Mirror symmetry for simple elliptic singularities with a group action

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktor der Naturwissenschaften
— Dr. rer. nat. —

genehmigte Dissertation

von
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geboren am 18. Mai 1985 in Moskau

2015

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Tag der Promotion: 01. Dezember 2014

to my mother

Kurzzusammenfassung

Diese Arbeit beschäftigt sich mit der Spiegelsymmetrie in der Singularitätentheorie. Dies beinhaltet so genannte Calabi-Yau/Landau-Ginzburg und Landau–Ginzburg/Landau–Ginzburg Spiegelisomorphismen. Diese Isomorphismen wurden für die einfach elliptischen Singularitäten von Milanov, Ruan, Krawitz und Shen erhalten. Es handelt sich um die Isomorphismen der Gromov–Witten–Theorie der Orbifolds $\mathbb{P}^1_{a_1,a_2,a_3}$ oder der FJRW–Theorie auf der A-Seite und Kyoji Saito's flache Strukturen einer einfach elliptischen Singularität auf der B-Seite. Die von Physikern vorgeschlagene Idee ist, dass die B-Seite in Bezug auf Symmetrien der Singularität "orbifolded" wird. Die Spiegelsymmetrie für das "orbifolded" B-Modell ist das Hauptthema dieser Arbeit. Aber die Objekte, die an der Spiegelsymmetrie teilnehmen, sind bis jetzt nicht vollständig definiert.

Wir stellen die Axiome für die Landau-Ginzburg-Modelle der A- und B-Seiten auf. Mit dieser Axiomatisierung bauen wir die Calabi-Yau/Landau-Ginzburg Spiegelsymmetrie für die Singularität $\tilde{E}_8$ mit der Symmetriegruppe $\mathbb{Z}_3$ auf. Wir berechnen auch das entsprechende B-Modell. Unserer Kenntnis nach ist dies das erste Beispiel für eine Landau-Ginzburg B-Modell, die "orbifolded " ist.

Als Ersatz für den Wechsel der primitiven Form stellen wir eine Aktion auf dem Raum der Frobenius–Mannigfaltigkeiten dar, die äquivalent zu der Transformation der primitiven Form bei einer einfach elliptischen Singularität ist. Durch die Anwendung dieser Aktion klassifizieren wir die Frobenius–Mannigfaltigkeiten, die die Axiome des früher eingeführen orbifolded A-Modells erfüllen. Mit Hilfe der Theorie der Modulformen zeigen wir, dass es nur eine solche Frobenius–Mannigfaltigkeit gibt, die die Landau-Ginzburg/Landau-Ginzburg Spiegelsymmetrie für die Singularität $\tilde{E}_8$ mit der Symmetriegruppe $\mathbb{Z}_3$ ergibt.

Als Nebenprodukt erhalten wir einige Ergebnisse zu der Gromov–Witten Theorie der Orbifolds $\mathbb{P}^1_{2,2,2,2}$ und $\mathbb{P}^1_{6,3,2}$.

Keywords: Spiegelsymmetrie, Frobenius–Mannigfaltigkeiten, primitive Formen, Gromov–Witten Theorie

Abstract

This thesis is devoted to the mirror symmetry in singularity theory. This involves the so-called Calabi-Yau/Landau-Ginzburg and Landau-Ginzburg/Landau-Ginzburg mirror symmetry isomorphisms. These isomorphisms were established for the simple elliptic singularities by Milanov, Ruan, Krawitz and Shen. This involve the isomorphisms of the Gromov–Witten theory of the orbifolds $\mathbb{P}^1_{a_1,a_2,a_3}$ or FJRW–theory on the A side and Kyoji Saito's flat structures of a simple elliptic singularity on the B side. However the idea originating from physics is that the B side could be "orbifolded" w.r.t. symmetries of the singularity. The mirror symmetry for the "orbifolded" B-model is the major topic of this thesis. However even the objects that take part in the mirror symmetry are not completely defined up to now.

We introduce the axiomatization for the Landau-Ginzburg models of A- and B-sides. Using it we establish the Calabi-Yau/Landau-Ginzburg mirror symmetry for the singularity $\tilde{E}_8$ with the symmetry group $\mathbb{Z}_3$. We also compute the corresponding B-model. This is the first example of an orbifolded Landau-Ginzburg B-model up to our knowledge.

As a substitute for the primitive form change we introduce an action on the space of Frobenius manifolds that is equivalent to the simple elliptic singularity primitive form change. Applying this action we classify those Frobenius manifolds that meet the requirements of the orbifolded A-model axiomatization introduced earlier. Using the theory of modular forms we show that there is only one such Frobenius manifold giving the Landau-Ginzburg/Landau-Ginzburg mirror symmetry for the singularity $\tilde{E}_8$ with the symmetry group $\mathbb{Z}_3$.

As a side-product we get several results on the Gromov–Witten theory of the orbifolds $\mathbb{P}^1_{2,2,2,2}$ and $\mathbb{P}^1_{6,3,2}$.

Keywords: Mirror symmetry, Frobenius manifolds, primitive forms, Gromov–Witten theory

Contents

CHAPTER 1

Introduction

Originating from physics the idea of Mirror symmetry has become nowadays a very beautiful subject in mathematics. Starting as a correspondence between two objects of the same kind, namely two Calabi-Yau threefolds (cf. [**29, 8**]), it is now generalized to the relation between the objects being of completely different origin and is even sometimes formulated in a very different form (cf. [**47, 23, 45**]).

Even so, an integral part of mirror symmetry in any form comes from singularity theory. Conceptually the mirror symmetry is an interchange of the A-model with the B-model. Following the idea of physicists (cf. [**8, 48**]) the B-model should be treated in a family over some base $\mathcal{S}$ with a mirror phenomena occuring at the "special points" $s \in \mathcal{S}$. Each special point should give a different "phase" of the $N = 2$ supersymmetric QFT. The B-model at such a point should be mirror dual to some A-model. Hence there could be several A-models of different kinds corresponding to one global B-model. We focus on the approach of Chiodo and Ruan [**10**] who introduce the rigorous program for the global mirror symmetry starting from the hypersurface singularity. In this case the global B-model is called Landau-Ginzburg B-model (cf. [**46**]). Physically interesting examples of a mirror symmetry appear when the Gromov–Witten theory is given by a variety that is Calabi-Yau. This type of mirror symmetry is called Calabi-Yau/Landau-Ginzburg (CY/LG for brevity) mirror symmetry.

In the case of simple elliptic singularities CY/LG mirror symmetry was established in [**42, 33, 34, 27**]. The other type of mirror symmetry that was proved in the same articles is the so-called Landau-Ginzburg/Landau-Ginzburg mirror symmetry. For the pair (W, G) consisting of the hypersurface singularity W and a symmetry group G the Landau-Ginzburg A-model was constructed in [**16**]. Such an A-model is nowadays called Fan-Jarvis-Ruan-Witten (or just FJRW) theory. In contrast with the B-model, the FJRW theory is not global and the state space has a completely different origin.

However, recent developments in physics suggest a more general understanding of the Landau-Ginzburg B-models, taking into account the symmetry group on the B side too (cf. [**20**]). Such B-models are called orbifolded B-models. Introducing the symmetry group on the B side affects also the A side in the way that we explain later in detail. The definition of an orbifolded LG B-model already appear to be a problem in this case. The work in this direction was done by R. Kaufmann in [**24**] from the physical and M. Krawitz in [**26**] from the mathematical points of view. However only the first steps towards the mirror symmetry were done. A rather different approach leading to the same predictions obtained by Kaufmann from the physical ideas was given by W. Ebeling and A. Takahashi in [**15**].

We address the problem of the global mirror symmetry for a orbifolded Landau-Ginzburg model in this thesis. In order to make precise the mirror correspondence we have to put all these models in some unifying framework. This is done by the Frobenius manifolds theory. We do not give full details of it referencing the classical textbooks [**11, 12, 19, 31**] original research paper of Saito [**39**] and modern treatment of it [**40**].

Frobenius manifolds. Let M be a domain in $\mathbb{C}^n$. Assume its tangent space $\mathcal{T}_M$ to be endowed with the constant non-degenerate bilinear form η,

$$\eta : \mathcal{T}_M \times \mathcal{T}_M \to \mathbb{C}.$$

Let $t_1, \ldots, t_n$ be coordinates on M. We associate the basis of $\mathcal{T}_M$ with the vectors $\partial/\partial t_i$ and consider η_{pq} as components of η in this basis. Viewed as the vector space it is called the *state space* of the Frobenius manifold.

Consider a complex-valued function $\mathcal{F} = \mathcal{F}(t_1, \ldots, t_n)$ on M. In what follows we assume $\mathcal{F}$ to be represented by a convergent power series in $t_1, \ldots, t_n$. The function $\mathcal{F}(\mathbf{t})$ is said to satisfy WDVV equation if for every fixed $1 \le i, j, k, l \le n$ we have:

$$\sum_{p,q} \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_p}\ \eta^{pq}\ \frac{\partial^3 \mathcal{F}}{\partial t_q \partial t_k \partial t_l} = \sum_{p,q} \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_k \partial t_p}\ \eta^{pq}\ \frac{\partial^3 \mathcal{F}}{\partial t_q \partial t_j \partial t_l}, \tag{1.1}$$

where $\eta^{pq} = (\eta^{-1})^{p,q}$.

Let $E \in \mathcal{T}_M$ be the vector field given in coordinates by:

$$E := \sum_{k=1}^{n} \left(d_k t_k \frac{\partial}{\partial t_k} + r_k \frac{\partial}{\partial t_k} \right), \quad d_k, r_k \in \mathbb{Q}.$$

Where we also assume that $r_k \neq 0$ only if $d_k = 0$. This vector field is called *Euler vector field.* We say that the function $\mathcal{F}$ has (conformal) degree $d \in \mathbb{Q}$ with respect to the Euler vector field E if the equality holds:

$$E \cdot \mathcal{F} = (3 - d)\mathcal{F} + \text{ quadratic terms.} \tag{1.2}$$

Using the function $\mathcal{F}$ define an algebra structure on $\mathcal{T}_M$. Let $c_{ij}^k(\mathbf{t})$ be the structure constants of the multiplication $\circ : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ defined by $c_{ij}^k(\mathbf{t}) := \sum_p c_{ijp}(\mathbf{t}) \eta^{pk}$, where

$$c_{ijk}(\mathbf{t}) := \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}, \quad 1 \le i, j, k \le n,$$

and $\eta^{ij} := \sum_{p,q} \eta_{pq} \delta^{pi} \delta^{qj}$. The structure constants $c_{ij}^k(\mathbf{t})$ define a commutative algebra structure by the construction, while the associativity is equivalent to the WDVV equation on $\mathcal{F}(\mathbf{t})$.

Assume in addition that $\mathcal{F}(\mathbf{t})$ is such that $\partial/\partial t_1$ is the unit of the algebra. Therefore we have:

$$\eta_{ij} = \frac{\partial^3 \mathcal{F}}{\partial t_1 \partial t_i \partial t_j}.$$

Then η_{ij} together with $c_{ij}^k(t)$ define the *Frobenius algebra* structure at any point $\mathbf{s} \in \mathcal{S}$:

$$\eta \left(\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right) = \eta \left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \circ \frac{\partial}{\partial t_k} \right).$$

Definition. The data η, $\circ$, E satisfying conditions as above define the rank n Frobenius manifold structure on M. The function $\mathcal{F}$ is called *Frobenius potential* of M. The number d is called *(conformal) dimension* and the coordinates $\mathbf{t}$ - *flat coordinates.*

Sometimes we are given first the function $\mathcal{F}$ satisfying WDVV equation without any underlying space M and holomorphicity property. In these ocasions $\mathcal{F}$ could anyway define a Frobenius manifold structure that we call *formal.*

NOTATION 1.1. *The algebra structure on $\mathcal{T}_M$ given by $c_{ij}^k\mid_{t=0}$ is called Frobenius algebra at the origin and denoted by $\mathcal{T}_M\mid_{t=0}$.*

Symmetries of the WDVV equation. Let $\mathcal{F}(\mathbf{t})$ be an arbitrary solution of the WDVV equation. It is clear that for $C \in \mathrm{GL}(n,\mathbb{C})$ and the change of variables $\tilde{\mathbf{t}} = C\mathbf{t}$ the function $\mathcal{F}(\tilde{\mathbf{t}})$ is solution to the WDVV equation too. Such changes of variables are called *symmetries* of the WDVV equation.

Consider two Frobenius manifolds whose potentials differ by such a *symmetry.* For $C \notin \mathrm{O}(n,\mathbb{C})$ the pairing defined by $\mathcal{F}(\tilde{\mathbf{t}})$ is different from that given by $\mathcal{F}(\mathbf{t})$ and the two Frobenius structures defined by them are different! In what follows we use the notion of isomorphism of two Frobenius manifolds from the point of view of the symmetries of the WDVV equation.

DEFINITION. Two Frobenius manifolds M and M' are said to be isomorphic if their potentials $\mathcal{F}(\mathbf{t})$ and $\mathcal{F}'(\tilde{\mathbf{t}})$ are connected by a linear change of variables $\tilde{\mathbf{t}} = C\mathbf{t}$ with $C \in \mathrm{O}(n,\mathbb{C})$:

$$\mathcal{F}(\mathbf{t}) = \mathcal{F}'(C\mathbf{t}).$$

Note that the WDVV equation has much more symmetries that just those given by a linear change of variables (cf. Appendix B in [**11**]).

Saito's flat structures. Let $W:\mathbb{C}^N \to \mathbb{C}$ be a holomorphic function defined on a neigborhood of $0 \in \mathbb{C}^N$. We further assume W to map the origin in $\mathbb{C}^N$ to the origin in $\mathbb{C}$ and define an isolated singularity at $0 \in \mathbb{C}$. This is equivalent to the fact that the germ of the hypersurface $X_0 := \{W = 0\} \subset (\mathbb{C}^N, 0)$ at the origin has at most an isolated singular point at the origin. Let $\mathbf{x} = (x_1,\dots,x_N)$ be coordinates on $\mathbb{C}^N$. Let $\mathbb{C}\{x\}$ stand for the ring of convergent power series. Consider the Milnor algebra of W

$$\mathcal{L}_W := \mathbb{C}\{x_1,\dots,x_N\}/(\partial_{x_1}W,\dots,\partial_{x_N}W).$$

Let $\mu := \dim \mathcal{L}_W$ be the Milnor number of $W(\mathbf{x})$. The fact that $W(\mathbf{x})$ defines an isolated singularity is equivalent to the finiteness of μ. A universal unfolding of $W(\mathbf{x})$ is the function $F:\mathbb{C}^N \times \mathbb{C}^\mu \to \mathbb{C}$ defined as follows:

$$F(\mathbf{x},\mathbf{s}) := W(\mathbf{x}) + \sum_{k=1}^{\mu} s_k\phi_k(\mathbf{x}),$$

where $\phi_i(\mathbf{x})$ are representatives of a basis of $\mathcal{L}_W$, and we assume also that $\phi_1(\mathbf{x})$ represents the unit of the Milnor algebra.

Let $W(\mathbf{x})$ be quasi-homogeneous. Namely there are $q_1,\dots,q_N \in \mathbb{Q}_{>0}$ such that:

$$W(\lambda^{q_1}x_1,\dots,\lambda^{q_N}x_N) = \lambda W(\mathbf{x}), \quad \forall\lambda \in \mathbb{C}^*.$$

Assign the weight q_i to the variable x_i and also the degree $\deg(\phi) := \sum a_i q_i$ to $\phi(\mathbf{x}) = x_1^{a_1}\dots x_N^{a_N}$. Assuming the unfolding $F(\mathbf{x},\mathbf{s})$ to satisfy the same quasi-homogeneity condition as $W(\mathbf{x})$ we assign the weights to the variables s_k:

$$\deg s_k := 1 - \deg\phi_k(\mathbf{x}).$$

Let $\mathcal{S} \subset \mathbb{C}^\mu$ and $B \subset \mathbb{C}^N$ be some sufficiently small balls centered at the origin. In particular we choose B such that $F(\mathbf{x},0)$ has only one critical point $\mathbf{x} = 0$ and choose $\mathcal{S}$ such that for any fixed $\mathbf{s} \in \mathcal{S}$ the function $F(\mathbf{x}, \mathbf{s}) : B \to \mathbb{C}$ has only isolated critical points. We will call $\mathcal{S}$ the *base space* of the singularity unfolding. Denote: $X := B \times \mathcal{S}$. In what follows we consider the unfolding F as a function germ at $(X, 0)$. Consider the map:

$$p : X \to \mathcal{S}, \quad (\mathbf{x}, \mathbf{s}) \to (\mathbf{s}).$$

Let $\mathcal{C}$ be the critical set of the unfolding F. Then $\mathcal{C}$ is the support of the sheaf:

$$\mathcal{O}_{\mathcal{C}} := \mathcal{O}_{X,0}/(\partial_{x_1} F, \dots, \partial_{x_N} F).$$

The sheaf $p_* \mathcal{O}_{\mathcal{C}}$ has a natural multiplication. For any two $\phi(\mathbf{x}, \mathbf{s}), \psi(\mathbf{x}, \mathbf{s})$ let $\phi, \psi \in p_* \mathcal{O}_{\mathcal{C}}$ be their residue classes modulo $(\partial_{x_1} F, \dots, \partial_{x_N} F)$. The multiplication of $p_* \mathcal{O}_{\mathcal{C}}$ reads:

$$\phi \circ_s \psi := \phi(\mathbf{x}, \mathbf{s}) \psi(\mathbf{x}, \mathbf{s}) \mod (\partial_{x_1} F, \dots, \partial_{x_N} F).$$

By the universality of the unfolding we have the isomorphism:

$$\mathcal{T}_{\mathcal{S}} \cong p_* \mathcal{O}_{\mathcal{C}}, \tag{1.3}$$

that endows $\mathcal{T}_{\mathcal{S}}$ with the multiplication structure depending on the point $s \in \mathcal{S}$. We will denote it by $\circ_s$ too.

Choosing a volume form $\omega = g(\mathbf{s}, \mathbf{x}) dx_1 \dots dx_N$ define the pairing η on $\mathcal{T}_{\mathcal{S}}$ as follows.

$$\eta_{kl}(\mathbf{s}) := \frac{1}{(2\pi\sqrt{-1})^N} \int_{\Gamma_\epsilon} \frac{\partial_{s_k} F \partial_{s_l} F}{\partial_{x_1} F \dots \partial_{x_N} F} \, \omega,$$

where Γ_ϵ is given by $|\partial_{x_1} F| = \dots = |\partial_{x_N} F| = \epsilon$ for small enough ϵ. This pairing is called the *residue pairing*.

THEOREM 1.1 (K. Saito). *For every quasi-homogeneous hypersurface singularity there is a volume form $\zeta(\boldsymbol{s}) d\boldsymbol{x}$ such that the residue pairing is flat.*

In general existence of a primitive form for a hypersurface singularity was proved by Morihico Saito in [**41**] (see also [**19**] for a subsequent explanation). Using Saito's theory of primitive forms it is possible to construct a Frobenius manifold structure on $\mathcal{S}$.

THEOREM 1.2 (Theorem 7.5 in [**40**]). *For any quasi-homogeneous isolated hypersurface singularity $W(\boldsymbol{x})$ and Saito's primitive form ζ of it the multiplication $\circ_t$ and residue pairing η define the structure of Frobenius manifold on the base space of the universal unfolding $\mathcal{S}$ (and hence of rank $\mu = \dim \mathcal{S}$).*

The Euler vector field E is defined by the equality:

$$(E(F)) \mid_{\mathcal{C}} = F \mid_{\mathcal{C}} .$$

Big advantage of the Frobenius manifold built on the base space of the singularity unfolding is that it could by defined at every point $\mathbf{s} \in \mathcal{S}$ be an appropriate choice of the primitive form "at the point s". Because of this fact Saito's Frobenius structure is called *global*. We will make this notion rigorous working with the simple elliptic singularities.

Orbifold GW theory. In [**1, 9**] the authors gave the treatment of Gromov–Witten theory (GW for brevity) for an orbifold $\mathcal{X}$. Roughly speaking their work allows us to mimic the "usual" GW theory to the case of an orbifold $\mathcal{X}$.

Fix $\beta \in H_2(\mathcal{X}, \mathbb{Z})$. The authors defined the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ of degree β stable orbifold maps from the genus g curve with n marked points to $\mathcal{X}$. Together with the suitable fundamental cycle $[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)]^{vir}$ one can introduce the correlators like in the usual GW theory.

The key object in the orbifold GW theory is the *inertia stack* of $\mathcal{X}$. Consider the diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and the fiber product w.r.t. to this map. Then the inertia stack is:

$$\mathcal{IX} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}.$$

In other words $\mathcal{IX}$ consists of the pair (x, σ), where $x \in \mathcal{X}$ and $\sigma \in \mathrm{Aut}(x)$. For the case of global quotient $\mathcal{X} = Y/G$ the inertia stack has a simple form:

$$\mathcal{IX} = \coprod_{(g)} Y^g/C(g),$$

where the summation is taken over the conjugacy classes (g) and $C(g)$ is the centralizer of g. The action of $C(g)$ factors through the action of $\langle g \rangle$ because the latter one acts trivially on Y^g. Hence consider:

$$\overline{C(g)} := C(g)/\langle g \rangle, \quad \text{and} \quad \overline{\mathcal{I}}\mathcal{X} := \coprod_{(g)} Y^g/\overline{C(g)}.$$

We will call $\overline{\mathcal{I}}\mathcal{X}$ the *rigidified inertia stack* of $\mathcal{X}$.

Definition. The orbifold cohomology of $\mathcal{X}$ is:

$$H^*_{orb}(\mathcal{X}) := H^*(\overline{\mathcal{I}}\mathcal{X}, \mathbb{Q}).$$

Define $ev_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) \to \overline{\mathcal{I}}\mathcal{X}$ – the map sending the stable orbifold map with n markings to its value at the i-th marked point.

Let $\gamma_i \in H^*_{orb}(\mathcal{X})$ be elements of the Chen-Ruan orbifold cohomology ring. The genus g n-point correlators of the orbifold GW theory of $\mathcal{X}$ are defined by:

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X}}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)]^{vir}} ev_1^*\gamma_1 \wedge \dots \wedge ev_n^*\gamma_n.$$

It is convenient to assemble the numbers obtained into a generating function called genus g potential of the (orbifold) GW theory. From now on assume that $\{\gamma_i\}$ is a basis of $H^*_{orb}(\mathcal{X}, \mathbb{Q})$. Take $\mathbf{t} := \sum_i \gamma_i t_i$ for the formal parameters t_i. The genus g potential reads:

$$\mathcal{F}^{\mathcal{X}}_g := \sum_{n,\beta} \frac{1}{n!} \langle \mathbf{t}, \dots, \mathbf{t} \rangle^{\mathcal{X}}_{g,n,\beta}.$$

The most important for us will be the genus zero potential. Due to the geometrical properties of the moduli space of curves we have the following proposition.

Proposition 1.3. *The genus* 0 *potential* $\mathcal{F}^{\mathcal{X}}_0$ *defines a formal Frobenius manifold structure of conformal dimension equal to* $\dim(\mathcal{X})$ *with the algebra at the origin isomorphic to* $H^*_{orb}(\mathcal{X})$.

Notation 1.2. *Denote by* $M^{GW}_{\mathcal{X}}$ *or simply by* $M_{\mathcal{X}}$ *the Frobenius manifold of the orbifold Gromov–Witten theory of* $\mathcal{X}$.

Berglund-Hübsch duality. Let W be a quasi-homogeneous polynomial in $x_1, \dots, x_N$. Associate to it the matrix $R = \{r_{ij}\}$ defined by the equality:

$$W(x_1, \dots, x_N) = \sum_{i=1}^{M} a_i \prod_{j=1}^{N} x_j^{r_{ij}}.$$

The matrix R allows one to define the dual polynomial W^T due to Berglund-Hübsch.

Definition. The quasi-homogeneous polynomial W as above defines an invertible singularity if the matrix R is square and invertible over $\mathbb{Q}$. The Berglund-Hübsch dual to W is the quasi-homogeneous polynomial $W^T(x_1, \dots, x_N)$ defined by:

$$W^T(x_1, \dots, x_M) := \sum_{i=1}^{N} a_i \prod_{j=1}^{N} x_j^{r_{ji}}.$$

We further assume that both W and W^T define isolated singularities at the origin $0 \in \mathbb{C}$.

The construction of Berglund-Hübsch is beautiful and essential but it doesn't guarantee good properties of the polynomial W^T.

Symmetries of the invertible singularity. In what follows we restrict ourselves to the quasi-homogeneous polynomials W defining invertible singularities only.

Definition.

- The maximal diagonal group of symmetries G_W is defined by:
$$G_W := \left\{ \alpha \in (\mathbb{C}^*)^N \mid W(\alpha \cdot \mathbf{x}) = W(\mathbf{x}) \right\},$$
where α acts coordinate-wise on $\mathbf{x}$.
- Define the exponential grading operator by:
$$J_W := (e^{2\pi i q_1}, \dots, e^{2\pi i q_N}).$$
Its cyclic group will be denoted by G_0:
$$G_0 := \langle J_W \rangle.$$

It is natural to consider the singularities W and W^T to be mirror to each other. However it is not enough. Correct mirror symmetry pairs are tuples (W, G) and (W^T, G^T), where G is a symmetry group of the singularity and G^T is in some way dual to G.

Depending on the side of the Mirror symmetry we will be working with two different types of the symmetry groups.

Definition. The group G is called A-admissible symmetry group of W if:

$$G_0 \subseteq G \subseteq G_W$$

and the group H is called B-admissible symmetry group of W if:

$$H \subseteq \mathrm{SL}_W := G_W \cap \mathrm{SL}(\mathbb{C}^N).$$

For an A-admissible group G introduce the notation:

$$\tilde{G} := G/G_0.$$

The definition of the dual group that agrees with the mirror symmetry assumption was first introduced by Berglund and Henningson in [**7**].

DEFINITION. For a subgroup $G \subset G_W$ define:

$$G^T := \mathrm{Hom}(G_W/G, \mathbb{C}^*).$$

EXAMPLE 1.4. For any invertible W we have: $(G_W)^T = \{id\}$.

More explicit definition of a dual group was given later by M. Krawitz [**26**]. The correctness of the definition is constituted by the following proposition.

PROPOSITION 1.5 (Lemma 3.3 in [**26**]). *Let W be invertible singularity and G - its A-admissible symmetry group. Then we have:*

$$(G^T)^T = G \quad and \quad G^T \subseteq G_{W^T} \cap \mathrm{SL}(\mathbb{C}^N).$$

Namely the dual group of an A-admissible symmetry group is a B-admissible symmetry group of the dual singularity.

The pair (W^T, G^T) is known nowadays under the name "Berglund-Hübsch-Krawitz dual" of (W, G).

DEFINITION. Let W and W' be invertible singularities and G, H - some A- and B- admissible groups of symmetries respectively. The pairs (W, G) and (W', H) are called *orbifolded* LG A- and B- models respectively.

Mirror symmetry with the trivial symmetry group. Let W define an invertible quasi-homogeneous singularity. Assume in addition that W satisfies the Calabi-Yau condition $\sum q_i = 1$. In this case the zero set of it is indeed a certain CY variety in some weighted projective space. Consider the GW theory of the orbifold X_{W,G_W} defined by:

$$X_{W,G_W} := \{W = 0\}/\tilde{G}_W.$$

The mirror symmetry conjectures read (cf. [**10**]).

CONJECTURE 1.1 (CY-LG mirror symmetry). *Up to a linear change of variables the Frobenius manifold potential of the GW theory of the orbifold $X_{W^T,G_{W^T}}$ coincides with the Frobenius manifold potential of W with the choice of the primitive form ζ at the special point.*

The choice of the primitive form ζ giving the CY-LG mirror isomorphism is called *primitive form at the large complex structure limit* or LCSL for brevity.

CONJECTURE 1.2 (LG-LG mirror symmetry). *There is a Frobenius manifold structure associated to the pair (W^T, G_{W^T}) such that up to a linear change of variables its potential $\mathcal{F}^A_{W^T,G_{W^T}}(\boldsymbol{t})$ coincides with the Frobenius manifold potential of W with the choice of the primitive form ζ at the special point.*

The choice of the primitive form ζ giving the LG-LG mirror isomorphism is called *primitive form at the Gepner point*.

Because Saito's flat structure appears in both theorems as the B-model while the A-model is in the two cases different, the B-model is called *global*. The next conjecture essentially supposes that two A-models of the same B-model are connected.

CONJECTURE 1.3 (CY/LG correspondence). *There is a group action $\hat{R}$ on the space of Frobenius manifolds relating two A-models:*

$$\hat{R} \cdot \mathcal{F}^A_{W^T,G_{W^T}} = \mathcal{F}^{GW}_{X_{W^T,G_{W^T}}}.$$

The action on the space of Frobenius manifolds (or more generally on the space of cohomological field theories) was developed by A. Givental in [**18**]. This is indeed some Givental's action $\hat{R}$ that has to be central in the CY-LG correspondence. We do not use this theory later on considering instead some group action on the space of Frobenius manifolds that arises from the analysis of the PDE's and is special for the class of Frobenius manifolds fixed.

Mirror symmetry with an arbitrary symmetry group. Let W be an invertible quasi-homogeneous singularity and G an A-admissible group of symmetries. Assume in addition that W satisfies the Calabi-Yau condition $\sum q_i = 1$. Consider the Gromov–Witten theory of the orbifold $X_{W,G}$ defined by:

$$X_{W,G} := \{W = 0\} / \tilde{G}.$$

Conjecture 1.4 (CY-LG mirror symmetry). *There is a family of Frobenius manifold structures $\mathcal{F}^B_{W,G}$ associated to the pair (W, G) with a B-admissible G such that up to a linear change of variables its potential taken at a "special phase" coincides with the Frobenius manifold potential of the GW theory of the orbifold X_{W^T,G^T}.*

Conjecture 1.5 (LG-LG mirror symmetry). *There is a Frobenius manifold structure $\mathcal{F}^A_{W^T,G^T}(t)$ with the A-admissible G^T associated with the pair (W^T, G^T) such that the Frobenius manifold structures $\mathcal{F}^B_{W,G}$ associated to the pair (W, G) taken at the "special phase" coincides with $\mathcal{F}^A_{W^T,G^T}(t)$ up to a linear change of variables.*

Conjecture 1.6 (CY/LG correspondence). *There is a group action $\hat{R}$ on the space of Frobenius manifolds relating two A-models:*

$$\hat{R} \cdot \mathcal{F}^A_{W^T,G^T} = \mathcal{F}^{GW}_{X_{W^T,G^T}}.$$

Essential candidate for LG A-model is the FJRW-theory. However it is not yet seen as the only possible candidate and another problem of it is that it is rather hard to compute even in the first essential examples.

We depict the global mirror symmetry in the following diagram.

A-side:	?FJRW theory of (W^T, G^T)	?	GW theory of $X_{W^T,G^T} := \{W^T = 0\}/\tilde{G}^T$
	0	p_k	∞
B-side:		Frobenius manifold of (W, G)	

Open problems. The major problem in the global mirror symmetry is that for a given singularity only the Gromov–Witten theory of $X_{W,G}$ is well-defined.

- The Frobenius manifold of the Landau-Ginzburg B-model of the pair (W, G) is not defined.
- There is no notion of the primitive form change for the Landau-Ginzburg B-model with non-trivial G.
- FJRW-theory is very hard to compute.

Summary of the results. The first achievement of this thesis is the *axiomatization* of the Frobenius manifold associated to the pair (W, G) that is given in Chapter 2. We use it later on in the thesis in order to establish the mirror symmetry with the non-trivial symmetry group for one particular example.

Let W_σ define a particular simple–elliptic singularity $\tilde{E}_8$:

$$W_\sigma(\mathbf{x}) := x_1^6 + x_2^3 + x_3^2 + \sigma x_1^4 x_2,$$

where σ is a complex parameter. For any $\xi \in \mathbb{C}\backslash\{1\}$ such that $\xi^3 = 1$ consider the symmetry group $G \cong \mathbb{Z}_3$ generated by:

$$h : (x_1, x_2, x_3) \to \left(\xi x_1, \xi^2 x_2, x_3\right).$$

The group $G = \langle h \rangle$ is a B-admissible symmetry group of W_σ for any $\sigma \in \mathbb{C}$. In the rest of the thesis we establish the CY-LG and LG-LG mirror symmetry theorems for the orbifolded LG B-model $(W_0, \mathbb{Z}_3)$.

On the way to the CY-LG mirror symmetry theorem we review in Chapter 3 Gromov–Witten potentials of the orbifolds $\mathbb{P}^1_{2,2,2,2}$ and $\mathbb{P}^1_{6,3,2}$. We prove the following proposition:

Proposition 1.6 (Proposition 3.13). *The genus* 0 *correlators of the Gromov–Witten theory of* $\mathbb{P}^1_{6,3,2}$ *having degree* 1 *and* 2 *insertions only are completely determined by the genus* 0 *correlators of the Gromov–Witten theory of* $\mathbb{P}^1_{2,2,2,2}$.

This is an interesting observation by its own building a connection between the Gromov–Witten potentials of the orbifolds that are connected by the stack-theoretic quotient.

The first main theorem of the thesis is given in Chapter 5. In particular we establish the CY-LG mirror symmetry for the pair $(\tilde{E}_8, \mathbb{Z}_3)$. According to our knowledge this is the first mirror symmetry theorem for the orbifolded Landau-Ginzburg models.

Theorem 1.7 (Theorems 5.1 and 5.2). *Consider the axiomatization of the orbifolded Landau-Ginzburg B-model given in Chapter 2. Then we have.*

- *There is a unique Frobenius manifold structure associated to the pair* $(\tilde{E}_8, \mathbb{Z}_3)$ *at the large complex structure limit.*
- *The genus* 0 *Gromov–Witten potential of the orbifold* $\mathbb{P}^1_{2,2,2,2}$ *coincides up to a linear change of variables with the Frobenius manifold structure associated to the pair* $(\tilde{E}_8, \mathbb{Z}_3)$ *taken at the large complex structure limit.*

We give explicitly the Frobenius manifold potential of the pair $(\tilde{E}_8, \mathbb{Z}_3)$ and also the change of variables connecting it to the genus 0 potential of $\mathbb{P}^1_{2,2,2,2}$. It is particularly interesting to note that this mirror pair could be read off the mirror theorem of W. Ebeling and A. Takahashi in [**15**] where the theory of orbifolded Gabrielov numbers of a singularity was developed.

The next theorem of the thesis establishes the connection between the Gromov–Witten theory of $\mathbb{P}^1_{2,2,2,2}$ and the Frobenius manifold structure on the space of a ramified covering. The latter one was given by B. Dubrovin in [**11**, Lecture 5].

Theorem 1.8 (Theorem 6.1). *The genus* 0 *Gromov–Witten potential of the orbifold* $\mathbb{P}^1_{2,2,2,2}$ *coincides up to a linear change of variables with the potential of the Frobenius submanifold of the space of degree* 8 *ramified coverings of the sphere by a torus having ramification profile* $(2,2,2,2)$ *over* $\infty \in \mathbb{P}^1$.

This theorem was published by the author in [**4**] and we repeat it in Chapter 6 for completeness.

Important step towards the *global* understanding of the orbifolded Landau–Ginzburg model is the notion of the primitive form change. We introduce in Chapter 7 the action $\mathcal{A}$ on the space of Frobenius manifolds that is equivalent to a primitive form change of a simple elliptic singularity. By the idea of the global mirror symmetry and also of the CY/LG, correspondence the Landau-Ginzburg A-model dual to the pair $(\tilde{E}_8, \mathbb{Z}_3)$ has to be in the orbit of $\mathbb{P}^1_{2,2,2,2}$ under this action.

Finally in Chapter 8 we establish the second main theorem of this thesis.

THEOREM 1.9 (Theorem 8.1). *There is a unique Frobenius manifold satisfying the axiomatization of the Frobenius manifold of the orbifolded LG A-model* $(\tilde{E}_8^T, \mathbb{Z}_3^T)$.

The last two chapters are based on the authors joint work with Atsushi Takahashi.

Acknowledgement. The author would like to thank his research advisor Wolfgang Ebeling for the constant support, attention to the project and numerous fruitful discussions.

The author is also indebted to Atsushi Takahashi who has proposed the way to attack some problems (and in particular to the CY-LG mirror symmetry conjecture) and the work done in the cooperation with whom became an important part of the thesis.

For the final edition of this thesis the author has strongly benefited from the reviewer comments of Claus Hertling and Nathan Priddis. Special thanks go to Fabien Clery for the help in modular forms and to Nicola Pagani for the help in Gromov–Witten theory.

The author is also grateful to all the members of the GRK "Analysis, Geometry, String theory" and all algebraic geometry institute members.

CHAPTER 2

Global mirror symmetry for simple elliptic singularities

The hypersurface simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ are defined by the following polynomials:

$$\begin{aligned} \tilde{E}_6: \quad & W_\sigma(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3 + \sigma x_1 x_2 x_3, \\ \tilde{E}_7: \quad & W_\sigma(\mathbf{x}) = x_1^4 + x_2^4 + x_3^2 + \sigma x_1^2 x_2^2, \\ \tilde{E}_8: \quad & W_\sigma(\mathbf{x}) = x_1^6 + x_2^3 + x_3^2 + \sigma x_1^4 x_2. \end{aligned} \tag{2.1}$$

They are of the form:

$$W_\sigma(\mathbf{x}) = W(x_1, x_2, x_3) + \sigma \phi_{-1}$$

where

- $W(x_1, x_2, x_3) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ is a quasi-homogeneous polynomial with the weights $q_i := 1/a_i$ satisfying $q_1 + q_2 + q_3 = 1$,
- $\phi_{-1} \in \mathcal{L}_W$ - the algebra element of degree 1,
- $\sigma \in \mathbb{C}$ - is a complex parameter.

We will call a simple elliptic singularity $W_\sigma(\mathbf{x})$ invertible if the polynomial $W(x_1, x_2, x_3)$ is invertible. Every simple elliptic singularity defines the elliptic curve E_σ called *elliptic curve at infinity* defined as follows.

$$E_\sigma : \{W_\sigma = 0\} \subset \mathbb{P}^2(c_1, c_2, c_3),$$

where $c_i = d/a_i$ and d is the least common multiple of the exponents a_1, a_2, a_3. Let $\Sigma \subset \mathbb{C}$ be such that E_σ is non-singular for $\sigma \in \Sigma$. Hence we get a flat family of elliptic curves over Σ. K. Saito gave in [**38**, Section 1.11] the formulae for the j-invariant of the elliptic curve E_σ:

$$\begin{aligned} \tilde{E}_6 : j(\sigma) &= -\frac{16\sigma^6}{\sigma^3 + 27}, \\ \tilde{E}_7 : j(\sigma) &= \frac{16(\sigma^2 + 12)^3}{(\sigma^2 - 4)^2}, \\ \tilde{E}_8 : j(\sigma) &= 1728 \frac{4\sigma^3}{4\sigma^3 + 27}. \end{aligned} \tag{2.2}$$

These formulae allow one to take $\tau = \tau(\sigma) \in \mathbb{H}$ to be the modulus of the elliptic curve E_σ such that we have:

$$j(\tau) = j(\sigma)$$

and consider the family of the elliptic curves parametrized by $\mathbb{H}$:

$$\mathcal{E} := \{\mathbb{C}^3 \times \mathbb{H} \mid W_\sigma(x_1, x_2, x_3) = 0\} \to \mathbb{H}. \tag{2.3}$$

1. Primitive form for the simple elliptic singularity

Let $\mathcal{S}$ be the base space of the universal unfolding $F(\mathbf{x}, \mathbf{s})$ of a simple elliptic W_σ. Because $\phi_{-1} \in \mathcal{L}_{W_\sigma}$, the parameter s_{-1} of the unfolding can be identified with σ and the base space turns out to be $\mathcal{S} = \Sigma \times \mathbb{C}^{\mu-1}$.

It was noticed already by K. Saito in [**39**] that every primitive form for a simple elliptic singularity is fixed by the data of E_σ (it was also explained in more details in [**33**]). In general the primitive form $\zeta = \zeta(\mathbf{s})$ depends on the point $\mathbf{s} \in \mathcal{S}$. It turns out directly from the axioms of the higher residue pairing and the axioms of a primitive form that for a simple elliptic singularity W_σ, out of all coordinates of $\mathbf{s}$, the primitive form $\zeta = \zeta(s_{-1})$ depends on s_{-1} only. In addition to this it satisfies the Picard-Fuchs equation of E_σ (where the identification $s_{-1} = \sigma$ is assumed). In what follows we denote the primitive forms of W_σ by $\omega = \omega(\sigma)$.

Consider the map:

$$\varphi : \mathbb{C}^3 \times \mathcal{S} \to \mathbb{C} \times \mathcal{S}, \quad \varphi(\mathbf{x}, \mathbf{s}) = (F(\mathbf{x}, \mathbf{s}), \mathbf{s}).$$

For any $\lambda \in \mathbb{C}$ and $\mathbf{s} \in \mathcal{S}$ consider $X_{\lambda,\mathbf{s}} := \varphi^{-1}(\lambda, \mathbf{s})$. Let $D \subset \mathbb{C} \times \mathcal{S}$ be the discriminant of F. Namely $D = \{(\lambda, \mathbf{s}) \mid X_{\lambda,\mathbf{s}} \text{ is singular}\}$. For $(\lambda, \mathbf{s}) \in (\mathbb{C} \times \mathcal{S})\backslash D$ the union X' of $X_{\lambda,\mathbf{s}}$ forms a smooth fibration that is called *Milnor fibration.*

The fibres $X_{\lambda,\mathbf{s}}$ are compactified by adding the elliptic curve E_σ. Consider the 2-form:

$$\Omega := \frac{dx_1 \wedge dx_2 \wedge dx_3}{dW_\sigma}.$$

It is holomorphic on $X_{\lambda,\mathbf{s}}$ but has simple poles along $E_\sigma \subset \overline{X}_{\lambda,\mathbf{s}}$. Then the form $\mathrm{res}_{E_\sigma}\Omega$ is a Calabi-Yau form of the elliptic curve E_σ. Moreover it has degree zero in $\mathbf{s}$ and hence depends on $s_{-1} = \sigma$ only (cf. [**30**]). Hence for any family $A_\sigma \in H_1(E_\sigma, \mathbb{Z})$:

$$\pi_A(\sigma) := \int_{A_\sigma} \mathrm{res}_{E_\sigma}\Omega$$

is solution to the Picard–Fuchs equation. Exact form of the Picard–Fuchs equation depends on W_σ. For example for $\tilde{E}_8$ it reads:

$$3\sigma(1-\sigma)\frac{d^2\pi_A}{d\sigma^2} + (2-5\sigma)\frac{d\pi_A}{d\sigma} - \frac{7}{48}\pi_A = 0.$$

THEOREM 2.1 (Chapter 3, Example 1 in [**39**]). *The primitive form for the simple elliptic singularity reads:*

$$\zeta = \zeta(\sigma) = \frac{d^3\boldsymbol{x}}{\pi_A(\sigma)},$$

for any $A_\sigma \in H_1(E_\sigma, \mathbb{C})$.

The proof of this theorem can be found in [**33**, Appendix A].

1.1. Special points. As it was noticed earlier the mirror phenomena should happen at the special points of the singularity unfolding. For the singularities $\tilde{E}_N$

these are $\{0,\infty\}\sqcup\{p_k\}$ given by:

$$\begin{aligned}
\tilde{E}_6: &\quad p_k &= -3\exp(\frac{2\pi\sqrt{-1}}{3}k), \quad 1\le k\le 3,\\
\tilde{E}_7: &\quad p_k &= 2\exp(\pi\sqrt{-1}k), \quad 1\le k\le 2,\\
\tilde{E}_8: &\quad p_k &= -\frac{3}{\sqrt[3]{4}}\exp(\frac{2\pi\sqrt{-1}}{3}k), \quad 1\le k\le 3.
\end{aligned}$$

It is not hard to see that the points p_k are exactly the points such that $j(p_k)=\infty$.

Proposition 2.2. *The j-invariant of the elliptic curve E_σ takes values $0, 1728, \infty$ at the special points.*

Proof. It is clear. $\square$

For $\tilde{E}_6$ and $\tilde{E}_7$ we have $j(\infty)=\infty$ however it is interesting to note that for $\tilde{E}_8$ we have: $\lim_{\sigma\to\infty} j(\sigma)=1728$.

Of course it is a subtle question to consider a primitive form, residue pairing $\eta_{ij}(\mathbf{s})$ and structure constants $c_{ij}^k(\mathbf{s})$ at the special point. We do not address this question here, but we will use explicitly computed flat coordinates by Milanov-Shen and Noumi-Yamada [**34, 36**].

2. Flat coordinates defined by the primitive form

The connection between the flat coordinates of a Frobenius manifold and the primitive form is established via the oscillatory integrals on the singularity side and deformed flat coordinates on the Frobenius manifolds side.

2.1. Deformed flat coordinates. Let $\mathcal{F}(t_1,\dots,t_n)$ be the potential of a Frobenius manifold M with the metric η and Euler field E. Let ∇ be the Levi–Civita connection of η. Consider the deformation of it:

$$\tilde{\nabla}_u v := \nabla_u v + z^{-1}\, u\circ v, \quad \forall u,v\in\mathcal{T}_M,$$

where z is a formal parameter. It could be extended to the connection on $M\times\mathbb{C}$ by setting:

$$\begin{aligned}
&\tilde{\nabla}_u\frac{d}{dz} := 0,\\
&\tilde{\nabla}_{\frac{d}{dz}}\frac{d}{dz} := 0,\\
&\tilde{\nabla}_{\frac{d}{dz}} v := z\partial_z v + E\circ v - \Theta(v),
\end{aligned}$$

for $\Theta(\partial/\partial t_i) = (1-d_i-d/2)\partial/\partial t_i$. The importance of this connection is approved by the following proposition due to Dubrovin.

Proposition 2.3 (Proposition 2.1 in [**12**]). *The curvature of the connection $\tilde{\nabla}$ defined on the Frobenius manifold M is equal to zero. Conversely if there is a Frobenius algebra structure on the tangent space of M with flat metric η, and Euler vector field E s.t.*

$$\mathcal{L}_E\eta = (2-d)\eta,$$

and the curvature of $\tilde{\nabla}$ vanishes, then M is a Frobenius manifold.

Flat coordinates of $\tilde{\nabla}$ could be chosen to be $(z, \tilde{t}_1(\mathbf{t}, z), \dots, \tilde{t}_n(\mathbf{t}, z))$. Then let ξ^k_α be defined by the equality $\sum_\alpha \xi^k_\alpha dt_\alpha = d\tilde{t}_k$. We have:

$$
\begin{aligned}
z\ \partial_\alpha \xi_\beta &= c^\gamma_{\alpha\beta}\xi_\gamma, \\
z\partial_z \xi_\beta &= E^\gamma c^\alpha_{\gamma\beta}\xi_\alpha - \Theta^\epsilon_\beta \xi_\epsilon.
\end{aligned} \tag{2.4}
$$

Define complex numbers $h_{\alpha,k}$ by the following equality:

$$\tilde{t}_\alpha = \sum_{k=0}^{\infty} h_{\alpha,k}(t)\ z^{-k}.$$

LEMMA 2.4 (Lemma 2.2 in [**12**]). *The coefficients $h_{\alpha,k}$ are determined recursively by the relations:*

$$
\begin{aligned}
h_{\alpha,0} &= \sum_\epsilon \eta_{\alpha,\epsilon} t_\epsilon, \\
\partial_\gamma \partial_\beta h_{\alpha,k+1} &= c^\epsilon_{\beta\gamma}\partial_\epsilon h_{\alpha,k}, \quad k \geq 0.
\end{aligned}
$$

where $1 \leq \alpha, \gamma, \beta \leq n$.

The functions $\tilde{t}_\alpha$ are called *deformed flat coordinates* of the Frobenius manifold. From the system of equations above the functions $\tilde{t}_\alpha$ are defined up to some matrix–valued function $G(z) = 1 + G_1 z^{-1} + G_2 z^{-2} + \dots$ not depending on t_ϵ. However different choices of this function $G(z)$ dont change the "geometry" of the Frobenius manifold and only give different *calibrations* of the same Frobenius manifold.

2.2. Oscillatory integrals. Let $F(\mathbf{x}, \mathbf{s})$ be the unfolding of the isolated quasi-homogeneous singularity $W(x_1, \dots, x_N)$ with the Milnor number μ. Fix some positive ρ, δ and ν. Let $B^N_\rho \subset \mathbb{C}^n$, $B^1_\delta \subset \mathbb{C}$ and $B^\mu_\nu \subset \mathbb{C}^\mu$ be respectively the balls of radii ρ, δ and ν centered at the origin. Consider the space:

$$\mathcal{X}_{\rho,\delta,\nu} := \left(B^N_\rho \times B^\mu_\nu\right) \cap \varphi^{-1}\left(B^1_\delta \times B^\mu_\nu\right) \subset \mathbb{C}^N \times \mathbb{C}^\mu.$$

Taking ρ such that $X_{0,0}$ is intersected transversally by ∂B^N_r for all $r:\ 0 < r \leq \rho$ and δ, ν such that $X_{\lambda,\mathbf{s}}$ is intersected transversally by ∂B^N_ρ for all $(\lambda, \mathbf{s}) \in B^1_\delta \times B^\mu_\nu$ we get the following proposition.

PROPOSITION 2.5 ([**2, 13**]). *There is $D \subset B^1_\delta \times B^\mu_\nu$ such that for $\mathcal{X}' := \mathcal{X}_{\rho,\delta,\nu} \backslash \varphi^{-1}(D)$ the map $\varphi : \mathcal{X}' \to (B^1_\delta \times B^\mu_\nu) \backslash D$ is a locally trivial fibration with a generic fibre homeomorphic to a bouquet of μ spheres of dimension $N-1$.*

For $m \in \mathbb{R}$ consider the space $\mathcal{X}^-_m \subset \mathcal{X}_{\rho,\delta,\nu}$ defined by:

$$\mathcal{X}^-_m := \{(\mathbf{x}, \mathbf{s}) \in \mathcal{X}' \mid \mathrm{Re}\left(F(\mathbf{x}, \mathbf{s})/z\right) \leq -m\}.$$

Because of the proposition above and exact sequence of the pair we get the following isomorphisms:

$$H_N(\mathcal{X}', \mathcal{X}^-_m) \cong H_{N-1}(X_{\lambda,\mathbf{s}}) \cong \mathbb{Z}^\mu. \tag{2.5}$$

Consider the cycles [1]:

$$\mathcal{A} \in \lim_{m\to\infty} H_N(\mathcal{X}', \mathcal{X}^-_m; \mathbb{C}) \cong \mathbb{C}^\mu.$$

[1] These cycles have first appeared in [**18**] and used later for example in [**33**] in the following form. For $(\mathbb{C}^N)_m := \{\mathbf{x} \in \mathbb{C}^N | \mathrm{Re}(F(\mathbf{s}, \mathbf{x})/z) \leq -m\}$. Define $\mathcal{A} \in \lim_{m\to\infty} H_N(\mathbb{C}^N, (\mathbb{C}^N)_{-m}; \mathbb{C}) \cong \mathbb{C}^\mu$. However such a definition should be considered more like a notation based on the fact that $\mathcal{X}'$ is contractible while the similarity of $\mathcal{X}^-_m$ and $(\mathbb{C}^N)_m$ is clear.

Introduce the oscillatory integrals:

$$\mathcal{J}_{\mathcal{A}}(\mathbf{s},z) := (-2\pi z)^{-N/2} z d_{\mathcal{S}} \int_{\mathcal{A}} e^{F(\mathbf{x},\mathbf{s})/z}\omega.$$

The outcome of Saito's theory of primitive forms is that in the flat coordinates $\mathbf{t}$ with the volume form given by the primitive form we have:

$$\begin{aligned} z\partial_t \mathcal{J}_{\mathcal{A}}(\mathbf{t},z) &= \partial_t \circ \mathcal{J}_{\mathcal{A}}(\mathbf{t},z), \\ (z\partial_z + E)\mathcal{J}_{\mathcal{A}}(\mathbf{t},z) &= \Theta \mathcal{J}_{\mathcal{A}}(\mathbf{t},z). \end{aligned} \tag{2.6}$$

where the 1-form $d_{\mathcal{S}} \in \Omega_{\mathcal{S}}$ is identified with the vector from $\mathcal{T}_{\mathcal{S}}$ by the pairing η, E is the Euler vector field and $\Theta : \mathcal{T}^*_{\mathcal{S}} \to \mathcal{T}^*_{\mathcal{S}}$ acts by: $\Theta(dt_i) = (\frac{1}{2} - d_i)dt_i$.

2.3. From the class A_σ to $\mathcal{A}$. We make the connection between the cycles $\mathcal{A}$ introduced above and $A_\sigma \in H_1(E_\sigma)$ that was used to define the primitive form for a simple elliptic singularity. Consider a tubular neighborhood of E_σ in $\overline{X}_{\lambda,\mathbf{s}}$. Its boundary in $\overline{X}_{\lambda,\mathbf{s}}$ defines an injective map $L : H_1(E_\sigma) \to H_2(X_{\lambda,\mathbf{s}})$. By the Cauchy-Leray theorem we have:

$$\pi_A(\sigma) = \int_{L(A)} \frac{d^3\mathbf{x}}{dF}.$$

By using the map L one can choose two of the generators of $H_2(X_{\lambda,\mathbf{s}})$ to be defined by the elliptic curve E_σ. In case of a hypersurface simple–elliptic singularity this means that two of the oscillatory integrals $\mathcal{J}_{\mathcal{A}_i}$ are defined by the elliptic curve E_σ (recall equation (2.5)). This leads to the proposition due to K. Saito.

Proposition 2.6 (Chapter 3, Example 1 in [**39**]). *For a simple elliptic singularity W_σ the degree zero flat coordinate t is fixed by the modulus of the elliptic curve E_σ:*

$$t = \frac{a\tau_0 + b}{c\tau_0 + d}, \quad a,b,c,d \in \mathbb{C}, ad - bc \neq 0,$$

and

$$\tau_0 = \frac{\int_{\gamma_1} \mathrm{res}_{E_\sigma}\Omega}{\int_{\gamma_2} \mathrm{res}_{E_\sigma}\Omega},$$

for γ_1, γ_2 - a free basis of $H_1(E_\sigma, \mathbb{Z})$.

Note that the cycles γ_1 and γ_2 do not necessarily give a symplectic basis. However this property is satisfied at the LCSL limit. It will appear explicitly in the next section.

3. Flat coordinates of $\tilde{E}_N$ at the LCSL

The flat coordinates at the LCSL (recall CY–LG mirror symmetry conjecture) for the simple elliptic singularities were given explicitly by Noumi-Yamada in [**36**].

For the simple elliptic singularity with the exponents a_1, a_2, a_3 consider $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{N}^3$ such that $0 \le \nu_i \le a_i - 2$. We will denote by I the (finite) set of all such ν. Then the unfolding of $\tilde{E}_N$ is given by:

$$F(\mathbf{x},\mathbf{s}) = W_{\tilde{E}_N}(\mathbf{x}) + \sum_{\nu \in I} s_\nu \phi_\nu(\mathbf{x}),$$

where

$$\phi_\nu(\mathbf{x}) := x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3},$$

and $W_{\tilde{E}_N}(\mathbf{x}) = W_\sigma(\mathbf{x}) \mid_{\sigma=0}$ for one of the polynomials from (2.1).

REMARK 2.1. At the start of this chapter we have defined the simple elliptic singularities $\tilde{E}_N$ via the polynomials depending also on the parameter σ. This dependence is not lost in this section because of the identification of σ with one of the parameters of the unfolding. In the formula above this is s_ν with $\nu = (a_1 - 2, a_2 - 2, a_3 - 2)$.

Let ϵ_i be the canonical basis in the lattice $\mathbb{Z}^3$. For $\alpha \in \mathbb{N}^I$ introduce the functions:

$$l(\alpha) := \sum_{\nu \in I} \alpha_\nu \nu \in \mathbb{N}^3, \quad \mathrm{wt}(\alpha) := \sum_{\nu \in I} \alpha_\nu \deg s_\nu \in \mathbb{Q}, \quad \deg s_\nu = 1 - \sum_{i=1}^3 \frac{\nu_i}{a_i}.$$

For any $\nu \in I$ consider the holomorphic functions on $\mathcal{S}$:

$$\psi_\nu^{(r)}(\mathbf{s}) := \sum_{\substack{\alpha \in \mathbb{N}^I \\ \mathrm{wt}(\alpha) = r - 1 + \deg s_\nu}} c_\nu\left(l(\alpha)\right) \prod_{\xi \in I} \frac{s_\xi^{\alpha_\xi}}{\alpha_\xi !}$$

where for fixed $\nu \in I$ and any $\mu \in (\nu + \sum_{i=1}^3 a_i \epsilon_i \mathbb{Z}) \cap \mathbb{N}^3$

$$c_\nu(\mu) := \prod_{i=1}^3 (-1)^{k_i} \frac{\Gamma\left(k_i + \frac{\nu_i + 1}{a_i}\right)}{\Gamma\left(\frac{\nu_i + 1}{a_i}\right)}, \quad k_i = \frac{\mu_i - \nu_i}{a_i}.$$

THEOREM 2.7 (Theorem 1.1 in [**36**]). *The flat coordinates of the singularity $\tilde{E}_N$ with the unfolding $F(\boldsymbol{x}, \boldsymbol{s})$ are:*

$$t_\nu = \frac{\psi_\nu^{(1)}(\boldsymbol{s})}{\psi_0^0(\boldsymbol{s})},$$

where the functions $\psi_0^0(\boldsymbol{s})$ are:

$$\tilde{E}_6 : \quad \psi_0^0(\boldsymbol{s}) = {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; -\frac{s_{111}^3}{27}\right),$$
$$\tilde{E}_7 : \quad \psi_0^0(\boldsymbol{s}) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \frac{s_{22}^2}{4}\right),$$
$$\tilde{E}_8 : \quad \psi_0^0(\boldsymbol{s}) = {}_2F_1\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; -\frac{4s_{41}^3}{27}\right).$$

PROOF. We give the idea of the proof for the reader's convenience.

Let $[\Gamma] \in H_3(\mathcal{X}', \mathcal{X}^-)$. Consider the level hypersurfaces $F/z = \lambda$. We can write the integrals $\mathcal{J}_\mathcal{A}(\mathbf{s}, z)$ using the Gelfand-Leray form.

$$\int_\Gamma e^{F/z} \omega = z \int_{-\infty}^0 e^\lambda \left(\int_{\partial_\lambda \Gamma} e^{\sum s_j \phi_j z^{d_j}} \frac{\omega}{df} \right) d\lambda.$$

Note that we used the quasi-homogeneity of F in order to change the dependence of the integral on z.

Expanding the exponent into a power series we can integrate out λ using the Gamma-functions. By using Lemma 2.4 we get the expression of the flat coordinates. $\square$

The flat coordinates of the Frobenius manifold $M_{\tilde{E}_N}$ are indexed by the same set I. The only degree zero variable is t_{0^*}, where for any $\nu \in I$ the index $\nu^* \in I$ is defined by $\eta_{\nu,\nu^*} = 1$.

In particular for $\tilde{E}_8$ the variables are indexed by $I = \{0, 10, 20, 30, 40, 11, 21, 31, 41\}$. Because $\phi_{41} = x_1^4 x_2$ the parameter σ is identified with the variable s_{41}. We also write $s_{-1} := s_{41}$.

The formulae of Noumi-Yamada give for $\tilde{E}_8$:

$$t_{-1} := t_{41} = s_{41} \frac{{}_2F_1\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; u\right)}{{}_2F_1\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; u\right)}, \quad u = -\frac{4s_{41}^3}{27}.$$

4. CY-LG and LG-LG with the trivial symmetry group

The Mirror conjectures with the symmetry group choice $G = \{id\}$ were proved to be true for the case of invertible simple elliptic singularities by Satake-Takahashi in [**42**] and Krawitz-Milanov-Ruan-Shen in [**27, 33, 34**] (except some special cases). In this case the dual group is the maximal group of symmetries $G^T = G_W$. The role of the LG A-model in this case was proved to be satisfied by the FJRW theory.

Theorem 2.8 (Theorem 3.6 in [**42**], Theorem 6.6 in [**33**] and Theorem 1.5 in [**34**]). *For the simple elliptic singularities $\tilde{E}_N$ there are isomorphisms:*

$$M_{\tilde{E}_6} \cong M^{GW}_{\mathbb{P}^1_{3,3,3}}, \quad M_{\tilde{E}_7} \cong M^{GW}_{\mathbb{P}^1_{4,4,2}}, \quad M_{\tilde{E}_8} \cong M^{GW}_{\mathbb{P}^1_{6,3,2}},$$

that constitute CY-LG mirror symmetry. The LG-LG mirror symmetry holds by the following isomorphisms:

$$M_{\tilde{E}_N} \cong M^{\mathrm{FJRW}}_{\tilde{E}_N, G_{max}} \quad N = 6, 7, 8.$$

Crucial part of the mirror isomorphisms is the choice of the primitive form for the singularity. We do not write here these details referring interested reader to the original papers.

This theorem was proved in the more general case of W_σ in [**34**] but however not in the full generality anyway.

4.1. Mirror symmetry for $\tilde{E}_8$ with the trivial group of symmetries.

We give here explicitly the CY-LG isomorphism for $\tilde{E}_8$ that was obtained in [**34**].

Theorem 2.9 (Theorem 1.5 in [**34**]). *Let $M_{\tilde{E}_8}$ be the Frobenius manifold structure associated to the unfolding of $\tilde{E}_8$ and $M_{\mathbb{P}^1_{6,3,2}}$ - the (orbifold) GW theory of $\mathbb{P}^1_{6,3,2}$. Then the following mirror isomorphism holds:*

$$M_{\mathbb{P}^1_{6,3,2}} \cong M_{\tilde{E}_8},$$

where the Frobenius structure of the RHS is given by the primitive form ζ_{LCSL} at the point $\sigma = \frac{3}{2}(-2)^{1/3}$ fixed by the period π_A (recall Theorem 2.1):

$$\pi_A(s_{-1}) = {}_2F_1\left(\frac{1}{12}, \frac{7}{12}; 1; 1 + \frac{4}{27} s_{-1}^3\right).$$

The degree 0 flat coordinates are identified by:

$$t_{-1} = 2\pi\sqrt{-1}t_{41}/6.$$

In [**34**] the authors give explicit formulae for this mirror isomorphism matching the generators of the Frobenius algebras. We will use these formulae later on and therefore write them down too.

Denote by Δ_{ij} the generators of the Chen-Ruan cohomology of $\mathbb{P}^1_{6,3,2}$. These are:

$$\Delta_0 = \{pt\},\ \Delta_{-1} = P,\ \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{21}, \Delta_{22}, \Delta_{31},$$

where P stands for the hyperplane class. These generators will appear more explicitly in the next section.

Recall the notation of Noumi-Yamada: $u := -\frac{4}{27}s_{-1}^3$. Define the functions $F^{(1)}_{1,\mathbf{r}}(u)$, $F^{(1)}_{2,\mathbf{r}}(u)$ for $\mathbf{r} \in I$ (see previous section) as follows.

$$\begin{cases} F^{(1)}_{1,\mathbf{r}}(u) = \ _2F_1\left(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}; \alpha_{\mathbf{r}} + \beta_{\mathbf{r}} - \gamma_{\mathbf{r}} + 1; 1-u\right), \\ F^{(1)}_{2,\mathbf{r}}(u) = \ _2F_1\left(\gamma_{\mathbf{r}} - \alpha_{\mathbf{r}}, \gamma_{\mathbf{r}} - \beta_{\mathbf{r}}; \gamma_{\mathbf{r}} - \alpha_{\mathbf{r}} - \beta_{\mathbf{r}} + 1; 1-u\right)\ (1-u)^{\gamma_r - \alpha_r - \beta_r}. \end{cases}$$

where the weights $\alpha_{\mathbf{r}}$, $\beta_{\mathbf{r}}$ and $\gamma_{\mathbf{r}}$ are given in Table 1.

Table 1. Weights of periods for $\tilde{E}_8$

$\phi_{\mathbf{r}}$	x_1	x_2	x_1^2	x_1x_2	x_1^3	$x_1^2x_2$	x_1^4	$x_1^3x_2$
$\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}}$	$\frac{1}{6}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{1}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{2}{3}$	$\frac{1}{6}, \frac{2}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{5}{6}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{3}$	$\frac{5}{12}, \frac{11}{12}, \frac{2}{3}$	$\frac{1}{3}, \frac{5}{6}, \frac{1}{3}$

The first part of the mirror isomorphism of Milanov-Shen is given by:

$$\Delta_0 \mapsto 1, \quad \Delta_{-1} \mapsto 36(1-u)\phi_{41}\pi_A^2,$$

where π_A is as in the theorem above. The remaining generators are identified by:

$$\begin{aligned} &\Delta_{11} \mapsto (1-u)^{1/6}\,\phi_{10}\,\pi_A, \\ &\Delta_{15} \mapsto (1-u)^{5/6}\,\phi_{31}\,\pi_A, \\ &\Delta_{21} \mapsto (1-u)^{1/3}\Big(F^{(1)}_{2,20}(u)\,\phi_{01} + (-2)^{-1/3}\ F^{(1)}_{2,01}(u)\,\phi_{20}\Big)\,\pi_A, \\ &\Delta_{12} \mapsto (1-u)^{1/3}\Big(F^{(1)}_{1,20}(u)\,\phi_{01} - 3(-2)^{-1/3}\ F^{(1)}_{1,01}(u)\,\phi_{20}\Big)\,\pi_A, \\ &\Delta_{31} \mapsto (1-u)^{1/2}\Big(F^{(1)}_{2,30}(u)\,\phi_{11} + (-2)^{-1/3}\ F^{(1)}_{2,11}(u)\,\phi_{30}\Big)\,\pi_A, \\ &\Delta_{13} \mapsto (1-u)^{1/2}\Big(F^{(1)}_{1,20}(u)\,\phi_{11} - 2(-2)^{-1/3}\ F^{(1)}_{1,11}(u)\,\phi_{30}\Big)\,\pi_A, \\ &\Delta_{22} \mapsto (1-u)^{2/3}\Big(F^{(1)}_{2,40}(u)\,\phi_{21} + (-2)^{-1/3}\ F^{(1)}_{2,21}(u)\,\phi_{40}\Big)\,\pi_A, \\ &\Delta_{14} \mapsto (1-u)^{2/3}\Big(F^{(1)}_{1,40}(u)\,\phi_{21} - \frac{5}{3}(-2)^{-1/3}\ F^{(1)}_{1,21}(u)\,\phi_{40}\Big)\,\pi_A. \end{aligned} \tag{2.7}$$

In the approach of Milanov, Ruan, Shen and Krawitz one fixes the primitive form by giving particular solution $\pi(\sigma)$ to the Picard-Fuchs equation. Such a solution defines particular family of cycles A_σ. Going this way it is possible to get some advantage from the analysis of the Picard-Fuchs equation, however it is a hard problem to recover the cycles A_σ.

CHAPTER 3

GW theory of elliptic orbifolds

The varieties that appear in this thesis on the A-side of the Mirror symmetry are so-called *elliptic* orbifolds. These are $\mathbb{P}^1$ with a finite set of, say k, isotropic points with the orbifold structure $(\mathbb{Z}/\mathbb{Z}_{a_1}, \dots, \mathbb{Z}/\mathbb{Z}_{a_k})$ where all $a_i \geq 2$. The term elliptic refers to the fact that these orbifolds can be realized as the global quotients of elliptic curves. This puts the constraint on the numbers a_i: $\sum_i 1/a_i = 2$. Therefore there are only 4 elliptic orbifolds: $\mathbb{P}^1_{2,2,2,2}$, $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$.

In order to establish the LG-CY mirror symmetry it is important to know the genus 0 potential of the elliptic orbifolds. However this can be done only for $\mathbb{P}^1_{2,2,2,2}$ and $\mathbb{P}^1_{3,3,3}$, what was found by Satake and Takahashi in [**42**].

In this chapter we review their result rewriting it in the way it will be needed later and also give closed formula for certain restriction of the genus 0 GW potential of $\mathbb{P}^1_{6,3,2}$.

1. GW theory of $\mathbb{P}^1_{2,2,2,2}$

The Frobenius manifold potential $M_{\mathbb{P}^1_{2,2,2,2}}$ was found explicitly in [**42**]. The rigidified inertia orbifold reads:

$$\overline{\mathcal{I}}\mathbb{P}^1_{2,2,2,2} = \mathbb{P}^1_{2,2,2,2} \coprod B(\mathbb{Z}_2) \coprod B(\mathbb{Z}_2) \coprod B(\mathbb{Z}_2) \coprod B(\mathbb{Z}_2).$$

Let $\Delta_0, \dots, \Delta_5$ be the basis of $H^*_{orb}(\mathbb{P}^1_{2,2,2,2})$ such that:

$$H^0_{orb}(\mathbb{P}^1_{2,2,2,2}) \simeq \mathbb{Q}\Delta_0,\ H^1_{orb}(\mathbb{P}^1_{2,2,2,2}) \simeq \bigoplus_{i=1}^{4} \mathbb{Q}\Delta_i,\ H^2_{orb}(\mathbb{P}^1_{2,2,2,2}) \simeq \mathbb{Q}\Delta_5.$$

The pairing is given by:

$$\eta(\Delta_0, \Delta_5) = 1, \quad \eta(\Delta_i, \Delta_j) = \frac{1}{2}\delta_{i,j},\ 1 \leq i, j \leq 4.$$

The genus zero potential can be written explicitly.

$$\begin{aligned}\mathcal{F}^{\mathbb{P}^1_{2,2,2,2}}(t_0, t_1, t_2, t_3, t_4, t_{-1}) &= \frac{1}{2}t_0^2 t_{-1} + \frac{1}{4}t_0(\sum_{i=1}^{4} t_i^2) + (t_1 t_2 t_3 t_4) f_0(q) \\ &+ \frac{1}{4}\left(\sum_{i=1}^{4} t_i^4\right) f_1(q) + \frac{1}{6}\left(\sum_{i<j} t_i^2 t_j^2\right) f_2(q), \quad q := \exp(t_{-1}).\end{aligned}$$

for the certain functions f_i depending on $\exp(t_{-1})$ only. The quasihomogeneity is fixed by the Euler field E:

$$E := t_0 \frac{\partial}{\partial t_0} + \sum_{i=1}^{4} t_i \frac{1}{2}\frac{\partial}{\partial t_i}, \quad E \cdot \mathcal{F}^{\mathbb{P}^1_{2,2,2,2}} = 2\mathcal{F}^{\mathbb{P}^1_{2,2,2,2}}.$$

The WDVV equation on $\mathcal{F}^{\mathbb{P}^1_{2,2,2,2}}$ is equivalent to the following system of ODE:

$$
\begin{aligned}
&q\frac{d}{dq}f_0(q) = f_0(q)\left(\frac{8}{3}f_2(q) - 24f_1(q)\right),\\
&q\frac{d}{dq}f_1(q) = -\frac{2}{3}f_0(q)^2 - \frac{16}{3}f_1(q)f_2(q) + \frac{8}{9}f_2(q)^2,\\
&q\frac{d}{dq}f_2(q) = 6f_0(q)^2 - \frac{8}{3}f_2(q)^2.
\end{aligned}
\tag{3.1}
$$

The following observation is immediate:

Proposition 3.1. *Let $(f_0(q), f_1(q), f_2(q))$ be represented by their Taylor series in the neighborhood of $q = 0$ and $f_0(0) = 0$. Then these Taylor series are completely determined by the numbers $(f_0(0), f_1(0), f_2(0))$.*

Proof. Introduce the notation

$$f_a(q) = \sum_{k\geq 0} c_k^{(a)} \frac{q^k}{k!}.$$

Lemma 3.2. *Let $f_0(q)$, $f_1(q)$, $f_2(q)$ satisfy (3.1). For any $n \geq 1$ introduce the notation:*

$$
\begin{aligned}
K_n^{(0)} &:= \sum_{p+q=n, p\neq n, q\neq n} \left(\frac{8}{3}c_p^{(0)}c_q^{(2)} - 24c_p^{(0)}c_q^{(1)}\right),\\
K_n^{(1)} &:= \sum_{p+q=n, p\neq n, q\neq n} \left(-\frac{2}{3}c_p^{(0)}c_q^{(0)} - \frac{16}{3}c_p^{(1)}c_q^{(2)} + \frac{8}{9}c_p^{(2)}c_q^{(2)}\right),\\
K_n^{(2)} &:= \sum_{p+q=n, p\neq n, q\neq n} \left(6c_p^{(0)}c_q^{(0)} - \frac{8}{3}c_p^{(2)}c_q^{(2)}\right).
\end{aligned}
$$

Then the coefficients of the functions $f_a(q)$ satisfy the following system of equations:

$$
\begin{cases}
nc_n^{(0)} = K_n^{(0)} + \frac{8}{3}(c_n^{(0)}c_0^{(2)} + c_0^{(0)}c_n^{(2)}) - 24(c_0^{(0)}c_n^{(1)} + c_n^{(0)}c_0^{(1)}),\\
nc_n^{(1)} = K_n^{(1)} - \frac{2}{3}c_0^{(0)}c_n^{(0)} - \frac{16}{3}(c_n^{(1)}c_0^{(2)} + c_0^{(1)}c_n^{(2)}) + \frac{8}{9}c_n^{(2)}c_0^{(2)}, \qquad n \geq 1.\\
nc_n^{(2)} = K_n^{(2)} + 6c_n^{(0)}c_0^{(0)} - \frac{8}{3}c_n^{(2)}c_0^{(2)}.
\end{cases}
$$

Proof. This is obtained by comparing the q-expansion of the LHS and RHS of the equation (3.1). $\square$

The equation (3.1) for q^0 gives:

$$9c_0^{(0)}c_0^{(0)} - 4c_0^{(2)}c_0^{(2)} = 0.$$

Hence we have $c_0^{(2)} = 0$. For $n = 1$ the lemma above gives:

$$c_1^{(0)} = -24c_1^{(0)}c_0^{(1)}.$$

Assume $c_1^{(0)} = 0$ then it easy to see that $f_0(q) \equiv 0$. Otherwise we deduce: $24c_0^{(1)} = -1$.

The statement of the lemma reads:

$$\begin{cases} (n-1)c_n^{(0)} = K_n^{(0)}, \quad c_1^{(0)} \neq 0, & \\ nc_n^{(1)} = K_n^{(1)} + \dfrac{2}{9n} K_n^{(2)}, & n \geq 1 \\ nc_n^{(2)} = K_n^{(2)}. & \end{cases}$$

This defines the recursive procedure reconstructing all the coefficients of the functions $f_0(q)$, $f_1(q)$, $f_2(q)$. □

REMARK. The conditions of the proposition could be relaxed, however we only need it in this form while the stronger result extends the proof seriously.

In order to write down the potential of $\mathbb{P}^1_{2,2,2,2}$ we have to give the functions $f_k(q)$ explicitly.

DEFINITION. The functions $\vartheta_i(z,\tau)$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ are represented by the following Fourier expansions:

$$\vartheta_1(z,\tau) = \sqrt{-1} \sum_{n=-\infty}^{\infty} (-1)^n e^{(n-1/2)^2 \pi \sqrt{-1}\tau} e^{(2n-1)\pi\sqrt{-1}z},$$

$$\vartheta_2(z,\tau) = \sum_{n=-\infty}^{\infty} e^{(n-1/2)^2 \pi \sqrt{-1}\tau} e^{(2n-1)\pi\sqrt{-1}z},$$

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{n^2 \pi \sqrt{-1}\tau} e^{2n\pi\sqrt{-1}z},$$

$$\vartheta_4(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{n^2 \pi \sqrt{-1}\tau} e^{2n\pi\sqrt{-1}z}.$$

They will be called Jacobi theta functions or just theta functions.

It is clear from their Fourier expansions that Jacobi theta functions satisfy the Heat Equation:

$$\frac{\partial^2 \vartheta_i(z,\tau)}{\partial z^2} = 4\pi\sqrt{-1} \frac{\partial \vartheta_i(z,\tau)}{\partial \tau}, \quad 1 \leq i \leq 4.$$

DEFINITION. The functions $\vartheta_i(\tau) := \vartheta_i(0,\tau)$ for $2 \leq i \leq 4$ will be called theta constants.

Note that $\vartheta_1(0,\tau) \equiv 0$. Therefore we do not consider it.

NOTATION 3.1. *In what follows we skip the argument for the theta constants whenever it is fixed and we denote:*

$$\vartheta_i'(\tau) := \frac{\partial}{\partial z} \vartheta_i(\tau).$$

DEFINITION. For any $\tau \in \mathbb{H}$ define:

$$X_k^{\infty}(\tau) := 2 \frac{\partial}{\partial \tau} \log \vartheta_k, \quad 2 \leq k \leq 4.$$

Slightly abusing the notation introduce the functions $X_k^{\infty}(q)$ defined for $q = \exp(\pi\sqrt{-1}\tau)$.

$$X_k^{\infty}(q) := \frac{1}{\pi\sqrt{-1}} X_k^{\infty}\left(\frac{\tau}{\pi\sqrt{-1}}\right).$$

The theorem of Satake and Takahashi reads.

THEOREM 3.3 (Theorem 2.1 in [**42**]). *The functions* $f_k(q)$ *giving genus* 0 *GW potential of* $\mathbb{P}^1_{2,2,2,2}$ *are the following:*

$$
\begin{cases}
f_0(q) := \frac{1}{8}X_3^\infty(q) - \frac{1}{8}X_4^\infty(q), \\
f_1(q) := -\frac{1}{12}X_2^\infty(q) - \frac{1}{48}X_3^\infty(q) - \frac{1}{48}X_4^\infty(q), \\
f_2(q) := -\frac{3}{16}X_3^\infty(q) - \frac{3}{16}X_4^\infty(q).
\end{cases} \tag{3.2}
$$

In terms of the functions $X_k^\infty(q)$ the WDVV equation on $\mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}}$ is equivalent to the system of PDE on $X_i^\infty(\tau)$ known as Halphen's system:

$$
\begin{cases}
\dfrac{d}{d\tau}(X_2^\infty(\tau) + X_3^\infty(\tau)) = 2X_2^\infty(\tau)X_3^\infty(\tau), \\
\dfrac{d}{d\tau}(X_3^\infty(\tau) + X_4^\infty(\tau)) = 2X_3^\infty(\tau)X_4^\infty(\tau), \\
\dfrac{d}{d\tau}(X_4^\infty(\tau) + X_2^\infty(\tau)) = 2X_4^\infty(\tau)X_2^\infty(\tau).
\end{cases} \tag{3.3}
$$

It is a well known fact that the functions X_i^∞ as above give solution of this system (see for example [**37**]). We do not give the proof here because it requires some additional properties of theta constants that are not important for us.

PROPOSITION 3.4. *Applying a linear change of variables the potential* $\mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}}$ *can be rewritten in the form:*

$$
\mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}}(t_{-1}, t_0, \tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \tilde{t}_4) = \frac{t_0^2 t}{2} + \frac{t_0}{4}\sum_{i=1}^4 (\tilde{t}_i)^2 - (\tilde{t}_1^2\tilde{t}_3^2 + \tilde{t}_2^2\tilde{t}_4^2)\frac{1}{16}X_3^\infty(t_{-1})
$$
$$
- (\tilde{t}_1^2\tilde{t}_4^2 + \tilde{t}_2^2\tilde{t}_3^2)\frac{1}{16}X_4^\infty(t_{-1}) - (\tilde{t}_3^2\tilde{t}_4^2 + \tilde{t}_1^2\tilde{t}_2^2)\frac{1}{16}X_2^\infty(t_{-1}) - \frac{1}{64}\sum_{i=1}^4 (\tilde{t}_i)^4\gamma^\infty(t_{-1})
$$

with the Euler vector field preserved:

$$
E(t_0, \tilde{t}_i) = E(t_0, t_i),
$$

and $\gamma^\infty(t_{-1}) = \frac{2}{3}\sum X_i^\infty(t_{-1})$.

PROOF. Apply to $\mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}}$ the change of variables $t_1 = (\tilde{t}_4 - \tilde{t}_3)/\sqrt{2}$, $t_2 = (\tilde{t}_4 + \tilde{t}_3)/\sqrt{2}$, $t_3 = (\tilde{t}_1 - \tilde{t}_2)/\sqrt{2}$, $t_4 = (\tilde{t}_1 + \tilde{t}_2)/\sqrt{2}$ that obviously preserves the WDVV equation and the metric η. Simple computations show:

$$
\begin{cases}
\frac{1}{3}f_2(q) + f_1(q) = -\frac{1}{12}\sum X_i^\infty(q) = -\frac{1}{8}\gamma^\infty(q), \\
\frac{2}{3}f_2(q) - f_0(q) = -\frac{1}{4}X_3^\infty(q), \\
\frac{2}{3}f_2(q) + f_0(q) = -\frac{1}{4}X_4^\infty(q), \\
3f_1(q) - \frac{1}{3}f_2(q) = -\frac{1}{4}X_2^\infty(q).
\end{cases}
$$

It is an easy computation to check that the Euler vector field is preserved too.

Rewriting the functions $X_k^\infty(q)$ via $X_k^\infty(t_{-1})$ and applying one more change of variables $\tilde{t}_{-1} = t_{-1}/\pi\sqrt{-1}$, $\tilde{t}_0 = t_0\sqrt{\pi\sqrt{-1}}$, $\tilde{t}_i = t_i/(\pi\sqrt{-1})^{1/4}$ gives the correct form of the potential. □

2. Computational aspects of the GW theory

Two powerful tools in the computational problems of the GW potential of elliptic orbifolds are the uniqueness theorem by Ishibashi, Shiraishi and Takahashi published in [**21**] and the quasi-modularity theorem of Milanov and Shen proved in [**35**]. The first theorem is useful for the explicit computations while the second theorem assures that it is enough to make the computations up to a certain finite limit.

2.1. Uniqueness theorem of the orbifold GW Frobenius manifold. Let the numbers $a_1, a_2, a_3 \in \mathbb{Z}$ be such that $2 \le a_1 \le a_2 \le a_3$. Introduce the numbers:

$$\chi_A := \sum_{i=1}^{3} \frac{1}{a_i} - 1, \quad \mu_A := \sum_{i=1}^{3} a_i - 1.$$

The uniqueness theorem of Ishibashi, Shiraishi and Takahashi reads:

Theorem 3.5 (Theorem 3.1 in [**21**]). *There exists a unique Frobenius manifold M with the potential $\mathcal{F}$ of rank μ_A and dimension one with flat coordinates*

$$\{t_1, t_{\mu_A}, t_{i,j}\}, \textit{ for } \quad 1 \le i \le 3,\ 1 \le j \le a_i - 1.$$

satisfying the following conditions:

(1) *The unit vector field e and the Euler vector field E are given by*

$$e = \frac{\partial}{\partial t_1}, \ E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}.$$

(2) *The non-degenerate symmetric bilinear form η on $\mathcal{T}_M$ satisfies*

$$\eta\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{\mu_A}}\right) = \eta\left(\frac{\partial}{\partial t_{\mu_A}}, \frac{\partial}{\partial t_1}\right) = 1,$$

$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}}, \frac{\partial}{\partial t_{i_2,j_2}}\right) = \begin{cases} \frac{1}{a_{i_1}} & i_1 = i_2 \textit{ and } j_2 = a_{i_1} - j_1, \\ 0 & \textit{otherwise.} \end{cases}$$

(3) *The Frobenius potential $\mathcal{F}$ satisfies $E\mathcal{F}|_{t_1=0} = 2\mathcal{F}|_{t_1=0}$,*

$$\mathcal{F}|_{t_1=0} \in \mathbb{C}\left[[t_{1,1}, \dots, t_{1,a_1-1}, \dots, t_{i,j}, \dots, t_{3,1}, \dots, t_{3,a_3-1}, e^{t_{\mu_A}}]\right].$$

(4) *Under the condition* (3), *we have*

$$\mathcal{F}|_{t_1 = e^{t_{\mu_A}} = 0} = \sum_{i=1}^{3} \mathcal{G}^{(i)}, \quad \mathcal{G}^{(i)} \in \mathbb{C}[[t_{i,1}, \dots, t_{i,a_i-1}]], \ 1 \le i \le 3.$$

(5) *Under the condition* (3), *in the frame $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \dots, \frac{\partial}{\partial t_{3,a_3-1}}, \frac{\partial}{\partial t_{\mu_A}}$ of $\mathcal{T}_M$, the product $\circ$ can be extended to the limit $t_1 = t_{1,1} = \dots = t_{3,a_3-1} = e^{t_{\mu_A}} = 0$. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to*

$$\mathbb{C}[x_1, x_2, x_3] \Big/ \left(x_i x_j, \ a_i x_i^{a_i} - a_j x_j^{a_j}\right)_{1 \le i < j \le 3},$$

where $\partial/\partial t_{i,j}$ are mapped to x_i^j for $i = 1, \dots, 3, j = 1, \dots, a_i - 1$ and $\partial/\partial t_{\mu_A}$ are mapped to $a_1 x_1^{a_1}$.

(6) *The term*

$$\left(\prod_{i=1}^{3} t_{i,1}\right) e^{t_{\mu_A}}$$

occurs with the coefficient 1 *in* $\mathcal{F}$.

Important in using this theorem is another statement of Ishibashi, Shiraishi and Takahashi, claiming that the GW theory of the orbifolded projective lines with three isotropic points satisfies the conditions of the theorem above:

THEOREM 3.6 (Theorem 4.2 in [**21**]). *The conditions of Theorem 3.5 are satisfied by the genus* 0 *potential of the GW theory of* $\mathbb{P}^1_{a_1,a_2,a_3}$.

For the elliptic orbifolds we have $\sum_i \frac{1}{a_i} = 1$ and $\chi_A = 0$. Therefore the GW potential of the elliptic orbifolds depends on $\exp(t_{-1})$ rather than on t_{-1} itself. It complies with the similar statement for the orbifold $\mathbb{P}^1_{2,2,2,2}$.

2.2. GW theory and modular forms. It appeared explicitly in the previous section that the potential of the GW theory of $\mathbb{P}^1_{2,2,2,2}$ is defined via elliptic functions. We make this connection more explicit in this subsection.

DEFINITION. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a finite index subgroup. Let $k \in \mathbb{N}_{\geq 0}$ and $f(\tau)$ - a holomorphic function on $\mathbb{H}$.

- $f(\tau)$ is called a *modular form of weight* k if it satisfies the following condition:
$$\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$
- $f(\tau)$ is called a *quasi-modular form of weight* k *and depth* m if there are functions $f_0(\tau), \dots, f_m(\tau)$, holomorphic in $\mathbb{H}$ s.t.:
$$\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{l=0}^{m} f_k(\tau)\left(\frac{c}{c\tau+d}\right)^k \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Examples of the (quasi)-modular forms are given by Eisenstein series. The group Γ in these cases is the full modular group $\mathrm{SL}(2,\mathbb{Z})$.

Let $E_{2k}(\tau)$ for $k \in \mathbb{Z}_+$ be the Eisenstein series defined by:

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi\sqrt{-1}n\tau},$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and B_{2k} is a Bernoulli number. In particular we have $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$.

Eisenstein series E_{2k} are known to satisfy the modularity condition when $k \geq 2$. For $k = 1$ it has to be adjusted by the additional summand. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ we have:

$$\begin{aligned} E_2(\tau) &= \frac{1}{(c\tau+d)^2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{6c}{\pi\sqrt{-1}(c\tau+d)}, \\ E_{2k}(\tau) &= \frac{1}{(c\tau+d)^{2k}} E_{2k}\left(\frac{a\tau+b}{c\tau+d}\right), \ k \geq 2. \end{aligned} \tag{3.4}$$

It is clear from the definition and equations above that E_{2k} for $k \geq 2$ are modular forms and E_2 is a quasi-modular form.

Particularly interesting for us will be the following subgroups $\Gamma(N) \subset \mathrm{SL}(2,\mathbb{Z})$, called *principal congruence subgroups*:

Definition. For any positive $N \in \mathbb{N}$ define:

$$\Gamma(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \pm I \ \mathrm{mod} N\}.$$

The following result was proved by Milanov and Shen from the analysis of Saito's theory of the simple elliptic singularities:

Theorem 3.7 (Theorem 1.2 and Corollary 1.3 in [**35**]). *Let $\mathcal{X} = \mathbb{P}^1_{a_1,a_2,a_3}$ be an elliptic orbifold and $\mathcal{F}_g(\boldsymbol{t})$ - the corresponding GW potential. Let $\boldsymbol{t}$ be decomposed as $\boldsymbol{t} = (\boldsymbol{t}', t_{-1})$ such that t_{-1} corresponds to the class of the hyperplane in $H^*_{orb}(\mathcal{X})$. Then the expansion in $\exp(t_{-1})$ of the coefficients of $\mathcal{F}_g$ in $\boldsymbol{t}'$ are Fourier series of the quasi-modular forms w.r.t. to $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, where*

$$\Gamma = \begin{cases} \Gamma(3), & \mathcal{X} = \mathbb{P}^1_{3,3,3}, \\ \Gamma(4), & \mathcal{X} = \mathbb{P}^1_{4,4,2}, \\ \Gamma(6), & \mathcal{X} = \mathbb{P}^1_{6,3,2}. \end{cases}$$

Remark. The Fourier expansion of the quasi-modular form is given in $\exp(\pi i t)$ while the potential of the orbifold GW theory of $\mathcal{X}$ is polynomial in $\mathbf{t}'$ and power series in $\exp(t_{-1})$. By applying the change of variables like in Proposition 3.4 the theorem states that the coefficients of $\mathcal{F}(\mathbf{t})$ w.r.t. $\mathbf{t}'$ are quasi-modular forms.

For us it is important to note the following corollary.

Corollary 3.8. *The GW potential of the elliptic orbifold is defined and holomorphic for $t_{-1} \in \mathbb{H}$. In particular it is not defined at the point $t_{-1} = 0$.*

The theorem of Milanov-Shen is very helpful for the computational purposes due to the following fact. Consider $f(\tau)$ and $g(\tau)$ - two modular forms of weight k w.r.t. the principle congruence subgroup $\Gamma(N)$. Consider the Fourier expansion of f and g:

$$f(\tau) = \sum_{p \geq 0} f_p q^p, \ \ g(\tau) = \sum_{p \geq 0} g_p q^p \quad \text{for} \quad q := \exp(\pi\sqrt{-1}\tau)$$

Proposition 3.9 (Section 3.3 in [**25**]). *Consider the number called Sturm bound:*

$$L_N := \frac{k}{12} N \prod_{m|N} \left(1 + \frac{1}{m}\right).$$

Then if $f_p - g_p = 0$ for all $p \leq L_N$, then $f(\tau) - g(\tau) \equiv 0$.

2.3. $\mathrm{SL}(2,\mathbb{C})$ **action on the space of Frobenius manifolds.** In order to make the statement of Theorem 3.7 more explicit we define the action of $A \in \mathrm{SL}(2,\mathbb{C})$ on the GW potential. In this thesis we only need it for the genus 0 part of the GW potential, however it was done in the full generality in [**5**].

2.3.1. *Inversion transformation of Dubrovin.* Let the WDVV potential $\mathcal{F}$ be given by:

$$\mathcal{F}(\mathbf{t}) = \frac{1}{2}t_1^2 t_n + t_1 \sum_{p\leq q} \frac{1}{|\mathrm{Aut}(p,q)|}\eta_{p,q}t_p t_q + H(t_2,\dots,t_n),$$

for some function $H = H(t_2,\dots,t_n)$.

Consider the function $\mathcal{F}^I$.

$$\mathcal{F}^I(\hat{\mathbf{t}}) = (t_n)^{-2}\left[\mathcal{F}(\mathbf{t}) - \frac{1}{2}t_1\sum_{i,j}\eta_{ij}t_i t_j\right], \tag{3.5}$$

where for $1 < \alpha < n$:

$$\hat{t}_1 := \frac{1}{2}\frac{\sum_{ij}\eta_{ij}t_i t_j}{t_n}, \quad \hat{t}_\alpha := \frac{t_\alpha}{t_n}, \quad \hat{t}_n := -\frac{1}{t_n}.$$

PROPOSITION 3.10 (Appendix B in [**11**]). *The function $\mathcal{F}^I$ is solution to the WDVV equation and we have:*

$$\frac{\partial^3 \mathcal{F}^I}{\partial\hat{t}_1\partial\hat{t}_a\partial\hat{t}_b} = \eta_{ab}.$$

2.3.2. *General* $\mathrm{SL}(2,\mathbb{C})$ *action.* The following proposition defines the action of $A \in \mathrm{SL}(2,\mathbb{C})$ on the space of Frobenius manifolds.

PROPOSITION 3.11. *Let* $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$*. The function $\mathcal{F}^A$ defined as follows:*

$$\mathcal{F}^A(\boldsymbol{t}) := \frac{1}{2}t_1^2 t_n + t_1\sum_{p\leq q}\frac{1}{|\mathrm{Aut}(p,q)|}\eta_{p,q}t_p t_q + \frac{c}{8(ct_n+d)}\left(\sum_{p\leq q}\frac{1}{|\mathrm{Aut}(p,q)|}\eta_{p,q}t_p t_q\right)^2$$
$$+ (ct_n+d)^2 H\left(\frac{t_2}{ct_n+d},\dots,\frac{t_{n-1}}{ct_n+d},\frac{at_n+b}{ct_n+d}\right).$$

is solution to the WDVV equation.

PROOF. The action of A on the variable t_n can be decomposed as follows.

$$A\cdot t_n = \frac{at_n+b}{ct_n+d} = T_2\cdot S_{c^2}\cdot I\cdot T_1\cdot t_n,$$

where T_1 is the shift $t_n \to t_n + \frac{a}{c}$, S_{c^2} is the scaling $t_n \to c^2 t_n$, T_2 is one more shift $t_n \to t_n + \frac{d}{c}$ and $I : t_n \to -1/t_n$. We "quantize" this action on the variable t_n to the action on the Frobenius manifold potential. It is clear that the shifts T_1 and T_2 preserve the pairing η_{pq} and the WDVV equation. The action of I is quantized via the Inversion transformation of Dubrovin (see Proposition 3.10). Combining all the actions together we get the proposition. □

The quasi-modularity condition of Theorem 3.7 is equivalent to the equality:

$$\mathcal{F}^A = \mathcal{F}, \quad \forall A\in\Gamma.$$

EXAMPLE 3.12 (Appendix C in [**11**]). Consider the rank 3 potential:

$$\mathcal{F}^3 := \frac{t_0^2 t_{-1}}{2} + \frac{1}{2}t_0 t_1^2 - \frac{t_1^4}{16}\frac{\pi\sqrt{-1}}{3}E_2(t_{-1}).$$

It follows immediately from the modularity property of E_2 that $\mathcal{F}^A = \mathcal{F}$ for any $A \in \mathrm{SL}(2, \mathbb{Z})$.

It is not hard to see that in fact the statement of Theorem 3.7 could be made more precise. Namely for i_1, j_1 and i_2, j_2 such that $\eta_{i_k j_k} \neq 0$ the coefficient $t_{i_1} t_{j_1} t_{i_2} t_{j_2}$ of the GW potential of an elliptic orbifold $\mathcal{X}$ is a quasi-modular form of weight 2 while all other coefficients are indeed modular forms rather than quasi-modular ones!

3. GW theory of $\mathbb{P}^1_{6,3,2}$

It is still an open problem to write down in a closed formula the genus 0 potential of the GW theory of $\mathbb{P}^1_{6,3,2}$. However it could be computed explicitly up to any order of the variables by applying Theorem 3.5. Together with Theorem 3.7 we reconstruct part of the genus 0 potential in a closed formula.

3.1. Potential of the GW theory of $\mathbb{P}^1_{6,3,2}$. Let $a_1, a_2, a_3 \in \mathbb{N}$. Consider the basis $\Delta_0, \Delta_{-1}, \Delta_{ij}$ of $H^*_{orb}(\mathbb{P}^1_{a_1,a_2,a_3})$ such that:

$$H^*_{orb}(\mathbb{P}^1_{6,3,2}) \simeq \bigoplus \mathbb{Q}[\Delta_0] \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{a_i-1} \mathbb{Q}[\Delta_{ij}] \bigoplus \mathbb{Q}[\Delta_{-1}].$$

The classes Δ_0 and Δ_{-1} correspond to the classes of the point and hyperplane in $H^*(\mathbb{P}^1)$ respectively. The pairing reads:

$$\eta(\Delta_0, \Delta_{-1}) = 1, \quad \eta(\Delta_{ij}, \Delta_{kl}) = \frac{1}{a_i} \delta_{i,k} \delta_{j+l,a_i}.$$

Let $\mathcal{F}_{\mathbb{P}^1_{6,3,2}}$ be the WDVV potential of $\mathbb{P}^1_{6,3,2}$. It is function of 10 variables t_0, t_{-1}, t_{ij} with the Euler vector field given by:

$$E = t_0 \frac{\partial}{\partial t_0} + \sum_{k=1}^{5} t_{1k} \frac{6-k}{6} \frac{\partial}{\partial t_{1k}} + \sum_{k=1}^{3} t_{2k} \frac{3-k}{3} \frac{\partial}{\partial t_{2k}} + t_{31} \frac{1}{2} \frac{\partial}{\partial t_{31}}, \qquad E \cdot \mathcal{F}_{\mathbb{P}^1_{6,3,2}} = 2 \mathcal{F}_{\mathbb{P}^1_{6,3,2}}.$$

From the quasi-homogeneity condition given we see that $\mathcal{F}_{\mathbb{P}^1_{6,3,2}}$ is a polynomial in t_0, t_{ij} and infinite as a power series in $\exp(t_{-1})$.

We have written Mathematica script implementing Theorem 3.5 that is available at [**3**]. Using it we have computed the genus 0 potential $\mathcal{F}_{\mathbb{P}^1_{6,3,2}}$ up to $\exp(40 t_{-1})$. We give it here up to the second order in $\exp(t_{-1})$ only. Later on we will need explicit computations of the restriction of this potential up to higher order in $\exp(t_{-1})$. Slightly abusing the notation we write it in the coordinates $\{t_0, t_1, \dots, t_8, t_9\} = \{t_0, t_{1k}, t_{2k}, t_{31}, t_{-1}\}$.

$$\begin{aligned}
\mathcal{F}_{\mathbb{P}^1_{6,3,2}} &= \frac{1}{2} t_0^2 t_9 + t_0 \left(\frac{t_3^2}{12} + \frac{t_2 t_4}{6} + \frac{t_1 t_5}{6} + \frac{t_6 t_7}{3} + \frac{t_8^2}{4} \right) + \frac{t_2^3}{36} + \frac{1}{6} t_1 t_2 t_3 - \frac{t_3^4}{288} + \frac{1}{12} t_1^2 t_4 - \frac{t_8^4}{96} \\
&- \frac{1}{36} t_2 t_3^2 t_4 - \frac{1}{72} t_2^2 t_4^2 - \frac{1}{72} t_1 t_3 t_4^2 + \frac{1}{432} t_3^2 t_4^3 + \frac{t_2 t_4^4}{1296} - \frac{t_4^6}{38880} - \frac{1}{72} t_2^2 t_3 t_5 - \frac{1}{72} t_1 t_3^2 t_5 - \frac{1}{36} t_1 t_2 t_4 t_5 \\
&+ \frac{1}{324} t_3^3 t_4 t_5 + \frac{1}{144} t_2 t_3 t_4^2 t_5 + \frac{t_1 t_4^3 t_5}{1296} - \frac{5 t_3 t_4^4 t_5}{10368} - \frac{1}{144} t_1^2 t_5^2 + \frac{1}{432} t_2 t_3^2 t_5^2 + \frac{1}{432} t_2^2 t_4 t_5^2 + \frac{1}{432} t_1 t_3 t_4 t_5^2 \\
&- \frac{5 t_3^2 t_4^2 t_5^2}{5184} - \frac{t_2 t_4^3 t_5^2}{2592} + \frac{t_4^5 t_5^2}{34560} + \frac{t_1 t_2 t_5^3}{1296} - \frac{t_3^3 t_5^3}{7776} - \frac{5 t_2 t_3 t_4 t_5^3}{7776} - \frac{t_1 t_4^2 t_5^3}{15552} + \frac{11 t_3 t_4^3 t_5^3}{93312} - \frac{t_2^2 t_5^4}{15552} - \frac{t_1 t_3 t_5^4}{31104} \\
&+ \frac{11 t_3^2 t_4 t_5^4}{186624} + \frac{t_2 t_4^2 t_5^4}{31104} - \frac{13 t_4^4 t_5^4}{2239488} + \frac{11 t_2 t_3 t_5^5}{933120} + \frac{t_1 t_4 t_5^5}{933120} - \frac{91 t_3 t_4^2 t_5^5}{11197440} - \frac{11 t_3^2 t_5^6}{11197440} - \frac{13 t_2 t_4 t_5^6}{16796160}
\end{aligned}$$

$$+\frac{91t_4^3t_5^6}{201553920}-\frac{t_1t_5^7}{235146240}+\frac{43t_3t_4t_5^7}{201553920}+\frac{t_2t_5^8}{201553920}-\frac{41t_4^2t_5^8}{2418647040}-\frac{17t_3t_5^9}{9674588160}$$
$$+\frac{809t_4t_5^{10}}{2612138803200}-\frac{809t_5^{12}}{344802322022400}+\frac{t_6^3}{18}-\frac{1}{36}t_6^2t_7^2+\frac{1}{648}t_6t_7^4-\frac{t_7^6}{19440}$$
$$+e^{t_9}t_1t_6t_8+\frac{1}{6}e^{t_9}t_3t_4t_6t_8+\frac{1}{6}e^{t_9}t_2t_5t_6t_8+\frac{1}{72}e^{t_9}t_4^2t_5t_6t_8+\frac{1}{72}e^{t_9}t_3t_5^2t_6t_8+\frac{e^{t_9}t_4t_5^3t_6t_8}{1296}+\frac{e^{t_9}t_5^5t_6t_8}{155520}$$
$$+\frac{1}{6}e^{t_9}t_1t_7^2t_8+\frac{1}{36}e^{t_9}t_3t_4t_7^2t_8+\frac{1}{36}e^{t_9}t_2t_5t_7^2t_8+\frac{1}{432}e^{t_9}t_4^2t_5t_7^2t_8+\frac{1}{432}e^{t_9}t_3t_5^2t_7^2t_8+\frac{e^{t_9}t_4t_5^3t_7^2t_8}{7776}$$
$$+\frac{e^{t_9}t_5^5t_7^2t_8}{933120}+\mathrm{O}(e^{2t_9}).$$

3.2. Restriction of the genus 0 GW theory of $\mathbb{P}^1_{6,3,2}$. Proposition 3.11 together with Theorem 3.7 and Proposition 3.9 allow us to write explicitly part of the genus 0 potential of $\mathbb{P}^1_{6,3,2}$.

Let $\mathcal{F}_0^{\mathbb{P}^1_{6,3,2}}(\mathbf{t})$ be the genus 0 GW potential of $\mathbb{P}^1_{6,3,2}$ with the variables t_α corresponding to the classes Δ_α as in Section 3.1:

$$\mathbf{t}=\{t_0,t_{-1},t_{11},t_{12},t_{13},t_{14},t_{15},t_{21},t_{22},t_{31}\}.$$

Consider the function $\mathcal{F}^4$:

$$\mathcal{F}^4(t_0,t_{-1},t_{13},t_{31}):=\mathcal{F}_0^{\mathbb{P}^1_{6,3,2}}\,|_{t_{11}=0,t_{12}=0,t_{14}=0,t_{15}=0,t_{21}=0,t_{22}=0}\ .$$

Proposition 3.13. *The function $\mathcal{F}^4$ satisfies the WDVV equation and has the following explicit expression:*

$$\mathcal{F}^4=\frac{1}{2}t_0^2t_{-1}+t_0\left(\frac{t_{13}^2}{12}+\frac{t_{31}^2}{4}\right)+\frac{1}{36}t_{13}^4f_1(3t_{-1})+\frac{1}{18}t_{13}^2t_{31}^2f_2(3t_{-1})$$
$$+\frac{1}{9}t_{13}t_{31}^3f_0(3t_{-1})+\left(\frac{1}{12}f_1(3t_{-1})+\frac{1}{18}f_2(3t_{-1})\right)t_{31}^4.$$

Proof. It is clear that setting certain non-zero degree variables to zero in the function solving WDVV we get a function solving the WDVV equation too.

Assuming the quasi-homogeneity of $\mathcal{F}_0^{\mathbb{P}^1_{6,3,2}}$ the function $\mathcal{F}^4$ has the form:

$$\mathcal{F}^4=\frac{1}{2}t_0^2t_{-1}+t_0\left(\frac{t_{13}^2}{12}+\frac{t_{31}^2}{4}\right)+t_{13}^4g_1(t_{-1})+t_{13}^2t_{31}^2g_2(t_{-1})$$
$$+t_{13}t_{31}^3g_3(3t_{-1})+t_{13}t_{31}^3g_4(t_{-1})+t_{31}^4g_5(t_{-1}),$$

for some functions $g_k(t_{-1})$.

By Theorem 3.7 we know that these functions are quasi-modular forms w.r.t. $\Gamma(6)$. However it is clear from Proposition 3.11 that the functions $g_3(t_{-1})$ and $g_4(t_{-1})$ are modular of weight 2 rather than quasi-modular.

Let $q=\exp(t_{-1})$. Using Theorem 3.5 we compute the first terms of $\mathcal{F}^4$ to be:

$$g_1(q)=\left(-\frac{1}{288}+\frac{q^{12}}{12}+\frac{q^{24}}{4}\right)+\mathrm{O}(q^{30}),\ g_2(q)=\left(\frac{q^6}{2}+q^{12}+2q^{18}\right)+\mathrm{O}(q^{30}),$$
$$g_3(q)=\left(\frac{q^3}{3}+\frac{4q^9}{3}+2q^{15}+\frac{8q^{21}}{3}+\frac{13q^{27}}{3}\right)+\mathrm{O}(q^{30}),\ g_4(q)=0+\mathrm{O}(q^{30}),$$
$$g_5(q)=\left(-\frac{1}{96}+\frac{q^6}{2}+\frac{5q^{12}}{4}+2q^{18}+\frac{11q^{24}}{4}\right)t_{31}^4+\mathrm{O}(q^{30}).$$

Recall Proposition 3.9. The Sturm bound L_N for $\Gamma(6)$ is: $L_6 = 4$. Comparing the Fourier expansion above with the Fourier expansion of $f_0(q)$ we get:

$$\frac{1}{9}f_0(q^3) \equiv g_3(q), \quad g_4(q) \equiv 0.$$

In the same way we see that the Fourier expansions of g_1, g_2, g_5 up to the order q^{30} coincide with those of $f_1(q^3)/36$, $f_2(q^3)/18$ and $f_1(q^3)/12 + f_2(q^3)/18$. However we can not apply Proposition 3.9 here.

LEMMA 3.14. *Consider the functions $\tilde{f}_k(q)$ such that the following equalities hold:*

$$g_1(q) = \tilde{f}_1(q^3)/36,\ g_2(q) = \tilde{f}_2(q^3)/18,\ g_5(q) = \tilde{f}_1(q^3)/12 + \tilde{f}_2(q^3)/18,$$

then the WDVV equation of $\mathcal{F}^4$ rewritten via $\tilde{f}_k(q)$ is equivalent to the system of PDE (3.1)*.*

PROOF. This is done via the simple computation. □

This lemma completes the proof of Proposition 3.13. □

3.2.1. *Idea of Proposition 3.13.* Obviously the proof of Proposition 3.13 is not the way it was observed. We explain the idea behind it.

Consider two orbifolds $\mathcal{X} = \mathbb{P}^1_{2,2,2,2}$ and $\mathcal{Y} = \mathbb{P}^1_{6,3,2}$. Both can be realized as global quotients of the elliptic curves:

$$\mathcal{X} = [\mathbb{E}/\mathbb{Z}_2]\,, \quad \mathcal{Y} = [\mathbb{E}/\mathbb{Z}_6]\,.$$

One can consider the stack-theoretical quotient $\mathcal{Y} = \mathcal{X}//\mathbb{Z}_3$. Therefore one could consider the existence of the map $f^* : H^*_{orb}(\mathcal{Y}) \to H^*_{orb}(\mathcal{X})$, that is definitely not an isomorphism. Consider instead the smaller subspace $H_s \subset H^*(\mathcal{IX})$ such that $H_s \cong f^*\left(H^*(\mathcal{IY})\right)$.

Let $\tilde{\gamma}_k \in H_s$ and also $\tilde{\gamma}_k = f^*(\gamma_k)$ for $\gamma_k \in H^*(\mathcal{IY})$. Consider the correlators:

$$\langle \tilde{\gamma}_1, \dots, \tilde{\gamma}_n \rangle^{\mathcal{X}}_{n,g,\beta} = \prod_{i=1}^{n} ev^*_k(\tilde{\gamma}_k).\left[\overline{M}_{g,n}(\mathcal{X},\beta)\right] \in \mathbb{C}.$$

Applying the projection formula we have:

$$\langle \tilde{\gamma}_1, \dots, \tilde{\gamma}_n \rangle^{\mathcal{X}}_{n,g,\beta} = \prod_{i=1}^{n} ev^*_k(\gamma_k).f_*\left[\overline{M}_{g,n}(\mathcal{X},\beta)\right]$$

Unfortunately it is not clear whether one could consider the push-forward of the virtual fundamental class in question.

In the case of the particular orbifolds written it is reasonable to consider the correspondence between the correlators involving Δ_0,Δ_{-1}, Δ_{13} and Δ_{31} from $H^*_{orb}(\mathbb{P}^1_{6,3,2})$ and $\tilde{\Delta}_0$, $\tilde{\Delta}_{-1}$, $\tilde{\Delta}_1$, $\tilde{\Delta}_2$, $\tilde{\Delta}_3$, $\tilde{\Delta}_4$ from $H^*_{orb}(\mathbb{P}^1_{2,2,2,2})$. Assuming the stack-theoretical quotient $\mathcal{Y} = \mathcal{X}//\mathbb{Z}_3$ these should be related as:

$$\Delta_0 \leftrightarrow \tilde{\Delta}_0,\ \Delta_{-1} \leftrightarrow 3\tilde{\Delta}_{-1}$$

because of the quotient order and

$$\Delta_{31} \leftrightarrow \tilde{\Delta}_1,\ \Delta_{13} \leftrightarrow \frac{1}{\sqrt{3}}\left(\tilde{\Delta}_2 + \tilde{\Delta}_3 + \tilde{\Delta}_4\right).$$

because the isotropic point of order 6 in $\mathbb{P}^1_{6,3,2}$ is "composed" of 3 isotropic points of order 2 of $\mathbb{P}^1_{2,2,2,2}$ while the forth "goes" to the order 2 point in $\mathbb{P}^1_{6,3,2}$.

CHAPTER 4

Frobenius structures of the orbifolded LG A- and B-models

On the A-side Fan, Jarvis and Ruan have constructed in [**16**] the cohomological field theory $\Lambda_{g,n}$ associated to the orbifolded LG A-model. It is often called the FJRW–theory where "W" stays for Witten, who proposed existence of such a cohomological field theory. Like the GW theory it also gives some Frobenius manifold by the restriction of the cohomological field theory potential to the genus 0. It is a natural candidate for the Frobenius manifold of the orbifolded LG A-model. However it is not yet the only recognized A-model and another problem is that it is very hard to compute.

Up to now there is no definition of a Frobenius structure associated to an orbifolded LG B-model. Some work has been done by Kaufmann and Krawitz in [**24, 26**] who establish only the Frobenius algebra at the origin of the orbifolded LG B-model. Namely only the cubic part in $\mathbf{t}$ of the potential $\mathcal{F}(\mathbf{t})$.

In what follows we introduce the *axiomatization* of a Frobenius manifolds of an orbifolded LG A- and B-model. Comparing to the work of Kaufmann and Krawitz we only consider the same state space as they did. We do not assume the Frobenius algebra structure of Krawitz. Comparing to the work of Fan, Jarvis and Ruan we consider a much larger class of Frobenius manifolds. We take the same state space as in FJRW–theory with only one additional assumption that is however essential and believed for FJRW–theory, but not proved in general.

In what follows we assume W to be an invertible singularity. Consider one more piece of notations.

Definition. Let G be a B-admissible group symmetry group of W. For any $g \in G$ define the fixed locus of g by:

$$\mathrm{Fix}(g) := \{\mathbf{x} \in \mathbb{C}^N \mid g \cdot \mathbf{x} = \mathbf{x}\},$$

and the natural number $N_g := \dim \mathrm{Fix}(g)$.

Let W_g be the restriction of W to the fixed locus of $g \in G$:

$$W_g := W\mid_{\mathrm{Fix}(g)}, \quad W_g : \mathbb{C}^{N_g} \to \mathbb{C}.$$

1. Orbifolded Landau-Ginzburg B-model

Definition. Let G be a B-admissible symmetry group of W. Define:

- For any $g \in G$ let $\mathcal{L}_{W_g}$ be the Milnor ring of W_g. The state space of the orbifolded B-model reads:
$$\mathcal{H} := \bigoplus_{g\in G} (\mathcal{L}_{W_g})^G.$$
- Let e be the unit of $\mathcal{L}_W$. Define:
$$\mathcal{H}^{tw} := \bigoplus_{g\in G,\ g\neq e} (\mathcal{L}_{W_g})^G.$$

Then we have:

$$\mathcal{H} = (\mathcal{L}_{W_e})^G \oplus \mathcal{H}^{tw}.$$

The space $(\mathcal{L}_{W_e})^G$ will be called the untwisted sector of $\mathcal{H}$ and $\mathcal{H}^{tw}$ - the twisted sector of $\mathcal{H}$.

- For all $g \in G$ let η_g be the residue pairing of W_g restricted to $(\mathcal{L}_{W_g})^G$. Define the pairing $\eta^{\mathcal{H}}$ on $\mathcal{H}$ by the following rules:

$$u, v \in \mathcal{H} \Rightarrow \eta^{\mathcal{H}}(u,v) = \begin{cases} \eta_g(u,v), \text{ for } u, v \in (\mathcal{L}_{W_g})^G, \\ \eta_g(u,v), \text{ for } u \in (\mathcal{L}_{W_g})^G, v \in (\mathcal{L}_{W_{g^{-1}}})^G, \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $M_{(W,G)}$ be the Frobenius manifold of the LG B-model. Let $\mathcal{F}(\mathbf{t})$ be its potential. We assume it to satisfy the following axioms:

- The trivial group: For $G = \{id\}$ we get Saito's Frobenius structure:

$$M_{(W,G)} = M_{(W,\{id\})} \cong M_W.$$

- Untwisted sector: Restricted to the untwisted sector we get Saito's structure on the G-invariant subspace:

$$M_{(W,G)} \mid_{t_{g,k}=0, \ g \neq e} \ \cong (M_{W,\zeta})^G ,$$

with some primitive form choice ζ.
- The state space:

$$\mathcal{T} M_{(W,G)} \mid_{\mathbf{t}=0} \cong \ \mathcal{H}.$$

- The grading: There is choice of flat coordinates $\mathbf{t}$ on $M_{(W,G)}$ indexed by $g \in G$:

$$\mathbf{t} = \{(t_{g,1}, \dots, t_{g,\mu_g}), \ \forall g \in G\},$$

where $\mu_g := \dim(\mathcal{L}_{W_g})^G$.
- The pairing: The pairing of $M_{(W,G)}$ coincides in the flat coordinates with the pairing $\eta^{\mathcal{H}}$ of $\mathcal{H}$:

$$\frac{\partial^3 \mathcal{F}}{\partial t_{\mathbf{1},1} \partial t_a \partial t_b} = \eta^{\mathcal{H}}_{a,b}.$$

- The primitive form: There is a notion of the primitive form change on $M_{(W,G)}$ so that it agrees with Saito's primitive form change when $G = \{id\}$.
- The special point: There is a notion of a "special point" for the action above such that it coincides with a special point of the unfolding for $G = \{id\}$.
- Equivariance: Let e be the unit of G and $g_i \in G$ for $1 \leq i \leq k$ be such that:

$$g_1 \cdot \dots \cdot g_k \neq e \in G.$$

Then we have:

$$\frac{\partial^k \mathcal{F}}{\partial t_{g_1,i_1} \dots \partial t_{g_k,i_k}} \mid_{\mathbf{t}=0} = 0 \quad \forall i_1, \dots, i_k.$$

The last condition is best understood from the point of view of the cohomological field theories, we are assuming that the potential $\mathcal{F}$ is a generating function of some correlators similar to the GW-theory. In this case the moduli space of curves should be endowed with an additional structure of a group element choice at every curve marking and the last axiom just says that only those correlators are non-zero, that are supported on the "balanced" curves (cf. [**22**] as an example).

2. Orbifolded Landau-Ginzburg A-model

Let G be A-admissible. For any $h \in G$ consider:

$$\mathcal{H}_h := \Omega^{N_h}(\mathbb{C}^{N_h})/(dW_h \wedge \Omega^{N_h-1}).$$

By fixing the volume form $\omega = dx_1 \wedge \cdots \wedge dx_{N_h}$ we have the isomorphism:

$$\mathcal{H}_h \cong \mathcal{L}_{W_h} \cdot \omega.$$

The action of G extends to the action on $\Omega(\mathbb{C}^{N_h})$. The state space of the A-model reads:

$$\mathcal{H}_{W,G} := \left(\bigoplus_{h \in G} \mathcal{H}_h \right)^G$$

As in the case of the B-model, the pairing is defined via the isomorphism $\mathcal{H}_h \cong \mathcal{H}_{h^{-1}}$.

Let $M^A_{W,G}$ be the Frobenius manifold of a LG A-model of the pair (W, G). We consider it to satisfy the following axioms:

- The state space:
$$\mathcal{T} M^A_{W,G} \mid_{\mathbf{t}=0} \cong \mathcal{H}_{W,G}.$$
- Rationality: The potential $\mathcal{F}(\mathbf{t})$ of the Frobenius manifold $M^A_{W,G}$ is defined over $\mathbb{Q}$,
- CY/LG correspondence: $M^A_{W,G}$ is connected to the GW theory of X_W via the action on the space of Frobenius manifolds such that for $G = G_W$ it corresponds to the primitive form change,

The first axiom is satisfied by the Fan-Jarvis-Ruan theory. The second axiom is supposed to be satisfied too, however it is not proved in general.

This is a natural idea that the objects that have some finite group action structure should be defined over $\mathbb{Q}$ if the initial object satisfies the same property. The last two axioms just assert that we look for those Frobenius manifolds that can appear in Mirror symmetry.

CHAPTER 5

LG-CY mirror symmetry for the orbifolded LG model

We address the problem of LG-CY mirror symmetry for the LG B-model of $(\tilde{E}_8, \mathbb{Z}_3)$.

Consider the following symmetry group G acting on $\mathbb{C}^3$ preserving the singularity $\tilde{E}_8$. For $\xi^3 = 1, \xi \neq 1$ it is defined by:

$$h : (x, y, z) \to (\xi x, \xi^2 y, z).$$

Define $G := \langle h \rangle$. Obviously $G \cong \mathbb{Z}_3$.

THEOREM 5.1. *The Frobenius manifold potential of the orbifolded LG B-model $(\tilde{E}_8, \mathbb{Z}_3)$ at the LCSL reads:*

$$\begin{aligned}\mathcal{F}^{\mathbb{Z}_3} = & \frac{1}{2} t_{\mathit{1},0}^2 t_{\mathit{1},3} + t_{\mathit{1},0} \left(\frac{t_{\mathit{1},1}^2}{12} + \frac{t_{\mathit{1},2}^2}{4} \right) + t_{\mathit{1},0} t_h t_{h^2} + \frac{1}{36} t_{\mathit{1},1}^4 f_1(3t_{\mathit{1},3}) \\ & + \frac{1}{18} t_{\mathit{1},1}^2 t_{\mathit{1},2}^2 f_2(3t_{\mathit{1},3}) + \frac{1}{9} t_{\mathit{1},1} t_{\mathit{1},2}^3 f_0(3t_{\mathit{1},3}) + t_{\mathit{1},2}^4 \left(\frac{1}{12} f_1(3t_{\mathit{1},3}) + \frac{1}{18} f_2(3t_{\mathit{1},3}) \right) \\ & + t_h t_{h^2} \left(\frac{2}{9} t_{\mathit{1},1}^2 f_2(t_{\mathit{1},3}) + 2 t_{\mathit{1},2}^2 f_1(t_{\mathit{1},3}) - \frac{2}{3} t_{\mathit{1},1} t_{\mathit{1},2} f_0(t_{\mathit{1},3}) \right) + t_h^2 t_{h^2}^2 \left(\frac{2}{3} f_2(t_{\mathit{1},3}) + 2 f_1(t_{\mathit{1},3}) \right) \\ & + (t_h^3 + t_{h^2}^3) \left(t_{\mathit{1},1} f_0(t_{\mathit{1},3}) + t_{\mathit{1},2} (3 f_1(t_{\mathit{1},3}) - f_2(t_{\mathit{1},3})) \right),\end{aligned}$$

where the functions $f_0(t), f_1(t), f_2(t)$ coincide with those in equation (3.2).

W. Ebeling and A. Takahashi construct explicitly in [**15**] a variety that is in some sense "mirror dual" to the pair (W, G) where W is an affine cusp singularity. To do this they introduce *orbifolded* Dolgachev and Gabrielov numbers. It was probably background idea of the authors to use these mirror pairs for the LG-CY mirror symmetry. We show on the particular example that this idea works. In particular for the pair $(\tilde{E}_8, \mathbb{Z}_3)$ the mirror variety of Ebeling-Takahashi turns out to be $\mathbb{P}^1_{2,2,2,2}$ and we have the theorem:

THEOREM 5.2. *The Frobenius manifold of the LG B-model $(\tilde{E}_8, \mathbb{Z}_3)$ is isomorphic to the Frobenius manifold of the GW theory of $\mathbb{P}^1_{2,2,2,2}$.*

1. Orbifolded Gabrielov and Dolgachev numbers

In [**15**] W. Ebeling and A. Takahashi develop the approach to Gabrielov and Dolgachev numbers of the orbifold LG models. We give here only part of their work in a simplified manner.

DEFINITION. Let $W(x_1, x_2, x_3)$ be an invertible polynomial with the matrix $R = \{r_{ij}\}$ as in Introduction. The group $\hat{G}_W$

$$\hat{G}_W := \left\{ \lambda = (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 \mid \prod_{j=1}^3 \lambda_j^{r_{1j}} = \prod_{j=1}^3 \lambda_j^{r_{2j}} = \prod_{j=1}^3 \lambda_j^{r_{3j}} \right\}$$

will be called *maximal abelian group of symmetries* of W.

Note that for $(\lambda_1, \lambda_2, \lambda_3) \in \hat{G}_W$ and $\lambda := \prod_{j=1}^3 \lambda_j^{r_{1j}}$ holds:

$$W(\lambda \cdot \mathbf{x}) = \lambda W(\mathbf{x}).$$

For any invertible polynomial $W(x_1, x_2, x_3)$ and A-admissible symmetry group G of it define the group $\hat{G}$ by the following commutative diagram of two exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] \arrow[d, hook] & \hat{G} \arrow[r] \arrow[d, hook] & \mathbb{C}^* \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_W \arrow[r] & \hat{G}_W \arrow[r] & \mathbb{C}^* \arrow[r] & 1
\end{tikzcd}$$

To the orbifolded LG A-model (W, G) Ebeling and Takahashi associate the stacky curve $\mathcal{C}_{(W,G)}$:

$$\mathcal{C}_{(W,G)} := [W^{-1}(0) \backslash \{0\} / \hat{G}].$$

It could be seen as a smooth curve of the genus $g_{(W,G)}$ with a finite number of isotropic points.

Definition. The orders $\alpha_1, \ldots, \alpha_r$ of the isotropic points of $\mathcal{C}_{(W,G)}$ will be called Dolgachev numbers of (W, G) and denoted by $A_{(W,G)}$.

At the first glance the curve $\mathcal{C}_{W,G}$ looks to be different from $X_{W,G}$ defined in the introduction. However it is clear that the group $\hat{G}$ is an extension of G that controls the quasi-homogeneity of W. Namely if one embeds the stacky curve $\mathcal{C}_{W,G}$ into the weighted projective space $\mathbb{P}^2(c_1, c_2, c_3)$ the numbers c_i will be defined by the extension $\hat{G}$ and both curves will be isomorphic.

Let $W(x_1, x_2, x_3)$ be invertible polynomial. It was found in [**14**] that there is a holomorphic coordinate change such that the polynomial $W(x_1, x_2, x_3) + ax_1x_2x_3$ for some $a \in \mathbb{C}^*$ transforms to

$$W_{Ferma} := x_1^{p_1} + x_2^{p_2} + x_3^{p_3} - xyz$$

for some $p_i \geq 2$.

Another set of numbers associated of the pair (W, G) is the following.

Definition. Let G be a B-admissible symmetry group of an invertible $W(x_1, x_2, x_3)$, and $K_i \subset G$ - maximal subgroups fixing x_i coordinate. Let p_1, p_2, p_3 be the exponents of W_{Ferma} as above. The numbers $(\gamma_1, \ldots, \gamma_s)$ are called *Gabrielov numbers* of the pair (W, G):

$$\Gamma_{W,G} := (\gamma_1, \ldots, \gamma_s) = \left(\frac{p_i}{|G/K_i|} * |K_i|, \ 1 \leq i \leq 3 \right),$$

where by $a * |K_i|$ we mean that the number a is repeated $|K_i|$ times and the numbers 1 are omitted.

Definition. Let $G \subset \mathrm{SL}_n(\mathbb{C})$ - finite subgroup.

- The element $g \in G$ of the order r represented by

$$g = (e^{2\pi i a_1/r}, \ldots, e^{2\pi i a_n/r}), \quad 0 \leq a_k < r$$

  is said to be of the age 1 if $\sum_k a_k = r$.

- The number of elements g of G such that the fixed locus of g is $\{0\}$ having age 1 is denoted by j_G:
$$j_G = |\{g \in G | \text{age}(g) = 1 \text{ and } \text{Fix}(g) = \{0\}\}| .$$

The mirror theorem of [**15**] states:

THEOREM 5.3 (Theorem 7.1 in [**15**]). *Let $W(x_1, x_2, x_3)$ be an invertible polynomial and G a B–admissible group of symmetries. Then we have:*
$$g_{W^T,G^T} = j_G, \quad A_{W^T,G^T} = \Gamma_{W,G}.$$

Important part of the mirror theorem of Ebeling-Takahashi is based on the stack $\mathcal{C}_{(W^T,G^T)}$. This is an argument to the treatment of the GW theory proposed in Section 3.1 of Chapter 3.

1.1. Gabrielov numbers of $(\tilde{E}_8, \mathbb{Z}_3)$. Consider the polynomial
$$W(x_1, x_2, x_3) = x_1^6 + x_2^3 + x_3^2.$$
Obviously it is self-dual: $W^T = W$ but we put the transpose mark anyway to make clear the side of the mirror on which the singularity appears.

Consider the symmetry group $\mathbb{Z}_3$ acting on W as at the beginning of the chapter. For CY-LG mirror symmetry we are interested in the GW theory of the curve $\mathcal{C}_{(W^T,\mathbb{Z}_3^T)}$. Theorem 5.3 above shows that it's enough to compute the Gabrielov numbers of the pair $(W, \mathbb{Z}_3)$ in order to get the orbifold structure of $\mathcal{C}_{(W^T,\mathbb{Z}_3^T)}$.

To compute the Gabrielov numbers $\Gamma_{W,\mathbb{Z}_3}$ note that:
$$K_x = K_y = \{id\} \text{ and } K_z = \mathbb{Z}_3.$$
We get:
$$\frac{p_x}{|\mathbb{Z}_3/K_x|} * |K_x| = \frac{6}{3} = 2, \quad \frac{p_y}{|\mathbb{Z}_3/K_y|} * |K_y| = \frac{3}{3} = 1, \quad \frac{p_z}{|\mathbb{Z}_3/K_z|} * |K_z| = \frac{2}{1} * 3 = (2, 2, 2).$$
Hence we conclude:
$$\Gamma_{\tilde{E}_8,\mathbb{Z}_3} = (2, 2, 2, 2).$$
It is not hard to see that $j_{\mathbb{Z}_3} = 0$. This suggests the orbifold $\mathbb{P}^1_{2,2,2,2}$ to be a candidate for the CY-LG mirror symmetry of $(\tilde{E}_8, \mathbb{Z}_3)$.

2. CY-LG mirror symmetry for $(\tilde{E}_8, \mathbb{Z}_3)$

The two theorems announced at the beginning of the chapter are proved in this section.

Consider the unfolding of $\tilde{E}_8$:
$$\begin{aligned} F(\mathbf{x}, \mathbf{s}) := {} & x^6 + y^3 + z^2 + s_{-1}x^4y + s_{31}x^3y \\ & + s_{21}x^2y + s_{11}xy + s_{30}x^3 + s_{20}x^2 + s_{10}x + s_{01}y + s_0. \end{aligned}$$

The state space of the B-model $(\tilde{E}_8, \mathbb{Z}_3)$ reads
$$\mathcal{H} := (\mathcal{L})^G \oplus \langle 1_h \rangle \oplus \langle 1_{h^2} \rangle.$$
The first summand stands for the invariant part of the Milnor algebra and is generated by:
$$(\mathcal{L})^G \cong \langle 1, x^4y, xy, x^3 \rangle_{\mathbb{C}}.$$

NOTATION 5.1. *Denote by $\mathcal{F}^{\mathbb{Z}_3}$ the potential of the Frobenius manifold $M_{\tilde{E}_8,\mathbb{Z}_3}$ written in coordinates $t_{\mathbf{1},0},\dots,t_{\mathbf{1},3},t_h,t_{h^2}$ such that:*

$$\frac{\partial}{\partial t_{\mathbf{1},k}} \leftrightarrow e_k \in (\mathcal{L})^G,\ \frac{\partial}{\partial t_h} \leftrightarrow 1_h \in \mathcal{H},\ \frac{\partial}{\partial t_{h^2}} \leftrightarrow 1_{h^2} \in \mathcal{H}.$$

2.1. The strategy of the proof. Let $M_{(\tilde{E}_8)^G} \subset M_{(\tilde{E}_8,\{Id\})}$ be a Frobenius manifold such that

$$\mathcal{T}M_{(\tilde{E}_8)^G}\mid_{\mathbf{t}=0} \cong (\mathcal{L})^G.$$

The main idea of our treatment can be represented by the following diagram.

$$\begin{tikzcd}
M_{(\tilde{E}_8,\{Id\})} \arrow[rr, "A"] & & M_{\mathbb{P}^1_{6,3,2}} & \\
& (M_{(\tilde{E}_8,id)})^G \arrow[ul, hook, "E"'] \arrow[rr, "J"] \arrow[dd, equal, "D"] & & M^4 \arrow[ul, hook', "K"'] \arrow[dd, "C"] \\
M_{(\tilde{E}_8,\mathbb{Z}_3)} \arrow[rr, dashed, "B"] & & M_{\mathbb{P}^1_{2,2,2,2}} & \\
& M_{(\tilde{E}_8)^G} \arrow[ul, hook, "F"'] \arrow[rr, "I"] & & N^4 \arrow[ul, hook', "L"']
\end{tikzcd}$$

where

- A is the Mirror isomorphism of Theorem 2.9 .
- E and F are embeddings of certain submanifolds in $M_{(\tilde{E}_8,\{Id\})}$ and $M_{(\tilde{E}_8,\mathbb{Z}_3)}$ respectively. The first submanifold is identified in Proposition 5.4 and the second corresponds to the restriction to the untwisted sector.
- D is the isomorphism of the orbifolded LG B-model untwisted sector axiom.
- J and K are obtained by the restriction of the isomorphism A to the submanifold $\left(M_{\tilde{E}_8}\right)^G$.
- C is the isomorphism of Proposition 3.13.
- I is obtained by commutativity $C \cdot J = I \cdot D$. It defines mirror isomorphism on the submanifolds of $M_{(\tilde{E}_8,\mathbb{Z}_3)}$ and $M_{\mathbb{P}^1_{2,2,2,2}}$.
- Finally we show that there is a unique extension of the isomorphism I to the full manifolds.

REMARK 5.1. In the diagram above we have completely skipped the primitive form for $M_{(\tilde{E}_8,\{\mathrm{id}\})}$. However particular choice of it — ζ_{LCSL} is assumed by the mirror isomorphism A. At the same time the "untwisted sector" axiom of a LG B–model asserts that the Frobenius manifold $M_{(\tilde{E}_8,\mathbb{Z}_3)}$ is a particular phase of the orbifolded LG B–model $(\tilde{E}_8,\mathbb{Z}_3)$ agreeing with the primitive form ζ_{LCSL} of $\tilde{E}_8$.

Our statement is that the mirror isomorphism B is completely defined by this diagram. Namely by the "unorbifolded" CY-LG mirror symmetry and by the axioms of the orbifolded LG B-model.

2.2. Analysis of the untwisted sector of $M_{\tilde{E}_8,\mathbb{Z}_3}$. The dependence of the flat coordinates of $M_{\tilde{E}_8}$ on the "natural" coordinates s_{ij} is rather complicated (recall Theorem 2.7). However the next proposition shows that for the "right" choice of the symmetry group this relation has a transparent meaning.

In what follows we assume the primitive form and the flat coordinates for $\tilde{E}_8$ to be fixed as in Theorem 2.9.

PROPOSITION 5.4. *The restriction to the invariant part of the Milnor algebra is obtained by setting to zero certain flat coordinates of* $M_{\tilde{E}_8}$.

PROOF. Note that the restriction to the invariant part of $\mathcal{L}$ in terms of the singularity unfolding is given by setting certain variables s_{ij} to zero:

$$\mathcal{L}_{\tilde{E}_8} \to (\mathcal{L}_{\tilde{E}_8})^G \iff s_{ij} = 0 \text{ for } (i,j) \in \mathcal{I}_{tw}.$$

It is easy to see that for $\tilde{E}_8$ the only non-invariant generators of the Milnor algebra are the following:

$$x, x^3y, y, x^2, x^2y, x^4 \notin (\mathcal{L}_{\tilde{E}_8})^{\mathbb{Z}_3}.$$

and hence $\mathcal{I}_{tw} = \{10, 31, 01, 20, 21, 40\}$.

The set I of all indices of s_{ij} in the unfolding reads:

$$I = \mathcal{I}_{tw} \sqcup \mathcal{I}_{inv}, \quad \text{for} \quad \mathcal{I}_{inv} = \{41, 11, 30, 0\}.$$

The degrees of the corresponding variables read:

$$\left\{\frac{5}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right\} \quad \text{and} \quad \left\{0, \frac{1}{2}, \frac{1}{2}, 1\right\}.$$

Consider the formulae by Noumi-Yamada (see Theorem 2.7). It is easy to see from the degrees computed that whenever $\nu \in \mathcal{I}_{tw}$ the summation in $\psi_\nu^{(1)}$ does not contain the elements $\alpha \in \mathcal{I}_{inv}$ except $\alpha = 41$. But the corresponding variable has degree 0, while all variables from $\mathcal{I}_{tw}$ have positive degrees. Hence in every summand of the function $\psi_\nu^{(1)}$ for $\nu \in \mathcal{I}_{tw}$ there is at least one multiple s_μ with $\mu \in \mathcal{I}_{tw}$. Hence we get:

$$\psi_\nu^{(1)}(\mathbf{s}) \mid_{s_\mu=0,\ \mu\in\mathcal{I}_{tw}} = 0, \quad \forall \nu \in \mathcal{I}_{tw}.$$

This concludes the proof of the proposition. □

In what follows we are going to work with the Frobenius submanifolds. Namely we consider the restriction of the Frobenius structure to the submanifold. This topic was studied by I. Strachan in [**44**] where the Frobenius structure was defined on the tangent space of the submanifold $M' \subset M$. Our approach is slightly different. Given the WDVV potential $\mathcal{F}(\mathbf{t}, \mathbf{t}')$ we consider the function $\mathcal{F}'(\mathbf{t}')$:

$$\mathcal{F}'(\mathbf{t}') := \mathcal{F}(\mathbf{t}, \mathbf{t}') \mid_{\mathbf{t}=0} . \tag{5.1}$$

Obviously it satisfies the WDVV equation too.

NOTATION 5.2. *For two Frobenius manifolds* M *and* M' *we will write:*

$$M' = M \mid_{t=0} \quad \text{or} \quad M' \subset M$$

if the potentials $\mathcal{F}'$ *and* $\mathcal{F}$ *are related as in* (5.1).

PROPOSITION 5.5. *There is a unique rank* 4 *Frobenius submanifold* $N^4 \subset M_{\mathbb{P}^1_{2,2,2,2}}$ *such that*

$$\mathcal{T}N^4 \mid_{t=0} \cong (\mathcal{L}_{\tilde{E}_8})^G.$$

PROOF. Let t_{ij} be corresponding to the generators Δ_{ij} of $H^*_{orb}(\mathbb{P}^1_{6,3,2})$ and be therefore variables in $M_{\mathbb{P}^1_{6,3,2}}$. From the isomorphism formula (2.7) of the CY-LG mirror symmetry we see that

$$M_{\tilde{E}_8} \mid_{s_{ij}=0,(i,j)\in\mathcal{I}_{tw}} \quad \cong \quad M_{\mathbb{P}^1_{6,3,2}} \mid_{t_{ij}=0,(i,j)\in\mathcal{J}}$$

for $\mathcal{J} = \{11, 12, 14, 15, 21, 22\}$ as in Proposition 3.13 and $\mathcal{I}_{tw}$ as in Proposition 5.4.

Introduce the notation:

$$M^4 := M_{\mathbb{P}^1_{6,3,2}} \mid_{t_{ij}=0,(i,j)\in\mathcal{J}} .$$

We have to find the image of M^4 under C from the main diagram.

The Frobenius potential $\mathcal{F}^4$ of M^4 was computed explicitly in Proposition 3.13. The quasi-homogeneity of the Frobenius manifold potential $\mathcal{F}^4$ is given by the Euler vector field E^4:

$$E^4 = t_0\frac{\partial}{\partial t_0} + \frac{1}{2}t_{13}\frac{\partial}{\partial t_{13}} + \frac{1}{2}t_{31}\frac{\partial}{\partial t_{31}},\ E^4\cdot\mathcal{F}^4 = 2\mathcal{F}^4.$$

The general rank 4 Frobenius submanifold N^4 in $M_{\mathbb{P}^1_{2,2,2,2}}(t_0,t_{-1},t_1,t_2,t_3,t_4)$ satisfying the same quasihomogeneity conditions is given by

$$N^4(t_0,\tilde{t}_1,\tilde{t}_2,\tilde{t}_{-1}) = M_{\mathbb{P}^1_{2,2,2,2}}(\mathbf{t}(\tilde{\mathbf{t}})),$$

with

$$t_i = \tilde{t}_1 a_i + \tilde{t}_2 b_i,\quad 1\le i\le 4,$$

where $a_i, b_i \in \mathbb{C}$, and

$$\tilde{t}_{-1} = \phi(t_{-1}),$$

for some function ϕ such that $\phi(0)=1$.

We have $N^4 \cong M^4$ if and only if

$$k\cdot\mathcal{F}^4 \mid_{t_{13}=\tilde{t}_1,t_{31}=\tilde{t}_2} \;=\; \mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}} \mid_{\mathbf{t}(\tilde{\mathbf{t}})},$$

for some non-zero number k that is fixed by equating pairings on both sides to be $k=3$. Writing this equality explicitly using also algebraic independence of f_0, f_1 and f_2 we get the system of equations. Up to a symmetry interchanging the 4 coordinates t_i it gives only two solutions:

$$a_1=a_2=a_3=0,\ b_0=0,\ a_0=\pm\frac{1}{\sqrt{3}}\ b_1=b_2=b_3=\pm\frac{1}{\sqrt{3}},$$

Obviously both solutions give the same Frobenius submanifold.

□

2.3. Proof of Theorem 5.1. Using the axioms of the LG B-model we get:

PROPOSITION 5.6. *Let $\mathcal{F}^{\mathbb{Z}_3}$ satisfy the axioms of the orbifolded LG B-model (see Chapter 4). Then $\mathcal{F}^{\mathbb{Z}_3}$ depends effectively on $t_{\mathbf{1},p}$ for $0\le p\le 3$, $t_h t_{h^2}$, t_h^3 and $t_{h^2}^3$ only:*

$$\mathcal{F}^{\mathbb{Z}_3} = \mathcal{F}^{\mathbb{Z}_3}(t_{\mathbf{1},0},t_{\mathbf{1},1},t_{\mathbf{1},2},t_{\mathbf{1},3},t_h t_{h^2},t_h^3,t_{h^2}^3).$$

PROOF. The statement follows immediately from the equivariance axiom. □

We are going to use in this section the explicit form of the WDVV potential $\mathcal{F}^4$. However we consider it in the new coordinates that are convenient for the orbifolded LG B-model:

$$\mathcal{F}^4(t_{\mathbf{1},0},\dots,t_{\mathbf{1},3}) = \mathcal{F}^4 \mid_{t_0=t_{\mathbf{1},0},\ t_{13}=t_{\mathbf{1},1},\ t_{31}=t_{\mathbf{1},2},\ t_{-1}=t_{\mathbf{1},3}} .$$

PROPOSITION 5.7. *We have:*

- *The potential $\mathcal{F}^{\mathbb{Z}_3}$ of $M_{(\tilde{E}_8,\mathbb{Z}_3)}$ reads:*

$$\mathcal{F}^{\mathbb{Z}_3} = \mathcal{F}^4 + t_{\mathbf{1},0}t_h t_{h^2} + H(t_{\mathbf{1},1},t_{\mathbf{1},2},t_{\mathbf{1},3},t_h,t_{h^2}),$$

where H is some function satisfying $H \mid_{t_h=0,t_{h^2}=0}\equiv 0$.

- *The potential $\mathcal{F}^{\mathbb{Z}_3}$ satisfies the following quasi-homogeneity condition:*

$$E^{\mathbb{Z}_3} \cdot \mathcal{F}^{\mathbb{Z}_3} = 2\mathcal{F}^{\mathbb{Z}_3},$$

where

$$E^{\mathbb{Z}_3} = t_0 \frac{\partial}{\partial t_0} + \frac{1}{2} t_{\mathbf{1},1} \frac{\partial}{\partial t_{\mathbf{1},1}} + \frac{1}{2} t_{\mathbf{1},2} \frac{\partial}{\partial t_{\mathbf{1},2}} + \frac{1}{2} t_h \frac{\partial}{\partial t_h} + \frac{1}{2} t_{h^2} \frac{\partial}{\partial t_{h^2}}.$$

Proof. The first part of the proposition easily follows from the pairing axiom and untwisted sector axiom of the LG B-model. Note that the quasi-homogeneity of $\mathcal{F}^4$ fixes the conformal dimension of the Frobenius manifold. Hence the Euler vector field $E^{\mathbb{Z}_3}$ of $\mathcal{F}^{\mathbb{Z}_3}$ reads:

$$E^{\mathbb{Z}_3} = E^4 + d_h t_h \frac{\partial}{\partial t_h} + d_{h^2} t_{h^2} \frac{\partial}{\partial t_{h^2}}.$$

From the pairing we get: $d_h + d_{h^2} = 1$. Applying also Proposition 5.6 we get the statement. $\square$

By the quasi-homogeneity condition on $\mathcal{F}^{\mathbb{Z}_3}$ we get:

$$\begin{aligned}
\mathcal{F}^{\mathbb{Z}_3} = & \frac{1}{2} t_{\mathbf{1},0}^2 t_{\mathbf{1},3} + t_{\mathbf{1},0} \left(\frac{t_{\mathbf{1},1}^2}{12} + \frac{t_{\mathbf{1},2}^2}{4} \right) + t_{\mathbf{1},0} t_h t_{h^2} \\
& + \frac{1}{36} t_{\mathbf{1},1}^4 f_1(t_{\mathbf{1},3}) + \frac{1}{18} t_{\mathbf{1},1}^2 t_{\mathbf{1},2}^2 f_2(t_{\mathbf{1},3}) + \frac{1}{9} t_{\mathbf{1},1} t_{\mathbf{1},2}^3 f_0(t_{\mathbf{1},3}) + t_{\mathbf{1},2}^4 \left(\frac{1}{12} f_1(t_{\mathbf{1},3}) + \frac{1}{18} f_2(t_{\mathbf{1},3}) \right) \\
& + t_h t_{h^2} \left(t_{\mathbf{1},1}^2 b_1(t_{\mathbf{1},3}) + t_{\mathbf{1},2}^2 b_2(t_{\mathbf{1},3}) + t_{\mathbf{1},1} t_{\mathbf{1},2} b_3(t_{\mathbf{1},3}) \right) + t_h^2 t_{h^2}^2 b_4(t_{\mathbf{1},3}) \\
& + t_h^3 \left(t_{\mathbf{1},1} b_5(t_{\mathbf{1},3}) + t_{\mathbf{1},2} b_6(t_{\mathbf{1},3}) \right) + t_{h^2}^3 (t_{\mathbf{1},1} b_7(t_{\mathbf{1},3}) + t_{\mathbf{1},2} b_8(t_{\mathbf{1},3})),
\end{aligned}$$

for some functions $b_i(t_{\mathbf{1},p})$.

In what follows we are going to analyze the WDVV equation for $\mathcal{F}^{\mathbb{Z}_3}$. Recall that the WDVV equation has four parameters (see (1.1)).

Notation 5.3. *Let M be a Frobenius manifold with the potential $\mathcal{F}$. For any t_i, t_j, t_k, t_l - coordinates on M denote:*

$$WDVV(\partial_i, \partial_j, \partial_k, \partial_l) := \sum_{p,q} \left(\frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_p} \ \eta^{pq} \ \frac{\partial^3 \mathcal{F}}{\partial t_q \partial t_k \partial t_l} - \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_k \partial t_p} \ \eta^{pq} \ \frac{\partial^3 \mathcal{F}}{\partial t_q \partial t_j \partial t_l} \right)$$

Proposition 5.8. *The functions $b_i(t)$ such that $\mathcal{F}^{\mathbb{Z}_3}$ satisfies the WDVV equation are unique up to the following transformation:*

$$t_h \to a t_h, t_{h^2} \to t_{h^2}/a, \quad a \in \mathbb{C}^*,$$

that is an obvious symmetry of the WDVV equation.

Proof. Let $b_8(t) \equiv 0$ and $b_7(t) \not\equiv 0$. Then from the WDVV$(\partial_{h^2}, \partial_{h^2}, \partial_{(\mathbf{1},2)}, \partial_{(\mathbf{1},2)})$ taking the coefficient in front of $t_{(\mathbf{1},2)} t_{h^2}$ we get:

$$b_7(t) f_0(t) \equiv 0,$$

what contradicts $f_0(t) \not\equiv 0$. The case $b_8(t) \not\equiv 0$ and $b_7(t) \equiv 0$ is done in a similar way.

If we have $b_8(t) \equiv 0$ and $b_7(t) \equiv 0$, it is not hard to show that either all other $b_k(t) \equiv 0$ too, or $f_0(t) \equiv 0$, what contradicts the exact formula for it.

Assume $b_8(t) \not\equiv 0$ and $b_7(t) \not\equiv 0$. It is a computational exercise to show that the WDVV equation on $\mathcal{F}^{\mathbb{Z}_3}$ is equivalent to the following system:

$$
\begin{aligned}
\frac{d}{dt_{\mathbf{1},3}} b_8(t_{\mathbf{1},3}) &= -\frac{8}{3} b_8(t_{\mathbf{1},3}) \frac{2(f_2(t_{\mathbf{1},3}))^2 - 6f_2(t_{\mathbf{1},3})f_1(t_{\mathbf{1},3}) - 3(f_0(t_{\mathbf{1},3}))^2}{f_2(t_{\mathbf{1},3}) - 3f_1(t_{\mathbf{1},3})}, \\
b_1(t_{\mathbf{1},3}) &= \frac{2}{9} f_2(t_{\mathbf{1},3}), \\
b_2(t_{\mathbf{1},3}) &= 2f_1(t_{\mathbf{1},3}), \\
b_3(t_{\mathbf{1},3}) &= -\frac{2}{3} f_0(t_{\mathbf{1},3}), \\
b_4(t_{\mathbf{1},3}) &= \frac{2}{3} f_2(t_{\mathbf{1},3}) + 2f_1(t_{\mathbf{1},3}), \\
b_5(t_{\mathbf{1},3}) &= \frac{8}{81} \frac{3f_0(t_{\mathbf{1},3})f_1(t_{\mathbf{1},3}) - f_2(t_{\mathbf{1},3})f_0(t_{\mathbf{1},3})}{b_8(t_{\mathbf{1},3})}, \\
b_6(t_{\mathbf{1},3}) &= \frac{8}{81} \frac{(f_2(t_{\mathbf{1},3}))^2 - 6f_1(t_{\mathbf{1},3})f_2(t_{\mathbf{1},3}) + 9(f_1(t_{\mathbf{1},3}))^2}{b_8(t_{\mathbf{1},3})}, \\
b_7(t_{\mathbf{1},3}) &= \frac{b_8(t_{\mathbf{1},3})f_0(t_{\mathbf{1},3})}{3f_1(t_{\mathbf{1},3}) - f_2(t_{\mathbf{1},3})}.
\end{aligned}
$$

For example the expression for $b_1(t_{\mathbf{1},3})$ is obtained from the WDVV$(\partial_h, \partial_{h^2}, \partial_{(\mathbf{1},1)}, \partial_{(\mathbf{1},2)})$, $b_7(t_{\mathbf{1},3})$ is expressed via $b_8(t_{\mathbf{1},3})$ by WDVV$(\partial_{h^2}, \partial_{h^2}, \partial_{(\mathbf{1},1)}, \partial_{(\mathbf{1},2)})$ and PDE on $b_8(t_{\mathbf{1},3})$ follows from WDVV$(\partial_{h^2}, \partial_{h^2}, \partial_{(\mathbf{1},2)}, \partial_{(\mathbf{1},2)})$.

The only unknown function of this system is the function $b_8(t_{\mathbf{1},3})$ that is subject to the PDE written in the first line. Using the PDE on $f_1(t)$ and $f_2(t)$ (see (3.2)) it can be rewritten:

$$\frac{d}{dt} b_8(t) = b_8(t) \frac{\frac{d}{dt}(f_2(t) - 3f_1(t))}{f_2(t) - 3f_1(t)}.$$

As far as $b_8 \not\equiv 0$ we can rewrite:

$$\frac{d}{dt} \log b_8(t) = \frac{d}{dt} \log(f_2(t) - 3f_1(t)).$$

This equation can be solved explicitly:

$$b_8(t) = c\,(3f_1(t) - f_2(t)), \quad c \in \mathbb{C} \backslash \{0\}.$$

The only freedom of the functions $b_1, \ldots, b_8$ is this factor of c in b_8. This ambiguity appears only in the functions b_8 and b_7 as the multiple c and in the functions b_5 and b_6 as the multiple $1/c$.

Note that this corresponds exactly to the rescaling $t_h \to t_h/c^{1/3}, t_{h^2} \to c^{1/3} t_{h^2}$ that is an obvious symmetry of the WDVV equation since $\eta(\partial_{t_h}, \partial_{t_{h^2}}) = 1$. $\square$

Taking $c = 1$ in the formulae above we get the potential of $M_{(\tilde{E}_8, \mathbb{Z}_3)}$. This concludes the proof of Theorem 5.1.

2.4. Proof of Theorem 5.2. We give the isomorphism $M_{(\tilde{E}_8, \mathbb{Z}_3)} \cong M_{\mathbb{P}^1_{2,2,2,2}}$ explicitly. Let the function $\mathcal{F}^{\mathbb{Z}_3}$ be the WDVV potential of $M_{(\tilde{E}_8, \mathbb{Z}_3)}$.

Consider $b_8(t_{\mathbf{1},3})$ to be the coefficient of $t_{h^2}^3 t_{\mathbf{1},2}$ in it.

$$b_8(t_{\mathbf{1},3}) := \left[t_{h^2}^3 t_{\mathbf{1},2}\right] \ \mathcal{F}^{\mathbb{Z}_3}.$$

Define $c := b_8(0)/(3f_1(0) - f_2(0))$.

Then the isomorphism is given by:

$$t_0 \to t_{\mathbf{1},0}, \quad t_1 \to -\frac{t_{\mathbf{1},1}}{\sqrt{3}}, \quad t_2 \to -\frac{t_{\mathbf{1},2}}{\sqrt{3}} + ct_h + \frac{2t_{h^2}}{3c}, \quad t_{-1} \to t_{\mathbf{1},3},$$
$$t_3 \to -\frac{\sqrt{3}}{3} t_{\mathbf{1},2} + \frac{\left(-1+\sqrt{-3}\right)c}{2} t_h - \frac{\left(1+\sqrt{-3}\right)}{3c} t_{h^2},$$
$$t_4 \to -\frac{\sqrt{3}}{3} t_{\mathbf{1},2} - \frac{\left(1+\sqrt{-3}\right)c}{2} t_h + \frac{\left(-1+\sqrt{-3}\right)}{3c} t_{h^2}.$$

Using the explicit expression of the $M_{(\tilde{E}_8,\mathbb{Z}_3)}$ potential given in Theorem 5.1 it is checked explicitly that applying this change of variables to $\mathcal{F}^{\mathbb{Z}_3}$ we get the WDVV potential of $\mathbb{P}^1_{2,2,2,2}$.

CHAPTER 6

GW theory of $\mathbb{P}^1_{2,2,2,2}$ and Hurwitz-Frobenius manifold

In [**11**, Lecture 5] B. Dubrovin introduced a Frobenius manifold structure on the space of the ramified covering of the sphere. Such Frobenius manifolds are now known as Hurwitz-Frobenius manifolds. Particular interest in the Hurwitz-Frobenius manifolds in this thesis originates from the following theorem that appeared first in [**4**].

Let z be a coordinate on the elliptic curve $\mathcal{E}_{2\omega_1,2\omega_2}$ with the periods $2\omega_1, 2\omega_2$. Consider the space of functions $\mathcal{H}_{1,(2,2,2,2)} := \{\lambda : \mathcal{E}_{2\omega_1,2\omega_2} \to \mathbb{P}^1\}$ having the following explicit form.

$$\lambda(z) := \sum_{i=1}^{4} \left(\wp(z-a_i; 2\omega_1, 2\omega_2) u_i + \frac{1}{2} \frac{\wp'(z-a_i; 2\omega_1, 2\omega_2)}{\wp(z-a_i; 2\omega_1, 2\omega_2)} s_i \right) + c, \tag{6.1}$$

with $\omega_1, \omega_2, a_i, u_i, s_i, c$ - parameters of λ. Consider the subspace $\mathcal{H}^R_{1,(2,2,2,2)} \subset \mathcal{H}_{1,(2,2,2,2)}$ consisting of λ as above such that:

$$\begin{aligned} &a_1 = 0, \quad a_2 = \omega_1 + \omega_2, \quad a_3 = \omega_1, \quad a_4 = \omega_2, \\ &s_1 = s_2 = s_3 = s_4 = 0. \end{aligned} \tag{6.2}$$

THEOREM 6.1 (Theorem 1 in [**4**]). *The space $\mathcal{H}^R_{1,(2,2,2,2)}$ has Frobenius manifold structure isomorphic to the Frobenius structure of the GW theory of $\mathbb{P}^1_{2,2,2,2}$.*

1. Space of ramified coverings

Consider the space of meromorphic functions

$$C \xrightarrow{\lambda} \mathbb{P}^1$$

on the compact genus g Riemann surface C. Fix the pole orders of λ to be $\mathbf{k} := \{k_1, \dots, k_m\}$:

$$\lambda^{-1}(\infty) = \{\infty_1, \dots, \infty_m\}, \quad \infty_p \in C,$$

so that locally at ∞_p we have $\lambda(z) = z^{k_p}$.

Such meromorphic functions define the ramified coverings of $\mathbb{P}^1$ by C with the ramification profile $\mathbf{k}$ over ∞. We further assume that λ has only simple ramification points at $P_q \in \mathbb{P}^1 \backslash \{0\}$. The degree of the ramified covering is computed to be $N = \sum k_p$ and using the Riemann-Hurwitz formula we can compute the dimension of this space of functions:

$$n = 2g - 2 + \sum_{p=1}^{m} k_p + m,$$

that is exactly the number of simple ramification points. The smooth part of the Hurwitz-Frobenius manifold is parametrized by the values of λ at the simple ramification points: $(\lambda(P_1), \dots, \lambda(P_n))$.

DEFINITION. Two pairs (C_1, λ_1) and (C_2, λ_2) as above are said to be Hurwitz-equivalent if $\lambda_1 = \psi \circ \lambda_2$ for some analytic map $\psi : C_1 \to C_2$.

In what follows we consider the pairs (C, λ) up to the equivalence introduced.

DEFINITION. We define the Hurwitz-Frobenius manifold $\mathcal{H}_{g;\mathbf{k}}$ to be the moduli space of pairs (C, λ) as above with the additional data:

- $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ - the choice of a symplectic basis in $H_2(C)$,
- $\{w_1, \ldots, w_n\}$ - uniformization parameter of λ at ∞_i

$$w_p^{k_p}(z) = \lambda(z), \quad z \in U(\infty_p).$$

1.1. Frobenius manifold structure on $\mathcal{H}_{g;\mathbf{k}}$. Following Dubrovin we define a Frobenius manifold structure on $\mathcal{H}_{g;\mathbf{k}}$. Let ϕ be a differential of the first kind on C. Define the multi-valued coordinate $v(P)$ on C as:

$$v(P) = \int_{\infty_1}^{P} \phi. \tag{6.3}$$

THEOREM 6.2 (Theorem 5.1 in [**11**]). *The following functions are flat coordinates on $\mathcal{H}_{g,\mathbf{k}}$:*

$$\begin{aligned}
t_{p;a} &:= \mathrm{res}_{\infty_p} (w_p)^{-a} v d\lambda, & m \geq p \geq 1,\ k_p > a \geq 1,\\
v_r &:= \int_{\infty_1}^{\infty_r} \phi, \quad V_r := -\mathrm{res}_{\infty_r} \lambda\phi, & m \geq r > 1,\\
B_q &:= \oint_{b_q} \phi, \quad C_q := \oint_{a_q} \lambda\phi. & g \geq q \geq 1.
\end{aligned}$$

Let ∂_k be the basis vectors in $T\mathcal{H}_{g,\mathbf{k}}$ w.r.t. the flat coordinates introduced and $\lambda' = \partial_v \lambda$. Define structure constants of the multiplication $c(\cdot,\cdot,\cdot)$ and pairing $\eta(\cdot,\cdot)$:

$$\begin{aligned}
\eta(\partial_k, \partial_l) &:= \sum \mathrm{res}_{\lambda'=0} \frac{\partial_k \lambda \partial_l \lambda \mathrm{d}v}{\lambda'},\\
c(\partial_k, \partial_l, \partial_m) &:= \sum \mathrm{res}_{\lambda'=0} \frac{\partial_k \lambda \partial_l \lambda \partial_m \lambda \mathrm{d}v}{\lambda'}.
\end{aligned} \tag{6.4}$$

The theorem of Dubrovin states that this multiplication and pairing define a Frobenius manifold structure on $\mathcal{H}_{g;\mathbf{k}}$ with the coordinates introduced above playing the role of flat coordinates. Namely, in these coordinates we have $\partial_k \eta_{lm} = 0$. For the particular choice of flat coordinates as above the only non-vanishing entries of η are:

$$\eta_{t_{p;a}, t_{q;b}} = \frac{1}{k_p} \delta_{p,q} \delta_{a+b,k_p}, \quad \eta_{v_p, V_q} = \frac{1}{k_p} \delta_{p,q}, \quad \eta_{B_p, C_q} = \frac{1}{2\pi\sqrt{-1}} \delta_{p,q}.$$

DEFINITION. The function $\mathcal{F}^H$ called Frobenius (or WDVV) potential is defined by:

$$\partial_k \partial_l \partial_m \mathcal{F}^H = c(\partial_k, \partial_l, \partial_m).$$

It is clear from the definition that the multiplication defined by the structure constants $c(\partial_k, \partial_l, \partial_m)$ is commutative and associative. From the second property it follows that $\mathcal{F}^H$ is a solution of the WDVV equation (1.1).

In what follows we are interested in the algebra structure defined by $\mathcal{F}^H$. Therefore we assume it up to the quadratic terms in the variables.

2. Elliptic functions and theta constants

We will use extensively the theory of elliptic functions in our treatment of the Hurwitz-Frobenius manifolds.

2.1. Elliptic functions. Consider the lattice $\Lambda = 2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z}$ with $\omega_2/\omega_1 \in \mathbb{H}$. We will denote by D its fundamental domain.

Definition. A meromorphic function f on $\mathbb{C}$ is called elliptic w.r.t. the lattice Λ if it satisfies the following periodicity properties:

$$f(z+2\omega_1) = f(z), \quad f(z+2\omega_2) = f(z).$$

Recall the Weierstrass elliptic function:

$$\wp\left(z; 2\omega_1, 2\omega_2\right) := \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

It is obvious from the definition that $\wp$ is indeed an elliptic function. Another important example is its derivative $\wp'$ that is an elliptic function with the same periods.

Proposition 6.3. *The space of elliptic functions on the elliptic curve $E = \mathbb{C}/\Lambda$ is generated by $\wp$ and $\wp'$:*

$$\mathcal{M}(\mathcal{E}) = \mathbb{C}(\wp, \wp').$$

For our purposes it is helpful to rewrite the expansion of $\wp$ and $\wp'$ in z and $\tau := \omega_2/\omega_1$:

$$\begin{aligned}\wp(z,\tau) &= z^{-2} + \frac{1}{20}g_2(\tau)z^2 + \frac{1}{28}g_3(\tau)z^4 + O(z^6),\\ \wp'(z,\tau) &= -2z^{-3} + \frac{2}{20}g_2(\tau)z + \frac{4}{28}g_3(\tau)z^3 + O(z^5),\end{aligned}$$

for $g_2(\tau), g_3(\tau)$ - modular invariants of the elliptic curve.

The connection between the two definitions of the function $\wp$ is given by the equality:

$$(2\omega_1)^2\wp(z; 2\omega_1, 2\omega_2) = \wp\left(\frac{z}{2\omega_1}; \tau\right). \tag{6.5}$$

Another important property of the elliptic functions is the following:

Proposition 6.4. *Let $f(z)$ be an elliptic function. Then the sum of its residues in the fundamental domain D of Λ is zero:*

$$\sum_{a\in D} \mathrm{res}_{z=a} f(z)\mathrm{d}z = 0.$$

Definition. The Weierstrass zeta-function is defined by:

$$\zeta(z; 2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right).$$

Its main property is:

$$-\zeta'(z; 2\omega_1, 2\omega_2) = \wp(z; 2\omega_1, 2\omega_2).$$

Note that it is not periodic w.r.t. Λ.

DEFINITION. The quasi-periods $2\eta_k$ are defined by:

$$2\eta_k = \zeta(2\omega_k + z) - \zeta(z), \quad \forall z \in \mathbb{C}.$$

The connection between the periods and quasi-periods of the lattice Λ is given via the Legendre identity:

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi\sqrt{-1}}{2}.$$

2.2. Theta constants and elliptic functions. In what follows we use the values of $\wp(v, \tau)$ at the middle points of the period rectangle edges.

DEFINITION. Let $\wp(z) = \wp(z; 2\omega_1, 2\omega_2)$. The numbers $e_1, e_2, e_3 \in \mathbb{C}$ are defined by:

$$e_1 := \wp(\omega_1), \quad e_2 := \wp(-\omega_1 - \omega_2), \quad e_3 := \wp(\omega_2).$$

A well-known fact from the elliptic curves theory is that:

PROPOSITION 6.5. *The points ω_1, ω_2 and $\omega_1 + \omega_2$ are all zeros of $\wp'(z)$ in the fundamental domain.*

The quantities e_i are expressed via the theta constants (cf. [**28**, section 6] [1]):

$$\begin{aligned} e_1 &= \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_2''}{\vartheta_2}, \\ e_2 &= \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_3''}{\vartheta_3}, \\ e_3 &= \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_4''}{\vartheta_4}. \end{aligned}$$

Using the heat equation we get:

$$\frac{\vartheta_p''}{\vartheta_p} = 4\pi\sqrt{-1}\frac{\partial_\tau\vartheta_p}{\vartheta_p} = 2\pi\sqrt{-1}X_p.$$

We will also use the expression:

$$\eta_1\omega_1 = -\frac{1}{12}\frac{\vartheta_1'''}{\vartheta_1'}.$$

An important property of the derivatives of the theta constants is the following:

$$\frac{\vartheta_1'''}{\vartheta_1} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4}.$$

Using it together with the heat equation we get:

$$\omega_1\eta_1 = -\frac{1}{12}\sum_{p=2}^{4}\frac{\vartheta_p''}{\vartheta_p} = -\frac{\pi\sqrt{-1}}{6}\sum_{p=2}^{4}X_p = -\frac{\pi\sqrt{-1}}{4}\gamma(\tau).$$

[1]Note the difference in the z coordinate normalization of [**28**] with ours.

3. Hurwitz-Frobenius $\mathcal{H}_{1,(2,2,2,2)}$

The Hurwitz-Frobenius manifold $\mathcal{H}_{1,(2,2,2,2)}$ parametrizes the meromorphic functions on the elliptic curve $\mathcal{E} = \mathbb{C}/(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$, $\lambda : \mathcal{E} \to \mathbb{P}^1$ with a certain additional data.

3.1. The moduli problem. In our setup the function λ is defined on $\mathcal{E}$, therefore it has to be an elliptic function. Due to the ramification fixed it has four order 2 poles. Using Proposition 6.3 we write the generic function of this form:

$$\lambda(z) = \sum_{k=1}^{4} \left(\wp(z - a_k; 2\omega_1, 2\omega_2)u_k + \frac{1}{2}\frac{\wp'(z - a_k; 2\omega_1, 2\omega_2)}{\wp(z - a_k; 2\omega_1, 2\omega_2)} s_k \right) + c, \tag{6.6}$$

from where we have the "moduli":

- a_k – positions of the poles on $\mathcal{E}$,
- u_k, s_k – behavior at the poles,
- c – the shift,
- $2\omega_1, 2\omega_2$ – the "moduli" of the elliptic curve itself.

This sums up to 14 parameters, but they are not completely free of relations. From the Riemann-Hurwitz formula we see that the dimension of the space of such functions $\mathcal{H} := \{\lambda\}$ as above is 12.

Because of being an elliptic function we have:

$$\sum_{z\in D} \mathrm{res}_z \lambda = 0 \quad \Rightarrow \quad \sum_{k=1}^{4} s_k = 0.$$

We assume $s_1 = 0$.

On the covering curve we have $\mathcal{E}_{(2\omega_1, 2\omega_2)} \cong \mathcal{E}_{1,\tau}$ for $\tau = \omega_2/\omega_1$. These two elliptic curves give equivalent ramified coverings w.r.t. the Hurwitz-equivalence.

Because of the automorphisms of the elliptic curve moving its origin we can also assume $a_1 = 0$.

Proposition 6.6. *The Hurwitz-Frobenius manifold $\mathcal{H}_{1,(2,2,2,2)}$ is the space of functions λ as above considered as functions of:*

$$a_2, a_3, a_4, \;\; s_2, s_3, s_4, \;\; u_1, u_2, u_3, u_4, \;\; \omega_2/\omega_1.$$

In what follows we denote for simplicity $\mathcal{H} := \mathcal{H}_{1,(2,2,2,2)}$ and we keep the notation a_1 assuming that it is equal to zero.

3.2. Flat coordinates. Following Dubrovin (see Theorem 6.2) we introduce flat coordinates on the space $\mathcal{H}_{1,(2,2,2,2)}$. To do this one has to fix a certain differential on the covering curve. We take:

$$\phi := \mathrm{d}v = \frac{\mathrm{d}z}{2\omega_1},$$

where z is the coordinate on $\mathcal{E}$.

PROPOSITION 6.7. *The ramified covering λ is given in flat coordinates by:*

$$\begin{aligned}\lambda(z) = &\sum_{k=2}^{4}\left(\frac{1}{4}\wp\left(v - v_k, \tau\right) t_k^2 + \frac{1}{2}\frac{\wp'\left(v - v_k, \tau\right)}{\wp\left(v - v_k, \tau\right)} V_k\right) \\ &+ \frac{1}{4}\wp(v, \tau) t_1^2 + \eta_1\omega_1 \sum_{k=1}^{4} t_k^2 + C_1.\end{aligned} \tag{6.7}$$

The Euler vector field of the Frobenius structure in these coordinates is given by:

$$E_{\mathcal{H}} = C_1 \frac{\partial}{\partial C_1} + \sum \frac{1}{2} t_i \frac{\partial}{\partial t_i} + \sum V_i \frac{\partial}{\partial V_i}. \tag{6.8}$$

PROOF. Using the formulae by Dubrovin (see Theorem 6.2) we compute the flat coordinates.

$$v_k = \frac{a_k}{2\omega_1}, \quad V_k = \frac{s_k}{2\omega_1}, \quad B_1 = \int_0^{2\omega_2} \frac{dz}{2\omega_1} = \tau,$$

where $\tau = \omega_2/\omega_1$ is the modulus of the elliptic curve.

Compute $t_k := t_{k,1}$:

$$t_k = \mathrm{res}_{a_k} \frac{z - a_k}{2\omega_1} \frac{z - a_k}{\sqrt{u_k}} \left(\frac{-2u_k}{(z - a_k)^3} + \text{h.o.t.} \right) = -\frac{\sqrt{u_k}}{\omega_1},$$

where the branch of the square root is fixed by the choice of the uniformization parameter w_k.

Note that: $\wp'/\wp = \frac{\partial}{\partial z}\log(\wp)$. The value of $\zeta(z)$ is not defined at $z = 0$, therefore we have to use the limit computing C_1:

$$\begin{aligned}C_1 =& \frac{1}{2\omega_1} \lim_{\epsilon \to 0} \left[-\sum_k \left(\zeta(z - a_k) u_k + \frac{1}{2} \log \wp(z - a_k) v_k \right) + zc \right]_{\epsilon}^{2\omega_1 - \epsilon} \\ =& \frac{1}{2\omega_1} \sum_k \left(\left(\zeta(-a_k) - \zeta(2\omega_1 - a_k)\right) u_k + \frac{1}{2}\left(\log \wp(2\omega_1 - a_k) - \log \wp(-a_k)\right) v_k \right) + c.\end{aligned}$$

Because of the periodicity of the Weierstrass functions we get:

$$C_1 = c - \frac{\eta_1}{\omega_1} \sum_{k=1}^{4} u_k.$$

Using equality (6.5) we get the proposition. □

In the rest of the chapter we will be working with the function $\lambda(z)$ written in flat coordinates. We will not write the variable τ all the time meaning implicitly that Weierstrass functions inside are $\wp(v, \tau)$.

3.3. Structure constants of $\mathcal{H}_{1,(2,2,2,2)}$. In this section we provide all the computations needed to prove Theorem 6.1. Basically we compute structure constants of $\mathcal{H}_{1,(2,2,2,2)}$ using the formulae (6.4).

In the majority of residues we have to compute we will be dealing with elliptic functions. These will be the cases when the derivative of λ – elliptic function itself – is an elliptic function too. When it is so we can consider the residues at the points v_i instead of looking for points where $\lambda' = 0$.

PROPOSITION 6.8. *Let $f(v)$ be an elliptic function and x_p – its set of poles such that $\lambda'(x_p) \neq 0$. Then we have:*

$$\sum_{y:\ \lambda'(y)=0} \mathrm{res}_{v=y} \frac{f(v)\mathrm{d}v}{\lambda'(v)} = -\sum_p \mathrm{res}_{v=x_p} \frac{f(v)\mathrm{d}v}{\lambda'(v)}.$$

PROOF. The poles of the function $f(v)/\lambda'(v)$ w.r.t. v are:$\{x_p\} \sqcup \{y : \lambda'(y) = 0\}$. The quotient $f(v)/\lambda'(v)$ is an elliptic functions and we have:

$$\sum_p \mathrm{res}_{x_p} \frac{f(v)\mathrm{d}v}{\lambda'(v)} + \sum_{y:\ \lambda'(y)=0} \mathrm{res}_{v=y} \frac{f(v)\mathrm{d}v}{\lambda'(v)} = 0,$$

□

The case when we can not apply this principle is $\partial_\tau \lambda$. For this we use the lemma due to Frobenius-Stickelberger [**17**]:

LEMMA 6.9. *Let $f(z; 2\omega_1, 2\omega_2)$ be elliptic functions with the periods $(2\omega_1, 2\omega_2)$, then the following function is elliptic too with the same periods:*

$$\eta_1 \frac{\partial f}{\partial \omega_1} + \eta_2 \frac{\partial f}{\partial \omega_2} + \zeta \frac{\partial f}{\partial z},$$

where $\zeta = \zeta(z; 2\omega_1, 2\omega_2)$.

PROOF. We give a brief proof.
Differentiating the equality $f(z + 2\omega_1) = f(z)$ w.r.t. ω_1 we get:

$$\frac{\partial}{\partial \omega_1} f(z + 2\omega_1) + 2\frac{\partial}{\partial z} f(z + 2\omega_1) = \frac{\partial}{\partial \omega_1} f(z).$$

Together with the expression of the quasi-period we have:

$$\begin{aligned}
\eta_1 \frac{\partial f(z)}{\partial \omega_1} + \eta_2 \frac{\partial f(z)}{\partial \omega_2} + \zeta(z) \frac{\partial f(z)}{\partial z} &= \eta_1 \frac{\partial f(z + 2\omega_1)}{\partial \omega_1} \\
+ 2\eta_1 \frac{\partial f(z + 2\omega_1)}{\partial z} + \eta_2 \frac{\partial f(z + 2\omega_1)}{\partial \omega_2} &+ (\zeta(z + 2\omega_1) - 2\eta_1) \frac{\partial f(z + 2\omega_1)}{\partial z} \\
= \eta_1 \frac{\partial f(z + 2\omega_1)}{\partial \omega_1} + \eta_2 \frac{\partial f(z + 2\omega_1)}{\partial \omega_2} &+ \zeta(z + 2\omega_1) \frac{\partial f(z + 2\omega_1)}{\partial z}.
\end{aligned}$$

□

Consider the function $f(z; 2\omega_1, 2\omega_2)$, applying the change of variables as in (6.5) we get for $f(v, \tau)$:

$$\eta_1 \frac{\partial f}{\partial \omega_1} + \eta_2 \frac{\partial f}{\partial \omega_2} + \zeta \frac{\partial f}{\partial z} = -2\pi\sqrt{-1}\ \partial_\tau f + \zeta \partial_v f - 2\eta_1 \partial_v f.$$

where we used the Legendre identity.

NOTATION 6.1. *Introduce the notation for the corresponding elliptic function:*

$$h_f(z, t) := -2\pi\sqrt{-1}\ \partial_\tau f + \zeta \partial_v f - 2\eta_1 \partial_v f.$$

4. Restriction of the potential

Definition. Define by $\mathcal{F}_R$ the potential obtained by the restriction of $\mathcal{F}^H$ of the Hurwitz-Frobenius manifold to the submanifold $\mathcal{H}^R_{1,(2,2,2,2)}$:

$$\mathcal{F}_{\mathcal{R}} := \mathcal{F}^H \mid_{\mathcal{A}},$$

for

$$\mathcal{A} := \Big\{v_1 = 0, \quad v_2 = \frac{\tau}{2} + \frac{1}{2}, \quad v_3 = \frac{1}{2}, \quad v_4 = \frac{\tau}{2}, \quad V_2 = V_3 = V_4 = 0\Big\}.$$

It follows from Proposition 6.7 that this restriction agrees with equation (6.2).

Proposition 6.10. *The summands of $\mathcal{F}^H$ including variables v_p and V_p do not contribute to the restricted potential $\mathcal{F}_R$.*

We prove the proposition by computing the structure constants of the Frobenius structure.

From the Euler vector field of $\mathcal{H}$ we know that the variable V_p is given a non-zero integer degree. Hence it contributes to the potential $\mathcal{F}^H$ polynomially. Namely there is a natural number N such that V_p^n for $n \geq N$ does not appear in the series expansion of $\mathcal{F}^H$.

It is obvious from the structure constants residue formula that $\mathcal{F}^H$ is well defined at $V_p = 0$.

Hence we only have to take care of the variable v_p that has degree 0 and could give a non-zero contribution to the restricted potential.

Notation 6.2. *Let $f(v) = \sum_{-\infty}^{\infty} a_k v^k$ be formal power series in v, and $p \in \mathbb{Z}$. Denote by:*

$$[v^p]\ f(v) := a_p.$$

We need first the lemma:

Lemma 6.11. *In flat coordinates we have the following expressions for the structure constants. For $k \neq p$ we have:*

$$\begin{aligned}
c(t_k, v_k, v_k) &= \frac{g_2(\tau)}{20}\frac{t_k}{2}\eta_1\omega_1 V_k,\\
c(t_k, t_k, v_p) &= \frac{1}{8}\wp'(a_p - a_k)t_k^2 + \frac{1}{4}\frac{\wp''(z-a_k)\wp(z-a_k) - (\wp'(z-a_k))^2}{\wp(z-a_k)^2}V_k\\
c(t_k, t_k, v_k) &= 0,\\
c(v_p, v_p, C_1) &= 0.
\end{aligned}$$

Proof. The derivative of λ w.r.t. v_k reads:

$$\begin{aligned}
\frac{\partial\lambda}{\partial v_k} &= -\frac{1}{4}\wp'(v - v_k)t_k^2 - \frac{1}{2}\frac{\wp''(v-v_k)\wp(v-v_k) - (\wp'(v-v_k))^2}{\wp(v-v_k)^2}V_k\\
&= \frac{1}{2}\frac{t_k^2}{(v-v_k)^3} - \frac{V_k}{(v-v_k)^2} + O(1).
\end{aligned}$$

It is clear that it's an elliptic function.

Structure constants $c(t_k, v_k, v_k)$. By definition we have:

$$c(t_k, v_k, v_k) = -\text{res}_{v_k} \frac{(\partial_{v_k}\lambda)^2(\partial_{t_k}\lambda)\text{d}v}{\lambda'}.$$

Note that the behavior of the functions λ' and $-\partial_{v_k}\lambda$ in the neighborhood of the point a_k coincide:

$$\begin{aligned}
c(t_k, v_k, v_k) &= \text{res}_{v_k}(\partial_{v_k}\lambda\ \partial_{t_k}\lambda)\text{d}v \\
&= [(v - v_k)]\partial_{v_k}\lambda \cdot [(v - v_k)^{-2}]\partial_{t_k}\lambda \\
&+ [(v - v_k)^{-3}]\partial_{v_k}\lambda \cdot [(v - v_k)^2]\partial_{t_k}\lambda + \frac{2V_k}{t_k^2}[(v - v_k)^{-3}]\partial_{v_k}\lambda \cdot [(v - v_k)]\partial_{t_k}\lambda
\end{aligned}$$

The first two lines sum to zero (basically because $\text{res}_{v_k}\wp'(v - v_k)\wp(v - v_k) = 0$) and from the Laurent expansion of $\wp$ we get:

$$c(t_k, v_k, v_k) = \frac{g_2(\tau)}{20}\frac{t_k}{2}\eta_1\omega_1 V_k.$$

Structure constants $c(t_k, t_k, v_p)$. For $p \neq k$ we have:

$$c(t_k, t_k, v_p) = -\text{res}_{v_k} \frac{(\partial_{t_k}\lambda)^2(\partial_{v_p}\lambda)\text{d}v}{\lambda'} = \frac{2}{t_k^2}[(v - v_k)^{-4}](\partial_{t_k}\lambda)^2\partial_{v_p}\lambda.$$

The function $\partial_{v_p}\lambda$ is regular at the point v_k for $k \neq p$ and we write:

$$c(t_k, t_k, v_p) = \frac{2}{t_k^2}\partial_{v_p}\lambda\ |_{v=v_k}\ [(v - v_k)^{-4}](\partial_{t_k}\lambda)^2 = \frac{2}{t_k^2}\frac{t_k^2}{4}\partial_{v_p}\lambda\ |_{v=v_k}\ .$$

Structure constants $c(t_k, t_k, v_k)$. Compute the residue for $p = k$:

$$c(t_k, t_k, v_k) = -\text{res}_{v_k} \frac{(\partial_{t_k}\lambda)^2(\partial_{v_k}\lambda)\text{d}v}{\lambda'} = \text{res}_{v_k}(\partial_{t_k}\lambda)^2.$$

The Laurent expansion of $\partial_{t_k}\lambda$ contains even degrees of $v - v_k$ only. Hence the residue vanishes.

Structure constants $c(C_1, v_p, v_p)$. We do not need to compute the residue for this structure constants because we have:

$$c(C_1, v_p, v_p) = \eta(v_p, v_p) = 0,$$

where we used the equalities for the metric in the flat coordinates.

The Lemma is proved. □

Note that by the choice of a_k in the restriction we have to express $\partial_{v_p}\lambda$ at one of the fundamental rectangle edge middle points:

$$\begin{aligned}
a_2 - a_1 = \omega_1 + \omega_2,\ a_3 - a_1 = \omega_1,\ a_4 - a_1 = \omega_2, \\
a_2 - a_3 = \omega_2,\ a_2 - a_4 = \omega_1,\ a_3 - a_4 = \omega_1 - \omega_2.
\end{aligned}$$

NOTATION 6.3. *For* $k \neq l$, $4 \geq k, l \geq 1$ *denote* $\{kl\}$ *by:*

$$\{13\} = \{24\} := 1, \quad \{12\} = \{34\} := 2, \quad \{23\} = \{14\} := 3.$$

In this notation we have:

$$e_{\{13\}} = e_{\{24\}} = e_1, \quad e_{\{12\}} = e_{\{34\}} = e_2, \quad e_{\{23\}} = e_{\{14\}} = e_3.$$

Proof of Proposition 6.10. We show that all the structure constants listed above vanish under the restriction. Note that we did not compute the structure constants $c(v_p, v_p, \tau)$ and $c(v_k, v_l, v_p)$. This is not needed because due to the homogeneity condition on the Hurwitz-Frobenius manifold potential the variables v_p and τ have degree zero. Therefore the summands of $\mathcal{F}^H$ giving these structure constants appear with the factor of other variables that are assigned non-zero degree. These are V_p and t_p. Therefore these summands contribute to the structure constants $c(t_p, \cdot, \cdot)$ or $c(V_p, \cdot, \cdot)$ whose vanishing we prove.

It is clear that all the summands that have a factor of V_k vanish. There are only two structure constants that we have to treat more carefully: $c(t_k, t_k, v_p)$ and $c(\tau, v_p, v_p)$. For the first one we have:

$$c(t_k, t_k, v_p) = -\frac{1}{2}\partial_{v_p}\lambda(v_k - v_p).$$

The points $a_k - a_l$ are precisely those where $\wp'(z; 2\omega_1, 2\omega_2)$ vanishes. And we get:

$$\partial_{v_p}\lambda(a_k - a_p) = \frac{V_k}{2e_{\{kp\}}}\wp''\left(\frac{1}{2\omega_1}(a_k - a_l)\right).$$

This expression vanishes by setting $V_k = 0$.

□

5. Proof of Theorem 6.1

To prove Theorem 6.1 we compute the structure constants in the variables t_k, C_1 and τ.

5.1. Structure constants including variables t_k, C_1 and τ only.

Proposition 6.12. *In flat coordinates we have:*

$$\begin{aligned}
c(\tau, C_1, C_1) &= \frac{1}{2\pi\sqrt{-1}},\\
c(t_k, t_k, C_1) &= \frac{1}{2},\\
c(t_k, t_k, t_k) &= 3t_k \cdot \omega_1\eta_1,\\
c(t_k, t_k, t_l) &= t_l\left(\frac{1}{4}\wp(a_k - a_l) + \eta_1\omega_1\right).
\end{aligned}$$

Proof.

Structure constant $c(\tau, C_1, C_1)$. By definition we have:

$$c(\tau, C_1, C_1) = \sum \mathrm{res}_{\lambda'=0}\frac{\partial_\tau\lambda(v)\mathrm{d}v}{\lambda'(v)},$$

Apply Lemma 6.9:

$$c(\tau, C_1, C_1) = -\frac{1}{2\pi\sqrt{-1}}\sum \mathrm{res}_{\lambda'=0}\frac{h_\lambda(v)\mathrm{d}v}{\lambda'(v)}$$

where we used in the last equation that the ζ-function has only one pole at $v = 0$. The function h_λ is elliptic and we can apply the proposition 6.8:

$$c(\tau, C_1, C_1) = \frac{1}{2\pi\sqrt{-1}}\sum \mathrm{res}_{v_p}\frac{h_\lambda \mathrm{d}v}{\lambda'} = \frac{1}{2\pi\sqrt{-1}}\sum \mathrm{res}_{v_p}\frac{\zeta\lambda' - 2\pi\sqrt{-1}\partial_\tau\lambda - 2\eta_1\lambda'}{\lambda'}\mathrm{d}v.$$

The function $\partial_\tau\lambda/\lambda'$ is regular at the points v_p and does not contribute to the residue:

$$c(\tau, C_1, C_1) = \frac{1}{2\pi\sqrt{-1}}\sum \mathrm{res}_{v_p}\zeta \mathrm{d}v = \frac{1}{2\pi\sqrt{-1}}\mathrm{res}_{v_1}\zeta\mathrm{d}v = \frac{1}{2\pi\sqrt{-1}}.$$

Structure constant $c(t_k, t_k, C_1)$.

$$c(t_k, t_k, C_1) = \sum \mathrm{res}_{\lambda'=0}\frac{(\partial_{t_k}\lambda)^2\partial_{C_1}\lambda\mathrm{d}v}{\lambda'} = -\mathrm{res}_{v_k}\frac{\left(\dfrac{2t_k}{4(v-v_k)^2}+h.o.t.\right)^2\mathrm{d}v}{\dfrac{-2t_k^2}{4(v-v_k)^3}+h.o.t.}$$

where we use $h.o.t.$ for the higher order terms.

$$c(t_k, t_k, C_1) = \frac{t_k^2}{4}\frac{2}{t_k^2} = \frac{1}{2}.$$

Structure constant $c(t_k, t_k, t_k)$.

$$c(t_k, t_k, t_k) = -\mathrm{res}_{v_k}\frac{(\partial_{t_k}\lambda)^3\,\mathrm{d}v}{\lambda'} = \frac{2}{t_k^2}[(v-v_k)^{-4}](\partial_{t_k}\lambda)^3.$$

The Taylor expansion of the functions in the numerator is:

$$\partial_{t_k}\lambda = \frac{t_k}{2}\left(\frac{1}{(v-v_k)^2} + 4\eta_1\omega_1 + O\left((v-v_k)^2\right)\right).$$

There are only two options to get a degree -4 factor from its third power. Distributed in three factors they read: $-1,-1,-2$ and $-2,-2,-0$. The first one is not possible because degree -1 in $v - v_i$ appears only as the multiple of the variable V_k.

$$c(t_k, t_k, t_k) = \frac{2}{t_k^2}\frac{3t_k^3}{4}2\eta_1\omega_1 = 3t_k\omega_1\eta_1.$$

Structure constant $c(t_k, t_k, t_l)$.

$$c(t_k, t_k, t_l) = -\mathrm{res}_{v_k}\frac{(\partial_{t_k}\lambda)^2\,(\partial_{t_l}\lambda)\,\mathrm{d}v}{\lambda'} = \frac{2}{t_k^2}[(v-v_k)^{-4}](\partial_{t_k}\lambda)^2\,(\partial_{t_l}\lambda)\,.$$

The factor $\partial_{t_l}\lambda$ is regular at the point v_k. Therefore we just take the value of it at the point v_k.

$$c(t_k, t_k, t_l) = \frac{2}{t_k^2}\frac{t_k^2}{4}\,(\partial_{t_l}\lambda)\,|_{v=v_k}\;.$$

□

5.2. Structure constants at the special point.

PROPOSITION 6.13. *For the potential $\mathcal{F}_R$ we have:*

$$\frac{\partial^3\mathcal{F}_R}{(\partial t_k)^2\partial t_l} = -t_l\frac{\pi\sqrt{-1}}{2}X_{kl}(\tau).$$

PROOF. Because of Proposition 6.10 the potential $\mathcal{F}_R$ is obtained by integrating the structure constants $c(\cdot,\cdot,\cdot)$ of $\mathcal{H}$ containing t_k, τ, C_1 only. We have:

$$\frac{\partial^3\mathcal{F}_R}{(\partial t_k)^2\partial t_l} = \frac{\partial^3\mathcal{F}^H}{(\partial t_k)^2\partial t_l}\,|_{\mathcal{A}} = \int c(t_k, t_k, t_l)dt_k^2dt_l.$$

The latter one can be computed using theta constants. For the structure constant under the integral we have:

$$\left(\frac{1}{4}\wp(v_k - v_l) + \eta_1\omega_1\right) = \left(\frac{1}{12}\frac{\vartheta_1'''}{\vartheta_1'} - \frac{1}{4}\frac{\vartheta_{\{kl\}}''}{\vartheta_{\{kl\}}}\right) - \frac{1}{12}\frac{\vartheta_1'''}{\vartheta_1'} = -\frac{1}{4}\frac{\vartheta_{\{kl\}}''}{\vartheta_{\{kl\}}}.$$

Where we used the convention of Notation 6.3 in the double-index subscript.

Using the heat equation for $\vartheta_{\{kl\}}$ we get the proposition. □

5.3. Restricted potential. Integrating the structure constants that we have computed we write down the potential of $\mathcal{H}^R_{1,(2,2,2,2)}$. It reads:

$$\mathcal{F}_R = \frac{C_1^2\tau}{2}\frac{1}{2\pi\sqrt{-1}} + C_1\sum_k \frac{t_k^2}{4} - \sum_{p>q}\frac{t_p^2t_q^2}{4}\frac{\pi\sqrt{-1}}{2}X_{pq}(\tau) - \sum_k \frac{t_k^4}{24}\frac{3\pi\sqrt{-1}}{4}\gamma(\tau).$$

Introduce:

$$t_{-1} := \pi\sqrt{-1}\tau, \quad \tilde{C}_1 := \frac{C_1}{\sqrt{2}\pi\sqrt{-1}}, \quad \tilde{t}_k := t_k\sqrt{\pi\sqrt{-1}}\cdot 2^{1/4}.$$

The potential changes:

$$\mathcal{F}_R = \frac{\tilde{C}_1^2 t_{-1}}{2} + \tilde{C}_1\sum_k \frac{\tilde{t}_k^2}{4} - \sum_{p>q}\frac{\tilde{t}_p^2\tilde{t}_q^2}{16}\frac{1}{\pi\sqrt{-1}}X_{pq}(t_{-1}) - \sum_k \frac{\tilde{t}_k^4}{64}\frac{1}{\pi\sqrt{-1}}\gamma(t_{-1}),$$

where $\gamma(t_{-1})$ and $X_k(t_{-1})$ are functions in $\exp(t_{-1})$. Hence under this change of variables the potential transforms to one written in the form of Proposition 3.4. Theorem 6.1 is proved.

Corollary 6.14. *Let t^{GW} be variables of the $\mathcal{F}_0^{\mathbb{P}^1_{2,2,2,2}}$. The isomorphism reads in coordinates:*

$$t_{-1}^{\mathrm{GW}} = \pi\sqrt{-1}\tau, \quad t_0^{\mathrm{GW}} = \frac{C_1}{\sqrt{-2}\pi}, \quad t_1^{\mathrm{GW}} = (t_4 - t_3)\frac{\sqrt{\pi\sqrt{-1}}}{2^{1/4}}$$

$$t_2^{\mathrm{GW}} = (t_4 + t_3)\frac{\sqrt{\pi\sqrt{-1}}}{2^{1/4}} \quad t_3^{\mathrm{GW}} = (t_1 - t_2)\frac{\sqrt{\pi\sqrt{-1}}}{2^{1/4}} \quad t_4^{\mathrm{GW}} = (t_1 + t_2)\frac{\sqrt{\pi\sqrt{-1}}}{2^{1/4}}.$$

CHAPTER 7

Primitive form change for the orbifolded LG model

The theory of primitive forms for the orbifolded LG models does not yet exist. Therefore we propose a substitute for it. Namely we consider the change of the primitive form using the CY-LG mirror isomorphism of Theorem 5.2 and a certain action on the space of Frobenius manifolds.

Consider the orbifolded LG model $(\tilde{E}_8, \mathbb{Z}_3)$ with the choice of the primitive form at the LCSL-point to be a "base point". We apply to it a certain action $\mathcal{A}^{(\tau_0,\omega_0)}$ on the space of Frobenius manifolds that changes the primitive form.

Looking for the Frobenius manifold structure of $(\tilde{E}_8, \mathbb{Z}_3)$ giving the LG-LG mirror isomorphism the general idea of mirror symmetry suggests concerning the primitive form at the *special point*. This notion is not defined for orbifolded LG models either. In analogy with the case of the trivial symmetry group we propose this property to be "translated" to $\tau_0 \in \mathbb{Q}\sqrt{-D}$ for $D \in \mathbb{N}_+$.

To summarize:

$$
\begin{array}{lcr}
(W, \{id\}) & \leftrightarrow & (W, G) \\
\text{primitive form change} & \leftrightarrow & \text{the action } \mathcal{A}^{(\tau_0,\omega_0)} \\
\text{special point} & \leftrightarrow & \tau_0 \in \mathbb{Q}\sqrt{-D}, \quad D \in \mathbb{Z}_+.
\end{array}
$$

We further support the use of the action $\mathcal{A}^{(\tau_0,\omega_0)}$ in the following sections.

1. Frobenius manifold $M_6^{(\tau_0,\omega_0)}$

Consider a Frobenius manifold M of rank six and dimension one with flat coordinates $t_1, \dots, t_5, t_6$ satisfying the following conditions:

- The unit vector field e is given by $\frac{\partial}{\partial t_1}$.
- The Euler vector field E is given by

$$E = t_1 \frac{\partial}{\partial t_1} + \sum_{k=2}^{5} \frac{1}{2} t_k \frac{\partial}{\partial t_k}.$$

- The Frobenius potential $\mathcal{F}$ is given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{1}{2} t_1^2 t_6 + \frac{1}{4} t_1 (\sum_{i=2}^{5} t_i^2) + (t_2 t_3 t_4 t_5) f_0(t_6)$$
$$+ \frac{1}{4}(t_2^4 + t_3^4 + t_4^4 + t_5^4) f_1(t_6) + \frac{1}{6}(t_5^2 t_2^2 + t_5^2 t_3^2 + t_5^2 t_4^2 + t_2^2 t_3^2 + t_2^2 t_4^2 + t_3^2 t_4^2) f_2(t_6),$$

  where $f_0(t)$, $f_1(t)$ and $f_2(t)$ are holomorphic functions in t on an open domain in $\mathbb{C}$.

1.1. Solutions of the WDVV equation.

PROPOSITION 7.1. *The WDVV equation for $\mathcal{F}$ is equivalent to the following differential equation*:

$$\left\{\begin{array}{l} f_0'(t) = \frac{8}{3}f_0(t)f_2(t) - 24f_0(t)f_1(t), \\ f_1'(t) = -\frac{2}{3}f_0(t)^2 - \frac{16}{3}f_1(t)f_2(t) + \frac{8}{9}f_2(t)^2, \\ f_2'(t) = 6f_0(t)^2 - \frac{8}{3}f_2(t)^2. \end{array}\right. \tag{7.1}$$

PROOF. It is easy to check that $\mathrm{WDVV}(\partial_3, \partial_4, \partial_3, \partial_4)$ (recall Notation 5.3) gives the second and the third equations while $\mathrm{WDVV}(\partial_2, \partial_4, \partial_3, \partial_3)$ gives the first one. □

Without loss of generality consider:

$$\left\{\begin{array}{l} f_0(t) := \frac{1}{8}X_3(t) - \frac{1}{8}X_4(t), \\ f_1(t) := -\frac{1}{12}X_2(t) - \frac{1}{48}X_3(t) - \frac{1}{48}X_4(t), \\ f_2(t) := -\frac{3}{16}X_3(t) - \frac{3}{16}X_4(t), \end{array}\right. \tag{7.2}$$

for some functions $X_i(t)$ with the same domain of holomorphicity as $f_i(t)$.

PROPOSITION 7.2. *The equations* (7.1) *are equivalent to the following differential equations*:

$$\left\{\begin{array}{l} \frac{d}{dt}(X_2(t) + X_3(t)) = 2X_2(t)X_3(t), \\ \frac{d}{dt}(X_3(t) + X_4(t)) = 2X_3(t)X_4(t), \\ \frac{d}{dt}(X_4(t) + X_2(t)) = 2X_4(t)X_2(t), \end{array}\right. \tag{7.3}$$

classically known as Halphen's system of equations.

PROOF. This is obtained by a straightforward calculation. □

The following proposition is a beautiful example of the action on the space of Frobenius manifolds that is defined purely analytically in contrast with the heavy machinery of the Givental action.

PROPOSITION 7.3. *Let the triplet* $(X_2(t), X_3(t), X_4(t))$ *be a solution of the Halphen's equations* (7.3). *For any* $A \in \mathrm{GL}(2, \mathbb{C})$ *define a triplet of functions* $X_i^A(t)$ *for* $2 \le i \le 4$ *on a suitable domain in* $\mathbb{C}$ *by*

$$X_i^A(t) := \frac{\det(A)}{(ct+d)^2} X_i\left(\frac{at+b}{ct+d}\right) - \frac{c}{ct+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7.4}$$

Then $(X_2^A(t), X_3^A(t), X_4^A(t))$ *is also a solution of the Halphen's equations* (7.3).

PROOF. This statement was proved in [**37**] for $A \in \mathrm{SL}(2, \mathbb{C})$. Consider

$$A' = \begin{pmatrix} a' & b \\ c' & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}), \quad a' = a/\det A, c' = c/\det A.$$

However it is clear that whenever the triplet $X_i(t)$ is a solution to Halphen's system then for any $a \in \mathbb{C}^*$ the triplet $aX_i(at)$ is a solution too. Taking $a = \det A$ we get the proposition. □

It is important to note that this $\mathrm{GL}(2, \mathbb{C})$-action is the inverse action of the $\mathrm{GL}(2, \mathbb{C})$-action on the set of solutions of the WDVV equations given in Appendix B in [**11**]. Indeed, we have the following.

PROPOSITION 7.4. *Consider Dubrovin's inversion I (recall equation* (3.5) *in Chapter 3) of the rank 6 Frobenius manifold with potential $\mathcal{F}$. Then we have:*

$$\mathcal{F}^I = \mathcal{F}^A, \text{ for } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where we have Inversion transformation of the LHS and the $\mathrm{GL}(2,\mathbb{C})$ *action* (7.4) *on the RHS.*

PROOF. Some calculations yield the statement. □

1.2. Action of $\mathcal{A}^{(\tau_0,\omega_0)}$ on $M_{\mathbb{P}^1_{2,2,2,2}}$. Recall that the Frobenius manifold of $\mathbb{P}^1_{2,2,2,2}$ is fixed by the particular solution to the Halphen's equations that was denoted in Chapter 3 by $X_k^\infty(\tau)$:

$$X_k^\infty(\tau) := 2\frac{\partial}{\partial\tau}\log\vartheta_k(\tau), \quad 2 \le k \le 4,$$

for $\vartheta_k(\tau)$ – Jacobi theta constants.

Consider the following $\mathrm{GL}(2,\mathbb{C})$-action $\mathcal{A}^{(\tau_0,\omega_0)}$ on the Frobenius manifold of $\mathbb{P}^1_{2,2,2,2}$.

$$\mathcal{A}^{(\tau_0,\omega_0)} := \begin{pmatrix} \dfrac{\bar\tau_0}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0\tau_0 \\ \dfrac{1}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0 \end{pmatrix}. \tag{7.5}$$

DEFINITION. Choose $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$.

(1) Using the $\mathrm{GL}(2,\mathbb{C})$-action (7.4) specified by $\mathcal{A}^{(\tau_0,\omega_0)}$ define the triplet of functions

$$X_k^{(\tau_0,\omega_0)}(t) := (X_k^\infty)^{\mathcal{A}^{(\tau_0,\omega_0)}}(t) \quad \text{for} \quad 4 \ge k \ge 2.$$

Then the functions $X_k^{(\tau_0,\omega_0)}(t)$ are holomorphic on

$$D^{(\tau_0,\omega_0)} := \{t \in \mathbb{C} \mid |t| < |-4\pi\omega_0^2\mathrm{Im}(\tau_0)|\}.$$

(2) Denote by $M_6^{(\tau_0,\omega_0)} := \mathbb{C}^5 \times D^{(\tau_0,\omega_0)}$ the Frobenius manifold given by the Frobenius potential

$$\mathcal{F}_6^{(\tau_0,\omega_0)} = \frac{t_1^2 t_6}{2} + \frac{t_1}{4}\sum_{k=2}^5 t_k^2 - (t_3^2t_4^2 + t_5^2t_2^2)\frac{1}{16}X_2^{(\tau_0,\omega_0)}(t_6) - (t_5^2t_3^2 + t_2^2t_4^2)\frac{1}{16}X_3^{(\tau_0,\omega_0)}(t_6)$$
$$-(t_5^2t_4^2 + t_2^2t_3^2)\frac{1}{16}X_4^{(\tau_0,\omega_0)}(t_6) - \frac{1}{64}\left(\sum_{k=2}^5 t_k^4\right)\left(\frac{2}{3}\sum_{k=2}^4 X^{(\tau_0,\omega_0)}(t)\right).$$

We will also write:

$$\mathcal{F}_6^{(\tau_0,\omega_0)} = \mathcal{A}^{(\tau_0,\omega_0)} \cdot \mathcal{F}_{\mathbb{P}^1_{2,2,2,2}}.$$

It is clear from the particular form of the potential $\mathcal{F}_6^{(\tau_0,\omega_0)}$ that the Euler vector field is preserved under the action of $\mathcal{A}^{(\tau_0,\omega_0)}$. Hence this purely algebraically defined $\mathrm{GL}(2,\mathbb{C})$-action on the set of solutions to Halphen's equations gives rise to the action on the Frobenius manifolds.

2. Primitive form change via $M_6^{(\tau_0,\omega_0)}$

Recall that the equation $W_\sigma = 0$ defines the family of elliptic curves over Σ. Let ζ_σ be the global primitive form of W_σ. It defines a certain period $\pi(\sigma)$ of the elliptic curve $E_\sigma := \{W_\sigma = 0\}$.

Consider $\lambda_\sigma \in \mathcal{H}_{1,(2,2,2,2)}$ such that:

$$\lambda_\sigma : E_\sigma \xrightarrow{8:1} \mathbb{P}^1, \quad \forall \sigma \in \Sigma,$$

and the ramification profile $\lambda_\sigma^{-1}(\infty)$ consists of 4 points of order 2. Taking the holomorphic form on the elliptic curve E_σ we get the family of Frobenius manifold structures on $\mathcal{H}_{1,(2,2,2,2)}$. Denote it by M^σ.

$$\begin{tikzcd} \{W_\sigma = 0\} \arrow[r, hook] \arrow[d] & \mathcal{H}^R_{1,(2,2,2,2)} \arrow[r, "\cong"] & M^\sigma \\ \Sigma & & \end{tikzcd}$$

We identify M^{σ_0} for every $\sigma_0 \in \Sigma$ with the particular Frobenius manifold $M_6^{(\tau_0,\omega_0)}$. This allows us to consider the action $\mathcal{A}^{(\tau_0,\omega_0)}$ on the rank 6 Frobenius manifolds as corresponding to the primitive form change of W_σ.

2.1. Change of the primitive form. We present here the approach of [**6**] to the change of the primitive form. In contrast to the approach of Milanov, Ruan, Shen and Krawitz we work with particular cycles of the elliptic curve fixing the primitive form. The advantage of our approach is that we can recover all primitive forms in this way. Also the geometry of the elliptic curves could be used here. However it was proved in [**5**] that both approaches coincide.

In this subsection we would prefer to work in the Hodge-theoretical setting. Let $H_\mathbb{Z} = \mathbb{Z}\alpha \otimes \mathbb{Z}\beta$ be the homology group $H_1(E_\sigma, \mathbb{Z})$ of the elliptic curve at infinity. Then

$$H^*_\mathbb{C} := (H_\mathbb{C})^* := (H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C})^* = \mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee,$$

where $\{\alpha^\vee, \beta^\vee\}$ is the dual basis of $\{\alpha, \beta\}$, can be identified with the cohomology group $H^1(E_\sigma, \mathbb{Z})$. Recall the family of elliptic curves $\mathcal{E}$ defined in (2.3). The relative holomorphic volume form $\Omega \in \Gamma(\mathbb{H}, \Omega^1_{\mathcal{E}/\mathbb{H}})$ is described in terms of $\alpha^\vee, \beta^\vee$ as

$$\Omega = x(\tau)\,(\alpha^\vee + \tau\beta^\vee)$$

for some nowhere vanishing holomorphic function $x(\tau)$ on $\mathbb{H}$.

Let the relative holomorphic volume form $\zeta^\infty = \alpha^\vee + \tau\beta^\vee$ be the *primitive form* associated to the choice of the vector $\alpha \in H_\mathbb{C}$, which satisfies

$$\int_\alpha \zeta^\infty = 1 \text{ and } \int_\beta \zeta^\infty = \tau.$$

There is a systematic way to obtain a primitive form by the use of the canonical opposite filtration to the Hodge filtration corresponding to a point $\tau_0 \in \mathbb{H}$ as follows.

Proposition 7.5. *For $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$, there exists a unique relative holomorphic volume form $\zeta \in \Gamma(\mathbb{H}, \Omega^1_{\mathcal{E}/\mathbb{H}})$ such that*

$$\int_{\alpha'} \zeta = 1, \quad \alpha' := \frac{1}{\omega_0(\bar\tau_0 - \tau_0)}\left(\bar\tau_0\alpha - \beta\right).$$

PROOF. Some calculation yields

$$\zeta = \omega_0 \frac{\bar\tau_0 - \tau_0}{\bar\tau_0 - \tau} \left(\alpha^\vee + \tau\beta^\vee \right).$$

□

This holomorphic volume form ζ is the primitive form uniquely determined by the choice of the vector $\alpha' \in H_{\mathbb{C}}$. We first fix $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$ so that we have

$$\int_\alpha \zeta = \omega_0 \text{ and } \int_\beta \zeta = \omega_0\tau_0 \quad \text{at } \tau = \tau_0.$$

Next we choose $\beta' \in H_{\mathbb{C}}$ so that $\int_{\beta'} \zeta = 0$ at $\tau = \tau_0$ and $(\alpha', \beta') = 1$. It is easy to see that

$$\beta' := -\omega_0 \left(\tau_0 \alpha - \beta \right).$$

The bases $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are connected by the action of the following $\mathrm{SL}(2,\mathbb{C})$ matrix on $H^1(E_\sigma, \mathbb{Z})$:

$$(7.6) \qquad A_{hom} := \begin{pmatrix} -\dfrac{\bar\tau_0}{2\sqrt{-1}\omega_0 \mathrm{Im}(\tau_0)} & \dfrac{1}{2\sqrt{-1}\omega_0 \mathrm{Im}(\tau_0)} \\ -\omega_0\tau_0 & \omega_0 \end{pmatrix}, \ \begin{pmatrix} \alpha' & \beta' \end{pmatrix} = A_{hom} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

However the connection between the flat coordinates is constituted by the action $\mathcal{A}^{(\tau_0,\omega_0)}$. Let t_{-1} be the flat coordinate "at infinity" associated to the basis $\{\alpha, \beta\}$. Then the flat coordinate t'_{-1}, corresponding to the basis $\{\alpha', \beta'\}$ transforms as follows (recall Proposition 2.6):

$$\frac{t'_{-1}}{2\pi\sqrt{-1}} = \int_{\beta'} \zeta = 2\sqrt{-1}\omega_0^2 \mathrm{Im}(\tau_0) \frac{\tau_0 - t_{-1}}{\bar\tau_0 - t_{-1}},$$

what is equivalent to:

$$t'_{-1} = \mathcal{A}^{(\tau_0,\omega_0)} \cdot t_{-1}.$$

2.2. The actions A_{hom} and $\mathcal{A}^{(\tau_0,\omega_0)}$. In this subsection we present the setting in which the action of $\mathcal{A}^{(\tau_0,\omega_0)}$ on the flat coordinate t_{-1} arises naturally as the action on the space of Frobenius manifolds.

Let $A \in \mathrm{SL}(2,\mathbb{C})$ and $\{\lambda : \mathcal{E}_\tau \to \mathbb{P}^1\} \in \mathcal{H}^R_{1,(2,2,2,2)}$. The ramified covering λ is written explicitly via the elliptic functions. This allows us to act by A on λ, what corresponds to the action of A on the lattice of the covering elliptic curve.

PROPOSITION 7.6. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. *Its action on the elliptic curve lattice gives rise to the action on the flat coordinates of* $\mathcal{H}^R_{1,(2,2,2,2)}$ *that reads in coordinates* $(\tau, C_1, t_1, t_2, t_3, t_4)$*:*

$$\hat\tau = \frac{a\tau + b}{c\tau + d}, \quad \hat C_1 = C_1 - \frac{\pi\sqrt{-1}}{2} \frac{c}{c\tau + d} \sum_{i=1}^4 t_i^2, \quad \hat t_i = \frac{t_i}{c\tau + d}, \quad 1 \le i \le 4$$

PROOF. Note that by the action of A we have:

$$\hat\omega_2 = a\omega_2 + b\omega_1, \quad \hat\omega_1 = c\omega_2 + d\omega_1.$$

Recall the expression of flat coordinates of $\mathcal{H}_{1,(2,2,2,2)}$ given in Proposition 6.7. Let $\hat C_1, \hat t_1, \ldots, \hat t_4$ and $\hat\tau$ be the flat coordinates on the A-transformed Frobenius structure of $\mathcal{H}^R_{1,(2,2,2,2)}$.

For the variables t_i, $1 \le i \le 4$ and τ we have:

$$\hat{\tau} = \frac{a\tau + b}{c\tau + d} = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1},\ \hat{t}_i = -\frac{\sqrt{\tilde{u}_i}}{\tilde{\omega}_1} = \frac{t_i}{c\tau + d}.$$

Consider the variable C_1.

$$\hat{C}_1 = c - \frac{\hat{\eta}_1}{\hat{\omega}_1} \sum_{i=1}^{4} u_i = c - \frac{c\eta_2 + d\eta_1}{c\omega_2 + d\omega_1} \sum_{i=1}^{4} u_i.$$

Using the Legendre identity we rewrite:

$$\hat{C}_1 = c - \frac{1}{\omega_1}\left(\eta_1 + \frac{\pi\sqrt{-1}c}{2}\frac{1}{c\omega_2 + d\omega_1}\right)\sum_{i=1}^{4} u_i = c - \frac{\eta_1}{\omega_1}\sum_{i=1}^{4} u_i - \frac{1}{\omega_1^2}\left(\frac{\pi\sqrt{-1}}{2}\frac{c}{c\tau + d}\right)\sum_{i=1}^{4} u_i,$$

and we get the identity. $\square$

This proposition allows us to apply on $\mathcal{H}^R_{1,(2,2,2,2)}$, viewed as a Hurwitz-Frobenius manifold, the action of the element $A_{hom} \in \mathrm{SL}(2,\mathbb{C})$ defined in (7.6) for the elliptic curves.

PROPOSITION 7.7. *The change of the primitive form from ζ^∞ to ζ_{σ_0} at the special point σ_0 is equivalent to the action $\mathcal{A}^{(\tau_0,\omega_0)}$ such that:*

$$j_W(\sigma_0) = j(\tau_0),$$

and τ_0 is imaginary quadratic.

PROOF. Let A_{hom} be the $\mathrm{SL}(2,\mathbb{C})$ matrix corresponding to the primitive form change. We can consider it as an operator on the space $\mathcal{H}^R_{1,(2,2,2,2)}$:

$$A_{hom} : \{\lambda : E_\infty \to \mathbb{P}^1\} \to \{\hat{\lambda} : E_{\tau_0} \to \mathbb{P}^1\},$$

where E_{τ_0} is an elliptic curve with the modulus τ_0 by the construction of A_{hom}. However from Theorem 2.1 the primitive form at the point σ_0 is fixed by the period of the elliptic curve E_{σ_0} that is isomorphic to $\mathcal{E}_{\tau_0}$.

Due to Proposition 7.6 the action of A_{hom} on the space $\mathcal{H}^R_{1,(2,2,2,2)}$ agrees with the action of the same $\mathrm{SL}(2,\mathbb{C})$ element on the flat coordinates of $\mathcal{H}^R_{1,(2,2,2,2)}$.

The following lemma gives the action on $M_{\mathbb{P}^1_{2,2,2,2}}$ induced by A_{hom}.

LEMMA 7.8. *The $\mathrm{SL}(2,\mathbb{C})$-action of A_{hom} on $\mathcal{H}^R_{1,(2,2,2,2)}$ induces the $\mathcal{A}^{(\tau_0,\omega_0)}$ action on $M_{\mathbb{P}^1_{2,2,2,2}}$.*

PROOF. Due to the proposition above the $\mathrm{SL}(2,\mathbb{C})$-action on $\mathcal{H}^R_{1,(2,2,2,2)}$ is well-defined and agrees with the action on the periods of the elliptic curve.

Consider the induced action of A_{hom} on $\mathcal{H}^R_{1,(2,2,2,2)}$.

Let ω_1', ω_2' be obtained by the A_{hom} action from ω_1, ω_2:

$$\begin{pmatrix}\omega_1' & \omega_2'\end{pmatrix} = A_{hom}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix},$$

and $\tau' = \omega_2'/\omega_1'$. It reads:

$$\tau' = 2\sqrt{-1}\omega_0 \mathrm{Im}\tau_0 \frac{\tau_0 - \tau}{\bar{\tau}_0 - \tau}.$$

The inverse of the change of variables reads:

$$\tau = \frac{-\tau'\bar{\tau}_0 + 2\sqrt{-1}\omega_0^2\tau_0 \mathrm{Im}\tau_0}{-\tau' + 2\sqrt{-1}\omega_0^2 \mathrm{Im}\tau_0}.$$

Assuming also addition scaling by $2\pi\sqrt{-1}$ that has to be applied on the $M_{\mathbb{P}^1_{2,2,2,2}}$ side (because of the scaling by $2\pi\sqrt{-1}$ in the isomorphism of Theorem 6.1) we get exactly the action of $\mathcal{A}^{(\tau_0,\omega_0)}$. $\square$

It is clear from the definition and the proposition above that A_{hom} acts on $\mathcal{H}^R_{1,(2,2,2,2)}$ by moving the origin. Namely the flat coordinates of $A_{hom} \cdot \mathcal{H}^R_{1,(2,2,2,2)}$ are defined in the neighborhood of τ_0 of the same Frobenius manifold - $\mathcal{H}^R_{1,(2,2,2,2)}$. Because of $M^\sigma \mid_{\sigma=\infty} \cong M_{\mathbb{P}^1_{2,2,2,2}}$ we have:

$$\mathcal{A}_{hom} \cdot \mathcal{H}^R_{1,(2,2,2,2)} = M^\sigma \cong M_6^{(\tau_0,\omega_0)}.$$

Putting the σ_k corresponding to the special points in the j-invariant formulae (2.2) we get that the j-invariant is equal to $0, 1728, \infty$. We have:

$$0 = j(\sqrt{-1}), \quad 1728 = j(\exp(\frac{2\pi\sqrt{-1}}{3})).$$

The complex numbers $\tau_0 = \sqrt{-1}$ and $\tau_0 = \exp(\frac{2\pi\sqrt{-1}}{3})$ are clearly imaginary quadratic.

This completes the proof of the proposition. $\square$

Note that by the proposition above we also outline the space of Frobenius manifolds that can appear in the CY-LG correspondence.

CHAPTER 8

LG-LG mirror symmetry for $(\tilde{E}_8, \mathbb{Z}_3)$

The main theorem of this chapter is the following LG-LG mirror theorem. Consider the axiomatization of the Frobenius structure associated to the LG A-model $(\tilde{E}_8^T, \mathbb{Z}_3^T)$ (see Chapter 4).

Theorem 8.1. *The Frobenius manifold of the orbifolded LG A-model* $(\tilde{E}_8^T, \mathbb{Z}_3^T)$ *is isomorphic to the Frobenius manifold* $M_6^{(\sqrt{-1},\omega_0)}$ *with*

$$\omega_0 := \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\pi^{\frac{3}{2}}}.$$

It is important to note that $\tilde{E}_8^T = \tilde{E}_8$. Namely the polynomial defining singularity $\tilde{E}_8$ is not altered by the Berglund-Hübsch duality.

We have shown in the previous chapter that the change of the primitive form is translated to the $\mathcal{A}^{(\tau_0,\omega_0)}$-action on the Frobenius structures. In order to prove the theorem, by the CY/LG correspondence axiom and mirror theorem 5.2 we should consider the orbit of the Frobenius manifold $M_{\mathbb{P}^1_{2,2,2,2}}$ under the $\mathcal{A}^{(\tau_0,\omega_0)}$ group action.

We classify the rank 6 Frobenius structures $\mathcal{A}^{(\tau_0,\omega_0)} \cdot M_{\mathbb{P}^1_{2,2,2,2}}$ satisfying the axioms of the LG A-model introduced in Chapter 4. The most restrictive axiom appears to be the rationality axiom.

Definition. Let $\mathbb{K} \subset \mathbb{C}$ be a field. We say that a rank μ Frobenius manifold M is *defined over* $\mathbb{K}$ if there exist flat coordinates $t_1, \dots, t_\mu$ such that the Frobenius potential $\mathcal{F}$ of M belongs to $\mathbb{K}\{t_1, \dots, t_\mu\}$ and is defined at the point $t_1 = \dots = t_\mu = 0$.

In order to classify Frobenius manifolds $M_6^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ we have to investigate the series expansions of the functions $X_k^{(\tau_0,\omega_0)}(t)$. Consider the function:

$$\gamma^{(\tau_0,\omega_0)}(t) := \frac{2}{3}\sum_{k=2}^{4} X_k^{(\tau_0,\omega_0)}(t).$$

Obviously we have:

$$X_k^{(\tau_0,\omega_0)}(t) \in \mathbb{K}\{t\} \;\Rightarrow\; \gamma^{(\tau_0,\omega_0)}(t) \in \mathbb{K}\{t\}.$$

We introduce the rank 3 Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ such that its potential is fixed by the function $\gamma^{(\tau_0,\omega_0)}$. Hence the rationality of $M_3^{(\tau_0,\omega_0)}$ turns out to be a necessary condition for the rationality of $M_6^{(\tau_0,\omega_0)}$.

1. Rank 3 Frobenius manifold $M^{(\tau_0,\omega_0)}$

Consider a Frobenius manifold M of rank three and dimension one with flat coordinates t_1, t_2, t_3 satisfying the following conditions:

- The unit vector field e is given by $\frac{\partial}{\partial t_1}$.
- The Euler vector field E is given by $E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{2} t_2 \frac{\partial}{\partial t_2}$.
- The Frobenius potential $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{t_2^4}{16} \gamma(t_3)$$

where $\gamma(t)$ is a holomorphic function in t on an open domain in $\mathbb{C}$.

The following proposition was first observed by Dubrovin.

PROPOSITION 8.2 (Appendix C in [**11**]). *The WDVV equation on $\mathcal{F}$ is equivalent to the following differential equation known as Chazy equation.*

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2. \tag{8.1}$$

PROOF. The WDVV equation (1.1) with the indices (t_2, t_2, t_3, t_3) yields the statement. $\square$

1.1. Eisenstein series and elliptic curves. Let E_{2k} for $k \in \mathbb{Z}_+$ be the Eisenstein series defined in Chapter 3, Section 2.2. Consider a family of elliptic curves parameterized by $\mathbb{H}$:

$$\pi : \mathcal{E} := \{(x, y, \tau) \in \mathbb{C}^2 \times \mathbb{H} \mid y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\} \longrightarrow \mathbb{H},$$

then the modular invariants of the elliptic curve $g_2(\tau)$ and $g_3(\tau)$ can be expressed via Eisenstein series:

$$g_2(\tau) := \frac{4\pi^4}{3} E_4(\tau), \quad g_3(\tau) := \frac{8\pi^6}{27} E_6(\tau). \tag{8.2}$$

Denote by $\mathcal{E}_{\tau_0}$ the fiber of π over a point $\tau_0 \in \mathbb{H}$.

DEFINITION. Let $\mathbb{K} \subset \mathbb{C}$ be a field. Choose a point $\tau_0 \in \mathbb{H}$. We say that an elliptic curve $\mathcal{E}_{\tau_0}$ is *defined over* $\mathbb{K}$ if there exist $g_2, g_3 \in \mathbb{K}$ such that the algebraic variety

$$E_{g_2, g_3} := \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}$$

is isomorphic to $\mathcal{E}_{\tau_0}$.

The derivatives of the Eisenstein series E_2, E_4, E_6 satisfy the following identities due to Ramanujan:

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \frac{dE_2(\tau)}{d\tau} &= \frac{1}{12} \left(E_2(\tau)^2 - E_4(\tau) \right), \\ \frac{1}{2\pi\sqrt{-1}} \frac{dE_4(\tau)}{d\tau} &= \frac{1}{3} \left(E_2(\tau)E_4(\tau) - E_6(\tau) \right), \\ \frac{1}{2\pi\sqrt{-1}} \frac{dE_6(\tau)}{d\tau} &= \frac{1}{2} \left(E_2(\tau)E_6(\tau) - E_4(\tau)^2 \right). \end{aligned} \tag{8.3}$$

We also consider the complex-valued real-analytic function $E_2^*(\tau)$ on $\mathbb{H}$ defined by

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)},$$

which is a so-called almost holomorphic modular form of weight two because of the following.

PROPOSITION 8.3. *We have*

$$E_2^*(\tau) = \frac{1}{(c\tau+d)^2} E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

PROOF. The formula

$$\left(\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^{-1} = \frac{|c\tau+d|^2}{\mathrm{Im}(\tau)}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{8.4}$$

yields the statement. □

DEFINITION. A polynomial $f(\tau)$ in $\mathrm{Im}(\tau)^{-1}$ over the ring of holomorphic functions on $\mathbb{H}$ satisfying

$$f(\tau) = \frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

is called an *almost holomorphic modular form* of weight k.

PROPOSITION 8.4 (cf. Paragraph 5.1. in [**49**]). *Let $f(\tau)$ be an almost holomorphic modular form of weight k. Then the almost holomorphic derivative of $f(\tau)$ defined by*

$$\partial_k f(\tau) := \frac{1}{2\pi\sqrt{-1}}\frac{\partial f(\tau)}{\partial \tau} - \frac{k}{4\pi \mathrm{Im}(\tau)} f(\tau), \tag{8.5}$$

is an almost holomorphic modular form of weight $k+2$.

PROOF. One can check this directly by using the equations (3.4) and (8.4). We briefly explain for the reader's convenience the modularity property of $\partial_2 E_2^*(\tau)$. We have:

$$\begin{aligned}
\partial_2 E_2^* &= \frac{1}{12}\left(E_2(\tau)^2 - E_4(\tau)\right) - \frac{3}{4\pi(\mathrm{Im}(\tau))^2} - \frac{1}{2\pi \mathrm{Im}(\tau)} E_2^*(\tau). \\
&= \frac{1}{12}\left(E_2(\tau)^2 - \frac{6E_2(\tau)}{\pi \mathrm{Im}(\tau)} + \frac{9}{(\pi \mathrm{Im}(\tau))^2}\right) - \frac{1}{12}E_4(\tau). \\
&= \frac{1}{12}E_2^*(\tau)^2 - \frac{1}{12}E_4(\tau).
\end{aligned}$$

Due to the modularity properties of E_4 and E_2^* the proposition follows. □

In what follows we will drop the subscript k in the derivative keeping in mind that it is always fixed as we are given a modular form of weight k to differentiate. We will use the notation ∂^p meaning:

$$\partial^p g := \partial_{k+2(p-1)} \dots \partial_k g,$$

for g - an almost holomorphic modular form of weight k.

PROPOSITION 8.5. *We have*

$$\partial E_2^*(\tau) = \frac{1}{12}\left(E_2^*(\tau)^2 - E_4(\tau)\right),$$

$$\partial^2 E_2^*(\tau) = \frac{1}{36}\left(E_6(\tau) - \frac{3}{2}E_2^*(\tau)E_4(\tau) + \frac{1}{2}E_2^*(\tau)^2\right).$$

PROOF. This follows from direct calculations using the equations (8.3). □

1.2. Solutions of the WDVV equation.

PROPOSITION 8.6. *Suppose that a holomorphic function* $\gamma(t)$ *on a domain in* $\mathbb{C}$ *is a solution of the differential equation* (8.1). *For any* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$, *define a holomorphic function* $\gamma^A(t)$ *on a suitable domain in* $\mathbb{C}$ *as*

$$\gamma^A(t) := \frac{\det(A)}{(ct+d)^2}\gamma\left(\frac{at+b}{ct+d}\right) + \frac{6c}{ct+d}. \tag{8.7}$$

Then $\gamma^A(t)$ *becomes a solution of the differential equation* (8.1).

PROOF. This is obtained by a straightforward calculation. □

Consider the holomorphic function $\gamma^\infty(\tau)$ defined on $\mathbb{H}$ by:

$$\gamma^\infty(\tau) := \frac{\pi\sqrt{-1}}{3}E_2(\tau) \tag{8.8}$$

It was noticed by Dubrovin that the function $\mathcal{F}^\infty$, holomorphic on $M^\infty := \mathbb{C}^2 \times \mathbb{H}$ given by

$$\mathcal{F}^\infty = \frac{1}{2}t_1^2\tau + \frac{1}{2}t_1t_2^2 - \frac{t_2^4}{16}\gamma^\infty(\tau)$$

defines on M^∞ a Frobenius structure of rank three and dimension one. This Frobenius manifold structure was studied extensively by Dubrovin.

PROPOSITION 8.7. *The holomorphic function* $\gamma^\infty(\tau)$ *satisfies the differential equation* (8.1) *and is invariant under the* $\mathrm{SL}(2,\mathbb{Z})$*-action* (8.7).

PROOF. This follows from a direct calculation using the modular property (3.4) of $E_2(\tau)$ and Ramanujan derivatives formulae. □

1.3. Action of $\mathcal{A}^{(\tau_0,\omega_0)}$ on M^∞.

DEFINITION. Choose $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$.

(1) Define a holomorphic function $\gamma^{(\tau_0,\omega_0)}(t)$ on
$$D^{(\tau_0,\omega_0)} := \{t \in \mathbb{C} \mid |t| < |-4\pi\omega_0^2\mathrm{Im}(\tau_0)|\}$$
applying the $\mathrm{GL}(2,\mathbb{C})$-action (8.7) specified by $\mathcal{A}^{(\tau_0,\omega_0)}$ (cf. (7.5)) to the function $\gamma^\infty(\tau)$.

(2) Define complex numbers $c_i(\tau_0,\omega_0)$, $i \in \mathbb{Z}_{\geq 0}$, by the coefficients of the Taylor expansion of $\gamma^{(\tau_0,\omega_0)}(t)$ at $t=0$:
$$\gamma^{(\tau_0,\omega_0)}(t) = \sum_{n=0}^{\infty}\frac{c_n(\tau_0,\omega_0)}{n!}t^n.$$

(3) Denote by $M_3^{(\tau_0,\omega_0)} := \mathbb{C}^2 \times D^{(\tau_0,\omega_0)}$ the Frobenius manifold given by the Frobenius potential
$$\mathcal{F}^{(\tau_0,\omega_0)} = \frac{1}{2}t_1^2t + \frac{1}{2}t_1t_2^2 - \frac{t_2^4}{16}\gamma^{(\tau_0,\omega_0)}(t).$$

(4) Associate the elliptic curve to the Frobenius manifold $M_3^{(\tau_0,\omega_0)}$, given in coordinates by:

$$y^2 = x^3 - \frac{3}{2}c_0(\tau_0,\omega_0)x^2 + \frac{3}{2}c_1(\tau_0,\omega_0)x - \frac{1}{4}c_2(\tau_0,\omega_0). \tag{8.9}$$

1.4. From rank 6 to rank 3. The following theorem giving a complete classification of the solutions of Halphen's system of equations (7.3) was proved by Ohyama:

THEOREM 8.8 (Theorem 2.1 in [**37**]). *Let the triplet of functions* $(X_2(t), X_3(t), X_4(t))$ *be holomorphic in the neighborhood of* $z \in \mathbb{H}$ *and satisfy* (7.3). *Then:*

(1) *If the numbers* $X_k(z)$ *are pairwise distinct:*
$$X_p(z) \neq X_q(z) \quad p \neq q,\ 2 \leq p, q \leq 4,$$
then $\exists A \in \mathrm{SL}(2, \mathbb{C})$ *such that* $X_k(t) = X_k^A(t)$.

(2) *Otherwise, if two of the values of* $X_k(t)$ *at* $t = z$ *coincide for different indices (say* $X_p(z) = X_q(z)$*), then:*
$$\begin{aligned} X_p(t) &= X_q(t) = -\frac{c}{ct+d}, \\ X_r(t) &= -\frac{c}{ct+d} + \frac{a}{(ct+d)^2}, \end{aligned} \tag{8.10}$$
for $(c:d) \in \mathbb{P}^1$ *and some complex* $a \in \mathbb{C}$ *that vanishes if*
$$X_r(z) = X_p(z) = X_q(z).$$

A well known connection between the Chazy equation and Halphen's system of equations is given by:

PROPOSITION 8.9. *Consider the third order equation in* ω*:*
$$\omega^3 - \frac{3}{2}\gamma(t)\omega^2 + \frac{3}{2}\gamma'(t)\omega - \frac{1}{4}\gamma''(t) = 0. \tag{8.11}$$

(1) *Every triple of holomorphic functions* $(X_2(t), X_3(t), X_4(t))$ *that solves Halphen's system of equations* (7.3) *is the triple of roots* $(\omega_1(t), \omega_2(t), \omega_3(t))$ *of the third order equation* (8.11) *in* ω *for some* $\gamma(t)$ *that is a solution of the Chazy equation.*

(2) *Let* $\Delta^Q = \Delta^Q(t)$ *be the discriminant of the third order equation* (8.11). *Then* $(\omega_1(t), \omega_2(t), \omega_3(t))$ *give a constant solution if and only if* $\Delta^Q(0) = 0$ *and a non-constant solution if and only if* $\Delta^Q(0) \neq 0$.

PROOF. Consider $\gamma(t) := \frac{2}{3}\sum X_i(t)$. From the equations (7.3) we have:
$$\begin{aligned} \gamma'(t) &= \frac{2}{3}\left(X_2(t)X_3(t) + X_3(t)X_4(t) + X_2(t)X_4(t)\right), \\ \gamma''(t) &= 4X_2(t)X_3(t)X_4(t), \\ \gamma'''(t) &= 8X_2(t)X_3(t)X_4(t)\left(X_2(t) + X_3(t) + X_4(t)\right) - 4\sum_{i<j}\left(X_i(t)X_j(t)\right)^2. \end{aligned}$$
This concludes the proof of the first part of the proposition. The second part of the proposition follows immediately from Theorem 8.8. □

DEFINITION. The function $\gamma(t)$:
$$\gamma(t) := \frac{2}{3}\sum X_i(t)$$
will be called the Chazy equations solution associated to the triple $(X_2(t), X_3(t), X_4(t))$.

PROPOSITION 8.10. *Let the rank 6 Frobenius manifold* $M_6^{(\tau_0,\omega_0)}$ *be defined over* $\mathbb{K} \subset \mathbb{C}$*, then the rank 3 Frobenius manifold* $M_3^{(\tau_0,\omega_0)}$ *is defined over* $\mathbb{K}$ *too.*

PROOF. Consider the Chazy equation solutions $\gamma^{(\tau_0,\omega_0)}$ given by $M_3^{(\tau_0,\omega_0)}$ and the triplet of Halphen's system solution $(X_i^{(\tau_0,\omega_0)}(t))$ given by $M_6^{(\tau_0,\omega_0)}$. We have to show that:

$$\gamma^{(\tau_0,\omega_0)}(t) = \frac{2}{3}\sum X_i^{(\tau_0,\omega_0)}(t).$$

Because both LHS and RHS are obtained by the same action it is enough to show this equality for the Chazy solution γ^∞ and Halphen's solution $(X_2^\infty, X_3^\infty, X_4^\infty)$:

$$\gamma^\infty = \frac{\pi\sqrt{-1}}{3}E_2(t) = \frac{2}{3}\sum X_i^\infty(t).$$

We know already that the function $\gamma' := \frac{2}{3}\sum X_k^\infty$ satisfies the Chazy equation. Therefore it is enough to check that the first three Fourier coefficients of γ' coincide with those of γ^∞. It can be easily done using the explicit formulae. The Fourier expansion of X_p^∞ reads (cf. Chapter 1 in [**28**]):

$$\frac{1}{2\pi\sqrt{-1}}X_2^\infty = \frac{1}{4} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2ke^{2k\pi\sqrt{-1}\tau}}{1-e^{2k\pi\sqrt{-1}\tau}},$$

$$\frac{1}{2\pi\sqrt{-1}}X_3^\infty = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2ke^{k\pi\sqrt{-1}\tau}}{1-e^{2k\pi\sqrt{-1}\tau}}, \qquad \frac{1}{2\pi\sqrt{-1}}X_4^\infty = -\sum_{k=1}^{\infty}\frac{2ke^{k\pi\sqrt{-1}\tau}}{1-e^{2k\pi\sqrt{-1}\tau}}.$$

By the equality

$$\sum_{k=1}^{\infty}e^{k\pi\sqrt{-1}\tau}\sigma_n(k) = \sum_{k=1}^{\infty}\frac{k^n e^{k\pi\sqrt{-1}\tau}}{1-e^{k\pi\sqrt{-1}\tau}}, \qquad n\in\mathbb{N}_+$$

and definition of $E_2(\tau)$ the proposition follows. □

2. Classification in rank 3

Proposition 8.10 gives a necessary condition for the Frobenius manifold $M_6^{(\tau_0,\omega_0)}$ to be defined over $\mathbb{Q}$. Because of it we start by classifying rank 3 Frobenius manifolds $M_3^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$.

2.1. Classification of $M^{(\tau_0,\omega_0)}$ over the field $\mathbb{K}$. We use two lemmas to give the classification of $M_3^{(\tau_0,\omega_0)}$ over $\mathbb{K}$.

LEMMA 8.11. *For every $\tau_0\in\mathbb{H}$ and $\omega_0\in\mathbb{C}^*$ the coefficients of $\gamma^{(\tau_0,\omega_0)}$ have the following expression via the Eisenstein series:*

$$\begin{aligned}
c_0(\tau_0,\omega_0) =& \frac{1}{6\omega_0^2}\left(E_2(\tau_0) - \frac{3}{\pi\mathrm{Im}(\tau_0)}\right),\\
c_1(\tau_0,\omega_0) =& \frac{c_0(\tau_0,\omega_0)^2}{2} - \frac{E_4(\tau_0)}{72\omega_0^4},\\
c_2(\tau_0,\omega_0) =& -c_0(\tau_0,\omega_0)^3 + 3c_0(\tau_0,\omega_0)c_1(\tau_0,\omega_0) + \frac{E_6(\tau_0)}{216\omega_0^6}.
\end{aligned}$$

PROOF. By definition of $\gamma^{(\tau_0,\omega_0)}$ we have:

$$\gamma^{(\tau_0,\omega_0)}(t) = -\frac{2}{t+4\omega_0^2\pi\mathrm{Im}(\tau_0)} + \frac{8\omega_0^2\pi^2\mathrm{Im}(\tau_0)^2}{3\left(t+4\omega_0^2\pi\mathrm{Im}(\tau_0)\right)^2}E_2\left(\frac{t\bar{\tau}_0 + 4\omega_0^2\pi(\tau_0)\mathrm{Im}(\tau_0)}{t+4\omega_0^2\pi\mathrm{Im}(\tau_0)}\right).$$

Setting $t = 0$ we get:

$$c_0(\tau_0, \omega_0) = \frac{1}{6\omega_0^2}\left(E_2(\tau_0) - \frac{3}{\pi \mathrm{Im}(\tau_0)}\right).$$

Using the formulae (8.3) we compute the first and second derivatives of $\gamma^{(\tau_0,\omega_0)}$:

For the first derivative we get:

$$c_1(\tau_0, \omega_0) = \frac{1}{72\omega_0^4}\left(\frac{9}{\pi^2 \mathrm{Im}(\tau_0)^2} - \frac{6E_2(\tau_0)}{\pi \mathrm{Im}(\tau_0)} + E_2(\tau_0)^2 - E_4(\tau_0)\right).$$

Using the expression for $c_0(\tau_0, \omega_0)$ we get the desired formula. The equality for the second derivative reads:

$$\begin{aligned} c_2(\tau_0, \omega_0) = \frac{1}{432\omega_0^6}\Big(&- \frac{27}{\pi^3 \mathrm{Im}(\tau_0)^3} + \frac{27E_2(\tau_0)}{\pi^2 \mathrm{Im}(\tau_0)^2} - \frac{9E_2(\tau_0)^2}{\pi \mathrm{Im}(\tau_0)} + E_2(\tau_0)^3 \\ &+ 3E_4(\tau_0)\left(\frac{3}{\pi \mathrm{Im}(\tau_0)} - E_2(\tau_0)\right) + 2E_6(\tau_0)\Big). \end{aligned}$$

Expressing the values of the Eisenstein series via $c_0(\tau_0, \omega_0)$ and $c_1(\tau_0, \omega_0)$ we get the statement of the lemma.

□

LEMMA 8.12. *Suppose that $\gamma(t)$ is a convergent power series in t given as $\gamma(t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$, then the Chazy equation is equivalent to the following recursion relation:*

$$c_{n+3} = \sum_{a=0}^{n} \binom{n}{a} \left(6c_a c_{n-a+2} - 9c_{a+1} c_{n-a+1}\right).$$

In particular, we have

$$c_3 = 6c_2 c_0 - 9c_1^2.$$

PROOF. This is obtained immediately by comparing the coefficients in t of the LHS and RHS in the Chazy equation. □

THEOREM 8.13. *Let $\mathbb{K} \subset \mathbb{C}$ be a field. Let $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$. The following are equivalent:*

(i) *The Frobenius manifold $M^{(\tau_0,\omega_0)}$ is defined over $\mathbb{K}$.*
(ii) *All the coefficients of the series expansion of $f^{(\tau_0,\omega_0)}(t)$ are in $\mathbb{K}$.*
(iii) *We have*

$$E_2^*(\tau_0) \in \mathbb{K}\omega_0^2, \quad E_4(\tau_0) \in \mathbb{K}\omega_0^4, \quad E_6(\tau_0) \in \mathbb{K}\omega_0^6.$$

(iv) *Let ∂ be the almost holomorphic derivative defined by (8.5). We have*

$$-\frac{1}{24}E_2^*(\tau_0) \in \mathbb{K}\omega_0^2, \quad -\frac{1}{24}\partial E_2^*(\tau_0) \in \mathbb{K}\omega_0^4, \quad -\frac{1}{24}\partial^2 E_2^*(\tau_0) \in \mathbb{K}\omega_0^6.$$

(v) *We have*

$$E_2^*(\tau_0) \in \mathbb{K}\omega_0^2, \quad \mathcal{E}_{\tau_0} \text{ is defined over } \mathbb{K}.$$

PROOF. By definition, the Frobenius manifold $M^{(\tau_0,\omega_0)}$ is defined over $\mathbb{K}$ if and only if there are flat coordinates $t_1, \widetilde{t}_2, \widetilde{t}_3$ such that the Frobenius potential is given by

$$\mathcal{F}^{(\tau_0,\omega_0)} = \frac{1}{2}\eta_1 t_1^2 \widetilde{t}_3 + \eta_2 \frac{1}{2} t_1 \widetilde{t}_2^2 + \widetilde{t}_2^4 \widetilde{f}(\widetilde{t}_3) \quad \text{for some } \eta_1, \eta_2 \in \mathbb{K} \text{ and } \widetilde{f}(\widetilde{t}_3) \in \mathbb{K}\{\widetilde{t}_3\}.$$

However, this immediately implies that $t_2^2 = \eta_2 \widetilde{t}_2^2$, $t_3 = \eta_1 \widetilde{t}_3$ and $\gamma^{(\tau_0,\omega_0)}(t_3) = 16\eta_2^{-2}\widetilde{f}(\widetilde{t}_3)$, and hence the equivalence between the conditions (i) and (ii).

Because of Lemma 8.12 the first three coefficients c_0, c_1 and c_2 are enough to determine all the coefficients $c_n, n \geq 3$, due to the lemma. To get (iii) it is enough to check that $c_i(\tau_0,\omega_0) \in \mathbb{K}$ for $2 \geq i \geq 0$.

Using Lemma 8.11 we have that (ii) is equivalent to (iii).

By Proposition 8.5 and Lemma 8.11 we get that (iii) is equivalent to:

$$\begin{aligned} E_2^*(\tau_0) &= 6c_0(\tau_0,\omega_0)\omega_0^2, \\ \partial E_2^*(\tau_0) &= 6c_1(\tau_0,\omega_0)\omega_0^4, \\ \partial^2 E_2^*(\tau_0) &= 6c_2(\tau_0,\omega_0)\omega_0^6. \end{aligned}$$

The last condition (v) is equivalent to (iii) by using again Lemma 8.11 and definition of $\mathcal{E}_{\tau_0}$. This proves the theorem. □

2.1.1. *Examples.*

Proposition 8.14 (cf. Lemma 3.2 in [**32**]). *The equation*

$$E_2^*(\tau) = 0 \tag{8.12}$$

holds if and only if $\tau \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ *or* $\tau \in \mathrm{SL}(2,\mathbb{Z})\rho$ *where* $\rho := \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

The values of the Eisenstein series at $\tau = \sqrt{-1}$ are

$$E_2(\sqrt{-1}) = \frac{3}{\pi},\ E_4(\sqrt{-1}) = 3\frac{\Gamma\left(\frac{1}{4}\right)^8}{64\pi^6},\ E_6(\sqrt{-1}) = 0. \tag{8.13}$$

If

$$\omega_0 \in \mathbb{Q}\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\pi^{\frac{3}{2}}}$$

then $c_0(\sqrt{-1},\omega_0) = c_2(\sqrt{-1},\omega_0) = 0$ and $c_1(\sqrt{-1},\omega_0) \in \mathbb{Q}$.

The values of the Eisenstein series at $\tau = \rho$ are

$$E_2(\rho) = \frac{2\sqrt{3}}{\pi},\ E_4(\rho) = 0,\ E_6(\rho) = \frac{27}{2}\frac{\Gamma\left(\frac{1}{3}\right)^{18}}{2^8\pi^{12}}. \tag{8.14}$$

If

$$\omega_0 \in \mathbb{Q}\frac{\Gamma\left(\frac{1}{3}\right)^3}{4\pi^2}$$

then $c_0(\rho,\omega_0) = c_1(\rho,\omega_0) = 0$ and $c_2(\rho,\omega_0) \in \mathbb{Q}$.

2.2. SL**-action on the set of Frobenius manifolds** $M_3^{(\tau_0,\omega_0)}$**.** Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}(2,\mathbb{R})$. The correspondence

$$\tau_0 \mapsto \tau_1 := \frac{a\tau_0 + b}{c\tau_0 + d}, \quad \omega_0 \mapsto \omega_1 := (c\tau_0 + d)\omega_0$$

defines a $\mathrm{SL}(2,\mathbb{R})$-action on the set $\{(\tau_0,\omega_0) \mid \tau_0 \in \mathbb{H},\ \omega_0 \in \mathbb{C}\setminus\{0\}\}$. This is exactly the $\mathrm{SL}(2,\mathbb{R})$-action induced by (7.4) since

$$A\begin{pmatrix} \dfrac{\bar\tau_0}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0\tau_0 \\ \dfrac{1}{4\pi\omega_0\mathrm{Im}(\tau_0)} & \omega_0 \end{pmatrix} = \begin{pmatrix} \dfrac{(a\bar\tau_0+b)}{4\pi\omega_0\mathrm{Im}(\tau_0)} & (a\tau_0+b)\omega_0 \\ \dfrac{(c\bar\tau_0+d)}{4\pi\omega_0\mathrm{Im}(\tau_0)} & (c\tau_0+d)\omega_0 \end{pmatrix} = \begin{pmatrix} \dfrac{\bar\tau_1}{4\pi\omega_1\mathrm{Im}(\tau_1)} & \omega_1\tau_1 \\ \dfrac{1}{4\pi\omega_1\mathrm{Im}(\tau_1)} & \omega_1 \end{pmatrix}.$$

2.2.1. $\mathrm{SL}(2,\mathbb{Z})$-*action.* Lemma 8.11 yields the following.

PROPOSITION 8.15. *Let $\tau_0,\tau_1 \in \mathbb{H}$ and $\omega_0,\omega_1 \in \mathbb{C}\setminus\{0\}$. The following are equivalent*:

1. *There is an isomorphism of Frobenius manifolds $M_3^{(\tau_0,\omega_0)} \cong M_3^{(\tau_1,\omega_1)}$.*
2. *The equality $\gamma^{(\tau_0,\omega_0)}(t) = \gamma^{(\tau_1,\omega_1)}(t)$ holds.*
3. *There exists an element $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ such that*

$$\tau_1 = \frac{a\tau_0+b}{c\tau_0+d}, \quad \omega_1^k = (c\tau_0+d)^k\omega_0^k,$$

where $k=4$ if $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$, $k=6$ if $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho$ and $k=2$ otherwise.

PROOF. It is almost clear that condition (i) is equivalent to (ii). By Lemma 8.11, condition (ii) is equivalent to the equations

$$\frac{E_2^*(\tau_0)}{\omega_0^2} = \frac{E_2^*(\tau_1)}{\omega_1^2}, \quad \frac{E_4(\tau_0)}{\omega_0^4} = \frac{E_4(\tau_1)}{\omega_1^4}, \quad \frac{E_6(\tau_0)}{\omega_0^6} = \frac{E_6(\tau_1)}{\omega_1^6}. \tag{8.15}$$

This implies that $j(\tau_0) = j(\tau_1)$ and hence

$$\tau_1 = \frac{a\tau_0+b}{c\tau_0+d}, \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{8.16}$$

Therefore we obtain

$$\frac{E_2^*(\tau_0)}{\omega_0^2} = \frac{(c\tau_0+d)^2E_2^*(\tau_0)}{\omega_1^2}, \quad \frac{E_4(\tau_0)}{\omega_0^4} = \frac{(c\tau_0+d)^4E_4(\tau_0)}{\omega_1^4}, \quad \frac{E_6(\tau_0)}{\omega_0^6} = \frac{(c\tau_0+d)^6E_6(\tau_0)}{\omega_1^6}.$$

If τ_0 is such that $E_2^*(\tau_0) \neq 0$ it is equivalent to $\omega_1^2 = (c\tau_0+d)^2\omega_0^2$. All values $\tau_0 \in \mathbb{H}$ such that $E_2^*(\tau_0) = 0$ are given in Proposition 8.14. These are $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ and $\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho$. Analyzing the system of equations (8.15) in these two cases we get:

$$\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1} \ \Rightarrow\ E_6(\tau_0) = 0, E_4(\tau_0) \neq 0 \ \Rightarrow\ \omega_1^4 = (c\tau_0+d)^4\omega_0^4,$$

and

$$\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho \ \Rightarrow\ E_4(\tau_0) = 0, E_6(\tau_0) \neq 0 \ \Rightarrow\ \omega_1^6 = (c\tau_0+d)^6\omega_0^6.$$

Hence condition (iii) is equivalent to condition (ii) by Lemma 8.11 and Lemma 8.12. □

2.2.2. $\mathrm{SL}(2,\mathbb{Q})$-*action and complex multiplication.*

DEFINITION. An elliptic curve $\mathcal{E}$ is said to have complex multiplication if its modulus τ is imaginary quadratic. Namely $\tau \in \mathbb{Q}(\sqrt{-D})$ for a positive integer D.

A profound result of the theory of elliptic curves is that elliptic curves over $\mathbb{Q}$ with complex multiplication are easily classified:

THEOREM 8.16 (cf. Paragraph II.2 in [**43**]). *Up to isomorphism there are only 13 elliptic curves defined over $\mathbb{Q}$ that have complex multiplication.*

We give the list of the Weierstrass models of these elliptic curves in Table 1.

Modulus τ	Weierstrass equation	j-invariant	Δ_E
$(-1+\sqrt{-3})/2$	$y^2 = 4x^3 + 1$	0	3^3
$\sqrt{-3}$	$y^2 = 4x^3 - 60x + 88$	$2^4 3^3 5^3$	$2^8 3^3$
$(-1+3\sqrt{-3})/2$	$y^2 = 4x^3 - 120x + 253$	$-2^{15} 3 5^3$	3^5
$\sqrt{-1}$	$y^2 = 4x^3 + 4x$	$2^6 3^3$	2^5
$2\sqrt{-1}$	$y^2 = 4x^3 - 44x + 64$	$2^3 3^3 11^3$	2^9
$(-1+\sqrt{-7})/2$	$y^2 = 4x^3 - \frac{35}{4}x - \frac{49}{8}$	$-3^3 5^3$	7^3
$\sqrt{-7}$	$y^2 = 4x^3 - 2380x + 22344$	$3^3 5^3 17^3$	$2^{12} 7^3$
$\sqrt{-2}$	$y^2 = 4x^3 - 120x + 224$	$2^6 5^3$	2^9
$(-1+\sqrt{-11})/2$	$y^2 = 4x^3 - \frac{88}{3}x - \frac{847}{27}$	-2^{15}	11^3
$(-1+\sqrt{-19})/2$	$y^2 = 4x^3 - 152x + 361$	$-2^{15} 3^3$	19^3
$(-1+\sqrt{-43})/2$	$y^2 = 4x^3 - 3440x + 38829$	$-2^{18} 3^3 5^3$	43^3
$(-1+\sqrt{-67})/2$	$y^2 = 4x^3 - 29480x + 974113$	$-2^{15} 3^3 5^3 11^3$	67^3
$(-1+\sqrt{-163})/2$	$y^2 = 4x^3 - 8697680x + 4936546769$	$-2^{18} 3^3 5^3 23^3 29^3$	163^3

TABLE 1. 13 elliptic curves over $\mathbb{Q}$ with complex multiplication.

COROLLARY 8.17. *The modulus τ_0 of the elliptic curve $\mathcal{E}_{\tau_0}$ with complex multiplication defined over $\mathbb{Q}$ is in the* $\mathrm{SL}(2,\mathbb{C})$ *orbit of one of:*

$$\sqrt{-D}, \quad D \in \{1,2,3,4,7\},$$

or

$$\frac{-1+\sqrt{-D}}{2}, \quad D \in \{3,7,11,19,27,43,67,163\}.$$

Imaginary quadratic $\tau_0 \in \mathbb{C}$ are amazing from the point of view of the theory of modular forms too:

PROPOSITION 8.18 (cf. Theorem A1 in [**32**]). *Let $\tau \in \mathbb{C}$ be imaginary quadratic and $\tau \notin \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$. Then we have:*

$$\frac{E_2^*(\tau)E_4(\tau)}{E_6(\tau)} \in \mathbb{Q}(j(\tau)),$$

where $j(\tau)$ is the value of the j-invariant of the elliptic curve $\mathcal{E}_\tau$.

DEFINITION. Let $\tau_0 \in \mathbb{H}$, $\omega_0 \in \mathbb{C}\backslash\{0\}$.

(1) The Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ is said to *have a symmetry* if there exists an element $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \{1\}$ such that

$$\left(\gamma^{(\tau_0,\omega_0)}\right)^A(t) = \gamma^{(\tau_0,\omega_0)}(t).$$

(2) The Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ is said to *have a weak symmetry* if there exists an element $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \{1,-1\}$ such that

$$\left(\gamma^{(\tau_0,\omega_0)}\right)^A(t) = \gamma^{(\tau_0,\omega_0')}(t) \quad \text{for some } \omega_0' \in \mathbb{C}\backslash\{0\}.$$

REMARK 8.1. It is important to note that weak symmetry is not a symmetry of the Frobenius manifold unless $\omega_0 = \omega_0'$, because the corresponding $\mathrm{SL}(2,\mathbb{R})$-action relates different points in the space of all Frobenius manifolds of rank three .

THEOREM 8.19. *Let $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}\backslash\{0\}$.*

(1) *The Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ has a symmetry if and only if τ_0 is in the $\mathrm{SL}(2,\mathbb{Z})$ orbit of $\sqrt{-1}$ or ρ.*
(2) *The Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ has a weak symmetry if and only if τ_0 is from the list given in Corollary 8.17.*

PROOF. By Proposition 8.15 $M^{(\tau_0,\omega_0)}$ has a symmetry if and only if $\frac{a\tau_0+b}{c\tau_0+d} = \tau_0$ and

$$\omega_0^4 = (c\tau_0+d)^4\omega_0^4 \quad \text{for} \quad \tau_0 \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1},$$

or

$$\omega_0^6 = (c\tau_0+d)^6\omega_0^6 \quad \text{for} \quad \tau_0 \in \mathrm{SL}(2,\mathbb{Z})\rho,$$

or otherwise

$$\omega_0^2 = (c\tau_0+d)^2\omega_0^2.$$

The last equation is satisfied if and only if $(c\tau_0+d)^2 = 1$. It has no solutions for $\tau_0 \in \mathbb{H}$ and $c, d \in \mathbb{Z}$. It is an easy exercise to show that there is a suitable $A \in \mathrm{SL}(2,\mathbb{Z})$ solving the first two equations. This proves (i).

Let $M^{(\tau_0,\omega_0)}$ be defined over $\mathbb{Q}$ and have a weak symmetry. By Theorem 8.13 the elliptic curve $\mathcal{E}_{\tau_0}$ is defined over $\mathbb{Q}$.

Due to Proposition 8.15 we have $\frac{a\tau_0+b}{c\tau_0+d} = \tau_0$. Hence τ_0 satisfies the equation:

$$c\tau_0^2 + \tau_0(d-a) - b = 0.$$

If $c = 0$ or the discriminant of this quadratic equation is equal to zero we get the contradiction with $\tau_0 \in \mathbb{H}$. Hence the elliptic curve $\mathcal{E}_{\tau_0}$ has complex multiplication. From Proposition 8.16 we know that there are only 13 such τ_0 up to the $\mathrm{SL}(2,\mathbb{Z})$-action. Hence τ_0 is from the given list.

Assume that τ_0 is the modulus of one of the elliptic curves from this list. From the rationality assumption on the elliptic curve $\mathcal{E}_{\tau_0}$ we have $j(\tau_0) \in \mathbb{Q}$. The case of $\tau_0 = \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ was treated in Example 2.1.1 and we can apply Proposition 8.18. Its statement reads:

$$\frac{E_2^*(\tau_0)E_4(\tau_0)}{E_6(\tau_0)} \in \mathbb{Q}.$$

At the same time, since the elliptic curve is defined over $\mathbb{Q}$, there exists $a \in \mathbb{C}\backslash\{0\}$ such that:

$$a^2 g_2(\tau_0) \in \mathbb{Q}, \quad a^3 g_3(\tau_0) \in \mathbb{Q}.$$

From the equations (8.2) we have:

$$a^2\pi^4 E_4(\tau_0) = a^2 g_2(\tau_0)\frac{3}{4} \in \mathbb{Q}, \quad a^3\pi^6 E_6(\tau_0) = a^3 g_3(\tau_0)\frac{27}{8} \in \mathbb{Q}.$$

We conclude:

$$a\pi^2 E_2^*(\tau_0) \in \mathbb{Q}.$$

Summing up:

$$E_2^*(\tau_0) \in \mathbb{Q}(a\pi^2)^{-1}, \quad E_4(\tau_0) \in \mathbb{Q}(a\pi^2)^{-2}, \quad E_6(\tau_0) \in \mathbb{Q}(a\pi^2)^{-3}.$$

Taking $\omega_0^2 := (a\pi^2)^{-1}$ we get $M^{(\tau_0,\omega_0)}$ defined over $\mathbb{Q}$ because of Theorem 8.13. □

REMARK 8.2. We can rephrase Theorem 8.19 (i) above as: a Frobenius manifold $M^{(\tau_0,\omega_0)}$ has a symmetry if and only if $\mathcal{E}_{\tau_0}$ has non-trivial automorphisms.

3. Classification in the rank 6

We classify rank 6 Frobenius manifolds $M_6^{(\tau_0,\omega_0)}$ using the elliptic curve associated to the rank 3 Frobenius manifold $M_3^{(\tau_0,\omega_0)}$.

3.1. Classification of $\mathcal{A}^{(\tau_0,\omega_0)} \cdot M_{\mathbb{P}^1_{2,2,2,2}}$ over $\mathbb{Q}$.

LEMMA 8.20. *Suppose $X_i(t)$ is a convergent power series in t given by $X_i(t) = \sum_{n=0}^{\infty} \frac{x_n^{(i)}}{n!} t^n$. The differential equation (7.3) is equivalent to the following recursion relation*:

$$\begin{cases} x_n^{(2)} + x_n^{(3)} = 2(n-1)! \sum_{p=0}^{n-1} x_p^{(2)} x_{n-1-p}^{(3)} \\ x_n^{(3)} + x_n^{(4)} = 2(n-1)! \sum_{p=0}^{n-1} x_p^{(3)} x_{n-1-p}^{(4)} \\ x_n^{(4)} + x_n^{(2)} = 2(n-1)! \sum_{p=0}^{n-1} x_p^{(4)} x_{n-1-p}^{(2)} \end{cases} \tag{8.17}$$

PROOF. This is obtained by a straightforward calculation. □

Therefore the first three coefficients $x_0^{(2)}$, $x_0^{(3)}$ and $x_0^{(4)}$ are enough to determine all coefficients $x_n^{(i)}$ due to the recursion relation (8.17).

Let $\gamma^{(\tau_0,\omega_0)}$ be the Chazy equation solution associated to the triple of Halphen's system solutions $(X_2^{(\tau_0,\omega_0)}, X_3^{(\tau_0,\omega_0)}, X_4^{(\tau_0,\omega_0)})$. Recall the notation of $c_n(\tau_0,\omega_0)$:

$$\gamma^{(\tau_0,\omega_0)}(t) = \sum_{n\geq 0} c_n(\tau_0,\omega_0) \frac{t^n}{n!}.$$

PROPOSITION 8.21. *Let $\gamma^{(\tau_0,\omega_0)}$ be the Chazy equation solution associated to the triple of Halphen's system solutions $(X_2^{(\tau_0,\omega_0)}, X_3^{(\tau_0,\omega_0)}, X_4^{(\tau_0,\omega_0)})$. Let $g_2(\tau_0)$ and $g_3(\tau_0)$ be the modular invariants of the elliptic curve $\mathcal{E}_{\tau_0}$. Then the equation* (8.11) *at $t = 0$:*

$$\omega^3 - \frac{3}{2} c_0(\tau_0,\omega_0)\omega^2 + \frac{3}{2} c_1(\tau_0,\omega_0)\omega - \frac{1}{4} c_2(\tau_0,\omega_0) = 0$$

transforms by the change of variable $\hat{\omega} = (2\omega_0\pi)^2 \left(\omega - \frac{1}{2} c_0(\tau_0,\omega_0)\right)$ to the following one:

$$4\hat{\omega}^3 - g_2(\tau_0)\hat{\omega} - g_3(\tau_0) = 0.$$

PROOF. Getting rid of the square term in ω the equation (8.11) at $t = 0$ reads:

$$(\omega - \frac{1}{2}c_0)^3 + \frac{3}{2}(\omega - \frac{1}{2}c_0)(c_1 - \frac{c_0^2}{2}) + \frac{3}{4}c_0c_1 - \frac{1}{4}c_2(\tau_0,\omega_0) - \frac{c_0^3}{4} = 0$$

Introduce $\tilde{\omega} = \omega - c_0/2$. Using Lemma 8.11 we get:

$$\tilde{\omega}^3 - \tilde{\omega}\frac{3}{2}\frac{E_4(\tau_0)}{72\omega_0^4} - \frac{1}{4}\frac{E_6(\tau_0)}{216\omega_0^6} = 0$$

Expressing E_4 and E_6 via the modular invariants g_2, g_3 we have:

$$\tilde{\omega}^3 - \frac{1}{2^6}\frac{g_2(\tau_0)}{\omega_0^4\pi^4}\tilde{\omega} - \frac{1}{2^8}\frac{g_3(\tau_0)}{\omega_0^6\pi^6} = 0$$

what is equivalent to:

$$4\tilde{\omega}^3 - \frac{1}{2^4}\frac{g_2(\tau_0)}{\omega_0^4\pi^4}\tilde{\omega} - \frac{1}{2^6}\frac{g_3(\tau_0)}{\omega_0^6\pi^6} = 0.$$

Applying the change of variables $\hat{\omega} := (2\omega_0\pi)^2\tilde{\omega}$ and multiplying both sides of the equation above by $(2\omega_0\pi)^8$ the proposition follows. $\square$

COROLLARY 8.22. *The discriminant Δ^Q of the cubic equation* (8.11) *is a non-zero constant multiple of the discriminant Δ of the elliptic curve $\mathcal{E}_{\tau_0}$.*

Recall that the numbers $e_1, e_2, e_3 \in \mathbb{C}$ are roots of the equation $4x^3 - g_2x - g_3 = 0$. The following proposition is well known:

PROPOSITION 8.23 (cf. Chapter 6.12 in [**28**]). *All the numbers e_i are real if and only if g_2 and g_3 are real and $\Delta > 0$. In this case the periods of the elliptic curve are:*

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}, \quad \omega_2 = \sqrt{-1}\int_{-e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}},$$

and the modulus $\tau = \omega_2/\omega_1 \in \sqrt{-1}\mathbb{R}$.

Note that the numbers e_i depend on the particular form of the defining equation of the elliptic curve. In particular they are different for $g_2' = a^2g_2$ and $g_3' = a^3g_3$ for some $a \in \mathbb{C}^*$ while both equations define isomorphic elliptic curves.

PROPOSITION 8.24. *Fix some $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}^*$. Let the rank 6 Frobenius manifold $M_6^{(\tau_0,\omega_0)}$ be defined over $\mathbb{R}$, then $\exists g_2, g_3 \in \mathbb{R}$ such that*

$$\mathcal{E}_{\tau_0} \cong \{y^2 = 4x^3 - g_2x - g_3\}, \text{ and } e_1, e_2, e_3 \in \mathbb{R}.$$

PROOF. As far as $M_6^{(\tau_0,\omega_0)}$ is defined over $\mathbb{R}$ the equation (8.11) has three real roots. We relate them to the zeros of the elliptic curve.

By Proposition 8.21 applying the change of variables $\hat{\omega} = (2\omega_0\pi)^2(\omega + 2c_0(\tau_0,\omega_0))$ the equation (8.11) at $t = 0$ gets the form

$$4\hat{\omega}^3 - g_2(\tau_0)\hat{\omega} - g_3(\tau_0) = 0,$$

Due to Theorem 8.13 and the rank 6 to rank 3 reduction the elliptic curve $\mathcal{E}_{\tau_0}$ is defined over $\mathbb{R}$. Hence we have:

$$\exists a \in \mathbb{C}^* \quad \text{such that} \quad g_2' := g_2a^2, g_3' := g_3a^3 \in \mathbb{R}.$$

By Theorem 8.13 we have:

$$\frac{1}{\omega_0^4}E_4(\tau_0) = \frac{3}{4}\frac{g_2'}{a^2\omega_0^4\pi^4} \in \mathbb{R}, \quad \frac{1}{\omega_0^6}E_6(\tau_0) = \frac{27}{8}\frac{g_3'}{a^3\omega_0^6\pi^6} \in \mathbb{R}.$$

We conclude that:

$$a\omega_0^2\pi^2 \in \mathbb{R}.$$

Consider the change of variables:

$$\tilde{\omega} = a\hat{\omega} = a(2\omega_0\pi)^2 \left(\omega + 2c_0(\tau_0, \omega_0)\right).$$

Due to $a\omega_0^2\pi^2 \in \mathbb{R}$ we see that $\tilde{\omega}$ is obtained from ω by a real linear change of variables.

The elliptic curve $\mathcal{E}_{\tau_0}$ is isomorphic to the following elliptic curve defined by the cubic equation with the real coefficients:

$$4\tilde{\omega}^3 - g_2'\tilde{\omega} - g_3' = 0.$$

Hence its roots e_1, e_2, e_3 differ from the roots of the equation (8.11) at $t = 0$ by a real change of variables. $\square$

Theorem 8.25. *The Frobenius manifold $M_6^{(\tau_0,\omega_0)}$ is defined over $\mathbb{R}$ if and only if $\tau_0 \in \sqrt{-1}\mathbb{R}$ and $\omega_0^2 \in \mathbb{R}$.*

Proof. Let the Frobenius manifold $M_6^{(\tau_0,\omega_0)}$ be defined over $\mathbb{R}$. By Proposition 8.24 and Proposition 8.23 the modulus of the elliptic curve $\tau_0 \in \sqrt{-1}\mathbb{R}$. It is easy to see from the Fourier series expansion that we have:

$$E_2^*(\tau_0) \in \mathbb{R}, \quad E_4(\tau_0) \in \mathbb{R}, \quad E_6(\tau_0) \in \mathbb{R}.$$

The only zeros of $E_4(\tau)$ are $\tau \in \mathrm{SL}(2,\mathbb{Z})\rho$ and $E_2^*(\tau)$ vanishes only on the $\mathrm{SL}(2,\mathbb{Z})$ orbits of $\sqrt{-1}$ and ρ (cf. Proposition 8.14). The case $\tau \in \mathrm{SL}(2,\mathbb{Z})\sqrt{-1}$ was considered in Subsection 2.1.1. Outside this orbit we get at least two numbers that are non-zero.

Using Lemma 8.11 again we see that $\omega_0^2 \in \mathbb{R}$.

On using Lemma 8.11 and Proposition 8.21 it is clear that for every $\tau_0 \in \sqrt{-1}\mathbb{R}$ and $\omega_0^2 \in \mathbb{R}$ the Frobenius manifold $M_6^{(\tau_0,\omega_0)}$ is defined over $\mathbb{R}$. $\square$

3.2. Proof of the LG-LG mirror theorem. We show that there is unique Frobenius manifold satisfying the assumptions on the LG A-model.

Recall the assumption that the LG A-model is defined over $\mathbb{Q}$. According to Propositions 8.23 and 8.24 we should only consider Frobenius manifolds with $\tau_0 \in \sqrt{-1}\mathbb{R}$.

Requiring the Frobenius manifold to be in the $\mathcal{A}^{(\tau_0,\omega_0)}$ orbit of $M_{\mathbb{P}^1_{2,2,2,2}}$ with the imaginary quadratic τ_0 we have by Proposition 8.10 the necessary condition of the rank 3 Frobenius manifold $M_3^{(\tau_0,\omega_0)}$ to be defined over $\mathbb{Q}$ with the imaginary quadratic τ_0.

By Theorem 8.19 τ_0 is from the table of Corollary 8.17 and purely imaginary. These are:

$$\tau_0 \in \mathrm{SL}(2,\mathbb{Z})\ \{\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}, \sqrt{-7}\}.$$

It is easy to check explicitly by using the Weierstrass form of these elliptic curves and Proposition 8.24 that these examples do not give rational solutions to the equation (8.11) at $t = 0$ except when $\tau_0 = \sqrt{-1}$.

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