K} P_k \rho_S P_k \tag{4.11}$$

and

$$T_{S\to S}(\rho_S) = \operatorname{tr}_X[U_{SX}(\rho_S \otimes \rho_X)U_{SX}^\dagger] \qquad \forall \rho_S\ . \tag{4.12}$$

Each description allows to derive a complementary channel: The first description Eq. (4.11) yields a channel $T_{S\to A} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_A)$ mapping states on S to states on some ancilla A. Importantly this channel is obtained from a minimal Stinespring dilation (see Chapter 2). The second description Eq. (4.12) yields a channel $T_{S\to X\tilde{X}} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_X) \otimes \mathcal{B}(\mathcal{H}_{\tilde{X}})$, where X is the system initially in a (not necessarily thermal) state ρ_X as above and $\tilde{X}$ is the purifying system of X.

Now due to the P_k being projections, we are able to specify the structure of the first complementary channel $T_{S\to A}$. By Stinespring's theorem we additionally know that the second complementary channel $T_{S\to X\tilde{X}}$ is related to the first by an isometry (see Chapter 2), thereby allowing us to translate the structure of $T_{S\to A}$ to $T_{S\to X\tilde{X}}$. This then allows us to infer the desired property on the channel

$$T_{S\to X}(\rho_S) = \operatorname{tr}_S\left[U_{SX}(\rho_S \otimes \rho_X)U_{SX}^\dagger\right] = \operatorname{tr}_{\tilde{X}}[T_{S\to X\tilde{X}}(\rho_S)]\,,$$

which is obtained by computing the partial trace of the second complementary channel $T_{S\to X\tilde{X}}$ (note that for the fixed input ρ_S, we have that $T_{S\to X}(\rho_S) = \rho'_X$).

Let us now make this idea more concrete and start the proof. We begin with deriving the first complementary channel $T_{S\to A}$ by using Eq. (4.11), where A denotes a quantum system with a K-dimensional Hilbert space $\mathcal{H}_A$ spanned by the orthonormal basis $\{|k\rangle\}$ and where K is the Kraus rank of the dephasing channel $T_{S\to S}$. Let the vector $|0\rangle_A$ be any fixed element of this basis. Consider the operator $\sum_k P_k \otimes |k\rangle_A\langle 0|$ mapping vectors from the space $\mathcal{H}_S \otimes \{|0\rangle_A\langle 0|\}$ to $\mathcal{H}_S \otimes \mathcal{H}_A$, i.e. the action of this operator is only defined on states of the form $|\psi\rangle_S \otimes |0\rangle_A$. We can always extend this operator to a unitary U_{SA} on S and A. Note that this unitary is a Stinespring dilation of $T_{S\to S}$, since

$$T_{S\to S}(\rho_S) = \operatorname{tr}_A[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|)U_{SA}^\dagger]\,.$$

Moreover, this is a *minimal* Stinespring dilation, since the dimension of A is equal to the Kraus rank K of $T_{S\to S}$ (see Chapter 2). We therefore find our first complementary channel $T_{S\to A}$ by taking the partial trace over S instead of A,

$$\begin{aligned} T_{S\to A}(\rho_S) &= \mathrm{tr}_S[U_{SA}(\rho_S \otimes |0\rangle_A\langle 0|)U_{SA}^\dagger] \\ &= \mathrm{tr}_S\Big[\sum_{k,k'} P_k \rho_S P_{k'} \otimes |k\rangle_A\langle k'|\Big], \\ &= \sum_k \mathrm{tr}[P_k\rho_S]|k\rangle_A\langle k|\,. \end{aligned} \tag{4.13}$$

The output of this channel for any input ρ_S can therefore always be written as a convex combination of a complete set of mutually orthogonal rank-1 projections with weights $\mathrm{tr}[P_k\rho_S]$ [58, 73, 84, 85]).

We continue with the second complementary channel $T_{S\to X\tilde{X}}$. From Eq. (4.12) we know that we can describe the dephasing channel $T_{S\to S}$ by a unitary U_{SX} and a probably mixed state ρ_X on X. Let $\tilde{X}$ be a purifying system of X with Hilbert space $\mathcal{H}_{\tilde{X}}$ such that $|\psi\rangle_{X\tilde{X}} \in \mathcal{H}_X \otimes \mathcal{H}_{\tilde{X}}$ is a pure state with $\mathrm{tr}_{\tilde{X}}[|\psi\rangle_{X\tilde{X}}\langle\psi|] = \rho_X$. Defining the unitary $U_{SX\tilde{X}} = U_{SX} \otimes \mathbb{1}_{\tilde{X}}$ we then find another Stinespring dilation of $T_{S\to S}$, namely

$$T_{S\to S}(\rho_S) = \mathrm{tr}_{X\tilde{X}}[U_{SX\tilde{X}}(\rho_S \otimes |\psi\rangle_{X\tilde{X}}\langle\psi|)U_{SX\tilde{X}}^\dagger]\,.$$

This yields the second complementary channel

$$T_{S\to X\tilde{X}}(\rho_S) = \mathrm{tr}_S[U_{SX\tilde{X}}(\rho_S \otimes |\psi\rangle_{X\tilde{X}}\langle\psi|)U_{SX\tilde{X}}^\dagger]\,.$$

This channel now provides the link between the desired channel $T_{S\to X}(\rho_S) = \mathrm{tr}_{\tilde{X}}[T_{S\to X\tilde{X}}(\rho_S)]$ and the first complementary channel $T_{S\to A}$ of which we know the structure by Eq. (4.13). More concretely, Stinespring's theorem tells us that there exists an isometry $V : \mathcal{H}_A \to \mathcal{H}_X \otimes \mathcal{H}_{\tilde{X}}$ which links the first and second complementary channels (see Eq. (2.2)),

$$T_{S\to X\tilde{X}}(\rho_S) = V\, T_{S\to A}(\rho_S) V^\dagger\,. \tag{4.14}$$

Hence, using Eqs. (4.13) and (4.14) we compute

$$T_{S\to X}(\rho_S) = \mathrm{tr}_{\tilde{X}}[T_{S\to X\tilde{X}}(\rho_S)] = \mathrm{tr}_{\tilde{X}}[V\,T_{S\to A}(\rho_S)V^\dagger] = \sum_k \mathrm{tr}[P_k\rho_S]\mathrm{tr}_{\tilde{X}}[V|k\rangle_A\langle k|V^\dagger]\,.$$

Defining the states $\sigma_{X,k} = \mathrm{tr}_{\tilde{X}}[V|k\rangle_A\langle k|V^\dagger]$ on X, this shows the first part of the statement, that is, for any fixed ρ_S, the state $\rho'_X = T_{S\to X}(\rho_S)$ on X is of the form

$$\rho'_X = \sum_k \mathrm{tr}[P_k\rho_S]\sigma_{X,k}\,, \tag{4.15}$$

where the $\sigma_{X,k}$ do not depend on the initial state ρ_S.

It remains to show that the states $\sigma_{X,k}$ have entropy $S(\sigma_{X,k}) = S(\rho_X)$ for all k. To show this, let k be fixed and denote by $\psi_k = P_k\psi_k$ a pure quantum state on S supported in $P_k\mathcal{H}_S$. Since Eq. (4.15) holds for all initial states on S, we may evaluate it for $\rho_S = \psi_k$ to find that

$$\rho'_X = \mathrm{tr}_S[U_{SX}(\psi_k \otimes \rho_X)U^\dagger_{SX}] = \sigma_{X,k} \ .$$

Conversely, we may evaluate Eqs. (4.11) and (4.12) for this choice of initial state to find that the final state on S,

$$\rho'_S = \mathrm{tr}_X[U_{SX}(\psi_k \otimes \rho_X)U^\dagger_{SX}] = \sum_{k'} P_{k'}\psi_k P_{k'} = \psi_k \ ,$$

is pure, which implies that the systems S and X must be uncorrelated. Hence, if the initial state on S is $\rho_S = \psi_k$, we have after the unitary interaction U_{SX}

$$\rho'_{SX} = U_{SX}(\psi_k \otimes \rho_X)U^\dagger_{SX} = \psi_k \otimes \sigma_{X,k} \ .$$

But this then implies the desired statement, since entropies are invariant under unitary operations and hence

$$S(\rho_X) = S(\rho'_{SX}) - S(\psi_k) = S(\sigma_{X,k}) \qquad \forall k \ .$$

□

To understand the importance of this statement, let us consider an implementation $(\rho_E, U_{SE}) \in \mathcal{U}_\mathcal{F}$ of the dephasing channel $\mathcal{F}$ as defined in Definition 3.12. Such a pair (ρ_E, U_{SE}) clearly satisfies the requirements of Lemma 4.10. This lemma therefore provides us with a detailed structure of the final state ρ'_E on E, basically stating that this state is a convex combination of states $\sigma_{E,k}$ with the same entropy as the initial state ρ_E, $S(\sigma_{E,k}) = S(\rho_E)$.

We can characterise the implementations (ρ_E, U_{SE}) of the dephasing channel $\mathcal{F}$ even further with the following lemma, which specifies the exact form of the unitaries U_{SE} [29].

Lemma 4.11. *Let $\mathcal{F}$ be the dephasing channel on a quantum system S with respect to projections $\{P_k\}$ and let K denote the Kraus rank of $\mathcal{F}$. Then any implementation $(\rho_E, U_{SE}) \in \mathcal{U}_\mathcal{F}$ of $\mathcal{F}$ has the property that the unitary interaction U_{SE} is of the form*

$$U_{SE} = \sum_{k=1}^{K} P_k \otimes \tilde{U}_{E,k} \ ,$$

> *with $\tilde{U}_{E,k}$ unitaries on E such that $\mathrm{tr}[\tilde{U}_{E,j}\rho_E\tilde{U}_{E,k}^\dagger] = \delta_{jk}$ for all j,k. Such unitaries can only exist if the Hilbert space dimension of E satisfies $\dim\mathcal{H}_E \geq \sqrt{K}$. Moreover, for $\rho_E = \mathbb{1}_E/\dim\mathcal{H}_E$ maximally mixed, such unitaries are known to exist.*

Proof. Let $(\rho_E, U_{SE}) \in \mathcal{U}_{\mathcal{F}}$ be an implementation of $\mathcal{F}$, i.e. according to Definition 3.12 the state ρ_E on the environment E is a thermal with respect to some Hamiltonian H_E at temperature T, and hence of full rank, and the unitary interaction U_{SE} on S and E satisfies

$$\mathrm{tr}_E[U_{SE}(\rho_S\otimes\rho_E)U_{SE}^\dagger] = \sum_k P_k\rho_S P_k \quad \forall\rho_S\,. \tag{4.16}$$

We can employ Lemma 4.10 to find that the state $\rho_E' = \mathrm{tr}_S[U_{SE}(\rho_S\otimes\rho_E)U_{SE}^\dagger]$ on E after the unitary interaction can be written, for all ρ_S, as

$$\rho_E' = \sum_k \mathrm{tr}[P_k\rho_S]\sigma_{E,k}\,, \tag{4.17}$$

where the states $\sigma_{E,k}$ are independent of ρ_S.

Let us fix k and evaluate Eqs. (4.16) and (4.17) for pure states $\rho_S = \psi_k \in \mathcal{B}(P_k\mathcal{H}_S)$ in the support of P_k. We find that the marginal on S remains pure,

$$\mathrm{tr}_E[U_{SE}(\psi_k\otimes\rho_E)U_{SE}^\dagger] = \psi_k\,, \tag{4.18}$$

whereas taking the partial trace of S yields

$$\mathrm{tr}_S[U_{SE}(\psi_k\otimes\rho_E)U_{SE}^\dagger] = \sigma_{E,k}\,.$$

These two conditions imply that

$$U_{SE}(\psi_k\otimes\rho_E)U_{SE}^\dagger = \psi_k\otimes\sigma_{E,k}\,.$$

Let us denote by $\rho_E = \sum_j\lambda_j|j\rangle_E\langle j|$ the spectral decomposition of ρ_E, where all eigenvalues $\lambda_j > 0$ are strictly positive since ρ_E was assumed to have full rank. Inserting the decomposition into Eq. (4.18) and using the positivity of the λ_j, we have that

$$\mathrm{tr}_E[U_{SE}(\psi_k\otimes|j\rangle_E\langle j|)U_{SE}^\dagger] = \psi_k \qquad \forall j\,.$$

Since unitaries preserve the spectrum, this implies that the action of U_{SE} can be written as

$$U_{SE}(|\psi_k\rangle_S\otimes|j\rangle_E) = |\psi_k\rangle_S\otimes\tilde{U}_{E,|\psi_k\rangle_S}|j\rangle_E\,, \tag{4.19}$$

where $\psi_k = |\psi_k\rangle_S\langle\psi_k|$ and $\tilde{U}_{E,|\psi_k\rangle_S}$ denotes a unitary operator on E which, in general, may depend on the state $|\psi_k\rangle_S$. However, due to linearity, we know that this operator may only depend on the label k of the subspace that ψ_k has support in. To see this rigorously, let $|\phi_k\rangle_S$ and $|\tilde{\phi}_k\rangle_S$ be two orthogonal vectors in $P_k\mathcal{H}_S$, i.e. $\langle\phi_k|\tilde{\phi}_k\rangle = 0$. From Eq. (4.19) we know that by linearity

$$\begin{aligned} U_{SE}\big((|\phi_k\rangle_S - |\tilde{\phi}_k\rangle_S)\otimes|j\rangle_E\big) &= (|\phi_k\rangle_S - |\tilde{\phi}_k\rangle_S)\otimes\tilde{U}_{E,|\phi_k\rangle_S-|\tilde{\phi}_k\rangle_S}|j\rangle_E \\ &= |\phi_k\rangle_S\otimes\tilde{U}_{E,|\phi_k\rangle_S}|j\rangle_E - |\tilde{\phi}_k\rangle_S\otimes\tilde{U}_{E,\tilde{\phi}_k}|j\rangle_E \ . \end{aligned}$$

Rearranging yields

$$0 = |\phi_k\rangle_S\otimes\big(\tilde{U}_{E,|\phi_k\rangle_S}|j\rangle_E - \tilde{U}_{E,|\phi_k\rangle_S-|\tilde{\phi}_k\rangle_S}|j\rangle_E\big) + |\tilde{\phi}_k\rangle_S\otimes\big(\tilde{U}_{E,|\phi_k\rangle_S-|\tilde{\phi}_k\rangle_S}|j\rangle_E - \tilde{U}_{E,\tilde{\phi}_k}|j\rangle_E\big)$$

for all j. This can only be satisfied if

$$\tilde{U}_{E,|\phi_k\rangle_S} = \tilde{U}_{E,|\phi_k\rangle_S-|\tilde{\phi}_k\rangle_S} = \tilde{U}_{E,|\tilde{\phi}_k\rangle_S} \ .$$

We therefore know that the unitary operators $\tilde{U}_{E,|\phi_k\rangle_S}$ and $\tilde{U}_{E,|\tilde{\phi}_k\rangle_S}$ are identical if the vectors $|\phi_k\rangle_S$ and $|\tilde{\phi}_k\rangle_S$ are orthogonal and within the same subspace $P_k\mathcal{H}_S$. Now let $\{|e_j^k\rangle_S\}_j$ be an orthonormal basis of $P_k\mathcal{H}_S$ such that we can write $|\psi_k\rangle_S = \sum_j \psi_j^k|e_j^k\rangle_S$. Evaluating Eq. (4.19) for this vector and using that $\tilde{U}_{E,|e_j^k\rangle_S} = \tilde{U}_{E,|e_{j'}^k\rangle_S} \equiv \tilde{U}_{E,k}$ for all j, j', we find by linearity

$$\begin{aligned} U_{SE}(|\psi_k\rangle_S\otimes|j\rangle_E) &= \sum_j \psi_j^k U_{SE}(|e_j^k\rangle_S\otimes|j\rangle_E) \\ &= \sum_j \psi_j^k|e_j^k\rangle_S\otimes\tilde{U}_{E,|e_j^k\rangle_S}|j\rangle_E \\ &= |\psi_k\rangle_S\otimes\tilde{U}_{E,k}|j\rangle_E \ . \end{aligned} \tag{4.20}$$

Hence, linearity is indeed sufficient to prove that the unitary operators $\tilde{U}_{E,k}$ may only depend on the label k, rather than on the full vector $|\psi_k\rangle_S$.

So far we evaluated Eqs. (4.16) and (4.17) for pure states $\rho_S = \psi_k$ only. For mixed states ρ_S however, we find from Eq. (4.20) that

$$U_{SE}(\rho_S\otimes\rho_E)U_{SE}^\dagger = U_{SE}\Big(\sum_{ij}P_i\rho_S P_j\otimes\rho_E\Big)U_{SE}^\dagger = \sum_{ij}P_i\rho_S P_j\otimes\tilde{U}_{E,i}\rho_E\tilde{U}_{E,j}^\dagger \qquad \forall\rho_S \ , \tag{4.21}$$

which together with Eq. (4.16) gives us that

$$\sum_k P_k\rho_S P_k = \mathrm{tr}_E[U_{SE}(\rho_S\otimes\rho_E)U_{SE}^\dagger] = \sum_{ij}\mathrm{tr}[\tilde{U}_{E,i}\rho_E\tilde{U}_{E,j}^\dagger]P_i\rho_S P_j \qquad \forall\rho_S \ .$$

This holds if and only if the unitaries $\tilde{U}_{E,k}$ satisfy

$$\mathrm{tr}[\tilde{U}_{E,i}\rho_E\tilde{U}_{E,j}^\dagger] = \delta_{i,j} \qquad \forall i,j \ , \tag{4.22}$$

that is, $\{\tilde{U}_{E,k}\}_k$ is a so-called orthonormal unitary operator basis with respect to the modified scalar product in Eq. (4.22). Such bases can only exist for large enough Hilbert space dimension, that is $d_E \equiv \dim\mathcal{H}_E \geq \sqrt{K}$ as shown in [55, 76]. For $\rho_E = \mathbb{1}_E/d_E$ maximally mixed, such bases exist [55, 76] as we exemplify below.

This concludes the desired characterisation of those unitaries U_{SE} which, for a given full rank (in particular thermal) state ρ_E, satisfy Eq. (4.16): The unitary must be of the form in Eq. (4.21), where the operators $\tilde{U}_{E,k}$ on E satisfy Eq. (4.22).

For the sake of completeness, let us quickly check that our explicit construction above of the unitary U_{SE} indeed forms an implementation together with $\rho_E = \mathbb{1}_E/d_E$. Indeed, we can compute

$$\begin{aligned}\mathrm{tr}_E[U_{SE}(\rho_S\otimes\rho_E)U_{SE}^\dagger] &= \mathrm{tr}_E\Big[\Big(\sum_j P_j\otimes\tilde{U}_{E,j}\Big)\Big(\rho_S\otimes\mathbb{1}_E/d_E\Big)\Big(\sum_k P_k\otimes\tilde{U}_{E,k}^\dagger\Big)\Big]\\ &= \frac{1}{d_E}\sum_{j,k} P_j\rho_S P_k \mathrm{tr}[\tilde{U}_{E,j}\tilde{U}_{E,k}^\dagger]\\ &= \sum_k P_k\rho_S P_k \qquad \forall\rho_S\in\mathcal{B}(\mathcal{H}_S)\,.\end{aligned}$$

For this state ρ_E, the unitaries $\tilde{U}_{E,k}$ can for example be chosen as distinct elements from the set of unitaries [55, 76]

$$\tilde{U}_{E,l,m} = \sum_{r=0}^{d_E-1} \mathrm{e}^{\frac{2\pi i}{d_E} rm}|l+r\rangle\langle r|\,, \qquad l,m=0,1,\ldots,d_E-1\,,$$

where the addition in $|l+r\rangle$ is taken modulo d_E. These d_E^2 operators can be understood as a discrete version of Weyl systems of unitary operators [76] and indeed satisfy, as one can easily compute,

$$\mathrm{tr}[\tilde{U}_{E,l,m}\tilde{U}_{E,s,t}^\dagger] = d_E\cdot\delta_{l,s}\delta_{m,t} \qquad \forall l,m,s,t\in\{0,\ldots,d_E-1\}\,.$$

□

We are now ready to state the main result of this section [29].

Theorem 4.12. ***Energy cost of dephasing channels.***

Let $\mathcal{F}$ be a dephasing channel with respect to projections $\{P_k\}$ on a quantum system S with Hilbert space $\mathcal{H}_S$ and Hamiltonian H_S initially in state $\rho_S\in\mathcal{B}(\mathcal{H}_S)$. Then the energy cost $E_\mathcal{F}$ of $\mathcal{F}$ is exactly given by

$$E_{\text{cost}}^{\mathcal{F}} = \Delta E_S\,,$$

where $\Delta E_S = \mathrm{tr}[H_S(\sum_k P_k\rho_S P_k - \rho_S)]$ is the average energy change on S. Thus, optimal implementations of a dephasing channel do not dissipate heat, $\Delta Q_\mathcal{F} = 0$.

Before presenting the proof, let us note that this result is stronger than the lower bound from Corollary 4.5 for two reasons: On the one hand, this result *exactly* quantifies the energy costs in contrast to Corollary 4.5. On the other hand, the exact costs of the dephasing channel are in general *strictly* larger than predicted by the lower bound from Corollary 4.5. This is simply due to the fact that the dephasing channel is unital, i.e. $\mathcal{F}(\mathbb{1}_S) = \mathbb{1}_S$, and that the von Neumann entropy is non-decreasing under unital channels, i.e. $S(\rho_S) \leq S(\mathcal{F}(\rho_S))$ for all ρ_S[5]. Hence, whenever the entropy of the measured system S is changed in the dephasing operation (which is if and only if ρ_S was not already classical with respect to the projections $\{P_k\}$), Theorem 4.12 provides a strict improvement of the lower bound in Corollary 4.5. The theorem therefore also proves that the general lower bound from Corollary 4.5 is not tight since this lower bound can, in general, not be attained for dephasing channels [45].

Proof. Let $\mathcal{F}$ be a dephasing channel with respect to projections $\{P_k\}$ on a quantum system S. Recall from Definition 3.13 that the energy cost of this channel is defined as

$$\begin{aligned} E_{\text{cost}}^{\mathcal{F}} &= \Delta E_S + \inf_{\mathcal{U}_{\mathcal{F}}} \Delta E_E \\ &= \Delta E_S + \inf_{\mathcal{U}_{\mathcal{F}}} \operatorname{tr}[H_E(\operatorname{tr}_S[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^{\dagger}] - \rho_E)] , \end{aligned} \tag{4.23}$$

i.e. as the minimal average energy change on all involved systems, i.e. system S and environment E, for all possible implementations $(\rho_E, U_{SE}) \in \mathcal{U}_{\mathcal{F}}$ of $\mathcal{F}$. Recall further from Definition 3.12 that a tuple (ρ_E, U_{SE}) is an implementation of $\mathcal{F}$ if and only if ρ_E is thermal with respect to Hamiltonian H_E at temperature T and U_{SE} is a unitary on S and E such that

$$\mathcal{F}(\rho_S) = \sum_k P_k \rho_S P_k = \operatorname{tr}_E[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^{\dagger}] \qquad \forall \rho_S \in \mathcal{B}(\mathcal{H}_S) .$$

The proof now consists of two parts: First, we show that the energy cost of the dephasing channel is lower bounded as $E_{\text{cost}}^{\mathcal{F}} \geq \Delta E_S$. Then, in a second step, we provide an explicit implementation (ρ_E, U_{SE}) of $\mathcal{F}$ with cost $E_{\text{cost}}^{(\rho_E, U_{SE})} = \Delta E_S$. These two statements together then imply the desired statement.

Let us start with the first part of the proof with the goal to prove the validity of the inequality $E_{\text{cost}}^{\mathcal{F}} \geq \Delta E_S$. For this, let us for the moment fix an implementation (ρ_E, U_{SE}) of $\mathcal{F}$. Note that the quantity $\operatorname{tr}_S[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^{\dagger}]$ appearing in Eq. (4.23)

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[5] For dephasing channels, we can even exactly quantify the entropy increase, since $S(\mathcal{F}(\rho_S)) = S(\rho_S) + D(\rho_S' \| \rho_S)$ (see e.g. Theorem 11.9 of ref. [43]).

is just the final state ρ'_E of E after the unitary interaction U_{SE}. Hence, we know from Lemma 4.10 that this state can be written as

$$\rho'_E = \sum_k \mathrm{tr}[P_k \rho_S] \sigma_{E,k} \qquad \forall \rho_S\,,$$

where the states $\sigma_{E,k}$ have entropy $S(\sigma_{E,k}) = S(\rho_E)$ and are independent of ρ_S. The energy change ΔE_E on E is therefore given as

$$\Delta E_E = \mathrm{tr}[H_E(\mathrm{tr}_S[U_{SE}(\rho_S \otimes \rho_E)U_{SE}^\dagger] - \rho_E)] = \sum_k \mathrm{tr}[P_k \rho_S]\mathrm{tr}[H_E(\sigma_{E,k} - \rho_E)]$$

But this is non-negative: The terms $\mathrm{tr}[P_k\rho_S] \geq 0$ are probabilities, whereas the terms $\mathrm{tr}[H_E(\sigma_{E,k} - \rho_E)]$ are non-negative because of the non-negativity of the relative entropy,

$$\begin{aligned} 0 \leq k_B T D(\sigma_{E,k} \| \rho_E) &= \mathrm{tr}[H_E \sigma_{E,k}] - k_B T S(\sigma_{E,k}) - \mathrm{tr}[H_E \rho_E] + k_B T S(\rho_E) \\ &= \mathrm{tr}[H_E(\sigma_{E,k} - \rho_E)]\,, \end{aligned}$$

where we used Eq. (4.2) and the fact that $S(\sigma_{E,k}) = S(\rho_E)$ and that ρ_E is thermal by assumption. Summarising, the energy change ΔE_E on E is non-negative *for all* possible implementations (ρ_E, U_{SE}), which together with Eq. (4.23) proves the desired inequality $E^{\mathcal{F}}_{\mathrm{cost}} \geq \Delta E_S$.

We now continue with the second part of the proof which aims at constructing an explicit implementation (ρ_E, U_{SE}) of $\mathcal{F}$ with cost $E^{(\rho_E, U_{SE})}_{\mathrm{cost}} = \Delta E_S$. For this implementation we choose a thermal environment E with Hilbert space dimension $d_E \equiv \dim \mathcal{H}_E \geq \sqrt{K}$, where K again denotes the Kraus rank of $\mathcal{F}$. Furthermore we choose the Hamiltonian of E be trivial, i.e. $H_E = 0$, implying that the thermal state on E is maximally mixed, since

$$\rho_E = \frac{\mathrm{e}^{-\frac{H_E}{k_B T}}}{Z_E} = \frac{\mathbb{1}_E}{d_E}\,.$$

From the proof of Lemma 4.11 we know that there exists a unitary U_{SE} on S and E such that (ρ_E, U_{SE}) is an implementation of $\mathcal{F}$ since we chose $d_E \geq \sqrt{K}$. But clearly, the average energy change ΔE_E on E vanishes for this implementation, since $H_E = 0$. By Eq. (4.23) this implies that for this implementation $E^{\mathcal{F}}_{\mathrm{cost}} = \Delta E_S$, which concludes the proof. □

4.4 Quantum instruments

The most detailed formalism of quantum measurements are quantum instruments, as they formalise the intuitive picture in which a measurement on a quantum state returns an outcome k and the measured state $\rho'_{S,k}$ (see Section 3.1). So far, not many results on the energy cost of quantum instruments have been established and even those still suffer from certain restrictions on their applicability (see Section 4.7).

In the previous sections, we formulated lower bounds on the energy cost of measurements, where the resetting step could be omitted: Since no measurement outcome is recorded for both unselective measurements and quantum channels, the memory need not be reset. For quantum instruments, however, (and also for POVMs which we discuss in the next section) this reasoning does not hold any more: These formalisms store the measurement outcomes k within the memory and the resetting step, needed to allow for further uses of the memory, adds significantly to the respective energy costs. Indeed, the importance of the resetting step is highlighted by the fact that *without* the resetting step, the energy cost of an implementation of a quantum instrument can take *any* value, allowing at least in principle to extract/dump arbitrary amounts of energy from/into the involved systems. For *efficient* quantum instruments (see Section 3.1), this was already known as we elaborate in Section 4.7. To generalise this statement to general quantum instruments is the first part of this chapter (Section 4.4.1).

In a next step we compute general lower bounds on the energy cost of the resetting step, which then allows us to provide lower bounds on the total energy cost of quantum instruments. Since implementations of quantum instruments must satisfy much stricter requirements compared to the other formalisms of measurement, we expect larger energetic expenses to arise from implementing a quantum instrument. This intuition is reflected by this general lower bound on the energy cost of quantum instruments, which is never less than the bounds we obtained for unselective measurements and quantum channels.

The lower bound for quantum instruments is new in its generality (see Section 4.7) — similarly to the lower bound for quantum channels, however, it is not tight. This is due to the fact (which was not noted before) that this bound can be proven for simple state transformations, i.e. without requiring the implementation to output the correct post-measurement states $\rho'_{S,k}$ on S for *all* input states ρ_S.

Remarkably, this bound allows for the possibility to extract energy directly from the measurement device. Indeed, we provide an explicit implementation of a particular quantum instrument where such energy extraction is accomplished, thereby falsifying a conjecture stated in ref. [64]. We elaborate on this in Section 4.4.3 of this chapter.

4.4.1 No energy restrictions without resetting

Let us consider a quantum instrument $\{M_{ki}\}$ on a quantum system S. The goal of this section is to show that for such a measurement the energy cost of measurement *without* resetting can take any value depending on the implementation [64]. Hence, in contrast to the previous formalisms of quantum measurement, the resetting step is essential to compute lower bounds on the energy cost of quantum instruments.

Recall from Section 3.3 that the energy cost of the quantum instrument $\{M_{ki}\}$ on system S with Hamiltonian H_S in state ρ_S without resetting is given as

$$E_{\text{meas}}^{\{M_{ki}\}} = \inf_{\mathcal{U}_{\{M_{ki}\}}} E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \Delta E_S + \inf_{\mathcal{U}_{\{M_{ki}\}}} (\Delta E_M + \Delta E_E) \ ,$$

where $E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ denotes the energy cost of the implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of $\{M_{ki}\}$. The average energy change $\Delta E_S = \mathrm{tr}[H_S(\rho'_S - \rho_S)]$ on S is fixed for all implementations so that the optimisation involves only the average energy changes of the remaining involved systems, i.e. memory M and environment E. Naively, one may think that this optimisation problem is essentially the same as the one for unselective measurements in Eq. (4.7), the only difference being that the unitaries U_{SME} of implementations in $\mathcal{U}_{\{M_{ki}\}}$ need to satisfy more restrictions. If this were true we could simply employ the exact energy cost result for unselective measurement to obtain a lower bound on $E_{\text{meas}}^{\{M_{ki}\}}$. Note however the following lemma [29].

> **Lemma 4.4.1.** *For all quantum instruments $\{M_{ki}\}$ there exists an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{M_{ki}\}}$ such that its energy cost $E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ without resetting can attain any given value. Hence, there does not exist a finite lower bound on the energy cost $E_{\text{meas}}^{\{M_{ki}\}}$ of a quantum instrument $\{M_{ki}\}$.*

Proof. Let $\{M_{ki}\}$ be a fixed quantum instrument. The energy cost $E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$ of an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of $\{M_{ki}\}$ depends crucially on the Hamiltonians of the memory M and environment E. While the Hamiltonian H_E is related to the state ρ_E through the thermality condition, the Hamiltonian H_M can, for any implementation, be chosen independently of ρ_M, with no condition on the energy spectrum whatsoever, since ρ_M need not be thermal. Moreover, we can always find an implementation such that the initial and final state on M are not identical, i.e. $\rho'_M \neq \rho_M$. Hence, for all quantum instruments, we can always find an implementation with a suitably chosen H_M such that the average energy change $\Delta E_M = \mathrm{tr}[H_M(\rho'_M - \rho_M)]$ on M, and hence the energy cost $E_{\text{meas}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$, attains any desired value. $\square$

Remarkably, including the resetting cost $E_{\text{res}}^{\{M_{ki}\}}$ into the description of a microscopic framework for quantum instruments allows us to obtain lower bounds on the total energy cost $E_{\text{cost}}^{\{M_{ki}\}}$ of the measurement, as we show in the following section.

4.4.2 General lower bound on the energy cost of quantum instruments

The need to reset the memory in order to allow for multiple iterations of a quantum instrument using the same implementation, leads to an additional cost $E_{\text{res}}^{\{M_{ki}\}}$ that we need to include into the total energy expense for quantum instruments. We focus now on quantifying this cost.

Recall from Definition 3.7 that an implementation of the resetting step unitarily couples the memory M and a thermal environment B at temperature T such that the post-measurement state ρ'_M on M after measurement is mapped to its initial state ρ_M prior to the measurement,

$$\operatorname{tr}_B[U_{MB}(\rho'_M \otimes \rho_B)U_{MB}^\dagger] = \rho_M \tag{4.24}$$

for all $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ on S. Importantly, and in contrast to ref. [44] (see Section 4.7), we do not allow the unitary U_{MB} to also act on the post-measurement state ρ'_S on S, because this state is typically not available due to feedback (see Section 3.3).

For this process we find the following proposition [29].

> **Proposition 4.13.** *Let* $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ *be an implementation of a quantum instrument* $\{M_{ki}\}$ *on a system* S *and denote by* $\rho'_M = \operatorname{tr}_{SE}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger]$ *the final state of the memory after the unitary interaction* U_{SME}. *Then the energy cost* $E_{\text{res}}^{\{M_{ki}\}}$ *of the resetting step that takes* ρ'_M *back to* ρ_M *as in Eq.* (4.24) *is exactly given by*
>
> $$E_{\text{res}}^{\{M_{ki}\}} = -\Delta E_M + k_B T \left(S(\rho'_M) - S(\rho_M)\right), \tag{4.25}$$
>
> *where* $\Delta E_M = \operatorname{tr}[H_M(\rho'_M - \rho_M)]$ *is the average energy decrease and* $S(\rho'_M) - S(\rho_M)$ *is the entropy decrease on* M.

Proof. From Theorem 4.4 we know that

$$E_{\text{res}}^{\{M_{ki}\}} \geq -\Delta E_M + k_B T \left(S(\rho'_M) - S(\rho_M)\right), \tag{4.26}$$

since, for any fixed state ρ'_M, any implementation of the resetting process also corresponds to an implementation of the state transformation $\rho'_M \mapsto \rho_M$. Tightness of this inequality follows from the tightness of the erasing channel: While the requirement Eq. (4.24) is slightly weaker than the requirement in Definition 4.6 for the erasing channel[6], any implementation of an erasing channel is also an implementation of the resetting step. Consequently, we know from Theorem 4.7 that there exists an implementation (ρ_B, U_{MB}) of the resetting process with energy cost arbitrarily close to the lower bound in Eq. (4.26) such that

$$E_{\text{res}}^{\{M_{ki}\}} = -\Delta E_M + k_B T \left(S(\rho'_M) - S(\rho_M) \right) ,$$

which concludes the proof.

$\square$

Incorporating this cost for the resetting step we find the following theorem for the energy cost of a quantum instrument [29].

Theorem 4.14. ***Energy cost for general quantum instruments.***
Let $\{M_{ki}\}$ be a quantum instrument on a system S with Hamiltonian H_S initially in state ρ_S. Then the energy cost of this instrument $\{M_{ki}\}$ is lower bounded as

$$E_{\text{cost}}^{\{M_{ki}\}} \geq \Delta E_S + k_B T \Big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \Big] , \tag{4.27}$$

where $\Delta E_S = \operatorname{tr}[H_S(\rho'_S - \rho_S)]$ is the average energy change on S.

This theorem indeed says that the addition of the resetting cost allows for a nontrivial lower bound on the energy cost of quantum instruments. The bound states that the energy costs of such a measurement are determined by the average entropy descrease $S(\rho_S) - \sum_k p_k S(\rho'_{S,k})$ in the measured system (in addition to the average energy change ΔE_S on S, which is constant for all implementations). This matches the intuition that the more information is obtained through measurement the more energy needs to be expended [29]. Indeed, the quantity $S(\rho_S) - \sum_k p_k S(\rho'_{S,k})$ already appeared in an old paper by Groenewold [89], where he provides arguments why this average entropy decrease is the natural quantity to quantify the amount of information gathered in a quantum measurement.

However, note that this intuition, stating that more information acquisition requires more energy, is not made mathematically precise in Theorem 4.14, since it only provides a lower bound and not the exact costs. For the special class of projective quantum instruments however, we are able to compute the exact energy cost (see Section 4.5), which allows us to prove this intuition to be indeed right for

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[6] The erasing channel must map *all* states ρ_S on S to the fixed target state. In contrast, the resetting step only needs to reset all states ρ'_M on M *that can be obtained in an implementation of the measurement*, in particular only those states ρ'_M which are classical with respect to the projections $\{Q_{M,k}\}$.

such measurements.

Before presenting the proof of Theorem 4.14, let us compare this result with the lower bound for quantum channels from Corollary 4.5. For this, recall from Section 3.1 that the quantum channel $\mathcal{F}$ naturally induced by a quantum instrument $\{M_{ki}\}$ is obtained by ignoring the measurement outcome k, i.e. by taking the average

$$\mathcal{F}(\rho_S) = \sum_k p_k \rho'_{S,k} = \sum_{ki} M_{ki} \rho_S M_{ki}^\dagger \,.$$

We know from Corollary 4.5 that such a channel comes with a cost $E_{\text{cost}}^{\mathcal{F}} \geq \Delta E_S + k_B T[S(\rho_S) - S(\rho'_S)]$. But recall from Eq. (2.5) that the von Neumann entropy is concave and therefore we have that $S(\rho'_S) = S(\sum_k p_k \rho'_{S,k}) \geq \sum_k p_k S(\rho'_{S,k})$. This implies that the bound in Theorem 4.14 is at least as large as the one for the corresponding quantum channel $\mathcal{F}$.

This is intriguing: Both the quantum instrument and the induced quantum channel above map the state ρ_S to the post-measurement state ρ'_S. Furthermore, since we included the resetting step into our definition of the energy cost, the memory of any implementation for either measurement is reset to its initial state. This observation therefore suggests that the sole act of *temporarily* storing the measurement outcomes requires an amount of energy which ultimately leads to higher costs for quantum instruments compared to those of the induced quantum channel. Again, since we are only comparing lower bounds, this statement is not yet mathematically precise. However, our exact result for the special case of projective quantum instruments rigorously proves this statement (see Section 4.5).

Proof of Theorem 4.14. According to Definition 3.17, the energy cost $E_{\text{cost}}^{\{M_{ki}\}}$ of a quantum instrument $\{M_{ki}\}$ on a quantum system S with Hamiltonian H_S in state ρ_S is defined as

$$E_{\text{cost}}^{\{M_{ki}\}} = \inf_{\mathcal{U}_{\{M_{ki}\}}} E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} \,,$$

where

$$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \Big(\Delta E_E + \Delta E_B \Big) \tag{4.28}$$

is the energy cost of an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$.

Let us now fix an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{Mki\}}$ of the quantum instrument. From Proposition 4.13 we know that

$$\inf_{(\rho_B, U_{MB})} \Delta E_B = k_B T \left(S(\rho'_M) - S(\rho_M) \right) \,.$$

Moreover, we can always lower bound the average energy change ΔE_E on the environment E for a fixed initial state ρ_S by evaluating Corollary 4.2 for the state transformation $\rho_{SM} \mapsto \rho'_{SM}$, which yields

$$\Delta E_E \geq k_B T \left(S(\rho_{SM}) - S(\rho'_{SM}) \right) . \tag{4.29}$$

Let us focus on the second entropic term $S(\rho'_{SM})$ in this inequality. Since by Condition 2 of Definition 3.14 the average post-measurement state ρ'_{SM} on S and M is classical with respect to the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$, we may write ρ'_{SM} as

$$\rho'_{SM} = \sum_k (\mathbb{1}_S \otimes Q_{M,k}) \rho'_{SM} (\mathbb{1}_S \otimes Q_{M,k}) = \sum_k p_k \rho'_{SM,k} ,$$

where we used Condition 1 to identify $p_k = \operatorname{tr}[(\mathbb{1}_S \otimes Q_{M,k})\rho'_{SM}]$ and defined the states $\rho'_{SM,k} = (\mathbb{1}_S \otimes Q_{M,k})\rho'_{SM}(\mathbb{1}_S \otimes Q_{M,k})/p_k$. The state ρ'_{SM} can therefore be written as a convex combination of mutually orthogonal states $\rho'_{SM,k}$. Hence, using Eq. (2.5), we get

$$S(\rho'_{SM}) = H(\{p_k\}) + \sum_k p_k S(\rho'_{SM,k}) . \tag{4.30}$$

Note that also the final state $\rho'_M = \sum_k p_k \rho'_{M,k}$ on M can be written as a convex combination of the mutually orthogonal states $\rho'_{M,k}$ by Condition 2 of Definition 3.14 such that, again by Eq. (2.5),

$$S(\rho'_M) = H(\{p_k\}) + \sum_k p_k S(\rho'_{M,k}) . \tag{4.31}$$

Inserting Eq. (4.30) into Eq. (4.29) we obtain

$$\Delta E_E \geq k_B T \left(S(\rho_{SM}) - H(\{p_k\}) - \sum_k p_k S(\rho'_{SM,k}) \right) . \tag{4.32}$$

In a last step we bound the quantity $S(\rho'_{SM,k})$ in terms of the marginal entropies using the non-negativity of the mutual information (see Eq. (2.6)),

$$S(\rho'_{SM,k}) \geq S(\rho'_{S,k}) + S(\rho'_{M,k}) .$$

Inserting this together with Eq. (4.31) into Eqs. (4.32) and (4.28) and rearranging the terms for each of the involved systems, we finally obtain

$$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} \geq \Delta E_S + k_B T \left(S(\rho_S) - \sum_k p_k S(\rho'_{S,k}) \right) .$$

Note that this inequality holds for all implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$. Hence the right hand side is also a lower bound on the energy cost $E_{\text{cost}}^{\{M_{ki}\}} = \inf E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})}$, which gives the desired statement.

□

There is a remarkable feature about this proof [29]: it does not use Condition 1 from Definition 3.14 to its full extent, that is, the fact that the implementations of a quantum instrument must output the correct state $\rho'_{S,k}$ for each k for *all* initial states ρ_S. Indeed, for the proof to be correct, it is sufficient that the tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ outputs $\rho'_{S,k}$ for each k correctly merely for the fixed input ρ_S for which the bound is to be evaluated. This naturally questions the tightness of this general lower bound for quantum instruments. It seems likely that one can obtain better lower bounds if this additional restriction, that the implementation must work properly on all input states, can be employed. It remains an open question how to get such lower bounds precisely.

For the special case of projective quantum instruments, however, the situation is different: As we show in Section 4.5, by using the additional requirement, we are able to obtain much stronger constraints on the energetic cost for such measurements. We thereby also prove that our lower bound for general quantum instruments from Theorem 4.14 is generally not tight.

4.4.3 Inefficient measurements allow for energy extraction

In Section 3.1 we claimed that the inefficiency I (see Definition 3.6) of a quantum instrument plays an important role for its energy cost. Now that we have proven our general Theorem 4.14 which provides a lower bound on the energy expense of any possible quantum instrument, we are ready to investigate the role of the inefficiency in more detail.

As described in Section 3.4 there are two resources present in our framework, namely the thermal states of the environments E and B and the initial state ρ_S (or more specifically, its purity) of the measured system S. The goal of this section is to show that when both resources are "consumed" in the implementation of a quantum instrument, energy (or even work, as explained in Section 3.4) can be *extracted* via unitary dynamics on these systems. We thereby falsify a conjecture from [64]. In particular, we show that the inefficiency of the quantum instrument determines the availability of the purity resource and hence bounds the amount of extractable energy.

To make these statements precise, let us clarify what we mean by "extractable" energy. Naively, one may just set $E_{\text{extr}}^{\{M_{ki}\}} = -E_{\text{cost}}^{\{M_{ki}\}}$, i.e. the amount of extractable energy $E_{\text{extr}}^{\{M_{ki}\}}$ for the instrument $\{M_{ki}\}$ is then defined as the maximal average energy decrease on all involved systems. This matches the intuition that this amount of energy can later be used, due to conservation of energy, by an external agent who conducts the process. However, it is easy to see that, for all quantum instruments, the average energy change $\Delta E_S = \operatorname{tr}[H_S(\rho'_S - \rho_S)]$ appearing in the lower bound in

Eq. (4.27) can take any value depending on the Hamiltonian H_S and the initial state ρ_S of the measured system S (unless the trivial case where $T_k = \mathrm{id}$ is the identity map for all k). The quantity $E_{\mathrm{extr}}^{\{M_{ki}\}}$ can hence also take any value.

What we are interested in instead is the process where heat is converted into work by means of measurement. To analyse this process we consider following quantity [29].

> **Definition 4.15.** Let $\{M_{ki}\}$ be a quantum instrument on a system S in state ρ_S. Let further $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be an implementation of $\{M_{ki}\}$. The amount of extracted energy for this implementation is defined as the average energy decrease on all involved systems except S, i.e.
>
> $$E_{\mathrm{extr}} = -\Delta E_E - \inf_{(\rho_B, U_{MB})} \Delta E_B \ ,$$
>
> where, as usual, $\Delta E_E = \mathrm{tr}[H_E(\rho'_E - \rho_E)]$ and ΔE_B analogously.
> The *extractable energy* $E_{\mathrm{extr}}^{\{M_{ki}\}}$ is then defined as
>
> $$E_{\mathrm{extr}}^{\{M_{ki}\}} = \sup_{\mathcal{U}_{\{M_{ki}\}}} E_{\mathrm{extr}} = - \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{M_{ki}\}}} \Big(\Delta E_E + \Delta E_B \Big) \, .$$

The quantity $E_{\mathrm{extr}}^{\{M_{ki}\}}$ determines the maximal amount of work that can be extracted *from the thermal baths E* and *B* in the measurement $\{M_{ki}\}$. Note that, based on the intuition based on the Second Law of Thermodynamics, such energy extraction from the thermal baths should only be possible if the other resource, i.e. the purity of the measured system, is expended.

It remained an open question whether $E_{\mathrm{extr}}^{\{M_{ki}\}}$ can become positive for any quantum instrument $\{M_{ki}\}$. In fact, it was conjectured in [64] that this is not the case, i.e. $E_{\mathrm{extr}}^{\{M_{ki}\}} \leq 0$ for all $\{M_{ki}\}$ and all initial states ρ_S. The conjecture therefore states that performing a quantum instrument on any state ρ_S does not allow to convert heat into work. Indeed, for efficient quantum instruments (see Definition 3.6) this is indeed true [64]. In the following however we falsify this conjecture [29]. We do so, by first formulating upper bounds on $E_{\mathrm{extr}}^{\{M_{ki}\}}$ that hold for all quantum instruments and all initial states. These bounds will already provide some intuition what properties of a quantum instrument will allow for energy extraction. Then, we provide a specific quantum instrument together with a specific implementation such that a positive amount of energy can indeed be extracted for most initial states.

We can immediately formulate two simple upper bounds on $E_{\mathrm{extr}}^{\{M_{ki}\}}$: The first is directly obtained from Theorem 4.14,

$$E_{\mathrm{extr}}^{\{M_{ki}\}} \leq k_B T \Big(\sum_k p_k S(\rho'_{S,k}) - S(\rho_S) \Big) \, , \tag{4.33}$$

however this bound strongly depends on the initial state ρ_S of the measured system S. Instead we prefer here to formulate a bound that solely depends on properties of the measurement operators $\{M_{ki}\}$.

The second upper bound satisfies this requirement: We obtain it by "optimising" Eq. (4.33) over all initial and final states. More concretely, we let the initial state ρ_S of S be pure, such that $S(\rho_S) = 0$ attains its minimal value, and $\rho'_{S,k}$ be maximally mixed for each k, implying that $S(\rho'_{S,k}) = \log d_S$ is maximal, where d_S denotes the Hilbert space dimension of S. Inserting this into Eq. (4.33) yields the upper bound

$$E_{\text{extr}}^{\{M_{ki}\}} \leq k_B T \log d_S \,. \tag{4.34}$$

This bound is clearly independent of the initial state, but also basically independent of the instrument.

We now formulate a bound by explicitly including the *inefficiency* I of the quantum instrument. Let us therefore recall that, as described in Section 3.1, a measurement is called *inefficient* if the Kraus rank $I(k)$ of the corresponding measurement channel T_k is greater than 1 for at least one outcome k. The *inefficiency* of the measurement is defined by its maximal Kraus rank, $I = \max_k I(k)$.

For general measurements we then find the following proposition [29].

Proposition 4.16. *Let $\{M_{ki}\}$ be a quantum instrument with inefficiency I on a quantum system S in state ρ_S. The amount of extractable energy is upper bounded as*

$$E_{\text{extr}} \leq k_B T \log I \,,$$

irrespective of the chosen implementation.

Whenever the inefficiency I is not larger than the Hilbert space dimension of the measured system, i.e. $I \leq d_S$, this upper bound is better than the one in Eq. (4.34).

Proposition 4.16 proves that extraction of energy through measurement is, if at all, only possible for inefficient measurements. Efficient measurements can never yield energy, $E_{\text{extr}} \leq 0$, which agrees with previous findings [64].

Proof. We consider a quantum instrument $\{M_{ki}\}$ with inefficiency I. From Theorem 4.14 we know that the amount of extracted energy can never exceed the average increase of von Neumann entropy of the measured system,

$$E_{\text{extr}} \leq k_B T \Big(\sum_k p_k S(\rho'_{S,k}) - S(\rho_S) \Big) \,,$$

irrespective of the chosen implementation. Hence, it suffices to prove that the right hand side satisfies

$$\sum_k p_k S(\rho'_{S,k}) - S(\rho_S) \leq \log I$$

for all quantum instruments $\{M_{ki}\}$ to show the desired statement. We therefore aim to show that the inefficiency I places an upper bound on the average entropy increase, and hence on the average purity decrease, on S.

For this, let us define the probability

$$p_{ki} = \mathrm{tr}\big[M_{ki}\rho_S M_{ki}^\dagger\big]$$

to obtain outcomes k *and* i in the hypothetical setting where we had access to i. Note that we obtain the probability $p_k = \mathrm{tr}\big[\sum_i M_{ki}\rho_S M_{ki}^\dagger\big]$ of receiving outcome k by just summing over i, i.e. $p_k = \sum_{i=1}^{I(k)} p_{ki}$. Indeed, $\{p_{ki}\}$ is a valid probability distribution, since $p_{ki} \geq 0$ and $\sum_{ki} p_{ki} = 1$. Moreover, for fixed k we may consider the set $\{p(i|k)\}_i$ of conditional probabilities $p(i|k) = \frac{p_{ki}}{p_k}$, which is also a probability distribution, since all elements are non-negative and $\sum p(i|k) = \sum_i \frac{p_{ki}}{p_k} = 1$ by construction. With these definitions, we find that the von Neumann entropy of the post-measurement $\rho'_{S,k}$ can always be bounded as

$$\begin{aligned}
S(\rho'_{S,k}) &= S\left(\sum_i \frac{M_{ki}\rho_S M_{ki}^\dagger}{p_k}\right)\\
&= S\left(\sum_i p(i|k)\frac{M_{ki}\rho_S M_{ki}^\dagger}{p_{ki}}\right)\\
&\leq H\big(\{p(i|k)\}_i\big) + \sum_i p(i|k)\, S\left(\frac{M_{ki}\rho_S M_{ki}^\dagger}{p_{ki}}\right)\\
&\leq \log I + \sum_i p(i|k)\, S\left(\frac{M_{ki}\rho_S M_{ki}^\dagger}{p_{ki}}\right), \qquad (4.35)
\end{aligned}$$

where we used Eq. (2.5) for the first inequality. To obtain the second inequality we used that the Shannon entropy of a probability distribution attains its maximum value for the uniform distribution, which in this case corresponds to $p(i|k) = 1/I(k)$ for all i.

Now let L denote a linear operator on S. Since $LL^\dagger$ and $L^\dagger L$ have the same spectrum, we know that $S(LL^\dagger) = S(L^\dagger L)$ for any such L. Now choose $L = M_{ki}\sqrt{\rho_S}$, which is linear, and therefore

$$S\left(\frac{M_{ki}\rho_S M_{ki}^\dagger}{p_{ki}}\right) = S\left(\frac{M_{ki}\sqrt{\rho_S}\sqrt{\rho_S}M_{ki}^\dagger}{p_{ki}}\right) = S\left(\frac{\sqrt{\rho_S}M_{ki}^\dagger M_{ki}\sqrt{\rho_S}}{p_{ki}}\right).$$

Inserting this into (4.35) yields

$$S(\rho'_{S,k}) \leq \log I + \frac{1}{p_k}\sum_i p_{ki} S\left(\frac{\sqrt{\rho_S}M_{ki}^\dagger M_{ki}\sqrt{\rho_S}}{p_{ki}}\right),$$

which implies for the average entropy increase on S that

$$\begin{aligned}
\sum_k p_k S(\rho'_{S,k}) - S(\rho_S) &\leq \sum_k p_k \left(\log I + \frac{1}{p_k} \sum_i p_{ki} S\left(\frac{\sqrt{\rho_S} M_{ki}^\dagger M_{ki} \sqrt{\rho_S}}{p_{ki}} \right) \right) - S(\rho_S) \\
&= \log I + \sum_{k,i} p_{ki} S\left(\frac{\sqrt{\rho_S} M_{ki}^\dagger M_{ki} \sqrt{\rho_S}}{p_{ki}} \right) - S(\rho_S) \\
&\leq \log I + S\left(\sum_{k,i} p_{ki} \frac{\sqrt{\rho_S} M_{ki}^\dagger M_{ki} \sqrt{\rho_S}}{p_{ki}} \right) - S(\rho_S) \qquad (4.36) \\
&= \log I + S(\rho_S) - S(\rho_S) \qquad (4.37) \\
&= \log I \, ,
\end{aligned}$$

where we used Eq. (2.5) to obtain the second inequality (4.36) and Eq. (3.2) to obtain equality (4.37) . This concludes the proof.

□

Proposition 4.16 only provides an upper bound on the amount of extractable energy. In order to show that such energy extraction is indeed possible, we need to find an explicit implementation of some quantum instrument such that the average energy decrease E_{extr} on all involved systems except S (see Definition 4.15) is strictly positive.

At first sight, this may seem like a daunting task, since finding an implementation is in general difficult — especially, an implementation with certain desired properties. Adding to that, the following lemma shows that the standard implementation (see Definition 3.18) of any quantum instrument can never be used to extract energy.

Lemma 4.17. *Let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be the standard implementation of a quantum instrument. Then the amount of extracted energy is never positive,*

$$E_{\text{extr}} \leq 0 \qquad \forall \rho_S \, .$$

Proof. Let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be as above. Then the Hamiltonian of E is set to $H_E = 0$ (see Definition 3.18), such that the extractable energy is given by

$$E_{\text{extr}} = - \sup_{(\rho_B, U_{MB})} \Delta E_B = k_B T \left(S(\rho_M) - S(\rho'_M) \right) \, , \qquad (4.38)$$

where we applied Proposition 4.13 to obtain the second equality. But the initial state of ρ_M is pure in the standard implementation. Hence, $S(\rho_M) = 0$ which indeed implies that no energy can be extracted, $E_{\text{extr}} \leq 0$, no matter what state ρ_S the measured system S was initially in. □

Lemma 4.17 states that even for the case of inefficient quantum instruments, the standard implementation can never yield energy, which further supports the conjecture from ref. [64] that $E_{\text{extr}} \leq 0$ always. Surprisingly however there exist implementations of some inefficient measurements that allow to extract useful energy, as we now show [29].

> **Proposition 4.18.** *For any given $I \in \mathbb{N}$ with $I \leq d_S$, there exists a quantum instrument $\{M_{ki}\}$ on S with inefficiency I and an initial state ρ_S on S, such that the amount of extractable energy is $E_{\text{extr}}^{\{M_{ki}\}} = k_B T \log I$.*

This shows that energy extraction is indeed possible for certain quantum instruments, which falsifies the conjecture stated in [64]. We emphasise that energy extraction is only possible for inefficient measurements (see Proposition 4.16), which allow us to decrease the average purity of the measured system S as quantified by $S(\rho_S) - \sum_k p_k S(\rho'_{S,k})$ [29]. The energy is therefore extracted from the purity of S, which intuitively explains why our result does not contradict the Second Law of Thermodynamics.

Proof. Let $I \in \mathbb{N}$ be given. We additionally require that $I \leq d_S$, since we already know from Eq. (4.34) that we can never extract more than $\log d_S$ of energy. Hence, we can only hope to prove the statement for values of I less than or equal to d_S. Our goal is now to find a quantum instrument $\{M_{ki}\}$ with inefficiency I, an implementation of $\{M_{ki}\}$, and an implementation of the subsequent resetting step such that we can extract, for some initial state ρ_S, an amount of energy arbitrarily close to $E_{\text{extr}} = k_B T \log I$.

The construction is depicted in Figure 4.1 and works as follows: We fix two orthonormal bases $\{|k\rangle\}_{k=1}^{d_S}$ and $\{|i\rangle\}_{i=1}^{d_S}$ of a Hilbert space $\mathcal{H}_S$ with dimension d_S and consider the measurement operators

$$M_{ki} = \frac{1}{\sqrt{I}} |i\rangle\langle k| \,, \qquad k = 1, \ldots, d_S, \; i = 1, \ldots, I \;.$$

It is easy to verify that this is indeed a quantum instrument by checking that $\sum_{ki} M_{ki}^{\dagger} M_{ki} = \mathbb{1}_S$. We also compute

$$p_k = \mathrm{tr}[\sum_i^I M_{ki} \rho_S M_{ki}^{\dagger}] = \langle k|\rho_S|k\rangle \frac{1}{I} \mathrm{tr}[\sum_i^I |i\rangle\langle i|] = \langle k|\rho_S|k\rangle$$

and

$$\rho'_{S,k} = \frac{1}{p_k} \sum_i^I M_{ki} \rho_S M_{ki}^{\dagger} = \frac{1}{I} \sum_i^I |i\rangle\langle i|$$

to see that this measurement outputs the same post-measurement state for all k independent of the input state ρ_S. Let us quickly verify that the measurement has indeed inefficiency I. Recall from Chapter 2, that a convenient way to determine the inefficiency of the quantum instrument $\{M_{ki}\}$ is to compute the Choi matrices τ_k for each channel $T_k(\rho_S) = \sum_i^I M_{ki} \rho_S M_{ki}^\dagger$,

$$\begin{aligned}\tau_k &= (T_k \otimes \mathrm{id}) \left(\frac{1}{d_S} \sum_{l,m} |l\,l\rangle\langle m\,m| \right) \\ &= \frac{1}{d_S} \sum_{lm} T_k(|l\rangle\langle m|) \otimes |l\rangle\langle m| \\ &= \frac{1}{d_S I} \sum_i^I |i\rangle\langle i| \otimes |k\rangle\langle k| \,.\end{aligned}$$

As mentioned after Proposition 2.2, the matrix rank of a Choi matrix equals the Kraus rank of the corresponding channel. Hence, we find that the Kraus rank of the channel T_k is given by $I(k) = \mathrm{rank}(\tau_k) = I$ for all k. The quantum measurement $\{M_{ki}\}$ has therefore indeed inefficiency $\max_k I(k) = I$ (by Definition 3.1).

We now fix a concrete implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of this instrument. From Theorem 4.17 we already know that the standard implementation will not allow us to extract energy. Instead, we need to find another implementation of $\{M_{ki}\}$, which is energetically more efficient. We do so by slightly varying the standard implementation. For this, recall from Eq. (4.38) that the amount of extractable energy for the standard implementation is given by

$$E_{\mathrm{extr}} = k_B T \left(S(\rho_M) - S(\rho_M') \right) . \tag{4.39}$$

If we like to extract energy using a variant of this implementation it therefore seems favourable to increase the mixedness of the initial state ρ_M and, at the same, increase the purity of the final state ρ_M' on the memory M. More concretely, we modify the standard implementation from Definition 3.18 by changing the initial state of the memory to

$$\rho_M = |0\rangle_{M_1}\langle 0| \otimes \frac{1}{I} \sum_i^I |i\rangle_{M_2}\langle i| \,,$$

where M_1 and M_2 are quantum systems with Hilbert spaces $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$ such that $\mathcal{H}_M = \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}$. Moreover, we adapt the unitary interaction U_{SME} of the standard implementation from Eq. (3.19) by changing only the term U_{SM} to $U_{SM} = U_{S\leftrightarrow M_2}(U_{SM_1} \otimes \mathbb{1}_{M_2})$, where

$$U_{SM_1}(|\psi\rangle_S \otimes |0\rangle_{M_1}) = \sum_k \langle k|\psi\rangle_S \, |k\rangle_S \otimes |k\rangle_{M_1} \qquad \forall |\psi\rangle_S \in \mathcal{H}_S$$

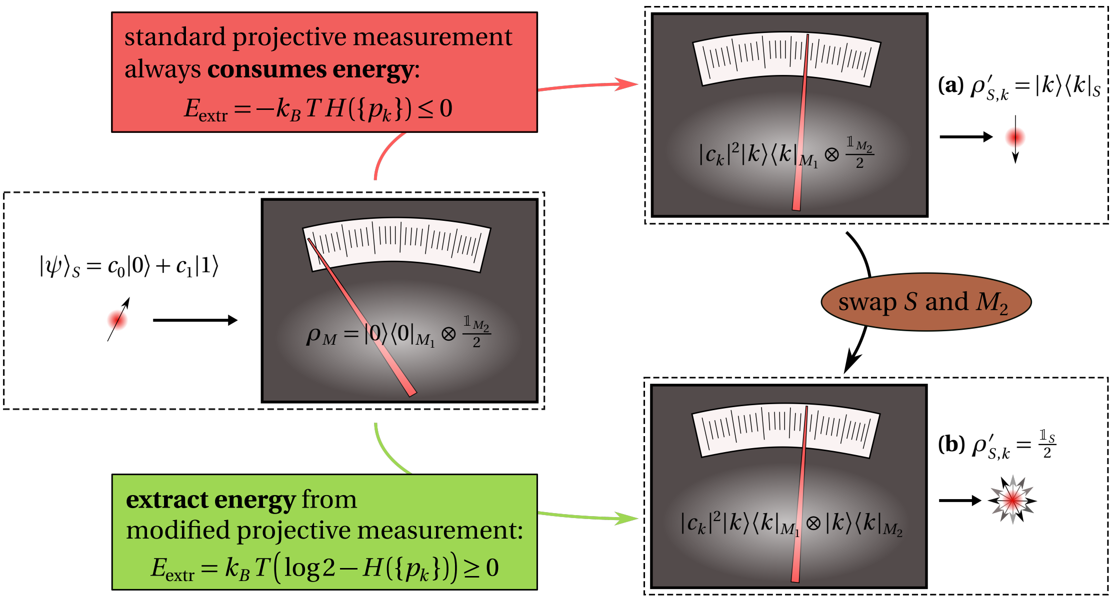

Figure 4.1: Inefficient measurements allow for energy extraction. A measurement in the spin-z basis is performed on a qubit initially in the pure state $\rho_S = |\psi\rangle\langle\psi|_S$ using a measurement device in state $\rho_M = |0\rangle\langle 0|_{M_A} \otimes \mathbb{1}_{M_B}/2$ [29]. **(a)** The measurement is a projective quantum instrument with measurement operators $M_k = |k\rangle\langle k|$. As stated in Proposition 4.16, this measurement always consumes energy, $E_{\text{extr}} \leq 0$. **(b)** The measurement is an inefficient quantum instrument with measurement operators $M_{ki} = |i\rangle\langle k|/\sqrt{2}$. It yields the same measurement statistics $\{p_k\}$ as the standard projective quantum instrument, but will always output the maximally mixed state $\rho'_{S,k} = \mathbb{1}_S/2$. Microscopically, this can be realised by a swap of the systems S and M_2 after the standard projective instrument. Such an inefficient measurement yields energy $E_{\text{extr}} \geq 0$, which falsifies a conjecture from ref. [64].

and $U_{S\leftrightarrow M_2}$ denotes the unitary swap between systems S and M_2. Let us verify that the tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is still an implementation of $\{M_{ki}\}$. Condition 0 is obviously satisfied. The mutually orthogonality requirement Eq. (3.13) in Condition 2 is also satisfied, since it is independent of the choice of ρ_M and U_{SM}. To see this, compute

$$\begin{aligned}
\rho'_{SM} &= \operatorname{tr}_E[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}] \\
&= \sum_{j,k} (\mathbb{1}_S \otimes Q_{M,j}) U_{SM}(\rho_S \otimes \rho_M) U^\dagger_{SM} (\mathbb{1}_S \otimes Q_{M,k}) \operatorname{tr}[V_{E,j}\rho_E V^\dagger_{E,k}] \\
&= \sum_{k} (\mathbb{1}_S \otimes Q_{M,k}) U_{SM}(\rho_S \otimes \rho_M) U^\dagger_{SM} (\mathbb{1}_S \otimes Q_{M,k}) \,,
\end{aligned}$$

which is block diagonal with respect to the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}_k$ irrespective of ρ_M and U_{SM}. It therefore remains to check Eq. (3.12) from Condition 1. We have

$$\begin{aligned}\operatorname{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E)U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger]\\ &= \operatorname{tr}_{ME}\Big[\frac{p_k}{I}\sum_i^I |i\rangle_S\langle i| \otimes |k\rangle_{M_1}\langle k| \otimes |k\rangle_{M_2}\langle k|\Big]\\ &= \frac{p_k}{I}\sum_i^I |i\rangle_S\langle i|\\ &= \sum_i^I M_{ki}\rho_S M_{ki}^\dagger \qquad \forall \rho_S \forall k\,.\end{aligned}$$

The tuple $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is thus indeed a valid implementation of the M_{ki}.

Let us now compute how much energy we can extract from this implementation. For this, note that we still choose $\rho_E = \mathbb{1}_E/d_E$ to be maximally mixed and $H_E = 0$, as for the standard implementation. Hence, the amount of extractable energy for our modified standard implementation is also given by Eq. (4.39). We determine the state on M after measurement,

$$\begin{aligned}\rho_M' &= \operatorname{tr}_S\Big[\sum_k (\mathbb{1}_S \otimes Q_{M,k})U_{SM}(\rho_S \otimes \rho_M)U_{SM}^\dagger(\mathbb{1}_S \otimes Q_{M,k})\Big]\\ &= \sum_k p_k |k\rangle_{M_1}\langle k| \otimes |k\rangle_{M_2}\langle k|\,.\end{aligned}$$

to find that the amount of extractable energy is given by

$$E_{\text{extr}} = k_B T(\log I - H(\{p_k\}))\,.$$

It follows that we can indeed extract $k_B T \log I$ of useful energy if the initial state ρ_S is any of the computational basis states $|k\rangle_S\langle k|$ such that $H(\{p_k\}) = 0$. □

4.5 Projective quantum instruments

The fact that our lower bound on the energy cost of general quantum instruments from Theorem 4.14 is also valid for simple state transformations (see discussion after the proof of Theorem 4.14) suggests that the exact costs of quantum instruments can actually be significantly larger. For an important subclass of quantum instruments, the so-called projective quantum instruments, we are able to show that this is indeed true. Our result on the energy cost of such measurements stated in Theorem 4.19 employs the requirement, that an implementation of a quantum instrument outputs the correct post-measurement states for *all* input states, to its full extent. This can also be seen from the fact that our results determine the *exact* energy cost for projective quantum instruments in contrast to the "lower bound"-like statements obtained before. Furthermore, this exact result gives the *strongest* energy requirements for measurement known so far. As a consequence, our results directly place severe physical limitations on the amount of projective quantum instruments which can be realised if only finite energy supply is available. We demonstrate these limitations in two important quantum applications in Chapter 5.

In the first part of this section we formulate our main result which exactly quantifies the energy cost of projective quantum instruments [29]. This is followed by several interesting theoretical consequences which follow from this result.

4.5.1 Energy cost of projective quantum instruments

Projective quantum instruments are the most prominent class of quantum measurements. As explained in Section 3.1, the specification of the measurement by means of a hermitian operator just corresponds to projective quantum instrument. This special subclass of quantum instruments hence corresponds to the "standard" formalism of quantum measurement, that most physicists employ.

Recall from Definition 3.3 that projective quantum instruments are defined as a quantum instrument $\{P_k\}$ with projective measurement operators, i.e. $P_k = P_k^\dagger = P_k^2$ for all k. Clearly, projective quantum instruments have inefficiency $I = 1$, which directly implies for the energy cost that $E_{\text{cost}}^{\{P_k\}} \geq \Delta E_S$, as was discussed in Section 4.4.3 (see Proposition 4.16).

The following theorem drastically improves on this statement, as it quantifies this cost exactly [29].

Theorem 4.19. ***Energy cost of a projective quantum instrument.***
Let $\{P_k\}$ be a projective quantum instrument on a quantum system S with Hamiltonian H_S initially in state ρ_S. Then the energy cost of $\{P_k\}$ is exactly given by

$$E_{\text{cost}}^{\{P_k\}} = \Delta E_S + k_B T H(\{p_k\}),$$

where $\Delta E_S = \text{tr}[H_S(\rho'_S - \rho_S)]$ denotes the average energy change on S and $H(\{p_k\})$ denotes the Shannon entropy of the outcome probability distribution $\{p_k\}$.

This result constitutes the strongest energy requirements for quantum measurement known so far [29]. It states that the energy cost of a projective quantum instrument is determined by the Shannon entropy of the outcome probability distribution (apart from the average energy change ΔE_S which is the same for all implementations). In contrast to all our previous results, the heat dissipation $\Delta Q_{\{P_k\}} = k_B T H(\{p_k\})$ for this formalism is therefore always non-negative. Moreover, whenever the initial state ρ_S was not already classical with respect to the projections $\{P_k\}$, the Shannon entropy $H(\{p_k\})$, and hence $\Delta Q_{\{P_k\}}$ is even *strictly* positive. For most initial states, projective quantum instruments are hence fundamentally irreversible in the sense that an amount $\Delta Q_{\{P_k\}}$ of energy is inevitably lost in the degrees of freedom of a thermal bath.

Since Theorem 4.19 determines the *exact* costs of projective quantum instruments, this result allows us to compute, for the first time [29], the exact energy cost of important applications such as quantum error correction as we show in Section 5.2.

The proof of Theorem 4.19 is based on the following lemma [29]:

Lemma 4.20. *Let $\{P_k\}$ be a projective quantum instrument on a quantum system S initially in state ρ_S. Let further $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be a tuple satisfying Condition 0 and 1 from Definition 3.14, i.e. ρ_M is a quantum state on a memory M, $\{Q_{M,k}\}$ are orthogonal projections on M such that $\sum_k Q_{M,k} = \mathbb{1}_M$, ρ_E is a thermal state on an environment E at temperature T and U_{SME} is a unitary on S, M, and E, satisfying*

$$P_k \rho_S P_k = \mathrm{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME} (\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \qquad \forall k, \rho_S\,.$$

Then the state $\rho'_{ME} = \mathrm{tr}_S[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger]$ on M and E after the unitary interaction U_{SME} can always be written as

$$\rho'_{ME} = \sum_k p_k \sigma_{ME,k}\,,$$

where $p_k = \mathrm{tr}[P_k \rho_S]$ is the probability to obtain outcome k and $\sigma_{ME,k}$ are quantum states, independent of ρ_S, with entropy $S(\sigma_{ME,k}) = S(\rho_M) + S(\rho_E)$ for all k and satisfying

$$\sigma_{ME,k} = (Q_{M,k} \otimes \mathbb{1}_E) \sigma_{ME,k} (Q_{M,k} \otimes \mathbb{1}_E) \qquad \forall k\,,$$

i.e. the $\sigma_{ME,k}$ are in particular mutually orthogonal.

Proof. Let $\{P_k\}$ be a projective quantum instrument and let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be a tuple satisfying Condition 0 and 1 from Definition 3.14. Since it is not required to satisfy Condition 2 as well, the tuple need not be an implementation of the projective quantum instrument. Let us write out Condition 1 for convenience,

$$P_k \rho_S P_k = \mathrm{tr}_{ME}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME} (\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \qquad \forall k, \rho_S\,. \tag{4.40}$$

Note that by summing this equation over k, we obtain

$$\sum_k P_k \rho_S P_k = \mathrm{tr}_{ME}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \qquad \forall k, \rho_S \,.$$

The left-hand side just corresponds to the dephasing channel $\mathcal{F}(\rho_S) = \sum_k P_k \rho_S P_k$ on S with respect to the projections $\{P_k\}$. Hence, the unitary U_{SME} satisfies Eq. (4.10) from Lemma 4.10 and we know that the final state on M and E after the unitary interaction U_{SME} can be written as

$$\rho'_{ME} = \mathrm{tr}_S[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] = \sum_k p_k \sigma_{ME,k} \,, \tag{4.41}$$

where $p_k = \mathrm{tr}[P_k \rho_S]$ and $\sigma_{ME,k}$ are states, independent of ρ_S, with von Neumann entropy $S(\sigma_{ME,k}) = S(\rho_M) + S(\rho_E)$ for all k.

It therefore remains to show that the states $\sigma_{ME,k}$ are also mutually orthogonal with respect to the projections $\{Q_{M,k} \otimes \mathbb{1}_E\}_k$, that is

$$\sigma_{ME,k} = (Q_{M,k} \otimes \mathbb{1}_E) \sigma_{ME,k} (Q_{M,k} \otimes \mathbb{1}_E) \qquad \forall k \,. \tag{4.42}$$

For this, let us fix k. Since Eq. (4.41) holds for all states ρ_S, we may choose $\rho_S = \psi_k = P_k \psi_k$ to be a pure state in the support of the projection P_k. We then obtain from Eq. (4.41) a concrete expression for $\sigma_{ME,k}$,

$$\sigma_{ME,k} = \mathrm{tr}_S[U_{SME}(\psi_k \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \,. \tag{4.43}$$

Now consider the quantity $(Q_{M,k} \otimes \mathbb{1}_E) \sigma_{ME,k} (Q_{M,k} \otimes \mathbb{1}_E)$, which is, for all k, a positive operator with unit trace. Indeed, we find by Eqs. (4.40) and (4.43) that

$$\begin{aligned}
&\mathrm{tr}[(Q_{M,k} \otimes \mathbb{1}_E) \sigma_{ME,k} (Q_{M,k} \otimes \mathbb{1}_E)] \\
&\qquad = \mathrm{tr}\big[(Q_{M,k} \otimes \mathbb{1}_E) \mathrm{tr}_S[U_{SME}(\psi_k \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger](Q_{M,k} \otimes \mathbb{1}_E)\big] \\
&\qquad = \mathrm{tr}\big[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME}(\psi_k \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger (\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E)\big] \\
&\qquad = \mathrm{tr}\big[P_k \psi_k P_k\big] \\
&\qquad = 1 \,.
\end{aligned}$$

Hence the projections $Q_{M,k} \otimes \mathbb{1}_E$ do not change the trace of $\sigma_{ME,k}$. This already implies the desired identity (4.42). To see this, we write $Q_k \equiv Q_{M,k} \otimes \mathbb{1}_E$ and compute

$$\begin{aligned}
1 = \mathrm{tr}[\sigma_{ME,k}] &= \mathrm{tr}[(Q_k + (\mathbb{1}_{ME} - Q_k)) \sigma_{ME,k} (Q_k + (\mathbb{1}_{ME} - Q_k))] \\
&= \mathrm{tr}[Q_k \sigma_{ME,k} Q_k] + \mathrm{tr}[(\mathbb{1}_{ME} - Q_k) \sigma_{ME,k} (\mathbb{1}_{ME} - Q_k)] \\
&= 1 + \mathrm{tr}[(\mathbb{1}_{ME} - Q_k) \sigma_{ME,k} (\mathbb{1}_{ME} - Q_k)]
\end{aligned}$$

which implies $(\mathbb{1}_{ME} - Q_k) \sigma_{ME,k} = \sigma_{ME,k} (\mathbb{1}_{ME} - Q_k) = 0$ and hence the mutual orthogonality criterion Eq. (4.42). □

With this lemma, we can now prove Theorem 4.19 [29].

Proof of Theorem 4.19. Recall from Definition 3.17 that the energy cost $E_{\text{cost}}^{\{P_k\}}$ of a projective quantum instrument $\{P_k\}$ is defined as

$$E_{\text{cost}}^{\{P_k\}} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{P_k\}}} \Big(\Delta E_E + \Delta E_B \Big) , \tag{4.44}$$

where the second infimum is taken over all possible implementations $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{P_k\}}$ (as defined in Definition 3.17) and where $\Delta E_E = \operatorname{tr}[H_E(\rho_E' - \rho_E)]$ denotes the average energy change on E (ΔE_B analogously).

We employ a strategy similar to the one used in the proof of our Theorem 4.14 for general quantum instruments. In this previous proof we applied Proposition 4.13 to find that the resetting cost on B is given as

$$\Delta E_B = k_B T \left(S(\rho_M') - S(\rho_M) \right) \tag{4.45}$$

and used Lemma 4.2 evaluated for the state transformation $\rho_{SM} \mapsto \rho_{SM}'$ to obtain the lower bound

$$\Delta E_E \geq k_B T \left(S(\rho_{SM}) - S(\rho_{SM}') \right) .$$

Note the formal similarity of these two statements. This is because both equations stem from the same general result, namely the lower bound in Lemma 4.2 for unselective measurements. The differences between the two equations is due to the different requirements on the unitaries U_{SME} for the measurement and U_{MB} for the resetting. The weak requirements on U_{MB} allow us to prove that the resetting cost is not only lower bounded by the above quantity, but actually equal to it. This is however not true for the measurement process, since the requirements on U_{SME} are much stricter. The lower bound from our Theorem 4.14 for general quantum instruments is therefore obtained *by not taking into account* the strict requirements on U_{SME}. Since our aim is to compute the exact cost for projective quantum instruments (in contrast to mere lower bounds), we must take into account these strict requirements.

We start by invoking the non-negativity of the relative entropy, to obtain the inequality (see Eq. (4.3))

$$\Delta E_E \geq k_B T \left(S(\rho_E') - S(\rho_E) \right) .$$

Combined with Eq. (4.45) we then have the following lower bound on the average energy change on the thermal environments E and B,

$$\begin{aligned} \Delta E_E + \Delta E_B &\geq k_B T \left(S(\rho_E') - S(\rho_E) + S(\rho_M') - S(\rho_M) \right) \\ &\geq k_B T \left(S(\rho_{ME}') - S(\rho_E) - S(\rho_M) \right) . \end{aligned} \tag{4.46}$$

Our goal is now to characterise the state ρ'_{ME} such that we can find a tight lower bound on $S(\rho'_{ME})$.

To this end, let us recall that for a given implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ the state ρ'_{ME} after measurement is defined as

$$\rho'_{ME} = \mathrm{tr}_S[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}^\dagger]\,,$$

where $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ satisfies Condition 0, 1, and 2 from Definition 3.14. From Lemma 4.20 we then know that ρ'_{ME} is of the form

$$\rho'_{ME} = \sum_k \mathrm{tr}[P_k \rho_S]\sigma_{ME,k} \quad \forall \rho_S\,,$$

where the states $\sigma_{ME,k}$ are mutually orthogonal with respect to the projections $(Q_{M,k} \otimes \mathbb{1}_E)$ and have entropy $S(\sigma_{ME,k}) = S(\rho_M \otimes \rho_E)$ for all k. Denoting $p_k = \mathrm{tr}[P_k \rho_S]$ we can therefore compute

$$\begin{aligned} S(\rho'_{ME}) &= H(\{p_k\}) + \sum_k p_k S(\sigma_{ME,k}) \\ &= H(\{p_k\}) + \sum_k p_k S(\rho_M \otimes \rho_E) \\ &= H(\{p_k\}) + S(\rho_M) + S(\rho_E)\,. \end{aligned}$$

Inserting this into Eq. (4.46) yields

$$\Delta E_E + \Delta E_B \geq k_B T \left(S(\rho'_{ME}) - S(\rho_E) - S(\rho_M)\right) = k_B T H(\{p_k\})\,,$$

which together with Eq. (4.44) implies that the energy cost of a projective measurement is lower bounded by

$$E_{\text{cost}}^{\{P_k\}} \geq \Delta E_S + k_B T H(\{p_k\}\,.$$

To conclude the proof, it remains to show that this lower bound is tight, i.e. that equality in the above statement can be attained. Interestingly, in contrast to our previous finding that the standard implementation is, in general, energetically more expensive than other implementations (see Theorem 4.17 and the proof of Proposition 4.18), we find in the case of projective measurements that the standard implementation *is* energetically optimal. To see this, let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ be the standard implementation of the projective quantum instrument as defined in Definition 3.18. According to Eq. (3.20), the energy cost of this implementation is given as

$$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \inf_{(\rho_B, U_{MB})} \Delta E_B\,.$$

We already have computed the infimum precisely in Proposition 4.13 such that we can write

$$E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \Delta E_S + k_B T \left(S(\rho'_M) - S(\rho_M)\right) .$$

Due to the definition of the standard implementation, we know that $S(\rho_M) = 0$ (note that the projective quantum instrument is efficient and hence the subsystem M_2, which is needed for inefficient quantum instruments, can be chosen to be one-dimensional, effectively setting $M \equiv M_1$). To compute $E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}$, we therefore determine the memory state after the measurement,

$$\rho'_M = \mathrm{tr}_{SE}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U_{SME}] = \sum_k p_k |k\rangle_M \langle k| ,$$

which implies that $S(\rho'_M) = H(\{p_k\})$. Hence, the energy cost of the standard implementation of the projective measurement $\{P_k\}$ is given by

$$E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} = \Delta E_S + k_B T H(\{p_k\}) .$$

This concludes the proof.

□

In the following we present some immediate consequences of our Theorem 4.19.

4.5.2 Energy cost for measuring quantum coherences

In this section we compare our general lower bound on the energy cost of quantum instruments from Theorem 4.14 to the exact energy cost of projective measurements quantified by Theorem 4.19. For this let us consider a projective quantum instrument $\{P_k\}$ on a quantum system S in state ρ_S. Evaluating the general lower bound from Theorem 4.14 yields

$$E_{\text{cost}}^{\{P_k\}} \geq \Delta E_S + k_B T \Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big) . \tag{4.47}$$

Note that since projective quantum instruments are efficient, this bound already follows from ref. [64]. Our exact result from Theorem 4.19 [29], i.e.

$$E_{\text{cost}}^{\{P_k\}} = \Delta E_S + k_B T H(\{p_k\}) , \tag{4.48}$$

drastically improves on such previous statements. Indeed, both results are consistent since the von Neumann entropy never decreases under unital channels. More concretely, for the dephasing channel with respect to the projections $\{P_k\}$, we have the following chain of inequalities

$$S(\rho_S) \leq S\Big(\sum_k P_k \rho_S P_k\Big) = H(\{p_k\}) + \sum_k p_k S(\rho'_{S,k}) , \tag{4.49}$$

where we used Eq. (2.5). Hence, the average entropy decrease $S(\rho_S)-\sum_k p_k S(\rho'_{S,k})$ from Eq. (4.47) is indeed a lower bound on the Shannon entropy $H(\{p_k\})$ from Eq. (4.48), the gap stemming precisely from the inequality in Eq. (4.49).

Consequently, the only cases where the lower bound Eq. (4.47) agrees with the exact cost Eq. (4.48) are the equality cases for Eq. (4.49), which can be easily characterised: Only when the initial state ρ_S was already diagonal with respect to the projections $\{P_k\}$ will the lower bound agree with the actual energy cost. That is, for projective measurements, the lower bound is optimal only when the initial state was classical to begin with [29]. Quantum coherences in the initial state however demand much more energy for the measurement than predicted by the lower bound in Eq. (4.47). Since this bound is obtained as a generalisation of the best bounds previously known, this implies that quantum coherences also demand much more energy than anticipated from existing literature [29].

Let us be clear about the meaning of this observation. Our Theorem 4.19 shows that the exact costs $E_{\text{cost}}^{\{P_k\}}$ (just like the probabilities p_k) are *independent* of coherences. The punchline of this section is hence: Quantum coherences do not matter energetically in a (projective) quantum measurement, in contrast to previous statements (see e.g. [66]) which would claim that quantum coherences *decrease* the energy cost, thereby suggesting a "quantum advantage" over classical physics in terms of the energy expense in measurements. This observation should hence be understood as pointing out the deficiencies of previous statements: Our results for projective quantum instruments clearly prove that coherences do not influence the energy costs of such measurements, as the exact cost depends only on the outcome probabilities [29].

4.5.3 Energy cost for jointly measurable projective measurements

Consider two jointly measurable projective quantum instruments $\{P_k\}$ and $\{Q_j\}$ on a quantum system S. In this case, "joint measurability" is just equivalent to requiring the projections to commute, i.e. $[P_k, Q_j] = 0$ for all j and k [90]. How much energy do we have to expend to conduct both measurements on a system S in state ρ_S? In Chapter 5 we will see how the answers we now obtain for this question will directly allow us to quantify the energy cost of quantum error correcting protocols [29].

Let us first specify the setting by imposing a temporal ordering on the measurements[7]: We assume that we first perform $\{P_k\}$ on ρ_S, which will output outcome k with probability $p_k = \mathrm{tr}[P_k\rho_S]$ together with the post-measurement state

---

[7] It is easy to see that, due to the joint measurability, our results will not depend on the temporal ordering. We merely choose it to fix notation and simplify the presentation.

$\rho'_{S,k} = P_k \rho_S P_k / p_k$. This is then followed by the measurement $\{Q_j\}$ on this post-measurement state $\rho'_{S,k}$ yielding outcome j with conditional probability $p(j|k) = \mathrm{tr}[Q_j \rho'_{S,k}]$ together with the post-measurement state

$$\rho''_{S,jk} = \frac{1}{p(j|k)} Q_j \rho'_{S,k} Q_j = \frac{1}{p_{jk}} Q_j P_k \rho_S P_k Q_j \,,$$

where $p_{jk} = p_k p(j|k) = \mathrm{tr}[Q_j P_k \rho_S]$ is the joint probability to obtain the pair (j,k) of outcomes.

Based on Theorem 4.19 there exist two equivalent ways to compute the energy cost of these two measurements: On the one hand, we can simply evaluate the energy cost for the two measurements independently, i.e. we first compute the energy cost $E^{\{P_k\}}_{\text{cost}}$ of performing $\{P_k\}$ on ρ_S,

$$E^{\{P_k\}}_{\text{cost}} = \mathrm{tr}\Big[H_S\big(\sum_k p_k \rho'_{S,k} - \rho_S\big)\Big] + k_B T H(\{p_k\}), \tag{4.50}$$

and add to this the energy cost $E^{\{Q_j\}}_{\text{cost},k}$ of performing $\{Q_j\}$ on $\rho'_{S,k}$,

$$E^{\{Q_j\}}_{\text{cost},k} = \mathrm{tr}\Big[H_S\big(\sum_j p(j|k) \rho''_{S,jk} - \rho'_{S,k}\big)\Big] + k_B T H(\{p(j|k)\}) \,, \tag{4.51}$$

averaged over all k, which gives

$$E^{\text{joint}}_{\text{cost}} = E^{\{P_k\}}_{\text{cost}} + \sum_k p_k E^{\{Q_j\}}_{\text{cost},k} \,. \tag{4.52}$$

On the other hand, we can look at the joint projective measurement $\{R_{jk}\}$ with measurement operators $R_{jk} = P_k Q_j$, which directly outputs the pair (j,k) of outcomes with probability $p_{jk} = \mathrm{tr}[Q_j P_k \rho_S]$ together with the post-measurement state $\rho''_{S,jk}$ from above[8]. The corresponding energy cost $E^{\{R_{jk}\}}_{\text{cost}}$ of this joint measurement is exactly equal to $E^{\text{joint}}_{\text{cost}}$. To see this, note that the average Shannon entropy $\sum_k p_k H(\{p(j|k)\})$ appearing in Eqs. (4.51) and (4.52) satisfies

$$\begin{aligned}
\sum_k p_k H(\{p(j|k)\}) &= -\sum_k p_k \sum_j p(j|k) \big(\log p_{jk} - \log p_k\big) \\
&= -\sum_{jk} p_{jk} \log p_{jk} + \sum_k p_k \log p_k \\
&= H(\{p_{jk}\}) - H(\{p_k\} \,,
\end{aligned}$$

---

[8] Note that $\{R_{jk}\}$ is *not* to be understood as an inefficient measurement as in Definition 3.2. In contrast, the measurement $\{R_{jk}\}$ is efficient and the notation simply indicates that the measurement outputs a pair (j,k) of measurement results.

where we employed the definition of the Shannon entropy from Eq. (2.3) and used that $p(j|k) = p_{jk}/p_k$ and $\sum_j p(j|k) = 1$. Inserting this together with Eqs. (4.50) and (4.51) into Eq. (4.52) yields that, indeed,

$$E_{\text{cost}}^{\text{joint}} = \text{tr}\Big[H_S\big(\rho_S'' - \rho_S\big)\Big] + k_B T H(\{p_{jk}\}) = E_{\text{cost}}^{\{R_{jk}\}} , \tag{4.53}$$

where $\rho_S'' = \sum_{jk} p_{jk} \rho_{S,jk}''$ denotes the average post-measurement state. We call these two equivalent ways of performing the measurements, i.e. either applying the measurements $\{P_k\}$ and $\{Q_j\}$ consecutively or directly measuring $\{R_{jk}\}$, the *joint measurement scheme*. This scheme, however, is often difficult to realise: Recall that the (energetically) optimal implementation of a quantum instrument generally depends on the initial state of the measured system (see Definition 3.17). For the measurement $\{Q_j\}$, however, this initial state $\rho_{S,k}'$ depends on the outcome k obtained in the previous $\{P_k\}$ measurement. As a consequence, an optimal implementation of $\{Q_j\}$ will generally employ a resetting step *conditioned on the outcome* k.

In contrast, one may instead employ another scheme to obtain the pair (j,k) of outcomes, which is easier to realise, but also expends more energy as we show below (see Corollary 4.21). This so-called *separate measurement scheme* simply uses two completely independent implementations of the two instruments $\{P_k\}$ and $\{Q_j\}$, respectively. More concretely, consider the same situation as above, where we first apply $\{P_k\}$ which is followed by the instrument $\{Q_j\}$. However, we now enforce that the implementations of $\{P_k\}$ and $\{Q_j\}$ are *independent* from each other, i.e. the resetting step of the implementation of $\{Q_j\}$ may *not* make use of the outcome k obtained in the measurement $\{P_k\}$.

It is instructive to think of these two measurement schemes in the following way: Implementations of the joint measurement $\{R_{jk}\}$ involve a single memory where both outcomes j and k are stored simultaneously, such that the resetting step may use information about k to erase outcome j. Hence, the resetting step within the joint measurement scheme can employ correlations between the two outcomes. In contrast, the separate measurement scheme involves two independent (perhaps even spatially separated) memories, one for each implementation of $\{P_k\}$ and $\{Q_j\}$, respectively, and the resetting step of either one implementation must be independent of the outcomes stored in the other memory.

Nevertheless, the separate measurement scheme outputs the pair (j,k) of outcomes with probability p_{jk} together with the post-measurement state $\rho_{S,jk}''$, just as the joint measurement scheme; the only difference lies in the restrictions on the resetting step for the separate measurement scheme.

The energy cost of the separate measurement scheme is then, analogously to

Eq. (4.52) for the joint measurement scheme, given as

$$E_{\text{cost}}^{\text{sep}} = E_{\text{cost}}^{\{P_k\}} + \sum_k p_k E_{\text{cost},k}^{\{Q_j\}}\,, \tag{4.54}$$

where only $E_{\text{cost}}^{\{Q_j\}}$ changed, due to the restricted resetting step, to

$$E_{\text{cost},k}^{\{Q_j\}} = \text{tr}\Big[H_S\big(\sum_j p(j|k)\rho''_{S,jk} - \rho'_{S,k}\big)\Big] + k_B T H(\{p_j\})\,, \tag{4.55}$$

the only difference being the Shannon entropy term $H(\{p_j\})$ (compared to $H(\{p(j|k)\})$ in Eq. (4.51)). Here, $p_j = \sum_k p_{jk}$ is the marginal distribution of p_{jk} which describes the probability to obtain outcome j in the $\{Q_j\}$ measurement if we do not have access to k. It is exactly the probability distribution $\{p_j\}$ which is stored in the memory of the implementation of $\{Q_j\}$ and which must be erased in the resetting step, hence the energy term $k_B T H(\{p_j\})$ in Eq. (4.55).

Interestingly, the next corollary shows that the joint measurement scheme, while experimentally more difficult to realise, is energetically more efficient than the separate measurement scheme [29].

> **Corollary 4.21.** *Let $\{P_k\}$ and $\{Q_j\}$ be two jointly measurable projective quantum instruments on a quantum system S. Then the joint measurement scheme requires at most as much energy as the separate measurement scheme, i.e.*
>
> $$E_{\text{cost}}^{\text{joint}} \leq E_{\text{cost}}^{\text{sep}}\,,$$
>
> *where $E_{\text{cost}}^{\text{joint}}$ and $E_{\text{cost}}^{\text{sep}}$ are as defined above (see Eq.* (4.52) *and Eq.* (4.54)*).*

Proof. Let $\{P_k\}$ and $\{Q_k\}$ be as above. In Eqs. (4.54) and (4.55) we already determined the cost of the separate measurement scheme to be

$$E_{\text{cost}}^{\text{sep}} = \text{tr}\Big[H_S\big(\rho''_S - \rho_S\big)\Big] + k_B T\big(H(\{p_j\}) + H(\{p_k\})\big)\,,$$

whereas the cost of the joint measurement scheme was (see Eq. (4.53))

$$E_{\text{cost}}^{\text{joint}} = \text{tr}[H_S(\rho''_S - \rho_S)] + k_B T H(\{p_{jk}\})\,.$$

To show that the separate measurement scheme consumes more energy than the joint measurement scheme, we therefore only need to show that

$$H(\{p_{jk}\}) \leq H(\{p_k\}) + H(\{p_j\}) \qquad \forall \rho_S\,.$$

To see this, recall from Eq. (2.6) that the mutual information links the Shannon entropy of a probability distribution to the Shannon entropy of the marginal distributions. In particular, we consider the mutual information $I(P:Q)_{\{p_{jk}\}}$ between the random variables P and Q with probability distributions $\{p_k\}$ and $\{p_j\}$, respectively. From Eq. (2.6) and the non-negativity of the mutual information we then have that

$$0 \leq I(P:Q)_{\{p_{jk}\}} = H(\{p_k\}) + H(\{p_j\}) - H(\{p_{jk}\}) ,$$

which implies the desired statement. □

Let us give some intuition for this result since the proof does not provide too many physical insights about the reasons why joint measurement schemes demand less energy than separate measurement schemes. A useful picture is the following: Remember that we can always choose the standard implementation, where $H_E = 0$ and hence $\Delta E_E = 0$, for an energetically optimal way to conduct each projective quantum instrument. Hence, what we need to compare are the energy changes ΔE_B of the thermal environment B employed in the resetting step. The statement of Corollary 4.21 can therefore be rephrased as saying that the average energy change on B for the joint measurement scheme is less than the sum of the average energy changes on B for the two instruments in the separate measurement scheme. This is not totally surprising due to the following observation: In the separate measurement scheme, both instruments are, by construction, implemented individually. This implies that also the resetting steps for both measurements must be independent from one another. Implementations of these resetting steps must therefore be conducted by independent unitaries, say $U_{MB}^{\{P_k\}}$ for the resetting step for $\{P_k\}$ and $U_{MB}^{\{Q_j\}}$ for $\{Q_j\}$. In contrast, the unitary $U_{MB}^{\{R_{kj}\}}$ for the joint measurement $\{R_{kj}\}$ need not be of the form $U_{MB}^{\{R_{kj}\}} = U_{MB}^{\{Q_j\}} U_{MB}^{\{P_k\}}$, hence the optimisation space for the joint measurement scheme is larger than the one for the separate measurement scheme.

As an illustration let us consider the simple example, where the two projective quantum instruments $\{P_k\}$ and $\{Q_j\}$ are identical, i.e. $P_k = Q_k$ for all k. Obviously, in this case we have $[P_k, Q_j] = 0$ for all j, k such that we can apply Corollary 4.21.

Employing the joint measurement scheme will then consume an amount

$$E_{\text{cost}}^{\text{joint}} = E_{\text{cost}}^{\{P_k\}}$$

of energy, where $E_{\text{cost}}^{\{P_k\}}$ is given as in Eq. (4.50). This means that the only the first measurement $\{P_k\}$ expends energy, whereas the second measurement $\{Q_j\}$ can be conducted at zero energy cost.

There are various equivalent ways to explain this: Firstly, the states $\rho'_{S,k}$ after $\{P_k\}$ are already classical with respect to the projections $\{Q_j\}$ and hence measuring $\{Q_j\}$ does not expend energy (see Section 4.5.2). Secondly, the second measurement does not yield any information, which would need to be erased in the resetting step. This is because, in the joint measurement scheme, the pair of outcomes (j,k) is perfectly correlated, since $p_{jk} = \delta_{j,k} p_k$ for all j and k, as one may easily compute. Indeed, inserting $p(j|k) = \delta_{j,k}$ into Eq. (4.51) directly shows that $E^{\{Q_j\}}_{\text{cost},k} = 0$ for all k.

In contrast, the energy cost of the separate measurement scheme is given as

$$E^{\text{sep}}_{\text{cost}} = 2 \cdot E^{\{P_k\}}_{\text{cost}}$$

and therefore exceeds the energy cost $E^{\text{joint}}_{\text{cost}}$ of the joint measurement scheme by a factor of 2. The reason for this is that the resetting step of the implementation of $\{Q_j\}$ may not employ the correlations stored in the joint probability distribution p_{jk}, i.e. it may only use the marginal distribution $\{p_j\}$ instead of the conditional probability distribution $\{p(j|k)\}_j$.

4.5.4 State-independent bound on heat dissipation from entropic uncertainty relations

Let us again consider two projective quantum instruments $\{P_k\}$ and $\{Q_j\}$ on a quantum system S. In contrast to the previous section, we do not demand these instruments to be jointly measurable. Imagine now the situation, where we want to perform only one of these instruments on a quantum system S in state ρ_S. We use a fair die to determine which of the instruments we use. This situation resembles what happens in measurement-based quantum computation, where for each computation step the next measurement basis is chosen depending on the previous measurement outcome (which occurs with probability 1/2). Another paradigmatic protocol, where several projective quantum instruments are conducted on the same state ρ_S, is quantum state tomography [16, 17].

According to our Theorem 4.19, the dissipated heat of these measurements averaged over several runs of this protocol, is given by

$$\Delta Q = \frac{1}{2}(\Delta Q^{\{P_k\}} + \Delta Q^{\{Q_j\}}) = \frac{1}{2} k_B T \big(H(\{p_k\}) + H(\{q_j\}) \big) . \tag{4.56}$$

We can use the non-negativity of the Shannon entropy to see that this quantity is always non-negative, i.e. $\Delta Q \geq 0$. But this observation still allows for the possibility that there exist implementations of $\{P_k\}$ and $\{Q_j\}$, respectively, such that the measurements can be conducted in a thermodynamically reversible manner, i.e. $\Delta Q = 0$. Remarkably, this is typically not the case, as we now show [29].

> **Corollary 4.22.** *Consider a quantum system S in state ρ_S. Assume further that we like to perform either the quantum instrument $\{P_k\}$ or the instrument $\{Q_j\}$, both with probability 1/2. Then the dissipated heat of*

these measurements averaged over several runs from Eq. (4.56) *is lower bounded as*

$$\Delta Q \geq -k_B T \log c \ ,$$

where $c = \max_{kj} ||P_k Q_j||_\infty$.

Importantly, the constant c is *independent* of the initial state ρ_S of the measured system. Moreover, whenever the instruments $\{P_k\}$ and $\{Q_j\}$ are such that any two projections P_k and Q_j do not share a common eigenvector, we have that $c > 0$. Applying such quantum instruments on a quantum system S is hence a fundamentally irreversible process, irrespective of the initial state ρ_S, during which energy is inevitably lost in the microscopic degrees of freedom of the thermal environments E and B.

Remarkably, the proof of Corollary 4.22 is based on a genuinely quantum feature, namely the existence of non-trivial uncertainty relations. Corollary 4.22 therefore shows that such uncertainty relations imply thermodynamic restrictions on the measurement process.

Proof. We already determined the average heat dissipated in the considered process to be

$$\Delta Q = \frac{1}{2}(\Delta Q^{\{P_k\}} + \Delta Q^{\{Q_j\}}) = \frac{1}{2} k_B T \big(H(\{p_k\}) + H(\{q_j\}) \big) \, ,$$

where $p_k = \mathrm{tr}[P_k \rho_S]$ and $q_j = \mathrm{tr}[Q_j \rho_S]$.

Applying the following so-called *entropic uncertainty relation* proven in refs. [91, 92],

$$H(\{p_k\}) + H(\{q_j\}) \geq -2 \log c \ ,$$

where $c = \max_{kj} ||P_k Q_j||_\infty$, yields

$$\Delta Q = -k_B T \log c \ .$$

This already concludes the proof.

Note however that improvements of such entropic uncertainty relations, i.e. state-independent lower bounds on $H(\{p_k\}) + H(\{q_j\})$ larger than c, will directly place even stronger thermodynamic restrictions on the heat expense of two projective quantum instruments. Indeed, it is known that the constant c is not optimal, meaning that such "better" state-independent lower bounds indeed exist (see e.g. [93–95]). It is however an open question how such bounds can be computed. □

4.6 POVMs

The last technical contribution of this chapter is a bound on the energy cost for the POVM formalism. Such measurements are the proper description for measurement settings where the post-measurement state is of no interest (see Section 3.1). As such, this class of measurements is less restrictive than quantum instruments and, consequently, we expect less restrictive energy requirements to physically implement a POVM.

Similar to the other formalisms of quantum measurement, we first compute a lower bound on the energy cost for general POVMs. This bound will not be tight, as we show by computing the *exact* cost of so-called PVMs, which typically exceeds our lower bound for general POVMs.

4.6.1 General lower bound

Before stating the lower bound, let us first point out similarities and differences to the other formalisms of measurement.

Note that since implementations of POVMs record the measurement outcome, the resetting step is non-trivial, similar to quantum instruments. Indeed, we can apply exactly the same reasoning as in Lemma 4.4.1 to show that including the resetting step is *essential* for the energy cost of POVMs to be finite. Since the resetting step needs to satisfy the same requirements as for quantum instruments, we can employ Proposition 4.13 to quantify the cost of the resetting step.

We also recall from Definition 3.20 that, in order to compute the energy cost of a POVM $\{E_k\}$, i.e.

$$E_{\text{cost}}^{\{E_k\}} = \inf_{(\rho_B, U_{MB})} \inf_{\mathcal{U}_{\{E_k\}}} \left(\Delta E_S + \Delta E_E + \Delta E_B \right), \tag{4.57}$$

we need to optimise the average energy $\Delta E_S = \mathrm{tr}[H_S(\rho_S' - \rho_S)]$ on S as well, since POVMs do not fix the post-measurement state ρ_S, in contrast to all other formalisms of measurement.

With this we can formulate the following lower bound on the energy cost of a general POVM.

Theorem 4.23. ***Lower bound on the energy cost of a POVM.***

Let $\{E_k\}$ be a POVM on a system S with Hamiltonian H_S initially in state ρ_S. Then the energy cost of this POVM is lower bounded as

$$E_{\text{cost}}^{\{E_k\}} \geq \Delta F_{S,\text{th}} ,$$

where $\Delta F_{S,\text{th}} = F(\rho_{S,\text{th}}) - F(\rho_S)$ denotes the free energy difference between *the thermal state*

$$\rho_{S,\mathrm{th}} = \frac{\mathrm{e}^{-\frac{H_S}{k_B T}}}{Z_S}$$

with $Z_S = \mathrm{tr}[\mathrm{e}^{-\frac{H_S}{k_B T}}]$ *and the given initial state* ρ_S.

Proof. Let $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME}) \in \mathcal{U}_{\{E_k\}}$ be an implementation of the POVM $\{E_k\}$. We define the energy cost of this implementation

$$E_{\mathrm{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \Delta E_E + \inf_{(\rho_B, U_{MB})} \Delta E_B \ ,$$

i.e. as the average energy of all involved systems optimised over all implementations (ρ_B, U_{MB}) of the resetting step, as usual. Applying our result for the resetting step, i.e. Proposition 4.13, we have that

$$E_{\mathrm{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta E_S + \Delta E_E + k_B T \left(S(\rho'_M) - S(\rho_M) \right) \ . \tag{4.58}$$

From Definition 3.19 we know that applying the given implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ on the state ρ_S corresponds to the state transformation $\rho_{SM} \mapsto \rho'_{SM} = \mathrm{tr}_E[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger]$. We may therefore employ Proposition 4.1 to find that

$$\Delta E_E \geq k_B T \left(S(\rho_{SM}) - S(\rho'_{SM}) \right) \ .$$

Since the von Neumann entropy never decreases under unital channels, we can apply the dephasing channel with respect to the projections $\{\mathbb{1}_S \otimes Q_{M,k}\}$ on the state ρ'_{SM} to find that

$$\begin{aligned} \Delta E_E &\geq k_B T \left(S(\rho_{SM}) - S\Big(\sum_k (\mathbb{1}_S \otimes Q_{M,k}) \rho'_{SM} (\mathbb{1}_S \otimes Q_{M,k}) \Big) \right) \\ &\geq k_B T \left(S(\rho_{SM}) - H(\{p_k\}) - \sum_k p_k S(\rho'_{SM,k}) \right) \ , \end{aligned}$$

where we used Condition 1 of Definition 3.19 to identify $\mathrm{tr}[(\mathbb{1}_S \otimes Q_{M,k}) \rho'_{SM}]$ with the probability $p_k = \mathrm{tr}[E_k \rho_S]$ and defined the states $\rho'_{SM,k} = (\mathbb{1}_S \otimes Q_{M,k}) \rho'_{SM} (\mathbb{1}_S \otimes Q_{M,k}) / p_k$. The second inequality then follows from Eq. (2.5). Non-negativity of the mutual information $I(S:M)_{\rho'_{SM,k}}$ then implies that

$$\begin{aligned} \Delta E_E &\geq k_B T \left(S(\rho_{SM}) - H(\{p_k\}) - \sum_k p_k S(\rho'_{SM,k}) \right) \\ &= k_B T \left(S(\rho_{SM}) - H(\{p_k\}) - \sum_k p_k \Big(S(\rho'_{S,k}) + S(\rho'_{M,k}) - I(S:M)_{\rho'_{SM,k}} \Big) \right) \\ &\geq k_B T \left(S(\rho_S) - \sum_k p_k \Big(S(\rho'_{S,k}) \Big) + S(\rho_M) - S(\rho'_M) \right) \ , \end{aligned}$$

where we used that $\sum_k p_k \rho'_{M,k} = \rho'_M$, such that by Condition 2 of Definition 3.19 and Eq. (2.5) we have that $S(\rho'_M) = H(\{p_k\}) + \sum_k p_k S(\rho'_{M,k})$.

Inserting this into Eq. (4.58) we obtain the following lower bound on the energy cost of an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$,

$$E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} \geq \Delta E_S + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big)\,.$$

This looks very similar to the general result for quantum instruments, Theorem 4.14. Indeed, we basically just provided a slightly different proof of this statement. Note that this derivation shows that we only need the weaker classicality condition, i.e. Condition 2 of Definition 3.19, to prove Theorem 4.14. Leaving this little remark aside, let us now calculate the energy costs of POVMs which from Eq. (4.57) we know to be given by

$$\begin{aligned} E_{\text{cost}}^{\{E_k\}} &= \inf_{\mathcal{U}_{\{E_k\}}} E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} \\ &\geq \inf_{\mathcal{U}_{\{E_k\}}} \left[\Delta E_S + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big)\right] \\ &= \inf_{\mathcal{U}_{\{E_k\}}} \left[\mathrm{tr}[H_S(\rho'_S - \rho_S)] + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big)\right]\,. \end{aligned}$$

Note that ρ_S, H_S, and the probabilities p_k are fixed; the only parts of this lower bound that still depend on the implementation are the states $\rho'_{S,k}$ (and consequently $\rho'_S = \sum_k p_k \rho'_{S,k}$). But for any POVM we can find implementations such that these states $\rho'_{S,k}$ can be chosen at will[9]. Hence, the optimisation over all possible implementations in $\mathcal{U}_{\{E_k\}}$ just translates to an optimisation over the whole state space of S, i.e.

$$E_{\text{cost}}^{\{E_k\}} \geq \min_{\rho'_{S,k}\in\mathcal{B}(\mathcal{H}_S)} \left[\mathrm{tr}[H_S(\rho'_S - \rho_S)] + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big)\right]\,.$$

We replaced the “infimum” with a “minimum”, since the state space is convex and hence the minimum can be attained. Employing the definition of the free energy

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[9] To see this, suppose that $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is an implementation of a given POVM. From this we can then simply construct an implementation, where the memory consists of K additional subsystems $M_{k=1}, M_{k=2}, ..., M_{k=K}$, each initially in the desired state $\rho'_{S,k}$, where K denotes the total number of measurement outcomes. The unitary interaction U_{SME} is the same as in the original implementation followed by a conditional unitary which reads out the outcome k stored in M and then swaps S with subsystem M_k. This implementation will always output the state $\rho'_{S,k}$ on S if the outcome was k.

$F(\rho_S) = \mathrm{tr}[\rho_S H_S] - k_B T S(\rho_S)$ from Eq. (2.9) and the identity Eq. (4.2) from the proof of Proposition 4.1 we find that

$$\begin{aligned}
E_{\text{cost}}^{\{E_k\}} &\geq \min_{\rho'_{S,k}} \left[\mathrm{tr}[H_S(\rho'_S - \rho_S)] + k_B T\Big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big) \right] \\
&= \min_{\rho'_{S,k}} \sum_k p_k \big[\mathrm{tr}[H_S \rho'_{S,k}] - k_B T S(\rho'_{S,k})\big] - F(\rho_S) \\
&= k_B T \sum_k p_k \min_{\rho'_{S,k}} [-\log Z_S + D(\rho'_{S,k} \| \rho_{S,\mathrm{th}})] - F(\rho_S) \\
&= -k_B T \log Z_S - F(\rho_S) \\
&= F(\rho_{S,\mathrm{th}}) - F(\rho_S)\,,
\end{aligned}$$

where we defined the thermal state

$$\rho_{S,\mathrm{th}} = \frac{\mathrm{e}^{-\frac{H_S}{k_B T}}}{Z_S}$$

with respect to the Hamiltonian of H_S at temperature T. This concludes the proof. □

The quantity $\Delta F_{S,\mathrm{th}}$ in Theorem 4.23 corresponds to the maximal work that can be extracted from the initial state ρ_S (see e.g. [46]). It is zero if the initial state ρ_S was already thermal, which is equivalent to saying that no work can be extracted from a thermal state. The statement of Theorem 4.23 can therefore be rephrased as saying that the cost of performing a POVM is never less than the amount of work that can be extracted in this process. POVM measurements hence respect the Second Law of Thermodynamics.

As for quantum channels and quantum instruments, the proof of Theorem 4.23 does not use the implementation requirement, i.e. Condition 2 of Definition 3.19, properly as it does not need the quantifier "$\forall \rho_S$". The bound in Theorem 4.23 is thus not tight as we can infer from our much stronger result for PVMs which we present in the following section.

4.6.2 Energy cost of PVMs

We now consider a subclass of POVMs for which we are able to compute the exact energy costs.

Definition 4.24. PVM

A PVM (projection valued measure) is a POVM $\{P_k\}$ where all POVM elements P_k are projections, i.e. $P_k = P_k^\dagger = P_k^2$ for all k. The requirement $\sum_k P_k = \mathbb{1}_S$ from Definition 3.5 then implies that the POVM elements are mutually orthogonal projections.

For such measurements, we find the following theorem.

Theorem 4.25. ***Energy cost of a PVM.***

Let $\{P_k\}$ be a PVM on a system S with Hamiltonian H_S initially in state ρ_S. Then the energy cost of this PVM is exactly given by

$$E_{\text{cost}}^{\{P_k\}} = \Delta F_{S,\text{th}} + k_B T \left(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S) \right) . \tag{4.59}$$

We note that this result is consistent with (and stronger than) the general bound in Theorem 4.23 since $S\Big(\sum_k P_k \rho_S P_k\Big) \geq S(\rho_S)$ for all initial states ρ_S [82].

Before presenting the proof, let us to comment on the applicability of this result. The basic goal of this thesis is to derive (lower bounds on) the energy cost of quantum measurements when the mathematical description of this measurement (e.g. the POVM elements in this case) *and* the initial state ρ_S are given[10]. Now, as mentioned above, a POVM is only suitable when the post-measurement state is ignored. The purpose of performing a POVM measurement can then only be to gather information about the initial state ρ_S. But this state is by construction already given and therefore also the probabilities for all outcomes k. A valid question is hence why one should be interested to formulate such lower bounds for POVMs. To answer this question, consider the case where we are given a set $\{\rho_S^1, \rho_S^2, ..., \rho_S^N\}$ of possible initial states of our system S and suppose we like to perform a POVM on this set. The POVM could for example be used to discriminate between these states with the goal to find out which state ρ_S^i the system is actually in (see e.g. [96–98]). Using our Theorem 4.25 we can bound the energetic cost of such POVMs in a non-trivial way: Denoting by $E(\rho_S^i)$ the lower bound Eq. (4.59) evaluated for the initial state ρ_S^i, we could simply compute the bound $\sum_i P_i E(\rho_S^i)$ on the average energy cost if we are additionally given the probabilities P_i with which the initial state ρ_S^i occurs. Alternatively, we can also characterise the bound $\min_i E(\rho_S^i)$ on the minimal amount of energy that we need to invest. Hence, depending on the figure of merit we are interested in, we can employ Theorem 4.25 to obtain non-trivial results that make sense even though we consider the initial state ρ_S as given.

Let us now proceed with the proof, which employs the following lemma.

Lemma 4.26. *Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded operator on a Hilbert space $\mathcal{H}$ such that $0 \leq A \leq P = P^\dagger = P^2$. Then $A = PA = AP = PAP$.*

Proof. Since $0 \leq A \leq P$ we have that

$$0 \leq (1-P)A(1-P) \leq (1-P)P(1-P) = 0 ,$$

---

[10] Of course, we also consider the Hamiltonian H_S as given, but this is not important for the argument.

where we used that P is a projection, i.e. $P = P^\dagger = P^2$. This implies that $(1-P)A(1-P) = 0$. Additionally, we use the following defining property of positive operators: An operator X is positive if and only if there exist operators Y such that $X = Y^\dagger Y$. Moreover, a positive operator $X \geq 0$ satisfies $X = Y^\dagger Y = 0$ if and only if $Y = 0$.

Since A is positive, we therefore know that there exist operators B such that $A = B^\dagger B$. We then have that $(1-P)B^\dagger B(1-P) = 0$. But $(1-P)B^\dagger B(1-P)$ is again a positive operator, hence $B(1-P) = 0$, from which we get that $A(1-P) = B^\dagger B(1-P) = 0$. This already yields the desired statement $A = PA = AP = PAP$.

□

We now prove Theorem 4.25.

Proof of Theorem 4.25. The proof is structured as follows: In the first part we show that the energy cost of the PVM $\{P_k\}$ is lower bounded as

$$E_{\text{cost}}^{\{P_k\}} \geq \Delta F_{S,\text{th}} + k_B T \left(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S) \right) .$$

In the second part of the proof we construct a specific implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of the PVM $\{P_k\}$ with a cost exactly equal to this lower bound. This then shows the desired equality statement.

We start the first part of the proof by recalling from Definition 3.19 that any implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of the PVM $\{P_k\}$ satisfies

$$\operatorname{tr}[P_k \rho_S] = \operatorname{tr}[(\mathbb{1}_S \otimes Q_{M,k} \otimes \mathbb{1}_E) U_{SME} (\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \qquad \forall k, \rho_S \tag{4.60}$$

and

$$[Q_{M,k}, \operatorname{tr}_{SE}[U_{SEM}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger]] = 0 \qquad \forall k, \rho_S \,. \tag{4.61}$$

Let us define the quantum channel $T_{S\to M} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_M)$ that takes ρ_S as an input and outputs the corresponding post-measurement state on the memory M by

$$T_{S\to M}(\rho_S) = \operatorname{tr}_{SE}[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \,. \tag{4.62}$$

Correspondingly, we define the channel $T_{S\to SE} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_E)$ which outputs the post-measurement state on S and E by

$$T_{S\to SE}(\rho_S) = \operatorname{tr}_M[U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E) U_{SME}^\dagger] \,. \tag{4.63}$$

We note that both requirements, Eqs. (4.60) and (4.61), are conditions on the channel $T_{S\to M}$ as we can rewrite them as

$$\operatorname{tr}[P_k \rho_S] = \operatorname{tr}[Q_{M,k} T_{S\to M}(\rho_S)] \qquad \forall k, \rho_S \tag{4.64}$$

and

$$[Q_{M,k}, T_{S\to M}(\rho_S)] = 0 \qquad \forall k, \rho_S\ . \tag{4.65}$$

Let now ρ_S be fixed throughout this proof. Recall from Eq. (4.1) that the energy cost of the implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ from above is defined as the average energy change on all involved systems, i.e.

$$\begin{aligned}
E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} &= \Delta E_S + \Delta E_E + \inf_{(\rho_B,U_{MB})} \Delta E_B \\
&= \Delta E_S + \Delta E_E + k_B T \left(S(\rho'_M) - S(\rho_M)\right) \\
&\geq \Delta E_S + k_B T \left(S(\rho'_E) - S(\rho_E) + S(\rho'_M) - S(\rho_M)\right) \\
&= \Delta F_S + k_B T \left(S(\rho'_S) + S(\rho'_M) + S(\rho'_E) - (S(\rho_S) + S(\rho_M) + S(\rho_E))\right) \\
&\geq \Delta F_S + k_B T \left(S(\rho'_{SE}) + S(\rho'_M) - (S(\rho_S) + S(\rho_M) + S(\rho_E))\right)\ ,
\end{aligned} \tag{4.66}$$

where we used Eq. (4.25) from Theorem 4.13 for the resetting in the first step, Eq. (4.3) from the proof of Proposition 4.1 in the second step, the definition of the free energy $\Delta F_S = F_S(\rho'_S) - F_S(\rho_S)$ with $F_S(\rho_S) = \mathrm{tr}[H_S \rho_S] - k_B T S(\rho_S)$ in the third step, and, in the last step, the non-negativity of the mutual information $I(S:E)_{\rho'_{SE}} = S(\rho'_S) + S(\rho'_E) - S(\rho'_{SE}) \geq 0$.

The general strategy of this first part of the proof is now to relate the entropies $S(\rho'_{SE})$ and $S(\rho'_M)$ of the final states to those of the initial states, $S(\rho_S)$, $S(\rho_M)$, and $S(\rho_E)$. To obtain this relation we may only use requirements Eqs. (4.64) and (4.65). Note that luckily these requirements directly constrain the state $\rho'_M = T_{S\to M}(\rho_S)$. Furthermore, the state $\rho'_{SE} = T_{S\to SE}(\rho_S)$ is related to the complementary channel of $T_{S\to M}$ (similar to the proof of Theorem 4.19 for projective quantum instruments) as we make more precise below in this proof. Before that however, we first gather four useful properties of the channel $T_{S\to M}$ and its complementary channel.

Define the completely positive, linear maps $T_k : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_M)$ via

$$T_k(\sigma_S) = Q_{M,k} T_{S\to M}(\sigma_S) Q_{M,k} \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H}) \tag{4.67}$$

such that we can write the channel $T_{S\to M}$ as a sum over these maps T_k since $\sum_k Q_{M,k} = \mathbb{1}_M$ by definition and hence by Eq. (4.65)

$$T_{S\to M}(\sigma_S) = \sum_k Q_{M,k} T_{S\to M}(\sigma_S) = \sum_k Q_{M,k} T_{S\to M}(\sigma_S) Q_{M,k} = \sum_k T_k(\sigma_S) \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,. \tag{4.68}$$

We now prove the first important identity needed for the following proof, namely that the von Neumann entropy of the final memory state can be written as

$$S(\rho'_M) = H(\{p_k\}) + \sum_k p_k S\left(T_k\left(\frac{P_k \rho_S P_k}{p_k}\right)\right)\ , \tag{4.69}$$

where we defined the probabilities $p_k = \mathrm{tr}[P_k\rho_S]$ as usual. To show this, note that Eq. (4.64) implies that

$$T_k^*(\mathbb{1}_M) = P_k \qquad \forall k \tag{4.70}$$

since

$$\mathrm{tr}[P_k\sigma_S] = \mathrm{tr}[Q_{M,k}T_{S\to M}(\sigma_S)] = \mathrm{tr}[T_k(\sigma_S)] = \mathrm{tr}[T_k^*(\mathbb{1}_M)\sigma_S] \qquad \forall k, \sigma_S\,.$$

Let us fix k for the moment and consider a bounded operator $B \in \mathcal{B}(\mathcal{H}_M)$ on M which is positive semi-definite, i.e. $B \geq 0$. Such an operator is always majorised by $\lambda_{\max}\mathbb{1}_M$, where $\lambda_{\max}$ denotes the maximal eigenvalue of B, i.e. $B \leq \lambda_{\max}\mathbb{1}_M$. Eq. (4.70) therefore implies that

$$0 \leq T_k^*(B) \leq T_k^*(B) + T_k^*(\lambda_{\max}\mathbb{1}_S - B) = \lambda_{\max}T_k^*(\mathbb{1}_M) = \lambda_{\max}P_k \qquad \forall B\,.$$

By Lemma 4.26 we then know that $T_k^*(B) = P_k T_k^*(B)P_k$ for all B which implies that

$$T_k(\sigma_S) = T_k(P_k\sigma_S P_k) \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,. \tag{4.71}$$

But then we have

$$\rho'_M = T_{S\to M}(\rho_S) = \sum_k T_k(\rho_S) = \sum_k T_k(P_k\rho_S P_k) = \sum_k p_k\,T_k(P_k\rho_S P_k/p_k)\,,$$

that is, ρ'_M is a convex combination of the states $T_k(P_k\rho_S P_k/p_k)$ with weights $p_k = \mathrm{tr}[P_k\rho_S]$. By Eq. (2.5) this indeed implies our first important identity, Eq. (4.69).

The second important identity is a direct consequence of Eqs. (4.68) and (4.71), from which we obtain that

$$T_{S\to M}\Big(\frac{P_k\rho_S P_k}{p_k}\Big) = \sum_j T_j\Big(P_j\frac{P_k\rho_S P_k}{p_k}P_j\Big) = T_k\Big(\frac{P_k\rho_S P_k}{p_k}\Big)\,. \tag{4.72}$$

We continue by proving the third important identity,

$$T_{S\to SE}(\sigma_S) = T_{S\to SE}\Big(\sum_k P_k\sigma_S P_k\Big) \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,, \tag{4.73}$$

which implies that the channel $T_{S\to SE}$, which for input state ρ_S outputs the state $\rho'_{SE} = T_{S\to SE}(\rho_S)$, is invariant under dephasing with respect to the projective POVM elements P_k. To show this, let us denote the Kraus operators of T_k by $K_{k,l}$ such that $T_k(\sigma_S) = \sum_l K_{k,l}\sigma_S K_{k,l}^\dagger$. Due to Eq. (4.67) we have that

$$T_{S\to M}(\sigma_S) = \sum_{k,l} K_{k,l}\sigma_S K_{k,l}^\dagger$$

and hence the operators $K_{k,l}$ are also the Kraus operators of $T_{S\to M}$. We can use Eq. (4.71) together with Eq. (4.67) to see that these Kraus operators must satisfy

$$K_{k,l} = Q_{M,k} K_{k,l} P_k \qquad \forall k,l\,. \tag{4.74}$$

Let us now define the channel $T^c_{S\to M} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_A)$ with $\mathcal{H}_A$ a Hilbert space of large enough dimension with orthonormal basis $\{|k,l\rangle\}$ via

$$\begin{aligned} T^c_{S\to M}(\sigma_S) &= \mathrm{tr}_M\Big[\sum_{k,l,k',l'} |k,l\rangle_A\langle k',l'| \otimes K_{k,l}\sigma_S K^\dagger_{k',l'}\Big] \\ &= \sum_{k,l,k',l'} |k,l\rangle_A\langle k',l'|\mathrm{tr}[K_{k,l}\sigma_S K^\dagger_{k',l'}] \\ &= \sum_{k,l,l'} |k,l\rangle_A\langle k,l'|\mathrm{tr}[K_{k,l}\sigma_S K^\dagger_{k,l'}]\,. \end{aligned}$$

We note two things about this channel: Firstly, it is a complementary channel of $T_{S\to M}$ since we can define the isometry $V : \mathcal{H}_S \to \mathcal{H}_A \otimes \mathcal{H}_M$ by $V = \sum_{k,l} |k,l\rangle_A \otimes K_{kl}$ such that $T^c_{S\to M}(\sigma_S) = \mathrm{tr}_M[V\sigma_S V^\dagger]$ and

$$T_{S\to M}(\sigma_S) = \sum_{k,l} K_{k,l}\sigma_S K^\dagger_{k,l} = \mathrm{tr}_A\left[\sum_{k,l,k',l'} |k,l\rangle_A\langle k',l'| \otimes K_{k,l}\sigma_S K^\dagger_{k',l'}\right] = \mathrm{tr}_A[V\sigma_S V^\dagger]\,.$$

Secondly, due to Eq. (4.74) we know that $T^c_{S\to M}$ is invariant under dephasing of the input state with respect to the projections P_k, that is

$$\begin{aligned} T^c_{S\to M}\Big(\sum_j P_j\sigma_S P_j\Big) &= \sum_{k,l,k',l'} |k,l\rangle_A\langle k',l'|\,\mathrm{tr}\Big[K_{k,l}\sum_j P_j\sigma_S P_j K^\dagger_{k',l'}\Big] \\ &= \sum_{k,l,l'} |k,l\rangle_A\langle k,l'|\,\mathrm{tr}\left[K_{k,l}\sigma_S K^\dagger_{k,l'}\right] \\ &= T^c_{S\to M}(\sigma_S) \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,. \end{aligned} \tag{4.75}$$

Due to Eq. (4.63), another complementary channel of $T_{S\to M}$ is given by the channel $T_{S\to SE\tilde{E}} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_E) \otimes \mathcal{B}(\mathcal{H}_{\tilde{E}})$ with

$$T_{S\to S\tilde{M}E\tilde{E}}(\sigma_S) = \mathrm{tr}_M[U_{SM\tilde{M}E\tilde{E}}(\sigma_S \otimes \psi_{M\tilde{M}} \otimes \psi_{E\tilde{E}})U^\dagger_{SM\tilde{M}E\tilde{E}}] \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,,$$

where $\tilde{M}$ and $\tilde{E}$ are the purifying systems of M and E, respectively, such that the pure state $\psi_{M\tilde{M}}$ satisfies $\mathrm{tr}_{\tilde{M}}[\psi_{M\tilde{M}}] = \rho_M$ and analogously for $\psi_{E\tilde{E}}$. Moreover, we set $U_{SM\tilde{M}E\tilde{E}} := U_{SME} \otimes \mathbb{1}_{\tilde{M}} \otimes \mathbb{1}_{\tilde{E}}$. Note that this channel is linked to our desired channel $T_{S\to SE}(\sigma_S) = \mathrm{tr}_{\tilde{M}\tilde{E}}[T_{S\to S\tilde{M}E\tilde{E}}(\sigma_S)]$ through the partial trace. Due to the Stinespring theorem, see Eq. (2.2), we also know that the two complementary channels are related by a partial isometry $V : \mathcal{H}_M \to \mathcal{H}_S \otimes \mathcal{H}_{\tilde{M}} \otimes \mathcal{H}_E \otimes \mathcal{H}_{\tilde{E}}$ such that

$$T_{S\to SE}(\sigma_S) = \mathrm{tr}_{\tilde{M}\tilde{E}}[V\,T^c_{S\to M}(\sigma_S)V^\dagger] \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,.$$

The invariance property Eq. (4.75) of $T^c_{S\to M}$ therefore directly translates to the same invariance property for $T_{S\to SE}$ as claimed in Eq. (4.73).

Before we finally compute the desired lower bound on the energy cost of the PVM $\{P_k\}$, let us note the fourth, and last, identity

$$\begin{aligned} S(T_{S\to SE}(\sigma_S)) + S(T_{S\to M}(\sigma_S)) &\geq S(U_{SME}(\sigma_S \otimes \rho_M \otimes \rho_E)U^\dagger_{SME}) \\ &= S(\sigma_S) + S(\rho_M) + S(\rho_E) \qquad \forall \sigma_S \in \mathcal{B}(\mathcal{H})\,, \end{aligned} \tag{4.76}$$

which is a simple consequence of the non-negativity of the mutual information $I(M:SE)_{U_{SME}(\sigma_S\otimes\rho_{ME})U^\dagger_{SME}}$.

Having collected all necessary identities to finish the proof, let us continue with Eq. (4.66), which was our intermediate lower bound on the energy cost of $\{P_k\}$. We focus on the entropic quantity $S(\rho'_{SE})$ on the right hand side of Eq. (4.66). Using identities Eq. (4.73), Eq. (2.5), Eq. (4.76), and Eq. (4.72), in exactly this order, we find that

$$\begin{aligned} S(\rho'_{SE}) &= S\big(T_{S\to SE}(\rho_S)\big) \\ &= S\left(T_{S\to SE}\left(\sum_k P_k \rho_S P_k\right)\right) \\ &\geq \sum_k p_k S\left(T_{S\to SE}\left(\frac{P_k\rho_S P_k}{p_k}\right)\right) \\ &\geq \sum_k p_k \left[S\left(\frac{P_k\rho_S P_k}{p_k}\right) + S(\rho_M) + S(\rho_E) - S\left(T_{S\to M}\left(\frac{P_k\rho_S P_k}{p_k}\right)\right)\right] \\ &= \sum_k p_k \left[S\left(\frac{P_k\rho_S P_k}{p_k}\right) + S(\rho_M) + S(\rho_E) - S\left(T_k\left(\frac{P_k\rho_S P_k}{p_k}\right)\right)\right]\,. \end{aligned}$$

Inserting this together with our first identity Eq. (4.69) into Eq. (4.66) we obtain the following lower bound on the energy cost of any implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of a PVM,

$$\begin{aligned} E^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}_{\text{cost}} &\geq \Delta F_S + k_B T \sum_k p_k S\left(\frac{P_k\rho_S P_k}{p_k}\right) + k_B T H(\{p_k\}) - k_B T S(\rho_S) \\ &= \Delta F_S + k_B T\left(S\Big(\sum_k P_k\rho_S P_k\Big) - S(\rho_S)\right)\,. \end{aligned} \tag{4.77}$$

The energy cost of the PVM as in Definition 3.20 is obtained from $E^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}_{\text{cost}}$ by taking the infimum over all possible implementations, i.e.

$$E^{\{P_k\}}_{\text{cost}} = \inf_{\mathcal{U}_{\{E_k\}}} E^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})}_{\text{cost}}\,.$$

We employ exactly the same reasoning as in the proof of Theorem 4.23 to replace the “infimum” with a “minimum” after inserting the lower bound Eq. (4.77) into this equation. This yields

$$E_{\text{cost}}^{\{P_k\}} \geq \min_{\rho'_S} \Delta F_S + k_B T \left(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S) \right),$$

which by the definition of the free energy $F(\rho_S) = \operatorname{tr}[\rho_S H_S] + k_B T S(\rho_S)$ and the identity Eq. (4.2) from the proof of Proposition 4.1 yields

$$E_{\text{cost}}^{\{P_k\}} \geq \Delta F_{S,\text{th}} + k_B T \left(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S) \right). \tag{4.78}$$

This concludes the first part of the proof.

It remains to show that this bound is tight. We do this by providing an explicit construction of an implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ of any given PVM $\{P_k\}$ whose costs are exactly given by

$$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})} = \Delta F_{S,\text{th}} + k_B T \left(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S) \right).$$

The basic idea to find such a construction is the following: We can always use the standard implementation (see Definition 3.18) of a projective, and even more importantly *efficient*, quantum instrument that also serves as an implementation of the projective POVM. Since the quantum instrument is efficient, we do not necessarily need the part M_2 of the memory M. Instead, we may use this part to prepare the thermal state $\rho_{S,\text{th}}$ on S after the interaction by an swapping unitary. Namely, we choose

$$\rho_M = |0\rangle_{M_1}\langle 0| \otimes \rho_{M_2}$$

with $\rho_{M_2} = \rho_{S,\text{th}}$ as the initial state of the bipartite memory M,

$$Q_{M,k} = |k\rangle_{M_1}\langle k| \otimes \mathbb{1}_{M_2}$$

as the dephasing projections on this memory,

$$\rho_E = \frac{\mathbb{1}_E}{d_E}$$

as the thermal state on E, implying trivial Hamiltonian $H_E = 0$, and

$$U_{SME} = U_{S\leftrightarrow M_2} \left(\sum_k \mathbb{1}_S \otimes Q_{M,k} \otimes V_{E,k} \right) \Big(U_{SM} \otimes \mathbb{1}_E \Big)$$

as the unitary interaction, similarly to Definition 3.18. Here, $U_{S\leftrightarrow M_2}$ denotes the swapping unitary between S and M_2, and U_{SM} is chosen as in the standard implementation (see after Eq. (3.16)).

Such an implementation outputs the state

$$\rho'_{SME} = U_{SME}(\rho_S \otimes \rho_M \otimes \rho_E)U^\dagger_{SME} = \rho_{S,\text{th}} \otimes \sum_k |k\rangle_{M_1}\langle k| \otimes \Big(P_k \rho_S P_k\Big)_{M_2}$$

such that the entropies on M before and after the measurement are given as

$$S(\rho_M) = S(\rho_{S,\text{th}})$$

and

$$\begin{aligned} S(\rho'_M) &= S\left(\sum_k |k\rangle_{M_1}\langle k| \otimes \Big(P_k \rho_S P_k\Big)_{M_2}\right) \\ &= H(\{p_k\}) + \sum_k p_k S(P_k \rho_S P_k / p_k) \\ &= S\Big(\sum_k P_k \rho_S P_k\Big), \end{aligned}$$

respectively. The energy cost of our implementation $(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$ is therefore

$$\begin{aligned} E_{\text{cost}}^{(\rho_M,\{Q_{M,k}\},\rho_E,U_{SME})} &= \Delta E_S + \Delta E_E + k_B T\Big(S(\rho'_M) - S(\rho_M)\Big) \\ &= \Delta E_S + k_B T\Big(S(\rho'_M) - S(\rho_M)\Big) \\ &= \Delta E_S + k_B T\Big(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_{S,\text{th}})\Big) \\ &= \Delta F_{S,\text{th}} + k_B T\Big(S\Big(\sum_k P_k \rho_S P_k\Big) - S(\rho_S)\Big). \end{aligned}$$

This proves the tightness of the bound in Eq. (4.78) and concludes the proof. □

4.7 Comparison with previous literature

As mentioned in the introduction of this chapter, there exists an extensive amount of literature related to the energy cost of quantum measurements. Whereas we mentioned some of these references already throughout this chapter, we provide in this section a more general comparison of our results with the main contributions in the existing literature. In particular, we discuss the relations between our framework and those of Reeb/Wolf [38], Kammerlander/Anders [66], Landauer [30], Bennett [32], Sagawa/Ueda [64, 99], Jacobs [100], Bedingham/Maroney [45], and Faist [44].

All of these works differ from the results in this thesis in at least one of the following points: The first of these points is the considered *formalism of measurement* (see Section 3.1), i.e. some of these works only deal with a specific formalism of measurement, mostly unselective measurements. Even if quantum channels or quantum instruments are considered, the corresponding works do not incorporate the *implementation requirement* that the measurement device must output the correct post-measurement states *for all* input states, which corresponds to Condition 2 in Definitions 3.12 and 3.14. The last point is that several of these prior works assume the *validity of thermodynamic laws* for quantum measurements, which is in contrast to our purely quantum-mechanical framework. This thesis can therefore be understood as a complement, a generalisation and a unification of previous results.

Reeb/Wolf [38]. This is the work on which our presentation of the energy cost of unselective measurements is based on. Since Section 4.2 is essentially already a summary of those results of their work which are important for our analysis, we keep the comparison short. In ref. [38], the authors formulate a purely quantum-mechanical framework which coincides with ours for unselective measurements (see Section 3.3) and which allows them to mathematically prove that the (minimal) heat dissipation of a state transformation $\rho_S \mapsto \rho'_S$ is given as

$$\Delta Q_{\rho_S \mapsto \rho'_S} = k_B T\big(S(\rho_S) - S(\rho'_S)\big)\,. \tag{4.79}$$

This corresponds exactly to Theorem 4.4, the derivation of which is completely based on [38]. The physical implications of this formal result have been conjectured and formulated in a less mathematical framework already by Landauer and Bennett, as explained below. Besides this, the authors also consider so-called finite-size corrections to Eq. (4.79), where the setting is restricted to allow only for a Hilbert space dimension of the thermal bath which is below some finite value. In this case, the optimal implementation of the unselective measurement, as sketched in the proof of Theorem 4.4, does not work any more and it was proven in [38] that more heat needs to be dissipated than predicted by Eq. (4.79).

Such finite-size corrections were not studied in this thesis, but would definitely be interesting also for the other formalisms of measurement. Ref. [38] does not describe any of these other formalisms.

Kammerlander/Anders [66]. In this paper, the authors provide a statement similar to the result Eq. (4.79) by [38], but in a different framework. Based on a set of thermodynamic assumptions, such as the validity of the First and Second Law of

Thermodynamics or the identification of thermodynamic and von Neumann entropies, they show that state transformations of the form $\rho_S \mapsto \rho'_S = \sum_k P_k \rho_S P_k$, where a given initial state ρ_S is mapped to the dephased state $\sum_k P_k \rho_S P_k$ with respect to eigenprojections $\{P_k\}$ of the Hamiltonian H_S of S, consume a (minimal) amount

$$\Delta W = \Delta E_S + k_B T\big(S(\rho_S) - S(\rho'_S)\big)$$

of thermodynamic work. This is in parallel to Theorem 4.4, since the energy cost $\Delta E^{\mathcal{M}}_{\text{cost}}$ of a quantum measurement $\mathcal{M}$ which we calculate in this thesis can simply be identified with the thermodynamic work as explained in Section 3.4. The main statement of this paper is that state transformations of the form $\rho_S \mapsto \rho'_S = \sum_k P_k \rho_S P_k$ allow to *extract* work in the overall process if the initial state ρ_S has coherences, i.e. if the state was not already classical with respect to the projections $\{P_k\}$. In addition to this, the authors provide a specific protocol, which realises the desired state transformation, consumes exactly the amount ΔW of work and is experimentally realisable with standard methods. Using this protocol therefore allows to realise the work extraction from such "non-classical" initial states.

In this thesis we add to these findings in several conceptual as well as quantitative aspects: Firstly, Theorem 4.4 (which is based on [38]) shows that such work extraction from coherences is possible for *any* state transformation $\rho_S \mapsto \rho'_S$ as long as the entropy $S(\rho'_S)$ of the final state is strictly larger than the entropy $S(\rho_S)$ of the initial state. Secondly, note that the particular state transformation $\rho_S \mapsto \rho'_S = \sum_k P_k \rho_S P_k$ considered in [66] can be realised by implementing the corresponding dephasing channel $\mathcal{F}(\rho_S) = \sum_k P_k \rho_S P_k$ or the projective quantum instrument $\{P_k\}$. Our Theorem 4.12 shows that coherences in the initial state are *completely irrelevant* for the energy cost of such measurements; this suggests that coherences do not matter as soon as we require the measurement device to operate correctly for *all* initial states. Our Theorem 4.12 also shows that one can *never extract work* when implementing a dephasing channel. Moreover, if we additionally like to store the measurement outcomes in order to allow for feedback, even more work needs to be expended as we showed in Theorem 4.19. Thirdly, we provide attainability proofs, similar to the explicit protocol of ref. [66], but valid for other formalisms of measurement as well.

Landauer [30] and Bennett [32]. These papers serve as representatives for a whole series of papers related to the Maxwell's demon problem (see also [31,33–38, 83]). Most of the results developed in this field of research are however only vaguely related to our mathematically rigorous framework, so we will keep the comparison short.

Ref. [30] was the first in which a link between information processing and thermodynamics was established. The main insight of this paper was that the process of information erasure is necessarily accompanied with a minimal heat expense. This is typically referred to as *Landauer's principle.* Ref. [32] applied these findings to the Maxwell's demon problem [31, 40], thereby clarifying that it is *not* the act of measurement that necessarily costs work. Instead, he argued that a measurement can always be implemented in a (thermodynamically) reversible manner, such that no work expense (or heat dissipation) is needed. In contrast, it is rather the obligatory act of resetting the memory which requires the work expense. These statements were refined (and criticised) in a large amount of literature. For example, one refinement is ref. [38], since the fact that "information erasure is accompanied with a minimal heat expense" can be understood as a special case of Theorem 4.4: information erasure implies an entropy decrease, $S(\rho_S) - S(\rho'_S) \geq 0$, which by Eq. (4.79) implies heat dissipation. We will therefore refer to the bound in Theorem 4.4 as the *Landauer bound* in Chapter 5. Another refinement of the arguments by Bennett is ref. [64], as we explain below.

Both refs. [30] and [32] do not treat formalisms other than state transformations.

Sagawa/Ueda [64]. This reference comes closest to our framework for quantum instruments. Indeed, the authors compute the lower bound

$$E_{\text{cost}}^{\{M_k\}} \geq \Delta E_S + k_B T \mathcal{I} \tag{4.80}$$

on the energy cost $E_{\text{cost}}^{\{M_k\}}$ for an *efficient* quantum instrument $\{M_k\}$. Here, the quantity $\mathcal{I} = S(\rho_S) + H(\{p_k\}) + \sum_k \operatorname{tr}[M_k \rho_S M_k \log M_k \rho_S M_k]$ denotes the so-called "quantum-classical mutual information" and is non-negative. In this work, the authors also argued that only the sum of energy expenses due to measurement *and* erasure should be considered, as the energy cost of each process individually can take any desired value (see also Lemma 4.4.1). This extends the arguments by Bennett.

The derivation of this lower bound is only valid for *efficient* quantum instruments[11]. In Section 3.1 we argued why efficient quantum instruments alone do not allow us to fully capture all relevant measurement scenarios (see also our energy extraction example in Section 4.4.3). Moreover, the naive generalisation of this bound to general, inefficient quantum instruments $\{M_{ki}\}$ by substituting $M_k \to M_{ki}$ is not correct, which can be directly seen from our general Theorem 4.14: In fact, as one may compute, the general lower bound $\Delta E_S + k_B T\big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\big)$ from Theorem 4.14 coincides with $\Delta E_S + k_B T \mathcal{I}$ for *efficient* measurements. Our Theorem 4.14 therefore corresponds to the correct generalisation to inefficient quantum instruments.

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[11] In ref. [64] it is claimed that the derivation also holds for inefficient measurement, but this claim was refuted in an erratum [99] later on.

Importantly, the results of [64] can be obtained without requiring the measurement device to output the correct post-measurement states *for all* input states. This is common to all the references we discussed so far. Our Theorem 4.19 for *projective* (and hence efficient) quantum instruments, where this requirement is explicitly employed, proves that much more energy must be expended than predicted by Eq. (4.80). We illustrate this difference in Chapter 5.

Jacobs [100]. In this reference, the author proved the validity of our Theorem 4.14, i.e. the correct generalisation of the result by Sagawa/Ueda [64] to inefficient measurements, in a slightly less general setting. Since this is not completely obvious from the paper directly, we briefly sketch the reasoning that allows us to obtain Theorem 4.14 from ref. [100]. In contrast to our microscopic description of the measurement process, this reasoning is based on thermodynamic assumptions, namely that the initial state ρ_S is thermal and that the Second Law of Thermodynamics holds in the quantum regime.

The reasoning is then as follows: Consider a general quantum instrument $\{M_{ki}\}$ applied on this thermal state ρ_S. This will output an outcome k with probability p_k together with the post-measurement state $\rho'_{S,k}$. It is not important *how* this process is actually happening. The amount of energy, that can subsequently be extracted in a feedback process that restores the initial thermal state ρ_S and makes use of both outcomes and post-measurement states, is then calculated in ref. [100] to be

$$E_{\text{ext,Jacobs}} = \Delta E_S + k_B T\Big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big] .$$

The overall process of measurement and feedback is cyclic on S, since the initial and final states agree. Assuming the Second Law of Thermodynamics, this implies that the measurement must have consumed an amount $E_{\text{cost}}^{\{M_{ki}\}}$ of energy lower bounded as

$$E_{\text{cost}}^{\{M_{ki}\}} \geq \Delta E_S + k_B T\Big[S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\Big] .$$

Otherwise, we would be able to extract energy in a cyclic process that involves a single heat bath.

Note that his relation is identical to the statement of Theorem 4.14. However, this derivation is based on thermodynamic assumptions, in contrast to our microscopic and purely quantum-mechanical modelling of the measurement process. Moreover, the derivation does not allow to investigate the possibility of energy extraction through measurement as we presented in Section 4.4.3. And lastly, it is completely unclear how the implementation requirement, that the measurement device should output the correct post-measurement states *for all* initial states, can be incorporated into this reasoning. Findings of the form of ref. [100] can therefore only be employed to compute *lower bounds* on the energy cost of quantum measurements, in contrast to our exact results.

Bedingham/Maroney [45]. In this work, the authors consider explicitly the consequences of the implementation requirement for quantum channels, i.e. Condition 2 from Definition 3.12, although in a weaker sense: Whereas unselective measurements require the measurement to output the correct state for only a given input state, the authors of Bedingham/Maroney consider a given set $\{\rho_{S,i}\}$ of initial quantum states $\rho_{S,i}$ which occur with probability p_i, for which they like the process to output the desired states $\{\rho'_{S,i}\}$. More concretely, the requirement is that

$$\operatorname{tr}_{AE}[U_{SAE}(\rho_{S,i}\otimes\rho_A\otimes\rho_E)U^{\dagger}_{SAE}]=\rho'_{S,i} \qquad \forall i\ , \tag{4.81}$$

where A is an catalyst system, whose initial state ρ_A must be identical to the post-measurement state $\operatorname{tr}_{SE}[U_{SAE}(\rho_{S,i}\otimes\rho_A\otimes\rho_E)U^{\dagger}_{SAE}]$ for all i. Furthermore, ρ_E is a thermal state of an environment E and U_{SAE} is a unitary, just as in our framework. Note that this requirement is identical (if we neglect the catalyst A) to our implementation requirement if the set $\{\rho_{S,i}\}$ is a spanning set of the whole state space of S, i.e. we want the device to output the correct state for all possible initial states.

For some operations satisfying Eq. (4.81) the authors find that the heat dissipation $\Delta Q=\operatorname{tr}[H_E(\rho'_E-\rho_E)]$ is lower bounded by a quantity which exceeds the heat dissipation $k_B T\big(S(\rho_S)-S(\rho'_S)\big)$ of the unselective measurement $\rho_S=\sum_i p_i\rho_{S,i}\mapsto\sum_i p_i\rho'_{S,i}$. More concretely, they find that

$$\Delta Q\geq k_B T\big(S(\rho_S)-S(\rho'_S)\big)+\epsilon_Q$$

where ϵ_Q is strictly positive for certain operations satisfying Eq. (4.81). This proves that the general lower bound on the energy cost of quantum channels in Corollary 4.5 is not tight.

We add to this statement by computing the *exact* energy cost (and hence also heat dissipation) of projective quantum channels.

Faist *et al* [44]. In this work, the authors develop a single shot framework, in which they analyse the energy cost of general quantum operations. Since their results require knowledge about single shot entropic quantities, such as smoothed min- and max-entropies, we will not discuss this reference in greater detail here as this would be beyond the scope of this thesis. Let us however emphasise that while a single shot analysis typically includes an average energy framework such as ours as a special case, it is *not* true that the results can in this thesis can be derived from the framework of Faist *et al.* For example, let us consider a projective quantum instrument $\{P_k\}$. One can show that applying the so-called i.i.d. limit to the results obtained in [44] for such a measurement translates to the general lower bound from Theorem 4.14, which is much weaker than the exact costs in our Theorem 4.19.

For further details on this (and other differences) between our framework and the one of Faist *et al.*, we refer to the Appendix of ref. [29].

Most of the references just discussed do not consider measurements that obey our strict implementation requirement, which is the main reason why we find that the energy expense of quantum measurement is much larger than previously known [29]. Moreover, we even add to those two references [44, 45], where variants of our implementation requirement for certain formalisms are employed. Furthermore, the important formalism of POVMs is not treated at all in the existing literature (not even in [29]).

CHAPTER 5

Applications

In this chapter we present two important measurement-based protocols in quantum theory and compute their energy consumption on the basis of the results from Chapter 4. The first is the quantum Zeno stabilisation protocol, a paradigmatic example of quantum control theory that aims at stabilising a quantum state against its free Hamiltonian time evolution. We will see that several formalisms of quantum measurement (see Chapter 3) can be used for quantum Zeno stabilisation, hence we compute the energy cost within each of them. Interestingly, if the "traditional" approach to quantum Zeno stabilisation based on quantum instruments is employed, we will find that perfect stabilisation requires infinite energy [29].

The second application is so-called stabilizer quantum error correction (exemplified via the five-qubit code). Such protocols are an essential tool for quantum computation as they enable reliable computation even if the employed physical devices are prone to errors. We will determine the exact energy cost for the five-qubit code. On the one hand this will allow us to compare some of our general results presented in Chapter 4 with the best lower bounds on the energy cost of quantum measurements known so far. On the other hand, our results will also show that the standard realisation of the five-qubit code based on the separate measurement scheme presented in Section 4.5.3, which is experimentally more feasible, consumes much more energy than the optimal implementation: The costs can be significantly lower if a realisation related to the joint measurement scheme from Section 4.5.3 is employed instead of the separate measurement scheme [29].

The purpose of this chapter is to underline the importance of our main results in this thesis for relevant experimentally realisable procedures that are key to, for example, the development of quantum computers and quantum technology in general.

5.1 Energy cost of quantum Zeno protocols

Our first application concerns the energy cost for quantum state stabilisation via Zeno measurements (see e.g. [20, 101, 102] and references therein). We start with a brief introduction to this protocol.

Consider a quantum system S with Hamiltonian $H_S = E\sigma_X$ with Pauli operator σ_X. The spectrum of H_S is thus $\{-E, +E\}$. Let further the initial state of S be $\rho_S = |0\rangle_S\langle 0|$ and hence be pure and not an eigenstate of H_S. The free time evolution of S is then given by $\rho_{S,t} = U_t \rho_S U_t^\dagger$ with $U_t = \mathrm{e}^{-itH_S}$, the state of S will therefore periodically oscillate in time t.

Now suppose that we like to prevent the system S from changing with time; instead, our aim is to stabilise the system against its free time evolution such that the state remains (close to) the initial state ρ_S. This task can be accomplished by applying a sequence of projective quantum instruments on S. To see this, let us denote by t_{tot} the total time span over which we like to stabilise ρ_S, and by $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ the projections of the quantum instrument. The protocol now works as follows: After letting the system evolve for a time interval $\delta t = t_{\mathrm{tot}}/N$, we apply the instrument on S, which is again followed by free time evolution, which is followed by another measurement, irrespective of the previous outcome, and so on and so forth. In total, we therefore perform N independent, projective measurements on S. After the n-th measurement, the state is of the form

$$\rho_S^{(n)} = p_n |0\rangle_S\langle 0| + (1-p_n)|1\rangle_S\langle 1| \tag{5.1}$$

and thus with probability $p_n = \mathrm{tr}\left[P_0 \rho_S^{(n-1)}\right]$ the desired state ρ_S. As will become clear from the proof of our next theorem, the probability p_n evaluates to $p_n \simeq 1 - n\left(\frac{E\delta t}{\hbar}\right)^2$ to leading order. As a consequence, we show that the fidelity $F \equiv F(\rho_S, \rho_S^{(N)}) \simeq 1 - (E t_{\mathrm{tot}}/\hbar)^2/N$ between the final state $\rho_S^{(N)}$ and the state ρ_S can be as close to 1 as desired if the number N of measurements is large enough (and the time intervals δt correspondingly small). A fast sequence of projective measurements therefore indeed allows to stabilise S over the complete time span t_{tot}.

For the above protocol to succeed, we actually do not need to store the outcomes of the individual measurements. In contrast, the protocol is solely based on the changes of the average post-measurement state on S. Accordingly, we do not necessarily need to employ projective quantum instruments — an implementation of the dephasing operation

$$\mathcal{F}(\rho_S) = P_0 \rho_S P_0 + P_1 \rho_S P_1 \qquad \forall \rho_S \in \mathcal{B}(\mathcal{H}_S)\,, \tag{5.2}$$

applied at each time step instead of the projective quantum instrument, would in principle suffice for quantum Zeno stabilisation.

We can be even less restrictive and just demand that the system undergo the sequence of state transformations

$$\rho_S \equiv \rho_S^{(0)} \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(0)} \xrightarrow{\{P_k\}} \rho_S^{(1)} \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(1)} \xrightarrow{\{P_k\}} \dots \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(N-1)} \xrightarrow{\{P_k\}} \rho_S^{(N)}\,, \tag{5.3}$$

where $\rho_S^{(n)}$ is given as above and $\rho_{S,\delta t}^{(n)} = U_{\delta t}\rho_S^{(n)}U_{\delta t}^{\dagger}$ denotes the time evolved state. Here, each state transformation induced by the measurement $\{P_k\}$ would not need to yield the correct final states for *all* input states, but only for the fixed input state, i.e. for example an implementation of the state transformation

$$\rho_{S,\delta t}^{(j)} \xrightarrow{\{P_k\}} \rho_S^{(j+1)}$$

for some j only needs to output the state $\rho_S^{(j+1)}$ for the fixed input $\rho_{S,\delta t}^{(j)}$ (see Definition 3.10).

All these three variants (quantum instrument, dephasing channel and sequence of individual state transformations), are capable of realising quantum Zeno stabilisation. The following theorem quantifies their energetic cost.

Theorem 5.1. ***Energy cost of quantum Zeno stabilisation.***
Suppose a quantum Zeno protocol as above is performed that stabilises the initial state ρ_S with target fidelity F, i.e. the system undergoes the sequence of states

$$\rho_S \equiv \rho_S^{(0)} \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(0)} \xrightarrow{\{P_k\}} \rho_S^{(1)} \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(1)} \xrightarrow{\{P_k\}} \dots \xrightarrow{U_{\delta t}} \rho_{S,\delta t}^{(N-1)} \xrightarrow{\{P_k\}} \rho_S^{(N)}\,, \tag{5.4}$$

as in Eq. (5.3), *with $\rho_S^{(N)}$ such that $F(\rho_S, \rho_S^{(N)}) = F$. Then:*

- *The energy cost for performing the individual state transformations in Eq.* (5.4) *is exactly given by*

$$E_{\text{Zeno}}^{\rho_S \mapsto \dots \mapsto \rho_S^{(N)}} = -k_B T H(\{F, 1-F\}) \leq 0 \tag{5.5}$$

- *The energy cost for applying the dephasing operation $\mathcal{F}$ at each of the N time steps to achieve the desired sequence of system states in Eq.* (5.4) *is exactly given by*

$$E_{Zeno}^{\mathcal{F}} = 0\,, \tag{5.6}$$

- *The energy cost for applying a projective quantum instrument at each of the N time steps to achieve the desired sequence of system states in Eq.* (5.4) *is exactly given by [29]*

$$E_{Zeno}^{\{P_k\}} = \sum_{n=1}^{N} H(\{p_n, 1-p_n\}) \simeq \frac{1}{2}\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 \left(\frac{3}{2} - \log[1-F]\right) \geq 0\,, \tag{5.7}$$

  where "$\simeq$" indicates that higher-order terms in F can be neglected for $F \to 1$ (and constant E and t_{tot}).

As expected, the strictest energy requirements occur when quantum instruments are used to realise the required state transformations. In this case the energy cost diverges logarithmically with the target fidelity approaching 1. Any restriction on the amount of available energy therefore ultimately *limits the accuracy* (as measured by the fidelity F) of quantum Zeno control.

Such drastic consequences do not occur, when the stabilisation protocol is realised using the dephasing operation or a sequence of implementations of simple state transformations. While the optimal implementation of the dephasing operation is energetically neutral irrespective of the target fidelity, the experimentally more challenging procedure to devise optimal implementations for each state transformation allows to even extract energy during the stabilisation procedure. The fact that energy can, at least in principle, be extracted is not surprising as we diminished the purity of the system state ρ_S during the stabilisation process unless we conducted it perfectly, i.e. with final fidelity $F = 1$: In this case, the purity is preserved and the expectation that no energy can be extracted is easily seen to be true from Eq. (5.5)[1]. Hence, the better the stabilisation protocol, the less energy can be extracted, ultimately leading to an energetically neutral behaviour in the limit of perfect stabilisation.

These results show dramatic numerical differences in the energy costs of the different formalisms of quantum measurement in a realistic and relevant application. The lowest energy requirements come at the cost of non-feasibility, whereas the "traditional" way to realise the desired sequence of states via quantum instruments require higher (and even unbounded) energy consumption.

Proof. It is the virtue of our energy cost results that they only depend on the system states before and after the measurement (and of course, the Hamiltonian of S).

---

[1] The purity of the initial state is of course also preserved when $F = 0$ and in this case too, no energy can be extracted. However, we are interested in the energy cost of Zeno stabilisation for "large" target fidelity $F \in [\frac{1}{2}, 1]$.

In order to compute the energy requirements for stabilising the state ρ_S by quantum Zeno measurements, we therefore just need to determine the states $\rho_S^{(n)}$, which the measured system undergoes throughout the stabilisation process, to be able to then directly apply our results from Chapter 4.

Let us therefore first determine the states $\rho_S^{(n)}$. We already know that they are always of the form

$$\rho_S^{(n)} = p_n |0\rangle_S\langle 0| + (1-p_n)|1\rangle_S\langle 1|\,, \tag{5.8}$$

as they result from applying the projections $\{P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|\}$. Between the projections, the system evolves freely, i.e. by the unitary $U = \exp(-i\delta t H_S/\hbar)$, such that the probabilities between subsequent measurements change as

$$p_{n+1} = \langle 0|U\rho_S^{(n)}U^\dagger|0\rangle = p_n \cos(E\delta t/\hbar)^2 + (1-p_n)\sin(E\delta t/\hbar)^2\,.$$

This is a recursion formula which, for initial condition $p_0 = 1$ (implying that the system started out in the pure state $\rho_S = |0\rangle_S\langle 0|$), has the solution

$$p_n = \frac{1}{2}(1+\cos(2E\delta t/\hbar)^n)\,.$$

We can always Taylor expand p_n in orders of δt, yielding

$$p_n = 1 - n\left(\frac{E\delta t}{\hbar}\right)^2 + \mathcal{O}(\delta t^4)\,. \tag{5.9}$$

In the limit of short periods δt of free time evolution, or equivalently a large amount N of measurements in the constant stabilisation time t_{tot}, the probability of S being in the desired state after the n-th measurement is to leading order given by $p_n \simeq 1 - n\left(\frac{E\delta t}{\hbar}\right)^2$ (as claimed right after Eq. (5.1)).

Let us also quickly compute the fidelity F between the initial and the final state on S defined as $F \equiv F(\rho_S, \rho_S^{(N)}) = \langle 0|\rho_S^{(N)}|0\rangle$ [103], which by Eq. (5.8) yields

$$F = \langle 0|\big(p_N|0\rangle_S\langle 0| + (1-p_N)|1\rangle_S\langle 1|\big)|0\rangle = p_N\,. \tag{5.10}$$

The target fidelity F of the stabilisation protocol hence corresponds to the probability that quantum system S ends in the desired state $\rho_S = |0\rangle_S\langle 0|$. From Eq. (5.9) we then also have $F \simeq 1 - (E\,t_{\text{tot}}/\hbar)^2/N$ (as claimed right after Eq. (5.1)).

With this, let us now compute the energy cost of the three different variants to achieve quantum Zeno stabilisation. We start with the first variant, the sequence of state transformations. According to Theorem 4.4, the energy cost of a single state transformation $\rho_{S,\delta t}^{(n-1)} \xleftrightarrow{\{P_k\}} \rho_S^{(n)}$ is exactly given by

$$E_{\text{cost}}^{\rho_{S,\delta t}^{(n-1)} \xleftrightarrow{\{P_k\}} \rho_S^{(n)}} = \Delta E_S^{(n)} + \Delta k_B T\Big(S\big(\rho_S^{(n-1)}\big) - S\big(\rho_S^{(n)}\big)\Big) = k_B T\Big(S\big(\rho_{S,\delta t}^{(n-1)}\big) - S\big(\rho_S^{(n)}\big)\Big)\,,$$

where we used that the average energy on S is not changed in each transformation: Indeed, as one may easily compute we have $\mathrm{tr}[H_S\rho_S^{(n)}] = 0$ implying $\Delta E_S^{(n)} = \mathrm{tr}[H_S(\rho_S^{(n)} - \rho_{S,\delta t}^{(n-1)})] = 0$ for all $n = 1,\dots,N$.

Moreover, the energy cost of each of the unitary state transformations stemming from the free time evolution is precisely zero, i.e.

$$E_{\text{cost}}^{\rho_S^{(n)} \xmapsto{\{U_{\delta t}\}} \rho_{S,\delta t}^{(n)}} = 0 \qquad \forall n = 1,\dots,N\,. \tag{5.11}$$

This directly follows from Theorem 4.4 and the fact that $\mathrm{tr}[H_S(\rho_{S,\delta t}^{(n)} - \rho_S^{(n)})] = 0$ for all $n = 1,\dots,N$ (since $[U_{\delta t}, H_S] = 0$) and that unitary transformations do not change the entropy, i.e. $S(\rho_S^{(n)}) = S(\rho_{S,\delta t}^{(n)})$ for all $n = 1,\dots,N$.

The overall energy cost $E_{\text{Zeno}}^{\rho_S \mapsto \ldots \mapsto \rho_S^{(N)}}$ of conducting the complete chain in Eq. (5.4) is then obtained by taking the sum over all individual steps,

$$E_{\text{Zeno}}^{\rho_S \mapsto \ldots \mapsto \rho_S^{(N)}} = \sum_{n=1}^{N} E_{\text{cost}}^{\rho_S^{(n-1)} \xmapsto{\{P_k\}} \rho_S^{(n)}} = -k_B T S\big(\rho_S^{(N)}\big)\,.$$

But due to Eqs. (2.5), (5.8), and (5.10) this implies that

$$E_{\text{Zeno}}^{\rho_S \mapsto \ldots \mapsto \rho_S^{(N)}} = -k_B T H(\{p_N, 1-p_N\}) = -k_B T H(\{F, 1-F\})$$

as claimed in Eq. (5.5).

This shows that optimal implementations of each state transformation allow to extract energy in a quantum Zeno stabilisation protocol, since the Shannon entropy is never negative. This concludes the first part of the proof.

We continue with the second variant, the dephasing channel $\mathcal{F}(\rho_S)$ from Eq. (5.2). We can employ Theorem 4.12 to find for the energy cost that, as claimed in Eq. (5.6),

$$E_{\text{Zeno}}^{\mathcal{F}} = \sum_{n=1}^{N} \Delta E_S^{(n)} = 0\,,$$

where

$$\Delta E_S^{(n)} = \mathrm{tr}\Big[H_S\big(\mathcal{F}\big(\rho_{S,\delta t}^{(n-1)}\big) - \rho_{S,\delta t}^{(n-1)}\big)\Big] = 0 \qquad \forall n = 1,\dots,N$$

and where we used Eq. (5.11).

The dephasing channel does not allow to extract energy; it can only be conducted at precisely zero energy cost using an optimal implementation.

We conclude the proof with the third variant, the projective quantum instruments. According to our Theorem 4.19, the n-th measurement consumes exactly an amount $k_B T H(\{p_n, 1-p_n\})$ of energy, where we again used that the average energy on S is constant, $\Delta E_S^{(n)} = 0$. The total energy required is thus given by the sum over all N steps,

$$E_{\text{Zeno}}^{\{P_k\}} = \sum_{n=1}^{N} H(\{p_n, 1-p_n\}) = -\sum_{n=1}^{N} \big(p_n \log p_n + (1-p_n)\log(1-p_n)\big) \,. \tag{5.12}$$

Unfortunately, this quantity does not tell us much about the scaling behaviour of the energy cost with increasing target fidelity F. We therefore determine the asymptotic behaviour of this sum for the limit $F \to 1$, i.e. for the limiting case of perfect stabilisation, which is equivalent to the limit $N \to \infty$. From Eq. (5.9) we know that, in this limit, we may set $p_n \simeq 1 - n\left(\frac{E\delta t}{\hbar}\right)^2$, where the symbol "$\simeq$" indicates that higher-order terms can be neglected for large enough N (or F close enough to 1). With this, let us first simplify the second sum in Eq. (5.12),

$$\begin{aligned}
-\sum_{n=1}^{N}(1-p_n)\log(1-p_n) &\simeq -\sum_{n=1}^{N} n\left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2 \log\left[n\left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2\right] \\
&= -\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 \left(\sum_{n=1}^{N} \frac{n}{N^2}\log\frac{n}{N} + \sum_{n=1}^{N}\frac{n}{N^2}\log\left[\frac{1}{N}\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2\right]\right) \\
&\simeq -\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 \left(\sum_{n=1}^{N} \frac{1}{N}\frac{n}{N}\log\frac{n}{N} + \frac{N(N+1)}{2N^2}\log\big[1-F\big]\right) .
\end{aligned}$$

Note that, in the limit $N \to \infty$, the (Riemann) sum $\sum_{n=1}^{N} \frac{1}{N}\frac{n}{N}\log\frac{n}{N}$ converges to the integral $\int_0^1 x\log x\,dx = -1/4$, such that

$$\begin{aligned}
-\sum_{n=1}^{N}(1-p_n)\log(1-p_n) &\simeq -\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 \left(-\frac{1}{4} + \frac{N(N+1)}{2N^2}\log\big[1-F\big]\right) \\
&\simeq -\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 \left(-\frac{1}{4} + \frac{1}{2}\log\big[1-F\big]\right) .
\end{aligned} \tag{5.13}$$

Let us continue with the first sum in Eq. (5.12). Here we use that in the limit $N \to \infty$ we have that $\log(p_n) \simeq -n(\frac{E\,t_{\text{tot}}}{\hbar N})^2$ which yields

$$\begin{aligned}
-\sum_{n=1}^{N} p_n \log p_n &= \sum_{n=1}^{N}\left(1 - n\left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2\right) n\left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2 \\
&= \left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2 \sum_{n=1}^{N} n - \left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^4 \sum_{n=1}^{N} n^2 \\
&= \left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^2 \left(\frac{1}{2}N(N+1)\right) - \left(\frac{E\,t_{\text{tot}}}{\hbar N}\right)^4 \left(\frac{1}{6}N(N+1)(2N+1)\right) \\
&\simeq \frac{1}{2}\left(\frac{E\,t_{\text{tot}}}{\hbar}\right)^2 .
\end{aligned} \tag{5.14}$$

Inserting Eqs. (5.13) and (5.14) into Eq. (5.12) we then find the desired statement from Eq. (5.7)

$$E_{\text{Zeno}}^{\{P_k\}} = \sum_{n=1}^{N} H(\{p_n, 1-p_n\}) \simeq \frac{1}{2}\left(\frac{E\, t_{\text{tot}}}{\hbar}\right)^2 \left(\frac{3}{2} - \log[1-F]\right)$$

in the limit $N \to \infty$ needed for high target fidelity $F \to 1$.

$\square$

5.2 Energy cost of quantum error correction

In this section we present a second application of our results from Chapter 4 in a relevant setting, namely measurement-based quantum error correction (QEC). QEC protocols are an essential ingredient in the development of quantum computing devices or quantum memories [11, 43, 104] (also see [105] for a fairly recent review on QEC). Their basic task is to protect information stored in quantum states against noise that would, without QEC, modify and destroy this information.

Intuitively speaking, QEC protocols apply measurements to determine how the information was changed to allow for subsequent feedback operations that will (try to) recover the original state prior to the noise. We specify the details of such a protocol below. Importantly, the performance of QEC protocols scales with the frequency of measurements applied. Hence, in typical situations a vast amount of measurements is performed to allow for reliable storage of quantum information. Our results from Chapter 4 quantify the energy consumption for each of these measurements and, similar to our findings for the quantum Zeno stabilisation protocol, they put severe limitations on the performance of QEC if only limited energy supply is available [29].

More specifically, we consider here error correction based on the so-called five qubit code [10, 11]. Our results may however be equally well applied to any other measurement-based QEC protocol, e.g. stabiliser QEC.

Let us first describe the way the five qubit code operates. For this, consider a state $|\psi\rangle_L \in \mathcal{H}_L$ of a qubit with Hilbert space $\mathcal{H}_L = \mathbb{C}^2$. We will call this qubit the *logical qubit* in the following. Let $\{|0\rangle_L, |1\rangle_L\}$ denote a basis of $\mathcal{H}_L$ such that we can write the state of the logical qubit as $|\psi\rangle_L = \alpha_0|0\rangle_L + \alpha_1|1\rangle_L$, where $\alpha_0, \alpha_1 \in \mathbb{C}$ and $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The state of the logical qubit is now affected by a noise channel $\mathcal{N}: \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L)$, which maps $|\psi\rangle_L\langle\psi|$ to the noisy state $\mathcal{N}(|\psi\rangle_L\langle\psi|)$. The basic goal of QEC is now to recover the original state $|\psi\rangle_L\langle\psi|$ from the noisy state $\mathcal{N}(|\psi\rangle_L\langle\psi|)$.

To this end, the space $\mathcal{H}_L$ of the logical qubit is embedded into the space $\mathcal{C}_5 \equiv (\mathbb{C}^2)^{\otimes 5}$ of five so-called *physical qubits*. Due to this embedding, one introduces redundancy into the encoded information, which is ultimately the reason why the protocol can correct the effect of noise.

The embedding is done by mapping the basis elements $|0\rangle_L$ and $|1\rangle_L$ of $\mathcal{H}_L$ onto basis elements of $\mathcal{C}_5$. The embedding map is given by the codewords [43]

$$\begin{aligned}|0\rangle_L =& \frac{1}{4}[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ &+ |01010\rangle + |00101\rangle - |11011\rangle - |00110\rangle \\ &- |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle \\ &-|01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle] \,, \\ |1\rangle_L =& \frac{1}{4}[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\ &+ |10101\rangle + |11010\rangle - |00100\rangle - |11001\rangle \\ &- |00111\rangle - |00010\rangle - |11100\rangle - |00001\rangle \\ &-|10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle] \,,\end{aligned}$$

where $\{|0\rangle, |1\rangle\}$ denotes a basis of the state space of a physical qubit. This mapping singles out a two-dimensional subspace of $\mathcal{C}_5$, isomorphic to $\mathcal{H}_L$, which we refer to as the *code space*.

Without the embedding, the noise channel $\mathcal{N}$ would immediately change the quantum information that is stored in the complex amplitudes α_0 and α_1 of $\psi\rangle$. With the embedding, the noise is instead acting on each of the five physical qubits. We assume that this happens independently on each physical qubit, i.e. the noisy state is given as $\mathcal{N}^{\otimes 5}(|\psi\rangle_L\langle\psi|)$. We will see below how this embedding helps to protect the quantum information.

Depending on the physical situation, different noise channels are appropriate: prominent examples are the amplitude damping channel, the dephasing channel and the depolarising channel. Here, we will focus on the amplitude damping channel, but our results from Chapter 4 may equally well be applied to determine the energy cost for other noise channels. The amplitude damping channel of strength $\gamma \in [0,1]$ is defined via

$$\mathcal{N}_\gamma(\rho) = J_1 \rho J_1^\dagger + J_2 \rho J_2^\dagger$$

with Kraus operators $J_1 = \sqrt{\gamma}|0\rangle\langle 1|$ and $J_2 = \sqrt{\mathbb{1} - J_1^\dagger J_1}$. The amplitude damping channel of strength $\gamma = 0$ is therefore identical to the identity channel on $\mathcal{H}_L$ and thus corresponds to the noiseless case, where the quantum information is not altered. In contrast, the amplitude damping channel of strength $\gamma = 1$ maps any state ρ of the physical qubit to the fixed state $|0\rangle\langle 0|$ and therefore is an erasing channel (see Definition 4.6).

Irrespective of the noise model, the five qubit code now employs a set of measurements that are applied on the noisy state $\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)$. The respective measurement outcomes then allow us to infer which "error" occured on the physical qubits. This error can then be corrected by suitable correction operations as will be explained below. The measurements are typically referred to as *syndrome measurements* and the collection of outcomes of all syndrome measurements as *syndromes.*

For the five qubit code, the set of syndrome measurements consists of the following four hermitian operators

$$\begin{aligned} \mathcal{S}^1 &= \sigma_X\otimes\sigma_Z\otimes\sigma_Z\otimes\sigma_X\otimes\mathbb{1} \quad , \quad \mathcal{S}^2 = \mathbb{1}\otimes\sigma_X\otimes\sigma_Z\otimes\sigma_Z\otimes\sigma_X \ , \\ \mathcal{S}^3 &= \sigma_X\otimes\mathbb{1}\otimes\sigma_X\otimes\sigma_Z\otimes\sigma_Z \quad , \quad \mathcal{S}^4 = \sigma_Z\otimes\sigma_X\otimes\mathbb{1}\otimes\sigma_X\otimes\sigma_Z \ , \end{aligned}$$

where σ_X and σ_Z denote the Pauli operators. Each of these operators $\mathcal{S}^j$ has eigenvalues $s^j\in\{-1,+1\}$. To each such outcome s^j of the measurement $\mathcal{S}^j$ corresponds a projection $P^j_{s^j}$ that specifies the probability

$$p^j_{s^j} = \mathrm{tr}[P^j_{s^j}\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)]$$

to obtain this outcome and the respective post-measurement state

$$\rho'^j_{L,s^j} = P^j_{s^j}\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)P^j_{s^j}/p^j_{s^j} \ .$$

A syndrome $s\equiv(s^1,s^2,s^3,s^4)$ is then the collection of the outcomes of all four measurements $\mathcal{S}^1,\ldots,\mathcal{S}^4$. For example, the syndrome $s=(-1,-1,-1,+1)$ means that all measurements, except the last, yielded the measurement outcome -1. Let us calculate the probability for this syndrome to occur. For this, we first note that all syndrome measurements commute with one another, which implies that also any projection $P^j_{s^j}$ of the measurement $\mathcal{S}^j$ commutes with any projection $P^k_{s^k}$ of the measurement $\mathcal{S}^k$, i.e. $[P^j_{s^j},P^k_{s^k}]=0$ for all j,k and all outcomes s^j,s^k. Hence, we find by

$$\mathrm{tr}[P^j_{s^j}\rho'] = \mathrm{tr}[P^j_{s^j}\sum_{s^k}P^k_{s^k}\rho P^k_{s^k}] = \mathrm{tr}[\sum_{s^k}P^k_{s^k}P^j_{s^j}\rho P^j_{s^j}] = \mathrm{tr}[P^j_{s^j}\rho]$$

that the probability to find outcome s^j in the syndrome measurement $\mathcal{S}^j$ is the same — independent of whether the system state is the initial state ρ or the average post-measurement state $\rho'=\sum_{s^k}P^k_{s^k}\rho P^k_{s^k}$ obtained by applying another syndrome measurement $\mathcal{S}^k$ before $\mathcal{S}^j$. Applying, in any order, the four syndrome measurements $\mathcal{S}^1,\ldots,\mathcal{S}^4$ successively on the noisy state $\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)$ will therefore yield the syndrome $s\equiv(s^1,s^2,s^3,s^4)$ with probability

$$p_s = \mathrm{tr}[P_s\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)] \ , \tag{5.15}$$

where we defined $P_s = P^1_{s^1}\, P^2_{s^2}\, P^3_{s^3}\, P^4_{s^4}$, together with the post-measurement state

$$\rho'_s = P_s \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)P_s/p_s\ , \tag{5.16}$$

The average post-measurement state is given by

$$\rho' = \sum_s P_s \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)P_s\ . \tag{5.17}$$

Let us explain in the following paragraphs how the information about the syndrome s can be employed to recover the original state $|\psi\rangle_L\langle\psi|$ from the noisy state $\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)$. For this, it is instructive to consider the *depolarising channel* due its simple structure. The depolarising channel $\mathcal{D}_\epsilon : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L)$ of error strength $\epsilon \in [0,1]$ is defined as

$$\mathcal{D}_\epsilon(\rho) = (1-\epsilon)\rho + \frac{\epsilon}{3}(\sigma_X\rho\sigma_X + \sigma_Y\rho\sigma_Y + \sigma_Z\rho\sigma_Z)\,.$$

The initial state ρ is therefore left unchanged with probability $1-\epsilon$; with probability $\epsilon/3$ however one of the Pauli operators is applied. The state $\mathcal{D}_\epsilon^{\otimes 5}(|\psi\rangle_L\langle\psi|)$ of the five physical qubits after applying the noise channel $\mathcal{D}_\epsilon^{\otimes 5}$ can therefore be written as [43]

$$\begin{aligned}\mathcal{D}_\epsilon^{\otimes 5}(|\psi\rangle_L\langle\psi|) = {} & (1-\epsilon)^5|\psi\rangle_L\langle\psi| \\ & + \sum_{j=1}^{5}(1-\epsilon)^4\frac{\epsilon}{3}\big(\sigma_X^j|\psi\rangle_L\langle\psi|\sigma_X^j + \sigma_Y^j|\psi\rangle_L\langle\psi|\sigma_Y^j + \sigma_Z^j|\psi\rangle_L\langle\psi|\sigma_Z^j\big) \\ & + \dots\,,\end{aligned} \tag{5.18}$$

where σ_k^j with $k \in \{X,Y,Z\}$ denotes "σ_k on the j-th tensor factor and identities everywhere else", e.g. $\sigma_X^4 = \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_X\otimes\mathbb{1}$. We refer to an operator σ_k^j as a *single qubit error*. The channel $\mathcal{D}_\epsilon^{\otimes 5}$ therefore leaves the initial state $|\psi_L\rangle$ unchanged with probability $(1-\epsilon)^5$ and applies a specific single qubit error with probability $(1-\epsilon)^4\frac{\epsilon}{3}$. The dots in Eq. (5.18) correspond to terms in which *multiple qubit errors* occur, i.e. for example operators such as $\sigma_Z\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_X\otimes\mathbb{1}$. This specific multiple qubit error occurs with probability $(1-\epsilon)^3(\frac{\epsilon}{3})^2$ as one may easily compute. From this we see that, for small error strength ϵ, the probability of multiple qubit errors is much smaller than the probability of a single qubit error.

Ideally, a syndrome s obtained through the syndrome measurements above allows to exactly specify the error that occured. For the five qubit code this is only true for single qubit errors [10, 11, 43], i.e. any single qubit error corresponds to a unique syndrome, whereas multiple qubit errors cannot be properly detected. The fact that the syndrome s is unique for each single qubit error allows us to simply invert the effect of such noise by a corresponding feedback protocol: Suppose for example that the syndrome is $s = (-1,-1,-1,+1)$, as above. There is only one single qubit error that leads to this syndrome, namely a Y error on the third qubit, $\mathbb{1}\otimes\mathbb{1}\otimes\sigma_Y\otimes\mathbb{1}\otimes\mathbb{1}$. Since all Pauli matrices square to the identity, we may simply apply the same unitary $\mathbb{1}\otimes\mathbb{1}\otimes\sigma_Y\otimes\mathbb{1}\otimes\mathbb{1}$ to return the system to its initial state prior to the noise. More concretely, let ρ' be the average post-measurement state of the five physical qubits after having applied the four syndrome measurements as in Eq. (5.17). Additionally, denote by V_s the (unitary) single qubit error corresponding to syndrome s, which is stored with correct probability p_s in a memory M. The feedback protocol then corresponds to applying the unitary

$$U_{\text{fb}} = \sum_s V_s \otimes Q_s$$

on the five qubits and the memory M, where Q_s is the projection to read out syndrome s from the memory. This unitary simply applies the correcting unitary V_s on the five physical qubits depending on the outcome s which yields the average "corrected" state

$$\rho'_{\text{fb}} = \sum_s V_s P_s \rho_S P_s V_s^\dagger \,.$$

on the physical qubits.

Summarising, the five qubit code works by encoding the state of a single qubit into five qubits, such that, if noise acts independently on each of these physical qubits, any single qubit error can be detected by syndrome measurements and then corrected by the feedback unitary U_{fb}. While we explained the functionality of the five qubit code only for the depolarising channel, it can be shown that the exact same protocol consisting of the syndrome measurements $\mathcal{S}^1,\ldots,\mathcal{S}^4$ followed by the feedback unitary U_{fb} is also suitable for correcting the effect of any other noise channel (at least to some extent), as long as each physical qubit is affected by the noise independently [43].

Let us now return to QEC based on the five qubit code under *amplitude damping noise*. Before computing the energy cost stemming from the measurements in this QEC protocol by invoking the results presented in Chapter 4, there are two important remarks to be made. First, in contrast to our first application concerning quantum Zeno stabilisation (see Section 5.1), there is only one appropriate formalism to realise the measurements: Essential to the theory of measurement-based QEC is that a syndrome is obtained. Hence, the measurement formalism must include a memory that stores the measurement outcomes, which rules out the state transformation and channel formalism. Also, we need to apply a feedback unitary on the measured system, hence we need the notion of a post-measurement state, which rules out the POVM formalism. Consequently, the measurements that are to be used with the five qubit code must be realised by (projective) quantum instruments.

Our second remark concerns the fact that, although we are set to work within the formalism of quantum instruments, there are still at least two ways to realise the four syndrome measurements. To see this, note that all four syndrome measurements commute and are, as a consequence, jointly measurable (see Section 4.5.3). More concretely, the joint quantum instrument $\mathcal{S}$ is given by the set $\{P_s \equiv P^1_{s^1}\, P^2_{s^2}\, P^3_{s^3}\, P^4_{s^4}\}_s$ of projections [43]. This joint instrument outputs syndrome s with the same probability p_s as in Eq. (5.15) together with post-measurement states ρ'_s as in Eq. (5.16).

Just as in Section 4.5.3, we may therefore either employ a *single* implementation of the joint quantum instrument $\mathcal{S}$ or use individual implementations for each of the four quantum instruments $\mathcal{S}^1$,..., $\mathcal{S}^4$. In Section 4.5.3 we called the former approach the *joint measurement scheme* and the latter approach the *separate measurement scheme*. In Corollary 4.21 we showed that the separate measurement scheme consumes at least as much energy as the joint measurement scheme.

Let us now exactly quantify the energy cost of the five qubit code under amplitude damping noise with strength γ for both the separate and the joint measurement scheme using the results presented in Chapter 4. For this, we compute the marginal probability distributions

$$p^j_{s^j} = \mathrm{tr}[P^j_{s^j}\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)]$$

for all four syndrome measurements to find that they are all identical, i.e. $p^1_{-1} = p^2_{-1} = p^3_{-1} = p^4_{-1} = \frac{1}{2}(1-r^3)$ and $p^1_{+1} = p^2_{+1} = p^3_{+1} = p^4_{+1} = \frac{1}{2}(1+r^3)$, where $r = 1-\gamma \in [0,1]$ corresponds to the probability that no error occurs. According to Theorem 4.19 and the discussion in Section 4.5.3, the separate measurement scheme then consumes

$$E^{\mathrm{sep}}_{\mathrm{cost}} = \Delta E + 4k_B T H\left(\left\{\frac{1}{2}(1-r^3), \frac{1}{2}(1+r^3)\right\}\right) \tag{5.19}$$

of energy, where $\Delta E = \mathrm{tr}[H_{\mathrm{phys}}(\rho' - \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|))]$ denotes the average energy change of the five physical qubits and H_{phys} denotes their joint Hamiltonian.

The joint measurement scheme, while more difficult to implement in a realistic setting, consumes less energy (see Corollary 4.21). Explicitly computing the Shannon entropy $H(\{p_s\}_s)$ of the joint probability distribution $\{p_s\}_s$ yields, by Theorem 4.19, the energy cost

$$\begin{aligned} E_{\text{cost}}^{\text{joint}} = \Delta E - 10 k_B T &\left[\frac{1+r^2+2r^3}{16} \log\left(\frac{1+r^2+2r^3}{16} \right) \right] \\ -5 k_B T &\left[\frac{(r-1)^2(2r+1)}{16} \log\left(\frac{(r-1)^2(2r+1)}{16} \right) \right] \\ -k_B T &\left[\frac{1+5r^2+10r^3}{16} \log\left(\frac{1+5r^2+10r^3}{16} \right) \right] \end{aligned} \tag{5.20}$$

with ΔE as before. These are the *exact* energy costs of QEC based on the five qubit code under amplitude damping noise.

Let us compare our above results Eqs. (5.19) and (5.20) to those statements on the energy requirements of the five qubit code that could have been obtained from general energy cost statements in previous literature. These existing results only provide lower bounds on the energy cost, in contrast to our exact results. More concretely, we consider the *Landauer bound* [30, 32, 38],

$$E_{\text{cost}}^{\text{joint}} \geq E_{\text{cost}}^{\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \mapsto \rho'_{\text{fb}}} = \Delta E + k_B T \left(S\left(\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \right) - S(\rho'_{\text{fb}}) \right) \equiv E_{\text{cost}}^{\text{Lan}} ,$$

which corresponds to the state transformation result from Theorem 4.2 evaluated for the state transformation $\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \mapsto \rho'_{\text{fb}}$. Note that this bound even includes the cost of the feedback unitary U_{fb} in addition to the cost $E_{\text{cost}}^{\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \mapsto \rho'}$ of the state transformation $\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \mapsto \rho'$. The latter cost corresponds to an even worse lower bound on the exact energy cost of the five qubit code, partly because the von Neumann entropy difference $S\left(\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \right) - S(\rho')$ appearing in $E_{\text{cost}}^{\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \mapsto \rho'}$ is always non-positive (see e.g. Theorem 11.9 of ref. [43]).

A better lower bound was provided by Sagawa and Ueda in ref. [64], which for the case at hand coincides with our general lower bound formulated in Theorem 4.14. These lower bounds agree here, since the syndrome measurement is an *efficient* quantum instrument (see Section 4.7 for a more detailed comparison between our general result and the statement by Sagawa and Ueda). Evaluated for the states $|\psi\rangle_L$ in the five qubit code, the bound reads

$$E_{\text{cost}}^{\text{joint}} \geq \Delta E + k_B T \left(S\left(\mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|) \right) - \sum_s p_s S(\rho'_s) \right) \equiv E_{\text{cost}}^{\text{SU}} .$$

Figure 5.1 shows these different energy costs as a function of the noise strength $\gamma \in [0, 1]$ [29]. For the sake of comparison, we choose the Hamiltonian of the physical qubits such that $\Delta E = 0$. This is legitimate since the contribution ΔE

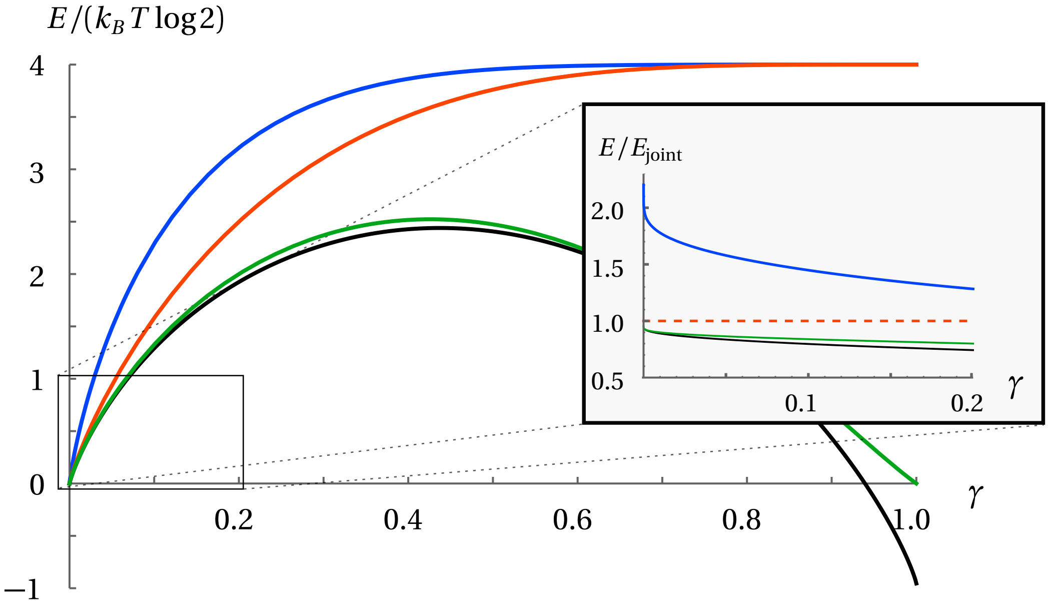

Figure 5.1: Energy cost of QEC with the five qubit code. The five qubit code can be realised using a separate (blue) or a joint (red) measurement scheme (see main text and also Section 4.5.3). For each scheme, the exact costs E_{sep} and E_{joint}, respectively, increase with the noise parameter γ. The energetically optimal joint measurement scheme consumes significantly less energy than the separate measurement scheme, which is typically more easily realised in experimental settings. Whereas the absolute difference between these costs is small for small noise scale γ, the relative difference $E_{\text{sep}}/E_{\text{joint}}$ (depicted in blue in the inset) increases for descreasing γ, showing that the separate measurement scheme can consume more than twice the energy cost of the joint measurement scheme. Applying the best previous estimates on the energy cost of projective quantum instruments to the five qubit code highlights the strength of our exact treatment of the energy cost: Both the lower bounds from Sagawa and Ueda $E^{\text{SU}}_{\text{cost}}$ (green) as well as the Landauer bound $E^{\text{Lan}}_{\text{cost}}$ (black) predict much weaker energy requirements than actually needed for the five qubit code (red curve) [29].

is common to all the above energy statements. Recalling the definition of heat dissipation from Definition 3.17, we may alternatively say that Figure 5.1 compares the (lower bounds on) thermodynamic heat dissipated in a single iteration of the QEC protocol. As expected, the energy cost of the separate measurement scheme exceeds the cost of the joint measurement scheme. The figure suggests that this effect is most prominent for $\gamma \approx 0.2$ — however, this is typically not the regime, QEC codes work in. In fact, to allow for so-called fault-tolerant quantum computation, the noise level γ must be below a certain threshold, which lies around the value of $\gamma \approx 0.05$ [6, 105]. To show the behaviour of the energy costs in this regime, we also depict the *relative* energy cost (with respect to E_{joint}) in the inset of Figure 5.1. This shows that in the sub-threshold regime, the cost E_{sep} can exceed E_{joint} by more than a factor of two.

Note that the results depicted in Figure 5.1 correspond to the heat dissipation of a *single* iteration of applying the syndrome measurement. To protect the quantum information over long time periods, this protocol is to be conducted multiple times, leading to a linear scaling of the corresponding heat dissipation with the number of iterations of the protocol. More drastically even, one may concatenate the encoding which maps one qubit into five qubits, to boost the performance of QEC protocols. That is, we may encode the logical qubit into five physical qubits, each of which is itself again encoded into five physical qubits, and so on and so forth. The QEC protocol would then work by applying, in a first step, syndrome measurements on all groups of five physical qubits within the lowest layer plus subsequent feedback. This is followed by syndrome measurements on all groups of five physical qubit within the next layer plus subsequent feedback and so on, until, in the last step, we perform syndrome measurements on the top layer qubits. Let us assume we use the energetically optimal, i.e. the joint, measurement scheme. Clearly, if the number of levels is l, then we need to perform $1+5+5^2+...+5^{l-1} = \frac{1}{4}(5^l-1)$ measurements. However, only the 5^{l-1} measurements on the groups of five qubits within the lowest (the l-th) layer are independent from each other — the outcome distributions from all other layers depend on the feedback done on the lowest layer. Hence, although these measurements will demand additional heat dissipation (recall that the dissipation is never negative for projective quantum instruments), we restrict our study only to the heat dissipation caused by first 5^{l-1} measurements within the lowest layer, which by our results above is exactly given as $5^{l-1}(E_{\text{joint}} - \Delta E)$. We conclude that such concatenated QEC protocols dissipate an amount of heat that scales *exponentially* with the number l of layers. Hence, even if the noise is only weakly affecting the information encoding qubits, i.e. $\gamma = 1-r$ is small, implying that the dissipated heat in a single iteration of the correction operation is correspondingly small, our results constitute that still a considerable amount of heat is dissipated for high performance QEC.

Let us finally comment on another intriguing feature of our results. As can be seen directly from Eqs. (5.19) and (5.20), the heat dissipation of the five qubit code is *independent* of the chosen initial state $|\psi\rangle_L$ *within* the code space. This is not to be expected if we just consider our general result, Theorem 4.19, on the energy cost of projective quantum instruments, where the probability distribution, and hence the heat dissipation, depends on the initial state. In Section 4.5.4 we showed how entropic uncertainty relations can be used to obtain non-trivial lower bounds on the heat dissipation, which were also state-independent. However, this method only worked for non-commuting projective quantum instruments and can hence not be applied here. The reason for the state-independence of our results here lies in the structure of the five qubit code and the noise channel. More concretely, the probability distribution $\left\{p_s^{\text{joint}}\right\}_s$ itself is already independent of the initial state $|\psi\rangle_L$. To see this, fix syndrome s and recall that the probability to obtain this syndrome is given by

$$p_s^{\text{joint}} = \operatorname{tr}\left[P_s \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)\right].$$

We compute

$$\begin{aligned} p_s^{\text{joint}} &= \operatorname{tr}\left[P_s \mathcal{N}_\gamma^{\otimes 5}(|\psi\rangle_L\langle\psi|)\right] \\ &= \operatorname{tr}\left[\left(\mathcal{N}_\gamma^{\otimes 5}\right)^*(P_s)|\psi\rangle_L\langle\psi|\right] \\ &= \langle\psi|_L\left(\left(\mathcal{N}_\gamma^{\otimes 5}\right)^*(P_s)\right)|\psi\rangle_L \qquad \forall|\psi\rangle_L \in \mathcal{H}_L\,, \end{aligned}$$

where we employed the definition of the adjoint map $(\mathcal{N}_\gamma^{\otimes 5})^*$. But this is equivalent to saying that the operator $P_L\left(\mathcal{N}_\gamma^{\otimes 5}\right)^*(P_s)P_L$ is proportional to the projector $P_L = |0\rangle_L\langle 0| + |1\rangle_L\langle 1|$ onto the code space. It is not difficult to see that the amplitude damping channel $\mathcal{N}$ and the syndrome measurement projections P_s satisfy this criterion for all syndromes s. This explains the state independence of the outcome probability distribution and of the energy cost of the five qubit code.

CHAPTER 6

Conclusion

In this thesis we determined the energy consumption of unselective measurements, of quantum channels, of quantum instruments, and of POVMs. On the one hand we both generalised and strengthened existing results in the literature. On the other hand, we obtained entirely new energy restrictions for quantum measurement which put drastic limitations on current and future quantum technology.

To determine the energy cost of quantum measurement, we developed a framework capable of describing all of the above four formalisms of quantum measurement (Chapter 3). This framework was based on *implementations*, a microscopic modelling of all the interactions of the quantum systems that are necessarily involved in the measurement process, namely the measured system S, the measurement device M itself and an environment E. Since the overall dynamics of all these systems is required to be unitary, we are guaranteed that we included *all* energetic expenses into our definition of the energy cost of an implementation.

Generally, there exist many implementations for any abstractly specified quantum measurement, and the energy cost depends on the implementation. We therefore determined the minimal energy cost over all possible implementations, thereby characterising the *fundamental* energy requirement for quantum measurement.

Most previous work was focused on the formalism of unselective measurements, where the measurement process is only characterised by the specification of the post-measurement state ρ'_S for one *fixed* input state ρ_S — the action of a device,

	lower bound on energy cost	tight?
unselective measurement:		
– general (Thm. 4.4)	$\Delta E_S + k_B T\big(S(\rho_S) - S(\rho'_S)\big)$	✓
quantum channel:		
– general (Cor. 4.5)	$\Delta E_S + k_B T\big(S(\rho_S) - S(\mathcal{F}(\rho_S))\big)$	✗
– erasing (Thm. 4.7)	$\Delta E_S + k_B T\big(S(\rho_S) - S(\mathcal{E}(\rho_S))\big)$	✓
– dephasing (Thm. 4.12)	$\Delta E_S\ (+0)$	✓
quantum instrument:		
– general (Thm. 4.14)	$\Delta E_S + k_B T\big(S(\rho_S) - \sum_k p_k S(\rho'_{S,k})\big)$	✗
– projective (Thm. 4.19)	$\Delta E_S + k_B T H\big(\{p_k\}\big)$	✓
POVM:		
– general (Thm. 4.23)	$\Delta F_{S,\mathrm{th}}\ (+0)$	✗
– PVM (Thm. 4.25)	$\Delta F_{S,\mathrm{th}} + k_B T\big(S\big(\sum_k P_k \rho_S P_k\big) - S(\rho_S)\big)$	✓

Table 6.1: Lower bounds on the energy cost for the four different measurement formalisms treated in this thesis. Some of these lower bounds are tight, i.e. they are identical to the *exact* energy cost of the respective measurements. Previous results from the existing literature appear in the red boxes, whereas the novel results from this thesis appear in the green boxes.

which realises this state transformation $\rho_S \mapsto \rho'_S$, on any input state other than ρ_S is *not* specified. This formalism of quantum measurement is quite general and allows one to formulate fundamental results such as Landauer's principle. Indeed, we re-derived this statement within our framework (Theorem 4.4).

However, unselective measurements do not properly capture realistic measurement devices which should conduct the desired measurement for *all* possible input states ρ_S. Imposing this implementation requirement is the main reason for the strength of many of our results.

In Table 6.1 we summarise our main general results from Chapter 4. The strongest result is our quantification of the exact cost $E_{\mathrm{cost}}^{\{P_k\}} = \Delta E_S + k_B T H\big(\{p_k\}\big)$ of a projective quantum instrument $\{P_k\}$ from Theorem 4.19, which gives the strictest energy requirements for quantum measurement known so far [29] — this is particularly interesting, since projective quantum instruments are excessively used in quantum research. We therefore explored several theoretical and practical consequences of this result: On the theoretical level we discussed the advantage of *quantum* treatments of the measurement process (Section 4.5.2), determined the effects of joint measurability on the energy cost (Section 4.5.3), and related the heat dissipation resulting from two incompatible projective quantum instruments to entropic uncertainty relations (Section 4.5.4) [29].

On the practical level we employed our result on the energy cost $E_{\text{cost}}^{\{P_k\}}$ of projective quantum instruments to determine the energy cost of quantum Zeno stabilisation (Section 5.1) and quantum error correction (Section 5.2). We found that control protocols on the basis of quantum Zeno stabilisation require an unbounded amount of energy in the limit of high target fidelity (Theorem 5.1). We computed, for the first time, the energy consumption of measurement-based quantum error correction and showed that a practical realisation scheme generally consumes significantly more energy than the optimal realisation (Figure 5.1) [29].

The results presented in this thesis suggest a host of interesting and relevant further questions on both the fundamental and the applied sides:

- **Exact costs for non-projective measurements.** We computed the exact energy costs for dephasing channels, projective quantum instruments and PVMs. All these formalisms share the property that their description is based on *projection* operators. This allowed us to rigorously determine the corresponding complementary channels in full generality, thereby employing the implementation requirement to its full extent. It is however not at all clear how this proof technique can be adapted to allow us to compute the energy cost of the general formalisms, i.e. for general quantum channels, quantum instruments and POVMs.

- **Finite-size effects.** In ref. [38] the authors considered corrections to the energy requirements for unselective measurement for the case where the Hilbert space dimension of the thermal bath E is upper bounded. In these cases, the energy cost is generally significantly larger than in the unrestricted case. In this thesis we did not consider such additional restrictions on the measurement process. It would be interesting to investigate such finite size effects for the other three measurement formalisms.

- **Measurements on infinite dimensional quantum systems.** All content in this thesis was formulated for finite dimensional Hilbert spaces and finite outcome sets. While it is known how the formalisms of quantum measurement translate to the infinite dimensional case, it is not at all clear how the energy requirements determined in this thesis need to be adapted. Such a generalisation of our framework is essential in order to treat measurements on bosonic systems such as the detection of photons in homodyne or heterodyne measurements common in quantum optics. We expect such a treatment to be possible using proof techniques similar to those developed in this thesis.

- **Single-shot approach.** All our results were developed in an "average" framework: The energy cost we determine corresponds to the average amount of energy needed to perform a quantum measurement. In contrast, it would be interesting to determine the exact energy requirements when a quantum measurement is performed only once.

- **Different output spaces.** This thesis concerned measurements which map states on the input space $\mathcal{H}_S$ to states on the *same* space $\mathcal{H}_S$. In particular, in our treatment of quantum instruments and POVMs, the output space was identical to the input space for all outcomes k. In a more general setting one could allow for *different output spaces* $\mathcal{H}_{S,k}$ for each outcome k.

All these questions focus on the *energy* cost of quantum *measurement*. However, it is natural to also consider resources other than energy which are required for quantum primitives other than measurement. As an example, it would be interesting to develop a general resource theory of *time* for quantum measurement, which answers the question of how much time is needed to perform a quantum measurement. Another example is the energy requirement for processes which need to operate correctly on a given set of initial states, which would interpolate between the state transformation concept, prominent in previous literature, and our strict implementation requirement where correct operations are required for *all* initial states.

We hope that the results developed in this thesis help to pursue future research in these directions.

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Glossary

S	quantum system that is being measured
M	memory system that stores the measurement outcome
E	thermal bath during measurement
B	thermal bath during resetting

$\mathcal{H}$	finite dimensional Hilbert space
$\mathcal{B}(\mathcal{H})$	linear operators on a Hilbert space $\mathcal{H}$
ρ_S	initial state of S
ρ'_S	average post-measurement state of S
$\rho'_{S,k}$	post-measurement state of S conditioned on k

$\{Q_{M,k}\}$	set of mutually orthogonal projections on M
U_{SME}	unitary on systems S, M, and E
$(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})$	implementation of a quantum instrument or a POVM
(ρ_E, U_{SE})	implementation of an unselective measurements or a quantum channel
$\mathcal{M}$	generic quantum measurement
$\mathcal{U}_{\mathcal{M}}$	set of implementations of $\mathcal{M}$
$\inf_{\mathcal{U}_{\mathcal{M}}}$	infimum over all implementations in $\mathcal{U}_{\mathcal{M}}$

$\rho_S \mapsto \rho'_S$	unselective measurement
$\mathcal{F}$	quantum channel
$\mathcal{E}$	erasing channel
$\mathcal{F}^c$	complementary quantum channel
$\{T_k\}$	quantum instrument
$I(k)$	Kraus rank of T_k
I	inefficiency of a quantum instrument
$\{M_{ki}\}$	quantum instrument
$\{E_k\}$	POVM
$\{P_k\}$	projective quantum instrument or PVM

H_S	Hamiltonian of S
T	temperature of a thermal environment
k_B	Boltzmann constant

ΔQ	heat dissipation
$E_{\text{res}}^{(\rho_B, U_{MB})}$	energy cost of an implementation of the resetting step
$E_{\text{cost}}^{(\rho_M, \{Q_{M,k}\}, \rho_E, U_{SME})}$	energy cost of an implementation of the measurement (with optimal resetting)
$E_{\text{cost}}^{\mathcal{M}}$	energy cost of the measurement $\mathcal{M}$
ΔE_S	average energy change on S
k	outcome
$\{p_k\}$	outcome probability distribution
$H(\{p_k\})$	Shannon entropy of $\{p_k\}$
$S(\rho)$	von Neumann entropy of state ρ
$I(X:X')_\rho$	mutual information between quantum systems X and X' in joint state ρ
$D(\rho\|\|\rho')$	relative entropy between quantum states ρ and ρ'
Z	partition function
$F(\rho)$	free energy of quantum state ρ
τ	Choi matrix
Ω	maximally entangled state
$\sigma_X, \sigma_Y, \sigma_Z$	Pauli matrices
$\mathbb{1}$	identity operator
$\otimes$	tensor product
$\oplus$	direct sum
tr	trace of an operator
tr_S	partial trace of an operator over quantum system S

Curriculum Vitae

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Academic career

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