On irreducible symplectic varieties of $K3^{[n]}$-type

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades
Doktor der Naturwissenschaften
Dr. rer. nat.
genehmigte Dissertation
von

Apostol D. Apostolov, M.Sc.,
geboren am 21. April, 1984
in Varna, Bulgarien

2014

Referent: Prof. Dr. Klaus Hulek

Koreferent: Prof. Eyal Markman, PhD

Tag der Promotion: 5. Februar, 2014

На моите родители

Zusammenfassung

Wir untersuchen die Modulräume von polarisierten irreduziblen symplektischen Mannigfaltigkeiten, welche deformationsäquivalent zu Hilbertschemata von n Punkten auf einer K3 Fläche sind. Solche Mannigfaltigkeiten haben eine nicht ausgeartete 2-Form und sind auch als irreduzible hyperkähler Mannigfaltigkeiten bekannt. Sie bilden einen der Baublöcke von den Ricci-flachen kompakten Kählermannigfaltigkeiten und in Dimension zwei stimmen diese mit den sogennanten K3 Flächen überein. Übliche Fragen bezüglich der Geometrie von Modulräumen beziehen sich auf die Anzahl irreduzibler Komponenten und Klassifikation.

Schlagwörter: irreduzible symplektische Mannigfaltigkeiten, Modulräume, Komponenten

Abstract

We study the moduli spaces of polarised irreducible symplectic manifolds, deformation equivalent to the Hilbert scheme of n points on a K3 surface. Such manifolds possess a non-degenerate 2-form and are also known as irreducible hyperkähler manifolds. They constitute one of the building blocks of Ricci-flat compact Kähler manifolds and in dimension two they coincide with the so called K3 surfaces. Standard questions about the geometry of moduli spaces pertain to the number of irreducible components and classification.

Keywords: irreducible symplectic manifolds, moduli spaces, components

Contents

Overview

In this work we are concerned with irreducible symplectic manifolds, deformation equivalent to the Hilbert scheme of n points on a K3 surface. Such manifolds can be considered as a higher-dimensional generalization of the notion of a K3 surface. Irreducible symplectic manifolds also arise as one of the building blocks of Ricci-flat compact Kähler manifolds.

The first chapter introduces most of the definitions and necessary theoretical preliminaries that we use throughout.

There are several ways to organize irreducible symplectic manifolds into moduli spaces. In the second chapter we consider the moduli space of polarized irreducible symplectic manifolds of $K3^{[n]}$-type, of fixed polarization type – cf. Ch. 1.2 for the definitions. We show that this moduli space is not always connected. More precisely, we fix positive integers n, d and $t|\gcd(2n-2, 2d)$ and we define $\Sigma_n^{d,t}$ to be the set of isometry classes of pairs (T, h), such that T is an even positive definite lattice of rank two and discriminant $4d(n-1)/t^2$, h is a primitive element of square $(h, h) = 2d$, and $h^\perp$ is generated by an element of square $2n-2$. The following proposition enumerates the set of connected components of the moduli space of polarized irreducible symplectic manifolds of $K3^{[n]}$-type, with polarization type of degree $2d$ and divisibility t (Eq. (2.2)) in terms of the set $\Sigma_n^{d,t}$:

Corollary 2.2.4

The number of connected components of the moduli space of polarized irreducible symplectic manifolds of $K3^{[n]}$-type, with polarization type of degree

$2d$ and divisibility t, is given by $|\Sigma_n^{d,t}|$.

The cardinality $|\Sigma_n^{d,t}|$ is itself computed in Prop. 2.3.1.

In the third chapter we investigate the relationship between different moduli spaces of polarized irreducible symplectic manifolds of $K3^{[n]}$-type. We fix positive integers n and d and an element $h_d \in \Lambda_{K3,n}$ (cf. Eq. (1.2)). To this datum we associate two modular varieties $\mathcal{G}_{h_d}$ and $\mathcal{F}_{h_d}$ which arise as arithmetic group quotients of the (polarized) *period domain* $\Omega_{h_d^\perp}$ (Eq. (1.3)), and a finite map $\pi : \mathcal{G}_{h_d} \longrightarrow \mathcal{F}_{h_d}$. Now the variety $\mathcal{F}_{h_d}$ has a certain modular interpretation – by Thms. 1.2.6-7 there is a period map which embeds any component of the moduli space of polarized irreducible symplectic manifolds of $K3^{[n]}$-type and polarization type given by h_d into $\mathcal{F}_{h_d}$. Under some conditions the map π is an isomorphism:

Theorem 3.1
Let $h_d \in \Lambda_{K3,n}$ be a primitive element of the form $h_d = fv + cl_{n-1}$ with

$$(h_d, h_d) = 2d > 0 \text{ and } div(h_d) = f.$$

Suppose that

$$\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1.$$

Let $\pi : \mathcal{F}_{h_d} \to \mathcal{G}_{h_d}$ be the map of modular varieties associated to h_d. If

$$f = 1 \text{ or } f = 2, \text{ and } f \neq n-1, 2n-2, 2d, d,$$

then π has degree 2; else π is an isomorphism.

Furthermore, if we fix an element of $PSL(2,\mathbb{Z})$, represented by an integer matrix $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we can produce two integers $\widetilde{n}$ and $\widetilde{d}$, and an element $h_{\widetilde{d}} \in \Lambda_{K3,\widetilde{n}}$, by a procedure described in Ch.3. Then we obtain the following proposition, which relates $\mathcal{G}_{h_d}$ and $\mathcal{G}_{h_{\widetilde{d}}}$:

Proposition 3.3
The vector $h_{\widetilde{d}} \in \Lambda_{K3,\widetilde{n}}$ *is primitive with*

$$div(h_{\widetilde{d}}) = \widetilde{f} \text{ and } (h_{\widetilde{d}}, h_{\widetilde{d}}) = 2\widetilde{d},$$

and there is an isomorphism $\mathcal{G}_{h_d} \to \mathcal{G}_{h_{\widetilde{d}}}$.

This means that whenever $\mathcal{G}_{h_d} \cong \mathcal{F}_{h_d}$ and $\mathcal{G}_{h_{\widetilde{d}}} \cong \mathcal{F}_{h_{\widetilde{d}}}$ (cf. Thm. 3.1 above), there is a birational map between the components of the corresponding moduli spaces – examples are listed at the end of Ch. 3.

In the fourth chapter we compute the so-called Hirzebruch-Mumford volume of the modular variety $\mathcal{F}_{h_d}$ in some cases:

Proposition 4.1
The HM volume of $\mathcal{F}_{h_d}$ *is given by*

$$\begin{aligned}\mathrm{vol}_{HM}(\widehat{O}^+(\Lambda_{K3,n}, h_d)) \; = \; & 2^{-(21+\rho(\delta_B)+\rho(\widetilde{\Delta}_B))}[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)]|\Delta_B|^{21/2}\delta_B^{-1} \\ & \cdot\pi^{-11}\Gamma(11)L(11, \left(\frac{\Delta_B}{*}\right))\frac{|B_2B_4...B_{20}|}{20!!}C_2(\Lambda_B) \\ & \cdot\prod_{p|\delta_B}(1+p^{-10})\cdot\prod_{p|\delta_B, p\nmid\widetilde{\Delta}_B}(1+\left(\frac{\widetilde{\Delta}_B}{p}\right)p^{-1})P_p(1)^{-1},\end{aligned}$$

where

$$C_2(\Lambda_B) := \begin{cases} (1+2^{-11})(1-(\frac{\Delta_B}{8})2^{-11})^{-1}, & \text{if } 2 \nmid \Delta_B; \\ \frac{2}{3}(1+(\frac{\widetilde{\Delta}_B}{8})2^{-1}), & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is even}; \\ \frac{2}{3}, & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is odd}, (\frac{\widetilde{\Delta}_B}{4}) = 1; \\ \frac{4}{3}, & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is odd}, (\frac{\widetilde{\Delta}_B}{4}) = -1; \\ 4, & \text{if } 2^j||\delta_B, 2^{2j+1}||\Delta_B; \\ 16, & \text{if } 2^{j_1}||\delta_B, 2^{j_1+j_2}||\Delta_B, j_2 > j_1+1; \end{cases}$$

The notation used above is introduced in Ch. 4. Such volumes can be used to estimate the growth of spaces of cusp forms as a function of their weight and also to determine the Kodaira dimension of certain modular varieties.

The fifth chapter is concerned with certain Hodge classes in the product of two IS manifolds. Suppose that we are given a Hodge isometry $\psi : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$ between the Beauville lattices of two irreducible symplectic manifolds X and Y, of $K3^{[n]}$-type. If ψ is a *parallel-transport operator* (cf. Def. 1.2.4), then ψ is induced by a bimeromorphic map between X and Y, by Verbitsky's Torelli Theorem (cf. Thm. 1.2.5). However there are isometries for which ψ is not a parallel-transport operator – examples can be constructed by using the results from Ch. 2. Our aim in the fifth chapter is to show that in some cases these isometries are also induced by algebraic classes, as predicted by the Hodge conjecture. Suppose that X is isomorphic to $\mathcal{M}_1 := \mathcal{M}_{\sigma_1}(v_1)$ and Y is isomorphic to $\mathcal{M}_2 := \mathcal{M}_{\sigma_2}(v_2)$. Here $\mathcal{M}_{\sigma_i}(v_i), i = 1, 2$ denote moduli spaces of σ_i-stable objects in $\mathcal{D}^b(S_i)$ (cf. Ch. 1.3), with Mukai vectors $v_i \in \widetilde{H}(S_i,\mathbb{Z})$ such that $(v_i, v_i) > 0$.

Proposition 5.2.2
Assume that S_1 is elliptic. Then ψ is induced by an algebraic correspondence which is a composition of rational Hodge isometries.

In the next statement we drop the assumption that the surface S_1 is elliptic at the expense of considering special Mukai vectors v_i.

Theorem 5.2.3
Suppose that $c_1(v_i) \in H^{1,1}(S_i,\mathbb{Z})$ vanish for $i = 1, 2$. Then the isometry ψ is induced by an algebraic correspondence, i.e. it is given by a cohomological correspondence, whose kernel is an algebraic class.

Other cases are treated in Props. 5.2.4 and 5.3.3.

In the last chapter we consider (noncommutative) deformations of twisted Fourier-Mukai kernels. By introducing the notion of *functors of Hodge type* (Def. 6.2) we are able to associate to each first-order commutative deformation of a moduli space of sheaves on a K3 surface a (generally noncommutative) deformation of the surface. In the following statement we prove a slight generalization of a theorem of Toda (cf. Thm. 6.3), concerning deformations of twisted Fourier-Mukai equivalences:

Theorem 6.5
Let X and Y be smooth projective varieties and let $\mathcal{A}, \mathcal{B}$ be Azumaya algebras over X, resp. Y. Let

$$\Phi_{\mathcal{E}} : \mathcal{D}^b_{coh}(\mathcal{A}) \to \mathcal{D}^b_{coh}(\mathcal{B})$$

be a Fourier-Mukai equivalence with kernel $\mathcal{E} \in \mathcal{D}^b_{coh}(\mathcal{A}^{op} \boxtimes \mathcal{B})$. Then for any first-order deformation $\mathbb{A}_\alpha, \alpha \in \mathrm{HH}^2(\mathcal{A})$, there is a deformation $\mathbb{B}_\beta, \beta \in \mathrm{HH}^2(\mathcal{B})$ and a deformation of $\mathcal{E}$ to an object of $\mathcal{D}^b_{coh}(\mathbb{A}^{op}_\alpha \times \mathbb{B}_\beta)$ such that the induced Fourier-Mukai transform

$$\Phi_{\widetilde{\mathcal{E}}} : \mathcal{D}^b_{coh}(\mathbb{A}_\alpha) \to \mathcal{D}^b_{coh}(\mathbb{B}_\beta)$$

is an equivalence.

ACKNOWLEDGEMENTS: I would like to thank my advisor Prof. K. Hulek for suggesting this interesting topic, for all the valuable comments and discussions, and for his support and patience. I am grateful to Dr. D. Ploog for his useful comments on the last two chapters of this thesis. I am especially thankful to Prof. E. Markman for his helpful suggestions, the beneficial discussions and motivating questions that form the basis of much of this work.

Chapter 1

Preliminaries

In this chapter we introduce most of the definitions and necessary theoretical preliminaries that we use throughout. The primary sources for this exposition are [GHS2] and [Mar1]. We work over the field of complex numbers.

1.1 Irreducible symplectic varieties

We begin by defining the main objects of our study.

Definition 1.1.1
*A complex manifold X is called an **irreducible symplectic (IS) manifold**, if the following hold*
(i) X is simply connected;
(ii) X is compact Kähler;
(iii) $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form.

Irreducible symplectic manifolds are also known as irreducible hyperkähler manifolds. Their importance stems from the fact that they constitute one of the building blocks of Ricci-flat compact Kähler manifolds. More precisely, we have the following decomposition theorem:

Theorem 1.1.2 ([Bog],[Bea])

Let X be a compact Kähler manifold with numerically trivial canonical bundle. Then there exists a finite étale cover $X' \to X$ such that

$$X' \cong \mathbb{T} \times \prod_{i=1}^{k} V_i \times \prod_{j=1}^{l} X_j,$$

where $\mathbb{T}$ is a compact, complex torus, the V_i are Calabi-Yau manifolds, and the X_j are IS manifolds.

The second cohomology group $H^2(X,\mathbb{Z})$ of an IS manifold carries an integral symmetric bilinear form $(\cdot,\cdot)_X$ of signature $(3, b_2(X)-3)$ called the *Beauville form* (or *Beauville-Bogomolov (BB) form*) (cf. [Bea]). Another important invariant of IS manifolds is the *Fujiki constant* $c_X \in \mathbb{Q}^+$ (cf. [Fuj]). The invariants are related by the equality

$$\int_X \alpha^{2n} = c_X(\alpha,\alpha)_X^n, \forall \alpha \in H^2(X,\mathbb{Z}),$$

where $2n = \dim_{\mathbb{C}} X$.

There are very few known examples of IS manifolds, namely

- K3 surfaces in dimension two;
- deformations of Hilbert schemes of n points on a K3 surface S, denoted by $S^{[n]}$; these are known as IS manifolds *of $K3^{[n]}$-type*; ([Bea])
- deformations of generalized Kummer varieties, constructed as zero fibers of maps of the form $\Sigma \circ f$, where $f : \mathbb{T}^{[n]} \to \mathbb{T}^{(n)}$ is the desingularization map from the Hilbert scheme of n points on a complex torus $\mathbb{T}$ to the n-th symmetric power of the torus, and $\Sigma : \mathbb{T}^{(n)} \to (\mathbb{T}, 0)$ is the sum map to the pointed torus $(\mathbb{T}, 0)$; ([Bea])
- O'Grady's examples in dimensions 6 and 10. ([OG1] and [OG2])

For a K3 surface S, the BB form is simply the intersection form and the lattice $(H^2(S,\mathbb{Z}), (\cdot,\cdot)_S)$ is isomorphic to the *K3 lattice*

$$\Lambda_{K3} := 3U \oplus 2E_8(-1), \tag{1.1}$$

where U is the standard hyperbolic lattice of rank two and E_8 is the unique even positive definite unimodular lattice of rank 8; the notation $\Lambda(m), m \in \mathbb{Z}$ means that the quadratic form associated to a lattice Λ is multiplied by m.

For IS manifolds of $K3^{[n]}$-type, we have the following

Proposition 1.1.3 (cf. [Bea, Prop. 6, Sect. 9])
The BB lattice of an IS manifold of $K3^{[n]}$-type is given by

$$\Lambda_{K3,n} := \Lambda_{K3} \oplus \langle l_{n-1} \rangle, \tag{1.2}$$

where $\langle l_{n-1} \rangle$ denotes a rank one lattice, generated by an element l_{n-1} of square $(l_{n-1}, l_{n-1}) = 2 - 2n$. The Fujiki constant is $(2n)!/2^n n!$.

1.2 Moduli of IS manifolds and period maps

There are several ways to organize IS manifolds into moduli spaces. In all cases one needs to add some extra structure to the data of an IS manifold. Fix a lattice Λ, which is isometric to $H^2(X_0, \mathbb{Z})$ with its Beauville form $(\cdot, \cdot)_{X_0}$, for some IS manifold X_0.

Definition 1.2.1
(i) A ***marked irreducible symplectic manifold*** (X, η) *consists of an IS manifold X together with the choice of an isomorphism*

$$\eta : H^2(X, \mathbb{Z}) \to \Lambda.$$

(ii) A ***polarisation*** *on an IS manifold X is an ample line bundle $\mathcal{L}$ on X.*

Since the irregularity of X is zero, a line bundle $\mathcal{L}$ can be identified with its first Chern class $h := c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ and we denote a *polarised IS manifold* by (X, h).

There is a (non-Hausdorff) coarse moduli space $\mathfrak{M}_\Lambda$, whose points represent equivalence classes $[(X, \eta)]$ of marked IS manifolds, whose second cohomology group is isometric to Λ, where $(X,\eta) \sim (X',\eta')$ if there is an isomorphism $g : X \xrightarrow{\sim} X'$ such that $\eta' = g^* \circ \eta$ (cf. [Huy2, Ch. 3]). Set

$$\Omega_\Lambda := \{[w] \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}) \mid (w,w) = 0, (w,\overline{w}) > 0)\} \tag{1.3}$$

to be the *period domain*, associated to Λ. There is a period map

$$P : \mathfrak{M}_\Lambda \to \Omega_\Lambda,$$

sending $[(X,\eta)]$ to the period point $[\eta(\sigma)] \in \Omega_\Lambda$, where σ is a generator of

$$H^0(X,\Omega_X^2) \simeq H^{2,0}(X,\mathbb{C}) \simeq \mathbb{C}\sigma.$$

The period map has the following properties

Theorem 1.2.2
(i) (local Torelli, cf. [Bea]) The map P is a local homeomorphism.
(ii) ([Huy1, Thm. 8.1]) The restriction of P to each connected component of $\mathfrak{M}_\Lambda$ is surjective.

In fact, for $K3$ surfaces, the period determines the isomorphism class of the surface. This is the content of the (weak) global Torelli theorem for K3 surfaces, dating back to Piateckii-Shapiro and Shafarevich ([PS]); cf. also [BR] for the analytic case.

Theorem 1.2.3 (cf. e.g. [GHS2, Thm. 2.4])
Two K3 surfaces S and S' are isomorphic if and only if there is an isometry of weight-two Hodge structures $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$.

Given a lattice Λ, $O(\Lambda)$ denotes the isometry group of Λ. Since the quotient set $\Omega_{\Lambda_{K3}}/O(\Lambda_{K3})$ parametrizes the pure Hodge structures of weight two on $\Lambda_{K3} \otimes \mathbb{C}$, coming from the second cohomology lattices of $K3$ surfaces, Thm. 1.2.3 gives a one-to-one correspondence between the set $\Omega_{\Lambda_{K3}}/O(\Lambda_{K3})$ and the set of isomorphism classes of $K3$ surfaces.

In higher dimensions, the Hodge structure on the BB lattice is no longer sufficient to determine the isomorphism class of an IS manifold. Nonetheless, it is still closely related to the geometry of IS manifolds, by the Global Torelli theorem of Verbitsky (cf. [Ver], [Mar1]). First we need to introduce some definitions.

Definition 1.2.4

(i) Let X_1, X_2 be IS manifolds. An isomorphism

$$f : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$$

is said to be a ***parallel-transport operator*** *from X_1 to X_2, if there exists a smooth, proper family of IS manifolds $\pi : \mathcal{X} \to T$ onto an analytic space, points $t_1, t_2 \in T$, isomorphisms*

$$\psi_i : X_i \to \mathcal{X}_{t_i}, i = 1, 2$$

and a continuous path

$$\gamma : [0, 1] \to T \text{ with } \gamma(0) = t_1, \gamma(1) = t_2$$

such that parallel transport in the local system $R^*\pi_*\mathbb{Z}$ along γ induces the homomorphism

$$\psi_2^* \circ f \circ (\psi_1^{-1})^* : H^*(\mathcal{X}_{t_1}, \mathbb{Z}) \to H^*(\mathcal{X}_{t_2}, \mathbb{Z}).$$

(ii) An isomorphism

$$f : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$$

is called a ***monodromy operator*** *if it is a parallel-transport operator from X to itself. The subgroup of $GL(H^*(X, \mathbb{Z}))$, generated by monodromy operators is denoted by $Mon(X)$.*

Let $\mathrm{Mon}^2(X)$ denote the image of $\mathrm{Mon}(X)$ in $\mathrm{GL}(H^2(X,\mathbb{Z}))$.

Now any IS manifold X determines an orientation class $or_X \in H^2(\widetilde{\mathcal{C}}_X,\mathbb{Z})$, i.e. a generator of $H^2(\widetilde{\mathcal{C}}_X,\mathbb{Z}) \cong \mathbb{Z}$, where $\widetilde{\mathcal{C}}_X$ is the cone

$$\{h \in H^2(X,\mathbb{R}) |\ (h,h) > 0\}$$

(cf. [Mar1, Ch. 4]). The cone $\widetilde{\mathcal{C}}_\Lambda$ in $\Lambda \otimes \mathbb{R}$ is defined analogously and the set of its orientations is denoted by $\mathrm{Orient}(\Lambda)$ – it is the set of two generators of $H^2(\widetilde{\mathcal{C}}_\Lambda,\mathbb{Z})$.

Let $O^+(H^2(X,\mathbb{Z}))$ (resp. $O^+(\Lambda)$) denote the subgroup of $O(H^2(X,\mathbb{Z}))$ (resp. $O(\Lambda)$) whose elements act trivially on $H^2(\widetilde{\mathcal{C}}_X,\mathbb{Z})$ (resp. $H^2(\widetilde{\mathcal{C}}_\Lambda,\mathbb{Z})$). Elements of $O^+(H^2(X,\mathbb{Z}))$ (resp. $O^+(\Lambda)$) are called *orientation-preserving*. In fact, $\mathrm{Mon}^2(X) \subset O^+(H^2(X,\mathbb{Z}))$, since monodromy operators are orientation-preserving isometries with respect to the BB form.

We can now state the following consequence of Verbitsky's results (cf. [Ver], [Huy3]), as formulated in [Mar1, Thm. 1.3]:

Theorem 1.2.5 (Hodge-theoretic Torelli)

(i) Let X_1 and X_2 be two IS manifolds which are deformation equivalent. If there is an isomorphism of Hodge structures

$$f : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$$

which is a parallel-transport operator, then X_1 and X_2 are bimeromorphic.

(ii) If, in addition, f maps a Kähler class of X_1 to a Kähler class of X_2, then X_1 and X_2 are isomorphic.

We can also consider moduli spaces of (primitively) polarized IS manifolds. These have the advantage that their connected components admit the structure of quasi-projective varieties. Moduli spaces of polarized varieties with trivial canonical bundle were constructed by Viehweg ([Vie]).

Let us fix an IS manifold X_0 with $H^2(X_0,\mathbb{Z}) \cong \Lambda$ and the $O(\Lambda)$-orbit $\overline{h}$

of a primitive vector $h \in \Lambda$, of degree $(h,h) > 0$. The orbit $\overline{h}$ is called a *polarization type*.

Viehweg's construction yields a moduli space of polarized IS manifolds of type $(X_0, \overline{h})$ – denote it by $\mathcal{V}_{X_0,\overline{h}}$. A point of $\mathcal{V}_{X_0,\overline{h}}$ represents an equivalence class of pairs (X, H), where X is a projective IS manifold, deformation equivalent to X_0, and $H \in H^2(X, \mathbb{Z})$ is the first Chern class of an ample line bundle and such that $\eta(H) \in \bar{h}$ with respect to a marking η of X.

Given $h \in \Lambda$ with $(h,h) > 0$, denote by $\Omega_{h^\perp}$ the period domain $\Omega_\Lambda \cap h_\Lambda^\perp$ and let $O(\Lambda, h)$ be the stabilizer of h in $O(\Lambda)$. This is a type IV symmetric domain and has two connected components – denote one of them by $\Omega^0_{h^\perp}$; then the subgroup of $O(\Lambda, h)$ fixing the connected components coincides with (cf. [GHS2, Sec. 3.3]):

$$O^+(\Lambda, h) := O(\Lambda, h) \cap O^+(\Lambda).$$

Let $\mathrm{Mon}^2(X, H)$ denote the stabilizer of H in $\mathrm{Mon}^2(X)$, and, given a marking η, put

$$\Gamma_h := \eta \mathrm{Mon}^2(X, H)\eta^{-1} \subset O^+(\Lambda, h).$$

The group Γ_h does not depend on the choice of marking – cf. [Mar1, Sec. 7.1].

Let $\mathcal{V}^0_{X_0,\overline{h}}$ be a component of $\mathcal{V}_{X_0,\overline{h}}$. There is a period map

$$P' : \mathcal{V}^0_{X_0,\overline{h}} \to \Omega^0_{h^\perp}/O^+(\Lambda, h).$$

It has the following properties:

Theorem 1.2.6
(i) ([GHS1, Thm. 1.5]) The map P' is a finite dominant morphism.

(ii) ([GHS2, Thm. 3.7]) The map P' factors through an open immersion ι:

$$\begin{tikzcd}
\mathcal{V}^0_{X_0,\overline{h}} \arrow[r, hook, "\iota"] \arrow[d, "P'"'] & \Omega^0_{h^\perp}/\Gamma_h \arrow[dl] \\
\Omega^0_{h^\perp}/O^+(\Lambda, h). &
\end{tikzcd}$$

Thm. 1.2.6.ii) requires Thm. 1.2.5 – cf. also [Mar1, Sec. 8].

For a lattice Λ, let $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$ denote the dual lattice. Note that there is a natural inclusion $\Lambda \subset \Lambda^\vee$ and put

$$\widehat{O}(\Lambda) := \{g \in O(\Lambda) \mid g|_{\Lambda^\vee/\Lambda} = \pm\mathrm{id}_{\Lambda^\vee/\Lambda}\} \tag{1.4}$$

and given $h \in \Lambda$, let $\widehat{O}(\Lambda, h)$ be the stabilizer of h in $\widehat{O}(\Lambda)$. If X is an IS manifold of $K3^{[n]}$-type, then $\mathrm{Mon}^2(X)$ is known to be

Theorem 1.2.7 ([Mar4, Thm 1.2])

$$Mon^2(X) = \widehat{O}^+(\Lambda_{K3,n}).$$

From now on, unless specified otherwise, we only consider IS manifolds deformation equivalent to Hilbert schemes of points on a K3 surface. Put $X_0 := S^{[n]}$, where S is a K3 surface. There are finitely many connected components of $\mathfrak{M}_{\Lambda_{K3,n}}$ parametrizing those marked pairs (X, η), where X is deformation equivalent to X_0 ([Mar1, L. 7.5]). Denote the set of these components by τ and let $\mathfrak{M}^\tau_{\Lambda_{K3,n}}$ denote their union. There is an $O(\Lambda_{K3,n})$-equivariant *refined period map*

$$\widetilde{P} : \mathfrak{M}^\tau_{\Lambda_{K3,n}} \to \Omega_{\Lambda_{K3,n}} \times \tau,$$

sending a marked pair $[(X, \eta)]$ to $([(X, \eta)], s)$, where $\mathfrak{M}^s_{\Lambda_{K3,n}}$ is the component containing $[(X, \eta)]$. The group $O(\Lambda_{K3,n})$ acts on $\Omega_{\Lambda_{K3,n}} \times \tau$ by changing a period point by an auto-isometry of $\Lambda_{K3,n}$, and a marking by post-composition with an auto-isometry of $\Lambda_{K3,n}$ (cf. [Mar1, Ch. 7.2] and Props. 2.1.1-2.1.2 in the next chapter). Now $\Omega_{h^\perp}$ has two connected components – the choice of h and $s \in \tau$ determines the choice of a connected component of $\Omega_{h^\perp}$, which we denote by $\Omega^{s,+}_{h^\perp}$ (cf. [Mar1, Ch. 4]). For $s \in \tau$, set

$$\mathfrak{M}^{s,+}_{h^\perp} := \widetilde{P}^{-1}((\Omega^{s,+}_{h^\perp}, s)).$$

The space $\mathfrak{M}^{s,+}_{h^\perp}$ contains those marked pairs $[(X, \eta)]$ for which $\eta^{-1}(h) \in H^2(X, \mathbb{Z})$ is of Hodge type $(1,1)$ and $(\eta^{-1}(h), \kappa) > 0$ for a Kähler class κ, i.e. it is the first Chern class of a big line bundle (cf. [Huy1, Cor. 3.10]). Let

$$\mathfrak{M}^{s,a}_{h^\perp} \subset \mathfrak{M}^{s,+}_{h^\perp}$$

be the subset, consisting of those $[(X, \eta)]$, for which $\eta^{-1}(h)$ is ample. $\mathfrak{M}^{s,a}_{h^\perp}$ is an open dense path-connected Hausdorff subset of $\mathfrak{M}^{s,+}_{h^\perp}$ ([Mar1, Cor. 7.3]). This result uses the Global Torelli theorem of Verbitsky ([Ver, Thm. 1.16]). Let $\widetilde{h}$ be an $O(\Lambda_{K3,n})$-orbit of pairs (h, s) with $(h, h) > 0$. Finally, form the disjoint union

$$\mathfrak{M}^a_{\widetilde{h}} := \coprod_{(h,s)\in\widetilde{h}} \mathfrak{M}^{s,a}_{h^\perp}.$$

$\mathfrak{M}^a_{\widetilde{h}}$ is a coarse moduli space for marked polarized triples - a point of $\mathfrak{M}^a_{\widetilde{h}}$ represents an isomorphism class $[(X, H, \eta)]$ of *polarized marked IS manifolds of type* $(X_0, \widetilde{h})$.

Denote the moduli space of polarized IS manifolds of $K3^{[n]}$-type by $\mathcal{V}_{X_0}$ – it is the union of the spaces $\mathcal{V}_{X_0,\overline{h}}$ over all polarization types $\overline{h}$, where $h \in \Lambda_{K3,n}$ and $(h, h) > 0$. Let $[(X, H)]$ be a point in $\mathcal{V}_{X_0}$ and let $\mathcal{V}^0$ be the connected component containing $[(X, H)]$. The next proposition relates this component to a quotient of the coarse moduli space of marked polarized triples:

Proposition 1.2.8 ([Mar1, L. 8.3]) *There exists a natural isomorphism*

$$\varphi : \mathcal{V}^0 \xrightarrow{\sim} \mathfrak{M}^a_{\widetilde{h}}/O(\Lambda_{K3,n})$$

in the category of analytic spaces.

1.3 Moduli of sheaves and stable objects on a $K3$ surface

For the theory of moduli spaces of sheaves on a K3 surface we follow the exposition in [Mar5, Ch. 1.1] and [HL]. Let S be a $K3$ surface. Let $K(S)$ denote the topological K-group of S and let

$$\chi : K(S) \to \mathbb{Z}$$

be the Euler characteristic. There is a bilinear form on $K(S)$, called the *Mukai pairing*:

$$(v, w) := -\chi(v^\vee \otimes w). \tag{1.5}$$

The group $K(S)$, together with the Mukai pairing is known as the *Mukai lattice*. It is isometric to the lattice

$$\widetilde{\Lambda} := 2E_8(-1) \oplus 4U. \tag{1.6}$$

There is a homomorphism

$$v(-) : K(S) \to H^*(S, \mathbb{Z}) \tag{1.7}$$

which sends a class $F \in K(S)$ to its *Mukai vector*

$$v(F) := ch(F)\sqrt{td_S} = (\mathrm{rk}(F), c_1(F), \chi(F) - \mathrm{rk}(F)). \tag{1.8}$$

Above we have used the cohomological grading on

$$H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

and the natural identifications of H^0 and H^4 with $\mathbb{Z}$. The homomorphism $v(-)$ is an isometry with respect to the Mukai pairing on $K(S)$ and the pairing on $H^*(S,\mathbb{Z})$ given by

$$((r_1,h_1,s_1),(r_2,h_2,s_2)) := (h_1,h_2)_S - r_1s_2 - r_2s_1. \tag{1.9}$$

Let us denote henceforth the group $H^*(S,\mathbb{Z})$ together with the pairing (1.9) by $\widetilde{H}(S,\mathbb{Z})$. The lattice $\widetilde{H}(S,\mathbb{Z})$ inherits a pure Hodge structure of weight two from the one on $H^2(S,\mathbb{Z})$. We have

$$\widetilde{H}^{1,1}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^{1,1}(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}).$$

Now let $v \in K(S)$ be a primitive class with the property $c_1(v) \in H^{1,1}(S,\mathbb{Z})$.

Definition 1.3.1

The class v is called ***effective****, if $(v,v) \geq -2, \mathrm{rk}(v) \geq 0$, and the following hold: if $\mathrm{rk}(v) = 0$, then $c_1(v)$ is the class of an effective (or trivial) divisor; if both $\mathrm{rk}(v)$ and $c_1(v)$ vanish, then $\chi(v) > 0$.*

If $\mathrm{rk}(v) = \chi(v) = 0$, we assume further that $c_1(v)$ generates the Neron-Severi group $NS(S)$.

An effective class v defines a locally finite set of walls (which may be empty, if $\rho(S) = 1$) in the ample cone $\mathrm{Amp}(S)$ of S. A class $H \in \mathrm{Amp}(S)$ is called *v-generic*, if it does not belong to any of these walls. Under the above assumptions on v there always exists a v-generic class H and we have the following theorem:

Theorem 1.3.2 (cf. [Muk2],[OG3, Main Thm.], [Yosh1, Thm. 8.1])

Let $v \in K(S)$ be a primitive, effective class and $H \in \mathrm{Amp}(S)$ be a v-generic polarisation. Then the moduli space $\mathcal{M}_H(v)$ of H-stable sheaves with Mukai vector v is non-empty of dimension $2n := (v,v) + 2$, it is smooth and projective, and it is of $K3^{[n]}$-type.

Let us put $v = (r,l,s)$. The space $\mathcal{M} := \mathcal{M}_H(v)$ is a fine moduli space, if there is a class $h \in H^{1,1}(S,\mathbb{Z})$ with $\gcd(r,(h,l),s) = 1$. In this case there exists a universal sheaf $\mathcal{E}$ on $S \times \mathcal{M}$, unique up to tensoring by the pullback of a line bundle on S. Otherwise, $\mathcal{M}$ is not fine, but there still exists a *quasi-universal family* $\widetilde{\mathcal{E}}$ *of similitude* $\rho > 1$:

Definition 1.3.3
(i) A flat family $\widetilde{\mathcal{E}}$ *on* $S \times T$ *is called a* ***quasi-family of similitude*** ρ $(\rho \in \mathbb{N})$*, if for each closed point* $t \in T$*, there is an element* $E \in \mathcal{M}_H(v)$ *such that* $\widetilde{\mathcal{E}}|_{\{t\}\times S} \cong E^{\oplus\rho}$.

(ii) Let p_T *denote the projection from* $S \times T$ *to* T*. Two quasi-families* $\widetilde{\mathcal{E}}$ *and* $\widetilde{\mathcal{E}}'$ *on* $S \times T$ *are called* ***equivalent****, if there exist vector bundles* V *and* V' *on* T *such that* $\widetilde{\mathcal{E}} \otimes p_T^* V' \cong \widetilde{\mathcal{E}}' \otimes p_T^* V$.

(iii) A quasi-family $\widetilde{\mathcal{E}}$ *is called* ***quasi-universal****, if, for every scheme* T' *and for any quasi-family* $\mathcal{T}$ *on* $T' \times S$*, there is a unique morphism* $f : T' \to T$ *such that* $f^*\widetilde{\mathcal{E}}$ *and* $\mathcal{T}$ *are equivalent.*

In fact, even in this case there is a universal sheaf, albeit not in an ordinary sense, but as a twisted sheaf – cf. the next section.

Denote the projection maps from $S \times \mathcal{M}$ to S, resp. $\mathcal{M}$ by p, resp. q. Let

$$\widetilde{\varphi} : \widetilde{H}(S, \mathbb{Q}) \to H^2(\mathcal{M}, \mathbb{Q})$$

be the map

$$\widetilde{\varphi} : \alpha \mapsto \frac{1}{\rho} c_1\{q_*(ch(\widetilde{\mathcal{E}}) \cup p^*\sqrt{\mathrm{td}(S)} \cup p^*\alpha)\}. \tag{1.10}$$

Denote by $\mathcal{D}_S$ the duality operator on $\widetilde{H}(S, \mathbb{Q})$ acting by multiplication by $(-1)^i$ on $H^{2i}(S, \mathbb{Q})$. The name comes from the fact that these operators map the chern character of a locally free sheaf to the chern character of its dual sheaf. The map $\mathcal{D}_S$ is a Fourier-Mukai transform with kernel $\pi_0 - \pi_2 + \pi_4$, where $\pi_{2i}, i = 0, 1, 2$ are the Künneth components of the class $[\Delta_S] \in H^4(S \times S, \mathbb{Z})$, Poincaré dual to the diagonal in $S \times S$. Moreover, by the Standard Conjectures for surfaces (cf. e.g. [Kl, Cor. 2A10]), this class is algebraic, being a linear combination of the algebraic classes $\pi_{2i}, i = 0, 1, 2$. Furthermore, the projection $H^*(\mathcal{M}, \mathbb{Q}) \to H^2(\mathcal{M}, \mathbb{Q})$ is also given by an algebraic kernel, by the Standard Conjectures, which have been proved in the case of moduli spaces of sheaves on a K3 surface (cf. [Ar, Cor. 7.9]; cf. also [ChM, Thm. 1.1]).

Recall the following theorem of O'Grady and Yoshioka:

Theorem 1.3.4 (cf. [OG3, Main Thm.], [Yosh3, Thm. 0.1])
Suppose that $(v, v) \geq 2$. Then the restriction of $\widetilde{\varphi} \circ \mathcal{D}_S$ to the sub-Hodge structure $v^\perp \subset \widetilde{H}(S, \mathbb{Z})$ is an integral Hodge isometry onto $H^2(\mathcal{M}, \mathbb{Z})$:

$$\widetilde{\varphi} \circ \mathcal{D}_S|_{v^\perp} : v^\perp \to H^2(\mathcal{M}, \mathbb{Z}) \subset H^2(\mathcal{M}, \mathbb{Q}). \tag{1.11}$$

The restriction $\widetilde{\varphi} \circ \mathcal{D}_S|_{v^\perp}$ is independent on the choice of quasi-universal family.

The IS manifolds, which are birational to a moduli space of sheaves on a K3 surface, also admit a modular interpretation in terms of Bridgeland stable objects in the derived category of the surface. Next, following [BM1] and [BM2, Ch. 2] we introduce Bridgeland stability conditions on $K3$ surfaces. The theory was originally developed in [Br1] and [Br2].

Definition 1.3.5
*Let $\mathcal{D}$ be a triangulated category. A **slicing** $\mathcal{P}$ of $\mathcal{D}$ is a collection of full, extension-closed subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$, satisfying:*

(i) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$*;*

(ii) If $\phi_1 > \phi_2$, then $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$*;*

(iii) For any object E in $\mathcal{D}$, there exists a collection of real numbers $\phi_1 >$

$\phi_2 > ... > \phi_n$ and a sequence of distinguished triangles

$$\begin{tikzcd}
E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \dots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$.

The above sequence is called the *Harder-Narasimhan filtration* of E – it is unique up to isomorphism. Each category $\mathcal{P}(\phi)$ is abelian and its objects are called *semistable of phase* ϕ; its simple objects are called *stable*.

Definition 1.3.6
Let $\mathcal{D}$ be a triangulated category and denote its K-group of by $K(\mathcal{D})$. A ***Bridgeland stability condition*** *on $\mathcal{D}$ is a triple $(\Lambda, Z, \mathcal{P})$, where*

(i) Λ is a finite rank ***lattice*** *(free abelian group), together with a surjection $v : K(\mathcal{D}) \twoheadrightarrow \Lambda$;*

(ii) The ***central charge*** *$Z : \Lambda \to \mathbb{C}$ is a group homomorphism;*

(iii) $\mathcal{P}$ is a slicing of $\mathcal{D}$, satisfying

(a)
$$\frac{1}{\pi}\arg(Z(E)) = \phi,$$
for all non-trivial $E \in \mathcal{P}(\phi)$;

(b) Given a norm $\parallel - \parallel$ on $\Lambda_{\mathbb{R}}$, there exists a constant $C > 0$ such that

$$|Z(E)| \geq C \parallel v(E) \parallel,$$

for all $E \in \mathcal{P}$.

The set of stability conditions $\mathrm{Stab}_{\Lambda}(\mathcal{D})$ with fixed Λ admits the structure of a complex manifold of dimension equal to the rank of Λ, by [Br1].

In the case of a $K3$ surface S, the set of stability conditions on $\mathcal{D}^b(S)$ with lattice $\widetilde{H}^{1,1}(S)$ is non-empty and Bridgeland has described a distinguished connected component of it, denoted by $\mathrm{Stab}^\dagger(S)$. ([Br2])

In addition, similarly to the case of sheaves discussed in the beginning of this section, given a Mukai vector $v \in \widetilde{H}^{1,1}(S)$, the space $\mathrm{Stab}^\dagger(S)$ admits a wall-and-chamber structure, described in [BM1, Prop. 2.3], and a stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ is called *v-generic* if it does not lie on any wall. Again, given a primitive effective vector v and a v-generic stability condition σ, there is a coarse moduli space $\mathcal{M}_\sigma(v)$ of σ-semistable objects in $\mathcal{D}^b(S)$, which is a smooth projective manifold ([BM1, Thm 1.3]). The space $\mathcal{M}_\sigma(v)$ does not change, if we change the stability condition inside the chamber of the wall-and-chamber decomposition containing σ. The analogue of Thm. 1.3.4 obtained by replacing $\mathcal{M}_H(v)$ by $\mathcal{M}_\sigma(v)$ has been proven in [BM1, Prop. 5.9]. Moduli spaces of H-stable sheaves are special cases of the above construction – associated to a v-generic polarization H, there is a Gieseker-type chamber $\mathcal{C} \subset \mathrm{Stab}(S)$ such that $\mathcal{M}_H(v)$ coincides with $\mathcal{M}_\sigma(v)$ for $\sigma \in \mathcal{C}$, cf. [BM2, Rmk. 2.12]. In particular, stability conditions allow for a modular interpretation of the birational models of the moduli spaces of sheaves on a K3 surface, by the following theorem:

Theorem 1.3.7 ([BM2, Thm. 1.2])

(i) Let σ and τ be generic stability conditions with respect to v. Then $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_\tau(v)$ are birational to each other.

(ii) Every smooth birational model (with trivial canonical bundle) of $\mathcal{M}_\sigma(v)$ appears as a moduli space $\mathcal{M}_\tau(v)$, for some stability condition $\tau \in \mathrm{Stab}(S)$.

1.4 Twisted sheaves

Our main sources for this section are [Cal1] and [HSt1] – cf. also [Lie]. Let X be a complex variety. The choice of a class $\alpha \in H^2_{an}(X, \mathcal{O}_X^*)$ (in the analytic topology on X) determines an isomorphism class of $\mathbb{G}_m$-gerbes $[X^\alpha \to X]$ on X. More precisely, a gerbe in this class is determined by the choice of a lift of α to a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X^*)\}$. So when we write X^α we abuse notation and we assume implicitly the choice of a 2-cocycle, representing α. When α is torsion, i.e. when α is an element of the *Brauer group* $\mathrm{Br}(X) := H^2_{tors}(X, \mathcal{O}_X^*)$, consisting of the torsion elements in the cohomology group $H^2_{an}(X, \mathcal{O}_X^*)$, the pair (X, α) is known as a *twisted variety.*

Definition 1.4.1
*In terms of an open covering $X = \bigcup_{i \in I} U_i$ and the cocycle $\{\alpha_{ijk}\}$ representing α, a (coherent) α-**twisted sheaf** E is given by the data of (coherent) sheaves E_i on U_i, and isomorphisms $\varphi_{ij} : E_i|_{U_{ij}} \to E_j|_{U_{ij}}$, which satisfy:*

(i) $\varphi_{ii} = \mathrm{id}$;

(ii) $\varphi_{ij} = \varphi_{ji}^{-1}$;

(iii) $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

The category of α-twisted coherent sheaves on X is abelian and is denoted by $\mathrm{Coh}(X, \alpha)$. Its K-group is denoted by $K(X, \alpha)$, its bounded derived category by $\mathcal{D}^b(X, \alpha)$. The category $\mathrm{Coh}(X, \alpha)$ is not a tensor category for non-trivial α. Nonetheless, there are bifunctors

$$- \otimes - : \mathrm{Coh}(X, \alpha) \times \mathrm{Coh}(X, \alpha') \to \mathrm{Coh}(X, \alpha.\alpha')$$

and their derived versions, constructed in [Cal1]. As in the untwisted case, there is a formalism of Fourier-Mukai transforms between derived categories of twisted varieties, i.e. to an object

$$\mathcal{E} \in \mathcal{D}^b(X \times Y, \alpha^{-1} \boxtimes \alpha')$$

one can associate a functor

$$\Phi_{\mathcal{E}} : \mathcal{D}^b(X, \alpha) \to \mathcal{D}^b(Y, \alpha').$$

Now given a complex manifold X and an element $B \in H^2(X, \mathbb{Q})$, let $B^{0,2}$

denote the $(0,2)$ part of B, i.e. the image of B under the natural map:

$$H^2(X,\mathbb{Q}) \longrightarrow H^2_{an}(X,\mathcal{O}_X) \cong H^{0,2}(X).$$

Next, put

$$\alpha_B := exp(B^{0,2}) \in H^2(X,\mathcal{O}_X^*)$$

where exp denotes the connecting homomorphism of the exponential exact sequence. The element B is also known as a *B-field lift* of α_B. Associated to B is a *twisted Chern character* map ([HSt1, Prop. 1.2]):

$$ch^B : K(X,\alpha_B) \to H^*(X,\mathbb{Q}) \tag{1.12}$$

and also a *twisted Mukai homomorphism*

$$v^B(-) := ch^B(-)\sqrt{td_X}.$$

The twisted Mukai vector can be used to define a homomorphism

$$\Phi_*^{B,B'} : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q}),$$

associated to an integral transform $\Phi : \mathcal{D}^b(X,\alpha_B) \to \mathcal{D}^b(Y,\alpha'_{B'})$. When $X = S$ is a K3 surface, we can use B to twist the Hodge structure on $\widetilde{H}(S,\mathbb{Z})$ by multiplying the subspace $\widetilde{H}^{2,0}(S) \subset H^*(S,\mathbb{C})$ with

$$\exp(B) = (1, B, \frac{B^2}{2}) \in \widetilde{H}(S,\mathbb{Q}) \subset H^*(S,\mathbb{C})$$

in the cohomology ring $H^*(S,\mathbb{C})$, i.e., by putting

$$\widetilde{H}^{2,0}(S,B) = \exp(B) \cdot \widetilde{H}^{2,0}(S).$$

Here $\exp(B)$ is not to be confused with $exp(B^{0,2})$. The $(1,1)$-classes $\widetilde{H}^{1,1}(S,B)$ are the ones orthogonal to $\widetilde{H}^{2,0}(S,B)$ with respect to the Mukai pairing (1.9). The lattice $\widetilde{H}(S,\mathbb{Z})$ with this Hodge structure is denoted by $\widetilde{H}(S,B,\mathbb{Z})$. We now can state the following theorem from [Yosh2], as formulated in [HSt2, Thm. 0.2]:

Theorem 1.4.2
Let S be a K3 surface with $B \in H^2(X, \mathbb{Q})$ and $v \in \widetilde{H}^{1,1}(S, B, \mathbb{Z})$ a primitive vector with $(v, v)_S = 0$ and $rk(v) > 0$. Then there exists a moduli space $\mathcal{M}_H(v)$ of H-stable (with respect to a v-generic polarization H) α_B-twisted sheaves E with $v^B(E) = v$ such that:
i) $\mathcal{M}_H(v)$ is a K3 surface.
ii) On $S' := \mathcal{M}_H(v)$ there is a B-field $B' \in H^2(S', \mathbb{Q})$ such that there exists a twisted universal family $\mathcal{E}$ on $(S \times S', \alpha_B^{-1} \boxtimes \alpha_{B'})$.
iii) The twisted sheaf $\mathcal{E}$ induces a Fourier-Mukai equivalence

$$\Phi_{\mathcal{E}} : \mathcal{D}^b(S, \alpha_B) \to \mathcal{D}^b(S', \alpha_{B'}).$$

Using the above theorem, Huybrechts and Stellari managed to prove a conjecture of Căldăraru (cf. [Cal2]):
Theorem 1.4.3 ([HSt2, Thm. 0.1])
Let S and S' be K3 surfaces, and $B \in H^2(X, \mathbb{Q}), B' \in H^2(X', \mathbb{Q})$ be B-fields. Any orientation-preserving Hodge isometry

$$g : \widetilde{H}(S, B, \mathbb{Z}) \to \widetilde{H}(S', B', \mathbb{Z})$$

is of the form $g = \Phi_*^{B,B'}$, for some equivalence

$$\Phi : \mathcal{D}^b(S, \alpha_B) \to \mathcal{D}^b(S', \alpha_{B'}).$$

Chapter 2

Connected Components

In this chapter we show that the moduli space of polarized irreducible symplectic manifolds of $K3^{[n]}$-type, of fixed polarization type, is not always connected. This can be derived as a consequence of Eyal Markman's characterization of polarized parallel-transport operators of $K3^{[n]}$-type. The chapter is based on [Ap, Ch. 1-3].

2.1 Monodromy Invariants

In this section we give a short overview of the results from [Mar1] that we use - they describe a method for finding an invariant of the components of the moduli space of polarized IS manifolds of $K3^{[n]}$-type. By studying the representation of the monodromy group on the cohomology of X, E. Markman came up with the following idea – let X be of $K3^{[n]}$-type and let $O_e(\Lambda_{K3,n}, \widetilde{\Lambda})$ (resp. $O_e(H^2(X,\mathbb{Z}), \widetilde{\Lambda})$) denote the set of primitive isometric embeddings of $\Lambda_{K3,n}$ (resp. $H^2(X,\mathbb{Z})$) into $\widetilde{\Lambda}$.

Now assume first that $n \geq 4$ and consider $Q^4(X,\mathbb{Z})$, which is the quotient of $H^4(X,\mathbb{Z})$ by the image of the cup product homomorphism

$$\cup : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}).$$

Now, $Q^4(X,\mathbb{Z})$ admits a monodromy invariant bilinear pairing which makes it isometric to the Mukai lattice. Moreover, the $\mathrm{Mon}(X)$-module

$$\mathrm{Hom}[H^2(X,\mathbb{Z}), Q^4(X,\mathbb{Z})]$$

contains a unique rank 1 saturated $\mathrm{Mon}(X)$ -submodule, which is a sub-Hodge structure of type $(1,1)$ ([Mar1, Thm. 9.3]). A generator of this module induces an $O(\widetilde{\Lambda})$-orbit of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ into the Mukai lattice, such that the image of $H^2(X,\mathbb{Z})$ under such an embedding is orthogonal to the image of the projection of $c_2(X) \in H^4(X,\mathbb{Z})$ in $Q^4(X,\mathbb{Z})$. As for the case $n = 2, 3$ – there is only a single $O(\widetilde{\Lambda})$-orbit of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice $\widetilde{\Lambda}$ anyway. This yields the following statement:

Theorem 2.1.1 ([Mar1, Cor. 9.5])
Let X be an IS manifold of $K3^{[n]}$-type, $n \geq 2$. X comes with a natural choice of an $O(\widetilde{\Lambda})$-orbit $[\iota_X]$ of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice $\widetilde{\Lambda}$. The subgroup $Mon^2(X)$ of $O^+(H^2(X,\mathbb{Z}))$ is equal to the stabilizer of $[\iota_X]$ as an element of the orbit space $O(H^2(X,\mathbb{Z}),\widetilde{\Lambda})/O(\widetilde{\Lambda})$.

Proposition 2.1.2 (cf. [Mar1, Cor. 9.10])
The set τ of connected components of the moduli space of marked IS manifolds of $K3^{[n]}$-type is in bijective correspondence to the orbit set

$$[O_e(\Lambda_{K3,n},\widetilde{\Lambda})/O(\widetilde{\Lambda})] \times \mathrm{Orient}(\Lambda_{K3,n}),$$

where $O(\widetilde{\Lambda})$ acts by post-composition on $O_e(\Lambda_{K3,n},\widetilde{\Lambda})$.

The bijection is given by mapping a component s to the pair

$$([\iota_X \circ \eta^{-1}], \eta_*(or_X)),$$

where $[(X,\eta)]$ is a point of $\mathfrak{M}^s_{\Lambda_{K3,n}}$.

Next we introduce parallel-transport operators in the polarized setting:

Definition 2.1.3
Let $(X_1, h_1), (X_2, h_2)$ be polarized IS manifolds. An isomorphism

$$f : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$$

is said to be a ***polarized parallel-transport operator*** *from (X, h_1) to (X, h_2), if there exists a smooth, proper family of IS manifolds $\pi : \mathcal{X} \to T$ onto an analytic space, points $t_1, t_2 \in T$, isomorphisms $\psi_i : X_i \to \mathcal{X}_{t_i}$, $i = 1, 2$, a continuous path $\gamma : [0,1] \to T$ with $\gamma(0) = t_1, \gamma(1) = t_2$, and a flat section h of $R^2\pi_*\mathbb{Z}$, such that f is a parallel-transport operator in the sense of Def. 1.2.4, $h_{t_i} = (\psi_i^{-1})^*(h_i)$, $i = 1, 2$, and h_t is an ample class in $H^{1,1}(\mathcal{X}_t, \mathbb{Z})$, $\forall t \in T$.*

We can now state the following characterization of polarized parallel-transport operators:

Theorem 2.1.4 (cf. [Mar1, Cor. 7.4., Thm. 9.8.])
Let (X_1, H_1) and (X_2, H_2) be polarized IS manifolds of $K3^{[n]}$-type. Set $h_i := c_1(H_i), i = 1, 2$. An isometry $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is a polarized parallel-transport operator from (X_1, H_1) to (X_2, H_2) if and only if g is orientation-preserving, $[\iota_{X_1}] = [\iota_{X_2}] \circ g$, and $g(h_1) = h_2$.

The above statement can be used to obtain a lattice-theoretic characterization of polarized parallel-transport operators in the following manner – choose a primitive isometric embedding

$$\iota_X : H^2(X, \mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$$

in the $O(\widetilde{\Lambda})$-orbit given by Thm. 2.1.1. For a primitive class $h \in H^2(X, \mathbb{Z})$, of degree $(h, h) = 2d > 0$, let $T(X, h)$ denote the saturation in $\widetilde{\Lambda}$, of the sublattice spanned by $\iota_X(h)$ and $\mathrm{Im}(\iota_X)^\perp$. $T(X, h)$ is a rank 2 positive definite lattice. Denote by $[(T(X, h), \iota_X(h))]$ the isometry class of the pair $(T(X, h), \iota_X(h))$, i.e. $(T(X, h), \iota_X(h)) \sim (T', h')$ iff there exists an isometry

$$\gamma : T(X, h) \to T'$$

such that $\gamma(\iota_X(h)) = h'$. Let $I(X)$ denote the set of primitive cohomology classes of positive degree in $H^2(X,\mathbb{Z})$. Let Σ_n be the set of isometry classes of pairs (T,h), consisting of an even rank 2 positive definite lattice T and a primitive element $h \in T$ such that $h^\perp \cong \langle 2n-2\rangle$. Let

$$f_X : I(X) \to \Sigma_n \tag{2.1}$$

be the function sending h to $[(T(X,h), \iota_X(h))]$. Note that $[(T(X,h), \iota_X(h))]$ does not depend on the choice of representative of the orbit $[\iota_X]$. Also $f_X(h) = f_{X'}(h')$, for any isomorphism $X \xrightarrow{\sim} X'$ mapping h' to h in cohomology. The function (2.1) is called a *faithful monodromy invariant function* in [Mar2] because it separates orbits for the action of the monodromy group of X on $I(X)$ (cf. [Mar2, Ch. 5.3]).

Proposition 2.1.5 ([Mar3, Lemma 0.4.]) *Let (X_1,H_1) and (X_2,H_2) be two polarized pairs of IS manifolds of $K3^{[n]}$-type. Set $c_1(H_i) = h_i$. Then $f_{X_1}(h_1) = f_{X_2}(h_2)$ if and only if there exists a polarized parallel-transport operator from (X_1,H_1) to (X_2,H_2).*

Proof. One direction is clear – suppose there exists a polarized parallel-transport operator

$$g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

from (X_1,H_1) to (X_2,H_2). In particular, $g(h_1) = h_2$. Since, by Thm. 2.1.4, $[\iota_{X_1}] = [\iota_{X_2}] \circ g$, there exists an isometry $\gamma \in O(\widetilde{\Lambda})$ such that

$$\gamma \circ \iota_{X_1} = \iota_{X_2} \circ g.$$

The map γ induces an isometry between the pairs $(T(X_1,h), \iota_{X_1}(h_1))$ and $(T(X_2,h), \iota_{X_2}(h_2))$, i.e.

$$f_{X_1}(h_1) = f_{X_2}(h_2).$$

Now assume that $f_{X_1}(h_1) = f_{X_2}(h_2)$. This means that the two pairs $(T(X_1,h), \iota_{X_1}(h_1))$ and $(T(X_2,h), \iota_{X_2}(h_2))$ are isometric. The idea is to construct the required parallel transport operator g from an isometry of $\widetilde{\Lambda}$. Now, $T(X_1,h)$ and $T(X_2,h)$ are primitively-embedded sublattices of signature $(2,0)$, of the same isometry class, in the unimodular lattice $\widetilde{\Lambda}$, of signature $(4,20)$. Therefore, [Nik1, Thm. 1.1.2.b)] implies that there exists a $\gamma \in O(\widetilde{\Lambda})$, such that

$$\gamma(T(X_1,h)) = T(X_2,h), \text{ and } \gamma(\iota_{X_1}(h_1)) = \iota_{X_2}(h_2).$$

Set $g = \iota_{X_2}^{-1} \circ \gamma \circ \iota_{X_1}$:

$$\begin{tikzcd} H^2(X_1,\mathbb{Z}) \arrow[r, hook, "\iota_{X_1}"] \arrow[d, "g"] & \widetilde{\Lambda} \arrow[d, "\gamma"] & \supset & T(X_1,h) \arrow[d, "\gamma|_{T(X_1,h)}"] \\ H^2(X_2,\mathbb{Z}) \arrow[r, hook, "\iota_{X_2}"] & \widetilde{\Lambda} & \supset & T(X_2,h) \end{tikzcd}$$

Indeed, the map g maps $H^2(X_1,\mathbb{Z})$ to $H^2(X_2,\mathbb{Z})$ because

$$\gamma(\iota_{X_1}(H^2(X_1,\mathbb{Z}))^{\perp}) = \iota_{X_2}(H^2(X_2,\mathbb{Z}))^{\perp}.$$

Then

$$[\iota_{X_2}] \circ g = [\iota_{X_2} \circ (\iota_{X_2}^{-1} \circ \gamma \circ \iota_{X_1})] = [\gamma \circ \iota_{X_1}] = [\iota_{X_1}],$$

since $\gamma \in O(\widetilde{\Lambda})$. Furthermore,

$$g(h_1) = \iota_{X_2}^{-1} \circ \gamma \circ \iota_{X_1}(h_1) = \iota_{X_2}^{-1} \circ \iota_{X_2}(h_2) = h_2.$$

Now assume that g is orientation-reversing. Choose $\alpha \in H^2(X_2,\mathbb{Z})$ satisfying $(\alpha,\alpha) = 2$, $(\alpha,h_2) = 0$. Define the isometry

$$\rho_\alpha(\lambda) := -\lambda + (\alpha,\lambda)\alpha, \lambda \in H^2(X_2,\mathbb{Z}).$$

Set $\tilde{g} := -\rho_\alpha \circ g$. Since ρ_α is an element of $\mathrm{Mon}^2(X_2)$ ([Mar1, Thm. 9.1]) and $\rho_\alpha(h_2) = -h_2$, $\tilde{g}$ is an orientation-preserving isometry between $H^2(X_1,\mathbb{Z})$ and $H^2(X_2,\mathbb{Z})$, satisfying

$$[\iota_{X_1}] = [\iota_{X_2}] \circ \tilde{g} \text{ and } \tilde{g}(h_1) = h_2.$$

Thm. 2.1.4 implies that $\tilde{g}$ is a polarized parallel-transport operator from (X_1,H_1) to (X_2,H_2). ■

2.2 Enumerating the components

Let μ_n denote the set of connected components of the moduli space space of polarized IS manifolds of $K3^{[n]}$-type $\mathcal{V}_{X_0}$. We are now ready to prove the following

Theorem 2.2.1
There is an injective map

$$f : \mu_n \to \Sigma_n,$$

given by mapping a connected component $\{\mathcal{V}^0\}$ *of* $\mathcal{V}_{X_0}$ *to* $f_X(h)$ *for some* $[(X, H)] \in \mathcal{V}^0$.

Proof. First of all, the map f is well-defined. Pick a point $[(X_1, H_1)] \in \mathcal{V}^0$. Choose a marking η_1 of X_1 and let $\mathfrak{M}^{a,s_1}_{h^\perp}$ be the component of the moduli space of marked polarized triples, containing the point $[(X_1, H_1, \eta_1)]$. Choose another point $[(X_2, H_2)] \in \mathcal{V}^0$ and a marking η_2' of X_2. Then $[(X_2, H_2, \eta_2')] \in \mathfrak{M}^a_{\widetilde{h}}$, where $\widetilde{h}$ is the $O(\Lambda_{K3,n})$-orbit of (h, s_1). Since $O(\Lambda_{K3,n})$ acts transitively on the set of components of $\mathfrak{M}^a_{\widetilde{h}}$, there is a marking η_2 such that $[(X_2, H_2, \eta_2)] \in \mathfrak{M}^{a,s_1}_{h^\perp}$. Choose a path

$$\gamma : [0, 1] \to \mathfrak{M}^{a,s_1}_{h^\perp}$$

such that $\gamma(0) = [(X_1, H_1, \eta_1)]$, $\gamma(1) = [(X_2, H_2, \eta_2)]$. For each $r \in [0, 1]$, $\gamma(r)$ has a Kuranishi neighborhood $U_r \subset \mathfrak{M}^{a,s_1}_{h^\perp}$ and a semi-universal family of deformations

$$\pi_r : \mathcal{X}_r \to U_r.$$

Upon shrinking U_r, we may assume that it is simply-connected. As in the proof of [Mar1, Cor. 7.4], we can use these to construct a polarized parallel transport operator from (X, H_1) to (X, H_2), namely $\gamma([0, 1])$ admits a finite subcovering $\{U_{r_i}\}_{i\in\{1,\ldots,m\}}$ and we can choose a partition $s_0 =$

$0, s_1, ..., s_{m-1}, s_m = 1$ of $[0,1]$ such that

$$\gamma(s_i) \in U_{r_i} \cap U_{r_{i+1}}, i = 1, ..., m-1$$

and $\gamma([s_{i-1}, s_i]) \subset U_{r_i}$. Now set

$$V_1 := U_{r_1}, V_i := V_{i-1} \bigcup_{\gamma(s_{i-1})} U_{r_i} \text{ for } i = 2, ..., m,$$

where $V_{i-1} \bigcup_{\gamma(s_{i-1})} U_{r_i}$ denotes the pushout of the inclusions of the point $\gamma(s_{i-1})$ in V_{i-1} and U_{r_i}. We can identify the fibers $\pi_{r_i}^{-1}(\gamma(s_i))$ and $\pi_{r_{i+1}}^{-1}(\gamma(s_i))$ in order to obtain a family

$$\pi : \mathcal{X} \to V_m$$

of IS manifolds, as well as a path

$$\tilde{\gamma} : [0,1] \to V_m$$

which is the concatenation of the paths $\gamma([s_{i-1}, s_i]), i = 1, ..., m-1$, and which yields a polarized parallel-transport operator from (X_1, H_1) to (X_2, H_2). Hence, by Prop. 2.1.5, $f_{X_1}(h_1) = f_{X_2}(h_2)$, and the map f is well-defined.

Now let $\mathcal{V}^1$ and $\mathcal{V}^2$ be two components of $\mathcal{V}_{X_0}$ such that

$$f(\{\mathcal{V}^1\}) = f(\{\mathcal{V}^2\}).$$

Choose points $[(X_i, H_i)] \in \mathcal{V}^i$, $i = 1, 2$. Since $f_{X_1}(h_1) = f_{X_2}(h_2)$, there exists a polarized parallel-transport operator from (X_1, H_1) to (X_2, H_2). Indeed, by Props. 2.1.4 and 2.1.5, there is an analytic family of IS manifolds

$$\pi : \mathcal{X} \to T$$

and a section $\tilde{h}$ of $R^2\pi_*\mathbb{Z}$ such that $(\mathcal{X}_{t_0}, \tilde{h}_{t_0}) \cong (X_1, h_1)$ and $(\mathcal{X}_{t_1}, \tilde{h}_{t_1}) \cong (X_2, h_2)$ for some $t_1, t_2 \in T$. Consider the path

$$\gamma : [0,1] \to T$$

with $\gamma(0) = t_1, \gamma(1) = t_2$ (cf. Def. 2.1.3). The long exact sequence associated to the exponential sequence yields:

$$\ldots \to R^1\pi_*\mathcal{O}_{\mathcal{X}} \to R^1\pi_*\mathcal{O}^*_{\mathcal{X}} \to R^2\pi_*\mathbb{Z} \to R^2\pi_*\mathcal{O}_{\mathcal{X}} \to \ldots$$

Since $\tilde{h}_t$ has type (1,1) at every point $t \in T$, the image of $\tilde{h}$ in $H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$ vanishes. The coherent sheaf $R^1\pi_*\mathcal{O}_{\mathcal{X}}$ also vanishes, since the irregularity of the fibers is zero. Hence h lifts locally to sections of the relative analytic Picard sheaf $R^1\pi_*\mathcal{O}^*_{\mathcal{X}}$. Then we may find a finite open cover $\{W_i\}_{i=1}^m$ of $\gamma([0,1])$ by simply connected sets W_i, markings $\tilde{\eta}^i$, and bundles $\mathcal{H}_i \in \mathrm{Pic}(\mathcal{X}|_{W_i}/W_i)$ which make $(\mathcal{X}|_{W_i}, \mathcal{H}_i, \tilde{\eta}^i)$ into families of marked polarized triples, such that the classifying maps α_i of the families map W_i to the same component $\mathfrak{M}^{a,0}_{h^\perp}$ of the moduli space of marked polarized triples $\mathfrak{M}^a_\chi$, for all $i = 1, \ldots, m$. Let

$$q : \mathfrak{M}^a_\chi \to \mathfrak{M}^a_\chi / O(\Lambda_{K3,n})$$

be the quotient map. Prop. 1.2.8 gives analytic isomorphisms

$$\varphi_i : \mathcal{V}^i \to \mathfrak{M}^a_\chi / O(\Lambda_{K3,n}), i = 1, 2.$$

Consider the points

$$[(X_2', H_2')] := \varphi_1^{-1}(q(\alpha_m(t_1))) \in \mathcal{V}^1 \text{ and } [(X_2, H_2)] \in \mathcal{V}^2.$$

Since they map to the same orbit of marked polarized triples in $\mathfrak{M}^a_\chi / O(\Lambda_{K3,n})$, there is an analytic isomorphism between (X_2', H_2') and (X_2, H_2). By GAGA (cf. [JPS]), it is induced by an algebraic isomorphism between (X_2', H_2') and (X_2, H_2). Hence $[(X_2', H_2')] = [(X_2, H_2)]$ as points in $\mathcal{V}^{\tilde{\tau}}$, which implies that $\mathcal{V}^1 = \mathcal{V}^2$. ■

Fix a K3 surface S and an isometry

$$(H^*(S, \mathbb{Z}), (\cdot, \cdot)_M) \cong \widetilde{\Lambda}.$$

Let X be of $K3^{[n]}$-type and let $h_d \in H^2(X, \mathbb{Z})$ be a primitive element of degree $2d$ and choose ι in the orbit $[\iota_X]$, so that $\mathrm{Im}(\iota)^\perp$ is generated by the vector

$$v := (1, 0, 1-n) \in \widetilde{\Lambda}.$$

Let $\langle 2d \rangle$ denote a rank 1 lattice generated by an element of length $2d$. For $r, s \in \mathbb{Z}$, (r, s) denotes the gcd of r and s. Set

$$t := \operatorname{div}(h_d) \tag{2.2}$$

to be the *divisibility* of h_d, i.e. the positive generator of the ideal $(h_d, H^2(X, \mathbb{Z}))$ in $\mathbb{Z}$. In particular, $\mathbb{Z}/t\mathbb{Z}$ is a subgroup of the discriminant group of $H^2(X, \mathbb{Z})$ and the latter has order $2n-2$. This means that t divides $2n-2$. The next proposition relates t to the index of $\langle \iota(h_d) \rangle \oplus \langle v \rangle$ in the saturation $T(X, h_d)$ with respect to ι_X. I thank an anonymous referee of [Ap] for suggesting a simpler proof of this proposition.

Proposition 2.2.2

The integer t is equal to the index of $\langle \iota(h_d) \rangle \oplus \langle v \rangle$ in $T(X, h_d)$.

PROOF. $\iota(h_d)$ can be written as

$$(c, tm\xi, c(n-1)) \text{ in } \widetilde{\Lambda},$$

where $c, m \in \mathbb{Z}, \xi \in H^2(S, \mathbb{Z})$ is primitive and

$$(m, \frac{2n-2}{t}) = 1.$$

Moreover, $(c, m) = 1$ by the primitivity of $\iota(h_d)$ in $v^{\perp}$. Now consider the class

$$u := \frac{\iota(h_d) - cv}{t} = (0, m\xi, \frac{c(2n-2)}{t}).$$

It is integral because of $t \mid 2n-2$, and it is primitive in $\widetilde{\Lambda}$ because of $(m, \frac{c(2n-2)}{t}) = 1$. Then it is easy to check that the lattice $\langle u, v \rangle$ is saturated in $\widetilde{\Lambda}$. First of all, pick a class

$$w := av + bu = (a, bm\xi, (1-n)(a - b\frac{2c}{t}))$$

which is primitive in $\langle u, v\rangle$; this means that $(a, b) = 1$. Suppose that $w = f\tilde{w}$, where $\tilde{w}$ is primitive in $\widetilde{\Lambda}$. Then f divides a, bm and $b\frac{c(2n-2)}{t}$. Furthermore, since $(a, b) = 1$, f divides $(m, \frac{c(2n-2)}{t}) = 1$, i.e.

$$f = 1 \text{ and } w = \tilde{w}.$$

This implies that $\langle u, v\rangle$ is saturated, i.e.

$$T(X, h_d) = \langle u, v\rangle.$$

Finally, since the index of $\langle \iota(h_d)\rangle \oplus \langle v\rangle$ in $\langle u, v\rangle$ is equal to t by a direct discriminant computation, we have shown the claim. ■

Now given positive integers d and $t \mid (2d, 2n-2)$, let

$$\mu_n^{d,t} \subset \mu_n$$

be the subset of connected components of $\mathcal{V}_{X_0}$ parametrizing polarized IS manifolds of $K3^{[n]}$-type, with polarization type of degree $2d$ and divisibility t. By [GHS1, Cor. 3.7], this data is sufficient to determine the polarization type whenever $(\frac{2n-2}{t}, \frac{2d}{t})$ and t are coprime. Define $\Sigma_n^{d,t}$ to be the set of isometry classes of pairs (T, h), such that T is an even positive definite lattice of rank two and discriminant $4d(n-1)/t^2$, h is a primitive element of square $(h, h) = 2d$, and $h^\perp$ is generated by an element of square $2n-2$.

Proposition 2.2.3
The image of the restriction $f|_{\mu_n^{d,t}}$ is $\Sigma_n^{d,t}$.

PROOF.
Let (T, h) be a representative of some isometry class in $\Sigma_n^{d,t}$. Let δ generate $h^\perp$ in T. By [Nik1, Thm. 1.1.2] we may choose a primitive isometric embedding

$$j : T \hookrightarrow \widetilde{\Lambda}.$$

Let

$$\tilde{\iota} : \Lambda_{K3,n} \hookrightarrow \widetilde{\Lambda}$$

be a primitive isometric embedding. Then the sublattice $\tilde{\iota}(\Lambda_{K3,n})^{\perp}$ is generated by a primitive element of degree $2n-2$ and we may find an embedding ι in the $O(\widetilde{\Lambda})$-orbit $[\tilde{\iota}]$ that

$$\iota(\Lambda_{K3,n})^{\perp} = \langle j(\delta) \rangle,$$

since $O(\widetilde{\Lambda})$ acts transitively on primitive elements. In particular, $j(h) \in \iota(\Lambda_{K3,n})$. The embedding ι determines a pair of components $t_1, t_2 \in \tau$ of the moduli space $\mathfrak{M}^{\tau}_{\Lambda_{K3,n}}$ of marked IS manifolds of $K3^{[n]}$-type, via the bijection

$$\tau \cong [O(\Lambda_{K3,n}, \widetilde{\Lambda})/O(\widetilde{\Lambda})] \times \mathrm{Orient}(\Lambda_{K3,n})$$

from Prop. 2.1.2. The composition $\iota \circ \eta$ belongs to the natural orbit $[\iota_X]$ for every marked pair (X, η) in these components $t_1, t_2 \in \tau$. Set

$$h_1 := \iota^{-1}(j(h)) \in \Lambda_{K3,n}.$$

Since $\mathfrak{M}^{a,t_1}_{h_1^{\perp}}$ is nonempty, we may choose a point in it, represented by a marked IS manifold (X, η). Then, by the definition of $\mathfrak{M}^{a,t_1}_{h_1^{\perp}}$,

$$h_2 := \eta^{-1}(h_1)$$

is an ample class in $H^2(X, \mathbb{Z})$. Let $H_2 \in \mathrm{Pic}(X)$ denote the corresponding polarization of X. Let $\mathcal{V}^0$ be the component of the moduli space of polarized pairs, containing $[(X, H_2)]$. Let χ be the $O(\Lambda_{K3,n})$-orbit of (h_1, t_1). Then $\mathfrak{M}^{a,t_1}_{h_1^{\perp}}$ is a component of $\mathfrak{M}^{a}_{\chi}$ and $\mathcal{V}^0 \cong \mathfrak{M}^{a}_{\chi}/O(\Lambda_{K3,n})$, by Prop. 1.2.8. In particular, by construction,

$$f_X(h_2) = f(\{\mathcal{V}^0\}) = [(T, h)],$$

i.e. $f|_{\mu_n^{d,t}}$ is surjective onto $\Sigma_n^{d,t}$. ■

We obtain as a corollary:

Corollary 2.2.4

The number of connected components of the moduli space of polarized IS *manifolds of $K3^{[n]}$-type, with polarization type of degree $2d$ and divisibility t, is given by $|\Sigma_n^{d,t}|$.*

PROOF.

Injectivity of $f|_{\mu_n^{d,t}}$ is given by Thm. 2.2.1; surjectivity onto $\Sigma_n^{d,t}$,by Prop. 2.2.3. Hence the number of connected components, i.e. the number of elements in the image of $f|_{\mu_n^{d,t}}$, is given by$|\Sigma_n^{d,t}|$. ■

2.3 Computations

In this section we compute the cardinality of the set $\Sigma_n^{d,t}$ and give examples. For an integer r, $\varphi(r)$ denotes the Euler φ-function, and $\rho(r)$ denotes the number of prime divisors of r.

Proposition 2.3.1

Let t be a divisor of $(2d, 2n-2)$, and set $D := 4d(n-1)/t^2$, $g := (2d, 2n-2)/t$, $\tilde{n} := (2n-2)/(2d, 2n-2), \tilde{d} := 2d/(2d, 2n-2)$, $w := (g, t)$, $g_1 := g/w$, $t_1 := t/w$. Put $w = w_+(t_1)w_-(t_1)$ where $w_+(t_1)$ is the product of all powers of primes (in the prime decomposition of w) dividing (w, t_1). Then

- $|\Sigma_n^{d,t}| = w_+(t_1)\varphi(w_-(t_1))2^{\rho(t_1)-1}$ *if $t > 2$ and one of the following sets of conditions hold:*
  g_1 is even, $(\tilde{d}, t_1) = (\tilde{n}, t_1) = 1$ and the residue class $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo t_1;
  ***OR** g_1, t_1, and $\tilde{d}$ are odd, $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2t_1$;*
  ***OR** g_1, t_1, and w are odd, $\tilde{d}$ is even, $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ and $-\tilde{d}/(4\tilde{n})$ is a quadratic residue modulo t_1;*

- $|\Sigma_n^{d,t}| = w_+(t_1)\varphi(w_-(t_1))2^{\rho(t_1/2)-1}$ *if $t > 2$, g_1 is odd, t_1 is even, $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2t_1$;*

- $|\Sigma_n^{d,t}| = 1$ *if $t \leq 2$ and one of the following sets of conditions hold:*
  g_1 is even, $(\tilde{d}, t_1) = (\tilde{n}, t_1) = 1$ and the residue class $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo t_1;

OR g_1, t_1, *and* $\tilde{d}$ *are odd,* $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ *and* $-\tilde{d}/\tilde{n}$ *is a quadratic residue modulo* $2t_1$*;*
OR g_1, t_1, *and* w *are odd,* $\tilde{d}$ *is even,* $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ *and* $-\tilde{d}/(4\tilde{n})$ *is a quadratic residue modulo* t_1*;*
OR g_1 *is odd,* t_1 *is even,* $(\tilde{d}, t_1) = (\tilde{n}, 2t_1) = 1$ *and* $-\tilde{d}/\tilde{n}$ *is a quadratic residue modulo* $2t_1$*;*

- $|\Sigma_n^{d,t}| = 0$, *else.*

PROOF. Denote by $D(T)$ the discriminant group of the lattice T. By definition, $|\Sigma_n^{d,t}|$ is equal to the number of isometry classes of primitive inclusions of $\langle 2d \rangle$ into lattices T of discriminant D, such that the orthogonal complement of $\langle 2d \rangle$ in a given T is isometric to $\langle 2n-2 \rangle$. By [Nik1, Thm. 1.5.1], these inclusions are classified by certain equivalence classes of monomorphisms

$$\gamma : H \to D(\langle 2n-2 \rangle) \cong \mathbb{Z}/(2n-2)\mathbb{Z},$$

where H is the unique subgroup of $D(\langle 2d \rangle) \cong \mathbb{Z}/2d\mathbb{Z}$ of order t such that Γ_γ (the graph of γ) is an isotropic subgroup of $D(\langle 2d \rangle) \oplus D(\langle 2n-2 \rangle)$ with respect to the discriminant form. The equivalence is given by: $\gamma \sim \gamma'$ if there exist

$$\psi \in O(\langle 2n-2 \rangle) \cong \mathbb{Z}/2\mathbb{Z}, \varphi \in O(\langle 2d \rangle) \cong \mathbb{Z}/2\mathbb{Z}$$

such that

$$\gamma \circ \overline{\varphi} = \overline{\psi} \circ \gamma'$$

($\overline{\varphi}$ and $\overline{\psi}$ are the induced isometries on the discriminant groups). Now the set of subgroups of the form Γ_γ can be identified with the set of generators in $D(\langle 2d \rangle) \oplus D(\langle 2n-2 \rangle)$ of the form $(\tilde{d}g, \tilde{n}g\varepsilon)$, where ε is a unit modulo the index t. The isotropy condition on Γ_γ reads as follows:

$$\frac{(\tilde{d}g)^2}{2d} + \frac{(\tilde{n}g\varepsilon)^2}{2n-2} = \frac{\tilde{d}g + \tilde{n}g\varepsilon^2}{t} = \frac{\tilde{d}g_1 + \tilde{n}g_1\varepsilon^2}{t_1} \equiv 0 \pmod{2\mathbb{Z}}.$$

In other words, we seek the number of classes ε modulo t, coprime to t, which satisfy the congruence

$$\tilde{d} + \tilde{n}\varepsilon^2 \equiv 0 \pmod{2t_1}.$$

The conditions for existence as well as the number of such solutions to a congruence of this form were already determined in the proof of Prop. 3.6 of [GHS1] in the context of computing orbits of vectors under the stable orthogonal group. Now, $O(\langle 2d \rangle)$ and $O(\langle 2n-2 \rangle)$ act on Γ_γ by 'reflecting', i.e. by flipping signs in the first, resp. second coordinate of the elements of the graph. In addition, Γ_γ is central-symmetric (i.e. $(a,b) \in \Gamma_\gamma \Leftrightarrow (-a,-b) \in \Gamma_\gamma$). For $2 \geq t$, the graphs are fixed by the action, for $t > 2$ there are no fixed graphs, hence, by the above considerations there are two graphs in each equivalence class, so we need to divide the numbers from [GHS1, Prop. 3.6] by two and we obtain the result. ■

In the following cases the polarization type is determined by the values of t and d, and the corresponding moduli spaces are indeed connected:

Proposition 2.3.2 *The moduli space of polarized IS manifolds of $K3^{[n]}$-type is connected in each of the following cases:*

- $t = 1$, any degree $2d > 0$ and dimension $2n > 2$ such that

  $$(n-1, d) = 1$$

  ('split' polarization);

- $t = 2$, any degree $2d > 0$ and dimension $2n > 2$ such that

  $$(n-1, d) = 1 \text{ and } d + n - 1 \equiv 0 (\text{mod } 4)$$

  ('non-split' polarization, cf. [GHS1, Ch.3]);

- $t = p^\alpha$ for a prime $p > 2$, any degree $2d$ and dimension $2n > 2$ such that $p^{2\alpha} | (2d, 2n-2)$.

Remarks.

(1) **Example:** In particular, Prop. 2.3.2 implies that the moduli space of polarized IS manifolds of $K3^{[2]}$-type, with fixed polarization type, is connected. However, the following example shows that fixing the polarization type need not imply connectedness of the moduli space. Let $d = pq$ and $n - 1 = mpq$, where p and q are different primes, and $-m$ is a quadratic residue modulo pq. Set $t = pq$. In this case, the polarization type is determined by t and d, since $(2m, 2) = 2$ and $t = pq$ are coprime. But $|\Sigma_n^{d,t}| = 2$, i.e. there are two connected components with this polarization type.

(2) Note that the formulas in Prop. 2.3.1 imply that the number of connected components $|\Sigma_n^{d,t}|$ can get arbitrarily large, as we vary the dimension $2n - 2$ and the degree $2d$.

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Chapter 3

Modular Varieties

In this chapter we investigate the relationship between different moduli spaces of polarized IS manifolds of $K3^{[n]}$-type. We start by introducing some notation:

The unit matrix of rank m is denoted by I_m.

Given a lattice Λ, $D(\Lambda) := \Lambda^\vee/\Lambda$ denotes the discriminant group of Λ; recall that any $s \in O(\Lambda)$ induces an isomorphism $\overline{s}$ of $D(\Lambda)$ and define

$$\widetilde{O}(\Lambda) := \{s \in O(\Lambda) \mid \overline{s} = \mathrm{id}_{D(\Lambda)}\},$$

$$\widehat{O}(\Lambda) := \{s \in O(\Lambda) \mid \overline{s} = \pm\mathrm{id}_{D(\Lambda)}\}.$$

Let $g(\Lambda)$ denote the genus of Λ – recall that a genus is an equivalence class of lattices, whereby two lattices belong to the same genus if and only if they have the same signatures and they become isometric after tensoring with the p-adic integers $\mathbb{Z}_p$, for every prime p.

Given a primitive element $h \in \Lambda$, $O(\Lambda, h), \widetilde{O}(\Lambda, h)$, and $\widehat{O}(\Lambda, h)$ denote the subgroups fixing h, in $O(\Lambda), \widetilde{O}(\Lambda)$, and $\widehat{O}(\Lambda)$, respectively.

The group $O(\Lambda)$ acts naturally on the period domain Ω_Λ, associated to Λ (cf. (1.3)) and given an arithmetic subgroup $\Gamma < O(\Lambda)$, we denote by $P\Gamma$ *the projectivization of* Γ, i.e. the image of Γ under the action $O(\Lambda) \to \mathrm{Aut}(\Omega_\Lambda)$.

Now let $n > 1, d \in \mathbb{N}$ and $f \mid (2d, 2n-2)$ be fixed and let $h_d \in \Lambda_{K3,n}$ be a vector with

$$(h_d, h_d) = 2d, \mathrm{div}(h_d) = f.$$

The conditions on n, d and f for the existence of h_d are listed in [GHS1, Prop. 3.6]. The vector h_d is of the form

$$fv + cl_{n-1},$$

where $v \in l_{n-1}^{\perp}$ – cf. Eq. (1.2) for the definition of l_{n-1}. Put

$$2b := (v, v).$$

Then, by [GHS1, Prop. 3.6.(iv)],

$$h_d^{\perp} \cong 2U \oplus 2E_8(-1) \oplus T_B, \tag{3.1}$$

where T_B is a rank two lattice with Gramm matrix

$$B := \begin{pmatrix} -2b & c\frac{2n-2}{f} \\ c\frac{2n-2}{f} & -2n+2 \end{pmatrix}.$$

The lattice T_B splits, i.e.

$$T_B = \langle -2d \rangle \oplus \langle -2(n-1) \rangle,$$

whenever $\operatorname{div}(h_d) = 1$ ([GHS1, Ex. 3.8]). Note that

$$(h_d, h_d) = 2d = f^2(v, v) + c^2(l_{n-1}, l_{n-1}) = 2bf^2 - 2c^2(n-1).$$

Then $2b > 0$ and the matrix B is negative definite; it is also integral, since $f | \gcd(2d, 2n-2)$, by the definition of f. In particular, $h_d^{\perp}$ has signature $(2, 20)$ and $\Omega_{h_d^{\perp}}$ is a type IV homogeneous domain with two connected components, on which the groups $\widetilde{O}(h_d^{\perp})$ and $\widehat{O}(\Lambda_{K3,n}, h_d)$ act. Furthermore, the groups $\widetilde{O}(h_d^{\perp})$ and $\widehat{O}(\Lambda_{K3,n}, h_d)$ interchange the two components of $\Omega_{h_d^{\perp}}$, i.e. the quotients $\Omega_{h_d^{\perp}}/\widehat{O}(\Lambda_{K3,n}, h_d)$ and $\Omega_{h_d^{\perp}}/\widetilde{O}(h_d^{\perp})$ are connected. This follows from the fact that the subgroups $\widehat{O}^+(\Lambda_{K3,n}, h_d) \subset \widehat{O}(\Lambda_{K3,n}, h_d)$ and $\widetilde{O}^+(h_d^{\perp}) \subset \widetilde{O}(h_d^{\perp})$, fixing the connected components, are proper subgroups of index two; for example, any reflection with respect to a vector of square two in $\widetilde{O}(h_d^{\perp})$ (or in $\widehat{O}(\Lambda_{K3,n}, h_d)$) interchanges the two components – cf.

[GHS3, Sect. 3.1].

The above quotients have the structure of quasi-projective varieties and we put

$$\mathcal{F}_{h_d} := \Omega_{h_d^\perp}/\widehat{O}(\Lambda_{K3,n}, h_d); \mathcal{G}_{h_d} := \Omega_{h_d^\perp}/\widetilde{O}(h_d^\perp). \tag{3.2}$$

From [GHS1, Lemma 3.2] it follows that we have a chain of natural subgroup embeddings of finite index

$$\widetilde{O}(h_d^\perp) \subset O(\Lambda_{K3,n}, h_d) \subset O(h_d^\perp). \tag{3.3}$$

In fact, it is immediate from the definitions that $\widetilde{O}(h_d^\perp) \subset O(\Lambda_{K3,n}, h_d)$ factors through the embedding $\widehat{O}(\Lambda_{K3,n}, h_d) \subset O(\Lambda_{K3,n}, h_d)$, i.e. the group $\widetilde{O}(h_d^\perp)$ embeds naturally as a finite index subgroup of $\widehat{O}(\Lambda_{K3,n}, h_d)$. In particular, to h_d we can associate a natural map

$$\pi : \mathcal{G}_{h_d} \longrightarrow \mathcal{F}_{h_d} \tag{3.4}$$

of finite degree, induced by the subgroup embedding. We are interested in finding a different set of numbers $\tilde{n} > 1, \tilde{d} \in \mathbb{N}$, and $\tilde{f} \mid \gcd(2\tilde{n}-2, 2\tilde{d})$, such that there exists a vector $h_{\tilde{d}} \in \Lambda_{K3,\tilde{n}}$ with

$$(h_{\tilde{d}}, h_{\tilde{d}}) = 2\tilde{d}, div(h_{\tilde{d}}) = \tilde{f}$$

and an isomorphism of modular varieties

$$\sigma : \mathcal{G}_{h_d} \xrightarrow{\sim} \mathcal{G}_{h_{\tilde{d}}};$$

furthermore, we would like to know whether this isomorphism descends to an isomorphism

$$\sigma' : \mathcal{F}_{h_d} \xrightarrow{\sim} \mathcal{F}_{h_{\tilde{d}}},$$

so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{G}_{h_d} \arrow[r, "\sigma"] \arrow[d, "\pi"] & \mathcal{G}_{h_{\tilde{d}}} \arrow[d, "\tilde{\pi}"] \\ \mathcal{F}_{h_d} \arrow[r, "\sigma'"] & \mathcal{F}_{h_{\tilde{d}}} \end{tikzcd} \tag{3.5}$$

The motivation comes from the modular interpretation of the variety $\mathcal{F}_{h_d}$, introduced in Ch. 1.2 – let $\mathcal{V}^0$ be a component of the moduli space of polarized IS manifolds of $K3^{[n]}$-type and polarization type given by h_d. Then there is an open immersion of algebraic varieties

$$\mathcal{V}^0 \hookrightarrow \mathcal{F}_{h_d},$$

by Thms. 1.2.6-7. Thus the existence of an isomorphism σ' as above means that there are birational maps between components of the moduli space of $K3^{[n]}$-type, of polarization type given by h_d, and the components of the moduli space of IS manifolds of $K3^{[\tilde{n}]}$-type, of polarization type determined by $h_{\tilde{d}}$.

In certain cases, the degree of the map π can be easily determined:

Theorem 3.1

Let $h_d \in \Lambda_{K3,n}$ be a primitive element of the form $h_d = fv + cl_{n-1}$ with

$$(h_d, h_d) = 2d > 0 \text{ and } div(h_d) = f.$$

Suppose that

$$\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1.$$

Let $\pi : \mathcal{F}_{h_d} \to \mathcal{G}_{h_d}$ be the map of modular varieties associated to h_d. If

$$f = 1 \text{ or } f = 2, \text{ and } f \neq n-1, 2n-2, 2d, d,$$

then π has degree 2; else π is an isomorphism.

Proof.

Put

$$K := h_d^{\perp} \oplus \langle h_d \rangle$$

and consider the following series of sublattices:

$$K < \Lambda_{K3,n} < \Lambda_{K3,n}^{\vee} < K^{\vee}.$$

Put

$$H := \Lambda_{K3,n}/K. \tag{3.6}$$

For any $a \in K^\vee$, let $\overline{a}$ denote the coset $a + K$, considered as an element of $D(K)$. Dividing out by K we obtain the following sequence of finite abelian groups:

$$H < \Lambda_{K3,n}^\vee/K < D(h_d^\perp) \oplus D(\langle h_d \rangle).$$

Note that

$$D(\Lambda_{K3,n}) \cong (\Lambda_{K3,n}^\vee/K)/H \cong \langle \bar{l}_{n-1}^\vee + H \rangle, \tag{3.7}$$

where $l_{n-1}^\vee := \frac{1}{2n-2} l_{n-1}$. Let p and q denote the projections from $D(h_d^\perp) \oplus D(\langle h_d \rangle)$ to $D(h_d^\perp)$, resp. $D(\langle h_d \rangle)$. We now list several facts that were shown in the course of the proof of [GHS1, Prop. 3.12], under the assumption $\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1$:

- (*) The groups H, $p(H)$ and $q(H)$ are cyclic subgroups of order $\frac{2d}{f}$ in $D(h_d^\perp) \oplus D(\langle h_d \rangle)$, $D(h_d^\perp)$, and $D(\langle h_d \rangle)$ respectively; the subgroup $q(H)$ is generated by $f\overline{h}_d^\vee \in D(\langle h_d \rangle)$, where $h_d^\vee := \frac{1}{2d} h_d$.

- We have that
$$D(h_d^\perp) \cong p(H) \oplus T, \tag{3.8}$$
where T is a cyclic group of order $\frac{2n-2}{f}$, generated by $f\bar{l}_{n-1}^\vee + c\overline{e}$; here $\overline{e}$ is an element of H. Furthermore,
$$\bar{l}_{n-1}^\vee = p(\bar{l}_{n-1}^\vee) - c\overline{h}_d^\vee.$$

- The group $O(\Lambda_{K3,n}, h_d)$ has a natural identification with the subgroup
$$\{\gamma \in O(h_d^\perp) \mid \overline{\gamma}\mid_{p(H)} = \mathrm{id}_{p(H)}\} \subset O(h_d^\perp). \tag{3.9}$$

- The natural map
$$\tau : O(\Lambda_{K3,n}, h_d) \longrightarrow O(T)$$
is a surjection with kernel $\widetilde{O}(\Lambda_{K3,n}, h_d)$ and
$$\widetilde{O}(\Lambda_{K3,n}, h_d) \cong \widetilde{O}(h_d^\perp). \tag{3.10}$$

In particular, since τ is surjective, we have that

$$O(\Lambda_{K3,n}, h_d) = \widehat{O}(\Lambda_{K3,n}, h_d) = \widetilde{O}(\Lambda_{K3,n}, h_d) = \widetilde{O}(h_d^{\perp})$$

holds whenever $O(T) = \{\mathrm{id}\}$. This happens whenever the cyclic group T is trivial, or has order two, i.e. whenever $f = n-1, 2n-2$ – cf. [GHS2, Cor. 3.13]. Hence the map π is an isomorphism for $f = n-1$ or $f = 2n-2$.

Now assume that $f \neq n-1, 2n-2$. Then $O(T)$ is not a singleton, because e.g. $\mathrm{id}_T \neq -\mathrm{id}_T$. In particular, the group $\widetilde{O}(\Lambda_{K3,n}, h_d)$ (the kernel of τ) is a proper subgroup of index two of the group $\tau^{-1}(\pm\mathrm{id}_T)$. Let us show that following inclusion holds:

$$\widehat{O}(\Lambda_{K3,n}, h_d) \subset \tau^{-1}(\pm\mathrm{id}_T). \tag{3.11}$$

If $\varphi \in \widetilde{O}(\Lambda_{K3,n}, h_d) \subset \widehat{O}(\Lambda_{K3,n}, h_d)$, then $\varphi \in \tau^{-1}(\pm\mathrm{id}_T)$ is obvious, because $\widetilde{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\mathrm{id}_T)$. If $\varphi \in \widehat{O}(\Lambda_{K3,n}, h_d) \setminus \widetilde{O}(\Lambda_{K3,n}, h_d)$, then φ acts as $-\mathrm{id}$ on the coset $\bar{l}_{n-1}^{\vee} + H$ (which generates $D(\Lambda_{K3,n})$ – cf. (3.7)) by definition. Furthermore, φ fixes h_d by definition, and hence also $D(\langle h_d \rangle)$. Hence φ also acts as $-\mathrm{id}$ on the coset $f\bar{l}_{n-1}^{\vee} + H + D(\langle h_d \rangle)$, which generates the group T (cf. (3.8)); hence $\varphi \in \tau^{-1}(\pm\mathrm{id}_T)$ and the inclusion (3.11) holds. This allows us to conclude that $\widetilde{O}(\Lambda_{K3,n}, h_d)$ is a proper subgroup of index two in $\widehat{O}(\Lambda_{K3,n}, h_d)$, whenever also the reverse inclusion holds, i.e. whenever $\widehat{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\pm\mathrm{id}_T)$; else $\widehat{O}(\Lambda_{K3,n}, h_d)$ coincides with $\widetilde{O}(\Lambda_{K3,n}, h_d)$. So now it remains to show the following

Claim: *The condition*

$$\widehat{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\pm\mathrm{id}_T)$$

is equivalent to the condition $f = 1$ *or* $f = 2$.

Proof. Suppose first that $\widehat{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\pm\mathrm{id}_T)$. Then $\widetilde{O}(\Lambda_{K3,n}, h_d)$ is a proper subgroup of index two in $\widehat{O}(\Lambda_{K3,n}, h_d)$, which means that there exists $\varphi \in \widehat{O}(\Lambda_{K3,n}, h_d)$ which acts as $-\mathrm{id}$ on the coset $\bar{l}_{n-1}^{\vee} + H$. Let $\overline{\varphi}$

denote the induced homomorphism of $D(K)$. Then $\overline{\varphi}$ satisfies

$$\overline{\varphi}(\bar{l}_{n-1}^{\vee}) = -\bar{l}_{n-1}^{\vee} + \overline{k} = -p(\bar{l}_{n-1}^{\vee} - \overline{k}) + c\overline{h}_d^{\vee} + q(\overline{k}), \tag{3.12}$$

where $\overline{k}$ is an element of H, the summand $-p(\bar{l}_{n-1}^{\vee} - \overline{k})$ is an element of the subgroup $D(h_d^{\perp})$, and the summand $c\overline{h}_d^{\vee} + q(\overline{k})$ is an element of $D(\langle h_d \rangle)$. Note that $q(\overline{k}) \in \langle f\overline{h}_d^{\vee} \rangle$ – cf. item (*) on p. 38. On the other hand,

$$\overline{\varphi}(\bar{l}_{n-1}^{\vee}) = \overline{\varphi}(p(\bar{l}_{n-1}^{\vee}) - c\overline{h}_d^{\vee}) = \overline{\varphi}(p(\bar{l}_{n-1}^{\vee})) - c\overline{h}_d^{\vee}, \tag{3.13}$$

where the summand $\overline{\varphi}(p(\bar{l}_{n-1}^{\vee}))$ is an element of the subgroup $D(h_d^{\perp})$, and the summand $-c\overline{h}_d^{\vee}$ is an element of $D(\langle h_d \rangle)$. Comparing the summands from Eqs. (3.12) and (3.13) in the cyclic group $D(\langle h_d \rangle)$ of order $2d$, we obtain the congruence

$$c + fs \equiv -c \pmod{2d}, \tag{3.14}$$

for some integer s. But this implies that $f \mid -2c$ (recall that $f \mid 2d$). Since c and f are coprime, this implies that $f = 1$ or $f = 2$.

Now for the other direction, suppose first that $f = 1$. Then $\bar{l}_{n-1}^{\vee} + c\overline{e}$ generates T – cf. (3.8). Let $\varphi \in O(\Lambda_{K3,n}, h_d)$ satisfy $\overline{\varphi}\mid_T = -\mathrm{id}_T$. In particular,

$$\overline{\varphi}(\bar{l}_{n-1}^{\vee} + c\overline{e}) = \overline{\varphi}(\bar{l}_{n-1}^{\vee}) + c\overline{e} = -\bar{l}_{n-1}^{\vee} - c\overline{e}.$$

Hence $\overline{\varphi}$ maps the coset $\bar{l}_{n-1}^{\vee} + H$ to $-\bar{l}_{n-1}^{\vee} + H$, i.e. $\overline{\varphi}$ is an element of $\widehat{O}(\Lambda_{K3,n}, h_d)$. Therefore $\widehat{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\pm\mathrm{id}_T)$.

Now suppose that $f = 2$. This implies that $d + n - 1 \equiv 0 \pmod 4$ – cf. [GHS1, Ex. 3.10]. Together with the condition $(f, (\frac{2d}{f}, \frac{2n-2}{f})) = (2, (d, n-1)) = 1$, this implies that $n-1$ and d are both odd. Now $2\bar{l}_{n-1}^{\vee} + c\overline{e}$ generates T – cf. (3.8). Let $\varphi \in O(\Lambda_{K3,n}, h_d)$ satisfy $\overline{\varphi}\mid_T = -\mathrm{id}_T$. We obtain

$$\overline{\varphi}(2\bar{l}_{n-1}^{\vee} + c\overline{e}) = \overline{\varphi}(2\bar{l}_{n-1}^{\vee}) + c\overline{e} = -2\bar{l}_{n-1}^{\vee} - c\overline{e}$$

and, hence $2\overline{\varphi}(\bar{l}_{n-1}^{\vee}) = -2\bar{l}_{n-1}^{\vee} - 2c\overline{e}$. We conclude that $\overline{\varphi}$ maps the coset $\bar{l}_{n-1}^{\vee} + H$ to either $-\bar{l}_{n-1}^{\vee} + H$, or $(n-2)\bar{l}_{n-1}^{\vee} + H$. Assume that $\overline{\varphi}$ maps

$\bar{l}_{n-1}^{\vee} + H$ to $(n-2)\bar{l}_{n-1}^{\vee} + H$. Since the homomorphism $\overline{\varphi}$ is an isometry, n should satisfy the congruence $(n-2)^2 \equiv 1 \pmod{2n-2}$. This congruence is equivalent to $n^2 \equiv 1 \pmod{2n-2}$, which is a contradiction to the fact that n is an even number (recall that d and $n-1$ are odd numbers by condition). Hence $\overline{\varphi}$ maps the coset $\bar{l}_{n-1}^{\vee} + H$ to $-\bar{l}_{n-1}^{\vee} + H$, i.e. $\overline{\varphi}$ is an element of $\widehat{O}(\Lambda_{K3,n}, h_d)$. Therefore $\widehat{O}(\Lambda_{K3,n}, h_d) = \tau^{-1}(\pm \mathrm{id}_T)$ and we have shown the claim. ■

The above claim implies that $\widetilde{O}(\Lambda_{K3,n}, h_d) \cong \widetilde{O}(h_d^{\perp})$ is a proper subgroup of index two in $\widehat{O}(\Lambda_{K3,n}, h_d)$, if $f = 1$, or $f = 2$, and $f \neq n-1, 2n-2$. Else if $f > 2$, or if $f = n-1, 2n-2$, then $\widetilde{O}(h_d^{\perp}) = \widehat{O}(\Lambda_{K3,n}, h_d)$ and the map π is an isomorphism.

Now suppose that $f = d = 1$, or $f = 2d = 2$. Then the cyclic group $p(H)$ is either trivial, or has order two. But then $-\mathrm{id}_{h_d^{\perp}}$ is an element of $\widehat{O}(\Lambda_{K3,n}, h_d)$, which implies that

$$P\widetilde{O}(h_d^{\perp}) \cong P\widehat{O}(\Lambda_{K3,n}, h_d), \tag{3.15}$$

i.e. the map π is an isomorphism.

Finally, if $f = 1$ or $f = 2$, and $f \neq n-1, 2n-2, d, 2d$, then $-\mathrm{id}_{h_d^{\perp}}$ is not an element of $\widehat{O}(\Lambda_{K3,n}, h_d)$, which implies that

$$[P\widehat{O}(\Lambda_{K3,n}, h_d) : P\widetilde{O}(h_d^{\perp})] = [\widehat{O}(\Lambda_{K3,n}, h_d) : \widetilde{O}(h_d^{\perp})] = 2,$$

i.e. the map π has degree 2. ■

In the following proposition, let $h_d \in \Lambda_{K3,n}$ be as above and let $h_{\tilde{d}} \in \Lambda_{K3,\tilde{n}}$ be a primitive vector with

$$(h_{\tilde{d}}, h_{\tilde{d}}) = 2\tilde{d} > 0, div(h_{\tilde{d}}) = \tilde{f}.$$

As in (3.1), we have isometries

$$h_d^{\perp} \cong \Lambda_B \text{ and } h_{\tilde{d}}^{\perp} \cong \Lambda_{\widetilde{B}},$$

where

$$\Lambda_B := 2U \oplus 2E_8(-1) \oplus T_B, \tag{3.16}$$

$$\Lambda_{\widetilde{B}} := 2U \oplus 2E_8(-1) \oplus T_{\widetilde{B}},$$

and T_B, $T_{\widetilde{B}}$ are negative definite lattices of rank 2. As in (3.6), we can define the subgroups $H < D(h_d^\perp) \oplus D(\langle h_d \rangle)$ and $\widetilde{H} < D(h_{\tilde{d}}^\perp) \oplus D(\langle h_{\tilde{d}} \rangle)$. Let $p(H)$ and $\tilde{p}(\widetilde{H})$ denote their projections to $D(h_d^\perp)$, resp. $D(h_{\tilde{d}}^\perp)$.

If there exists an isometry

$$S : \Lambda_B \to \Lambda_{\widetilde{B}},$$

then S induces, by extension of scalars, an isomorphism

$$S' : \mathbb{P}(\Lambda_B \otimes \mathbb{C}) \to \mathbb{P}(\Lambda_{\widetilde{B}} \otimes \mathbb{C}),$$

which maps the period domain $\Omega_{h_d^\perp}$ to $\Omega_{h_{\tilde{d}}^\perp}$. Moreover, since S is an isometry, we have

$$S\widetilde{O}(\Lambda_B)S^{-1} = \widetilde{O}(\Lambda_{\widetilde{B}}),$$

which means that σ is compatible with the action of $\widetilde{O}(\Lambda_B)$, resp. $\widetilde{O}(\Lambda_{\widetilde{B}})$ on $\Omega_{h_d^\perp}$, resp. $\Omega_{h_{\tilde{d}}^\perp}$. In particular, S' descends to an isomorphism

$$\sigma : \mathcal{G}_{h_d} \to \mathcal{G}_{h_{\tilde{d}}}.$$

We then say that σ is *induced by S*.

Proposition 3.2
There exists an isomorphism

$$\sigma : \mathcal{G}_{h_d} \longrightarrow \mathcal{G}_{h_{\tilde{d}}},$$

induced by an isometry

$$S : \Lambda_B \longrightarrow \Lambda_{\widetilde{B}},$$

if and only if $g(\Lambda_B) = g(\Lambda_{\widetilde{B}})$;

PROOF. If there is an isometry

$$S : \Lambda_B \longrightarrow \Lambda_{\widetilde{B}},$$

then it induces isometries $S \otimes \mathbb{Z}_p$ for every p – prime; furthermore Λ_B and $\Lambda_{\widetilde{B}}$ must have the same signatures. Hence $g(\Lambda_B) = g(\Lambda_{\widetilde{B}})$.

Now let $g(\Lambda_B) = g(\Lambda_{\widetilde{B}})$. But Λ_B is an indefinite lattice rank ≥ 3, and it contains a copy of the hyperbolic lattice, hence its genus contains only one isometry type (cf. [GHS3, Section 3.1]). Thus the condition $g(\Lambda_B) = g(\Lambda_{\widetilde{B}})$ implies that there is an isometry

$$S : \Lambda_B \longrightarrow \Lambda_{\widetilde{B}}.$$

The map S induces an isomorphism

$$\sigma : \mathcal{G}_{h_d} \longrightarrow \mathcal{G}_{h_{\tilde{d}}}.$$

■

It is easy to determine $g(\Lambda_B)$ from the matrix B – cf. Prop. 4.1 in the next chapter.

We can use the next proposition to produce many examples of pairs $h_d \in \Lambda_{K3,n}$ and $h_{\tilde{d}} \in \Lambda_{K3,\tilde{n}}$, for which there is an isomorphism $\mathcal{G}_{h_d} \to \mathcal{G}_{h_{\tilde{d}}}$. In the following proposition, let h_d and B be given as on p. 35 and put

$$\Delta_B := \det B.$$

Let q and s be coprime natural numbers such that the number

$$2\widetilde{n} - 2 := 2bq^2 - 2c\frac{2n-2}{f}qs + (2n-2)s^2$$

is positive. Let p and r be integers satisfying $ps - qr = 1$.

Put

$$L_1 := -2bq + c\frac{2n-2}{f}s \text{ and } L_2 := c\frac{2n-2}{f}q - (2n-2)s.$$

Note that

$$2\widetilde{n}-2=-qL_1-sL_2,$$

by definition. This means that $(L_1,L_2)\mid 2\widetilde{n}-2$ and $\widetilde{f}:=\frac{2\widetilde{n}-2}{(L_1,L_2)}$ is a positive integer. Furthermore,

$$2(L_1,L_2)\mid \widetilde{f}\Delta_B=\widetilde{f}(4\widetilde{b}(\widetilde{n}-1)-(pL_1+rL_2)^2)$$

and $\Delta_B>0$, hence $\widetilde{d}:=\widetilde{f}\frac{\Delta_B}{2(L_1,L_2)}$ is also a positive integer. Put

$$\widetilde{c}:=\frac{pL_1+rL_2}{(L_1,L_2)},$$

$$2\tilde{b}:=2bp^2-2c\frac{2n-2}{f}pr+(2n-2)r^2.$$

Let

$$w\in 3U\oplus 2E_8(-1)\subset\Lambda_{\widetilde{n}-1}$$

be a primitive vector with $(w,w)=2\widetilde{b}$. Put

$$h_{\tilde{d}}:=\tilde{f}w+\tilde{c}l_{2\tilde{n}-2}$$

and

$$\widetilde{B}:=\begin{pmatrix}-2\widetilde{b} & \widetilde{c}\frac{2\widetilde{n}-2}{\widetilde{f}}\\ \widetilde{c}\frac{2\widetilde{n}-2}{\widetilde{f}} & -2\widetilde{n}+2\end{pmatrix}$$

and note that $\Delta_B=\Delta_{\widetilde{B}}$.

Proposition 3.3

The vector $h_{\widetilde{d}}\in\Lambda_{K3,\widetilde{n}}$ is primitive with

$$div(h_{\widetilde{d}})=\widetilde{f}\text{ and }(h_{\widetilde{d}},h_{\widetilde{d}})=2\widetilde{d},$$

and there is an isomorphism $\mathcal{G}_{h_d}\to\mathcal{G}_{h_{\widetilde{d}}}$.

Proof. To show primitivity of $h_{\widetilde{d}}$, we need to show that $(\widetilde{c},\widetilde{f})=1$. Since

$$\widetilde{c}=\frac{pL_1+rL_2}{(L_1,L_2)}\text{ and }\widetilde{f}=-\frac{qL_1+sL_2}{(L_1,L_2)},$$

this condition is equivalent to

$$(pL_1 + rL_2, qL_1 + sL_2) = (L_1, L_2).$$

Now

$$(L_1, L_2) \mid (pL_1 + rL_2, qL_1 + sL_2)$$

is obvious. Furthermore, note that

$$s(pL_1 + rL_2) - r(qL_1 + sL_2) = (ps - qr)L_1 = L_1$$

Hence

$$(pL_1 + rL_2, qL_1 + sL_2) \mid L_1.$$

Similarly,

$$(pL_1 + rL_2, qL_1 + sL_2) \mid L_2.$$

Hence

$$(pL_1 + rL_2, qL_1 + sL_2) \mid (L_1, L_2).$$

It follows that

$$(L_1, L_2) = (pL_1 + rL_2, qL_1 + sL_2).$$

Now $div(h_{\widetilde{d}})$ is equal to the coefficient of w, which is $\widetilde{f}$ by construction. Furthermore

$$\begin{aligned}
(h_{\widetilde{d}}, h_{\widetilde{d}}) &= \widetilde{f}^2(w,w) + \widetilde{c}^2(l_{2\widetilde{n}-2}, l_{2\widetilde{n}-2}) = \widetilde{f}^2 \frac{\Delta_{\widetilde{B}}}{2\widetilde{n}-2} = \widetilde{f}\frac{2\widetilde{n}-2}{(L_1, L_2)} \cdot \frac{\Delta_B}{2\widetilde{n}-2} \\
&= \widetilde{f}\frac{\Delta_B}{(L_1, L_2)} = 2\widetilde{d}.
\end{aligned}$$

Note that

$$\widetilde{c}\frac{2\widetilde{n}-2}{\widetilde{f}} = pL_1 + rL_2.$$

Hence

$$\widetilde{B} = \begin{pmatrix} -2\widetilde{b} & pL_1 + rL_2 \\ pL_1 + rL_2 & -2\widetilde{n} + 2 \end{pmatrix}.$$

Put

$$S := I_{20} \oplus \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

The conditions imply that $\widetilde{B} = S^t B S$. Hence S^{-1} is an isometry $\Lambda_B \to \Lambda_{\widetilde{B}}$. This implies that $g(\Lambda_B) = g(\Lambda_{\widetilde{B}})$. Therefore, there is an isomorphism

$$\mathcal{G}_{h_d} \longrightarrow \mathcal{G}_{h_{\widetilde{d}}},$$

by Prop. 3.2. ∎

Examples.

1) Put

$$2d > 2, n = 2$$

and let $h_d \in \Lambda_2$ be of split polarization type, i.e. $f = 1, c = 0$. Then, since $n - 1 = f = 1$, $\mathcal{F}_{h_d} \cong \mathcal{G}_{h_d}$, by Thm 3.1. Put $q = 1, s = 0$. Then

$$2\widetilde{n} - 2 = 2d, 2\widetilde{d} = 2\widetilde{f} = 2.$$

Put $p = 0, r = 1$. Then $\widetilde{c} = 0$ and we obtain a primitive vector $h_{\widetilde{d}} \in \Lambda_{2d}$ of split polarization type. Thm 3.1 gives an isomorphism $\mathcal{G}_{h_{\widetilde{d}}} \to \mathcal{F}_{h_{\widetilde{d}}}$. By Prop. 3.3, $\mathcal{G}_{h_d} \cong \mathcal{G}_{h_{\widetilde{d}}}$. Thus we obtain an isomorphism $\mathcal{F}_{h_d} \cong \mathcal{F}_{h_{\widetilde{d}}}$.

2) Put

$$2d > 2, d \equiv -1 \pmod 4, n = 2$$

and let $h_d \in \Lambda_2$ be of non-split polarization type, i.e. $f = 2, c = 1$. Put $q = 1, s = 0$. Then

$$2\widetilde{n} - 2 = \frac{d+1}{2}, 2\widetilde{d} = d\frac{d+1}{2}, \widetilde{f} = \frac{d+1}{2}.$$

Put $p = 0, r = 1$. Then $\widetilde{c} = 1$ and we obtain a primitive vector $h_{\widetilde{d}} \in \Lambda_{\frac{d+1}{2}}$ with $div(h_{\widetilde{d}}) = \widetilde{f}$. If $d = 3$, then $\widetilde{f} = 2\widetilde{n} - 2 = 2$, else if $d > 3$, then $\widetilde{f} > 2$. In both cases, $\mathcal{G}_{h_{\widetilde{d}}} \cong \mathcal{F}_{h_{\widetilde{d}}}$, by Thm. 3.1. Thus Props. 3.1 and 3.3 give an isomorphism $\mathcal{F}_{h_d} \cong \mathcal{F}_{h_{\widetilde{d}}}$, i.e. we obtain birational maps between the component of the moduli space of $K3^{[2]}$-type, of polarization type given by h_d and the components of the moduli space of $K3^{[\frac{d+1}{4}+1]}$-type, of polarization type given by $h_{\widetilde{d}}$.

3) Let $n, d \in \mathbb{N}$ be fixed, and let $h_d \in \Lambda_{K3,n}$ be a primitive vector with $div(h_d) = f$. Assume further that $f > 2$. Put $q = f, s = c$. Then

$$2\widetilde{n} - 2 = 2bf^2 - (2n-2)c^2 = 2d, \widetilde{f} = f, \widetilde{d} = n - 1.$$

As in 1) we obtain an isomorphism $\mathcal{F}_{h_d} \cong \mathcal{F}_{h_{\widetilde{d}}}$ and note that it also interchanges the degree of the polarization with the dimension, i.e. we obtain birational maps between the components of the moduli space of $K3^{[n]}$-type, of polarization type given by h_d and the components of the moduli space of $K3^{[d+1]}$-type, of polarization type given by h_{n-1}.

4) It can also happen that $\mathcal{G}_{h_d} \cong \mathcal{G}_{h_{\widetilde{d}}}$, whereas $\mathcal{F}_{h_d} \ncong \mathcal{F}_{h_{\widetilde{d}}}$. Let $n, d \in \mathbb{N}$ be fixed with $n > 2, d > 1$, and let $h_d \in \Lambda_{K3,n}$ be a primitive vector with $div(h_d) = 1$. Put $q = 2n - 2, s = 1$. Then

$$2\widetilde{n} - 2 = (2n-2)(4d(n-1)+1), \widetilde{f} = 2n - 2 > 2, \widetilde{d} = 2d(n-1).$$

In particular, $\mathcal{G}_{h_{\widetilde{d}}} \to \mathcal{F}_{h_{\widetilde{d}}}$ has degree two and $\mathcal{G}_{h_{\widetilde{d}}} \cong \mathcal{F}_{h_{\widetilde{d}}}$, by Thm. 3.1. Furthermore, $\mathcal{G}_{h_d} \cong \mathcal{G}_{h_{\widetilde{d}}}$, by Prop. 3.3 and we obtain (generically) a two-to-one correspondence between the components of the respective moduli spaces.

5) Let us elaborate somewhat on the case of IS fourfolds with a polarisation h_d is of non-split type, of degree $h_d^2 = 6$. Let p, q, r, s be as in 2) above. Then

$$2\widetilde{n} - 2 = 2, 2\widetilde{d} = 6, \widetilde{f} = 2, \widetilde{c} = 1,$$

and we obtain a primitive vector $h_{\widetilde{d}} \in \Lambda_2$, of non-split type, i.e. we obtain an involution $I_{20} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of the period domain $\Omega_{h_{\widetilde{d}}^{\perp}}$ that descends to an involution $\overline{S}$ of the modular variety $\mathcal{F}_{h_d}$. The latter contains as an open subvariety the moduli space of IS fourfolds of non-split polarization type of degree six. This moduli space is related to the moduli space of cubic fourfolds. Let us recall the basic facts about cubic fourfolds, following [BD] and [AdT, Ch. 1]. The Fano variety of lines $F(X)$ on a cubic fourfold X, together with its Plücker polarization $h_d \in \mathrm{Pic}(F(X))$ is a polarized IS fourfold of $K3^{[2]}$-type, with a non-split polarization of degree six. The associated Abel-Jacobi map induces a Hodge isometry

$$\varphi : H^4_{\mathrm{prim}}(X, \mathbb{Z}) \to h_d^{\perp} \subset H^2(F(X), \mathbb{Z}),$$

where $H^4_{\mathrm{prim}}(X, \mathbb{Z})$ is taken with the weight two Hodge structure, induced by $H^{3,1}(X)$. By the Global Torelli theorem for cubic fourfolds, there is an injective period map from each component of the moduli space of marked cubic fourfolds to a component of $\Omega_{h_d^{\perp}}$ – cf. [Voi] and [Ch].

The moduli space contains a countable set of distinguished divisors $\mathcal{C}_d$, indexed by numbers d, subject to some numerical conditions, given in [Has, Thms. 1.0.1-2]. These were studied by Hassett in [Has], and are characterized by the fact that, for each fourfold $[X] \in \mathcal{C}_d$, there is a $T \in H^{2,2}(X, \mathbb{Z})$ such that $T^{\perp} \subset H^4_{\mathrm{prim}}(X, \mathbb{Z})$ is Hodge-isometric to the primitive cohomology $l_d^{\perp} \subset H^2(S, \mathbb{Z})$ of some degree d polarized K3 surface (S, l_d). On a different note, Kuznetsov ([Kuz]) has associated a semi-orthogonal decomposition

$$\{\mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$$

of the derived category $\mathcal{D}^b(X)$ of a cubic fourfold – here the admissible subcategory $\mathcal{A}_X$ can be interpreted as a noncommutative deformation of the bounded derived category of a K3-surface. The associated IS fourfold $F(X)$ can be interpreted as a moduli space of sheaves in $\mathcal{A}_X$ – cf. [Add, Ch. 4.2]. The loci of points $[(X, \eta)]$ in the moduli space, where $\mathcal{A}_X$ is actually equivalent to the derived category $\mathcal{D}^b(S)$ of a K3 surface are also known as the *geometric* or *Kuznetsov loci*. It was conjectured in [Kuz] that these are exactly the loci for which the fourfold X is rational. It was only recently proven that the Kuznetsov loci coincide (at least generically) with the Hassett loci introduced above – cf. [AdT, Thm. 1.1].

By using the involution $\overline{S}$ above we can associate to a (generic) cubic fourfold, a "dual" fourfold X' with the property that the sub-Hodge structures

$h_d^\perp$ and $h_{\widetilde{d}}^\perp$ are Hodge-isometric, but there is no polarized Hodge isometry between $H^2(F(X),\mathbb{Z})$ and $H^2(F(X'),\mathbb{Z})$, mapping h_d to $h_{\widetilde{d}}$. In this respect, this example is similar to the example of the involution of the moduli space of IS fourfolds with a degree two polarization (double EPW-sextics), studied by O'Grady in [OG4].

Chapter 4

Hirzebruch-Mumford Volumes

In this chapter we compute the so-called Hirzebruch-Mumford volumes of the modular varieties $\mathcal{F}_{h_d}$ in some cases. Given an even indefinite lattice Λ of signature $(2, m)$ and an arithmetic subgroup $\Gamma \subset O^+(\Lambda)$, the Hirzebruch-Mumford volume of the space $\mathcal{D}_\Lambda/\Gamma$, denoted by $\mathrm{vol}_{\mathrm{HM}}(\Gamma)$, is defined as a quotient of the volume of $\mathcal{D}_\Lambda/\Gamma$ and the volume of the compact dual space $\mathcal{D}_\Lambda^{(c)}$ with respect to suitably chosen volume forms on those spaces – cf. [GHS3, Ch. 1]. One major application of this invariant consists in estimating the growth of spaces of cusp forms on $\mathcal{D}_\Lambda/\Gamma$ as a function of their weight. The HM-volume is also used to determine the Kodaira dimension of modular varieties – cf. e.g. the results in [GHS4], [GHS5]. In [GHS3, Thm. 3.1], the following formula for the HM-volume is given:

$$\mathrm{vol}_{HM}(\Gamma) = 2[PO(\Lambda) : P\Gamma]|\det \Lambda|^{m+3/2} \prod_{k=1}^{m+2} \pi^{-k/2}\Gamma(k/2) \prod_p \alpha_p(\Lambda)^{-1}. \tag{4.1}$$

In the above formula $\alpha_p(\Lambda)$ are local densities of the quadratic space over $\mathbb{Q}_p$ associated to the lattice $\Lambda \otimes \mathbb{Z}_p$ for p – prime (cf. [GHS3, Ch. 3.2] for the definition). We recall a formula for $\alpha_p(\Lambda)$ from [Kit, Ch. 5] below. First we introduce some notation and definitions. A lattice T over $\mathbb{Z}_p$ is called p^j*-modular*, if the matrix $p^{-j}(v_i, v_j)_T$ is invertible over $\mathbb{Z}_p$ – here $(v_i, v_j)_T$

denotes a Gram matrix for T. The *norm* of T is the following ideal in $\mathbb{Z}_p$:

$$\mathrm{norm}(T) := \{\sum a_x(x,x)_T \mid x \in T, a_x \in \mathbb{Z}_p\}.$$

For a fixed p, the lattice $\Lambda \otimes \mathbb{Z}_p$ admits a Jordan decomposition $\bigoplus_{j\in\mathbb{Z}} \Lambda_j$, where Λ_j is a p^j-modular lattice of rank $n_j \geq 0$. For $p \neq 2$ define:

$$\begin{aligned}
w &:= \sum_j j n_j((n_j+1)/2 + \sum_{k>j} n_k), \\
P_p(n) &:= \prod_{i=1}^{n}(1-p^{-2i}), \\
P_p(\Lambda) &:= \prod_j P_p([n_j/2]), \\
E_p(\Lambda) &:= \prod_{j,\Lambda_j\neq 0}^{n} (1+\chi(N_j)p^{-n_j/2})^{-1},
\end{aligned}$$

where $\chi(N_j)$ is a character of the quadratic space associated to the unimodular scaling N_j of the lattice L_j; the *character* χ of a regular quadratic space W over the finite field $\mathbb{Z}/p\mathbb{Z}$ is a function, defined by:

$$\chi(W) := \begin{cases} 0, & \text{if } \dim W \text{ is odd;} \\ 1, & \text{if } W \text{ is a hyperbolic space;} \\ -1, & \text{else.} \end{cases}$$

Now we have the following formula for the local density for $p \neq 2$, given in [GHS3, Ch. 3.2, (10)]:

$$\alpha_p(\Lambda) = 2^{s-1}p^w P_p(\Lambda)E_p(\Lambda), \tag{4.2}$$

where s is the number of non-zero terms Λ_j in the Jordan decomposition of Λ.

Now consider the case $p = 2$. A unimodular lattice N over $\mathbb{Z}_2$ is called *even*, if it is trivial, or if norm(N)=$2\mathbb{Z}_2$, and *odd* otherwise. Every unimodular lattice N over $\mathbb{Z}_2$ decomposes as a sum $N^{even} \oplus N^{odd}$ of odd and even sublattices such that $\mathrm{rk}(N^{odd}) \leq 2$. Now define

$$
\begin{aligned}
P_2(\Lambda) &:= \prod_j P_2(\mathrm{rk} N_j^{even}/2); \\
E_j(\Lambda) &:= \frac{1}{2}(1 + \chi(N_j^{even}) 2^{-\mathrm{rk} N_j^{even}/2}), \text{if } N_j \text{ and } N_{j+1} \text{ are even,} \\
& \quad \text{unless } N_j^{odd} \cong \langle \epsilon_1 \rangle \oplus \langle \epsilon_2 \rangle \text{ with } \epsilon_1 \equiv \epsilon_2 (\mathrm{mod}\ 4); \\
E_j(\Lambda) &:= \frac{1}{2}, \text{ otherwise;} \\
E_2(\Lambda) &:= \prod_j E_j(\Lambda)^{-1}.
\end{aligned}
$$

Now we have the following formula for the local density for $p = 2$:

$$
\alpha_2(\Lambda) = 2^{n-1+w-q} P_2(\Lambda) E_2(\Lambda), \tag{4.3}
$$

where w is defined in the same way as for the case $p \neq 2$, and $q := \sum_j q_j$, where

$$
q_j := \begin{cases} 0, & \text{if } N_j \text{ is even;} \\ n_j, & \text{if } N_j \text{ is odd \ and } N_{j+1} \text{ is even;} \\ n_j + 1, & \text{if } N_j \text{ and } N_{j+1} \text{ are odd.} \end{cases}
$$

From now on we consider the lattice Λ_B for a fixed negative definite matrix B of rank two – cf. (3.16). It has signature $(2, 20)$. Define

$$
\delta_B := \gcd(B)
$$

– here $\gcd(B)$ denotes the gcd of the entries of the matrix B;

$$
\widetilde{\Delta}_B := \Delta_B / \delta_B^2.
$$

In the following, $\zeta(m)$ denotes the Riemann zeta function;
$(\frac{m}{n})$ denotes the Hilbert residue symbol; for a fixed m, there is an associated character to the Hilbert residue symbol – it is given by mapping n to $(\frac{m}{n})$; we denote this character by $(\frac{m}{*})$.
$L(s, \chi)$ denotes the Dirichlet L-series with respect to the character χ.
The notation $p^j || n$ means that $p^j \mid n$ and $(p^j, \frac{n}{p^j}) = 1$.

Proposition 4.1

The HM volume of $\mathcal{F}_{h_d}$ is given by

$$\begin{aligned}
\mathrm{vol}_{HM}(\widehat{O}^+(\Lambda_{K3,n},h_d)) &= 2^{-(21+\rho(\delta_B)+\rho(\widetilde{\Delta}_B))}[PO(\Lambda_B):P\widehat{O}^+(\Lambda_{K3,n},h_d)]|\Delta_B|^{21/2}\delta_B^{-1} \\
&\cdot\pi^{-11}\Gamma(11)L(11,\left(\frac{\Delta_B}{*}\right))\frac{|B_2B_4...B_{20}|}{20!!}C_2(\Lambda_B) \\
&\cdot\prod_{p|\delta_B}(1+p^{-10})\cdot\prod_{p|\delta_B,p\nmid\widetilde{\Delta}_B}(1+\left(\frac{\widetilde{\Delta}_B}{p}\right)p^{-1})P_p(1)^{-1}, \qquad (4.4)
\end{aligned}$$

where

$$C_2(\Lambda_B):=\begin{cases}(1+2^{-11})(1-(\frac{\Delta_B}{8})2^{-11})^{-1}, & \text{if } 2\nmid\Delta_B; \\ \frac{2}{3}(1+(\frac{\widetilde{\Delta}_B}{8})2^{-1}), & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is even}; \\ \frac{2}{3}, & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is odd}, (\frac{\widetilde{\Delta}_B}{4})=1; \\ \frac{4}{3}, & \text{if } 2^j||\delta_B, 2\nmid\widetilde{\Delta}_B, N_j \text{ is odd}, (\frac{\widetilde{\Delta}_B}{4})=-1; \\ 4, & \text{if } 2^j||\delta_B, 2^{2j+1}||\Delta_B; \\ 16, & \text{if } 2^{j_1}||\delta_B, 2^{j_1+j_2}||\Delta_B, j_2>j_1+1;\end{cases}$$

PROOF. By the classical theory on the classification of lattices over $\mathbb{Z}_p$ (cf. [OM, Ch. IX]), we need to compute a Jordan splitting of $\Lambda_B\otimes\mathbb{Z}_p$ by the procedure described in [OM, Sect. 94] and we obtain that the lattice Λ_B decomposes in the following way over $\mathbb{Z}_p$:

- $\Lambda_B\otimes\mathbb{Z}_p$ is a unimodular lattice of rank 22, if $p\nmid\Delta_B$;
- $\Lambda_B\otimes\mathbb{Z}_p$ is a direct sum of a unimodular lattice of rank 20 and a p^j-modular lattice of rank 2, if $p^j||\delta_B, p\nmid\widetilde{\Delta}_B$;
- $\Lambda_B\otimes\mathbb{Z}_p$ is a direct sum of a unimodular lattice of rank 21 and a p^j-modular lattice of rank 1, if $p\nmid\delta_B, p^j||\widetilde{\Delta}_B$;
- $\Lambda_B\otimes\mathbb{Z}_p$ is a direct sum of a unimodular lattice of rank 20 a p^{j_1}-modular lattice of rank 1 and a p^{j_2}-modular lattice of rank 1, if $p^{j_1}||\delta_B, p^{j_1+j_2}||\Delta_B, j_2>j_1$;

Then, by applying formula (4.2), the local density $\alpha_p(\Lambda_B)$ is given by

$$\alpha_p(\Lambda_B) = \begin{cases} P_p(11)(1+(\frac{\Delta_B}{p})p^{-11})^{-1}, & \text{if } p \nmid \Delta_B; \\ 2p^{3j}P_p(10)P_p(1)(1+p^{-10})^{-1}(1+(\frac{\widetilde{\Delta}_B}{p})p^{-1})^{-1}, & \text{if } p^j || \delta_B, p \nmid \widetilde{\Delta}_B \\ 2p^j P_p(10), & \text{if } p \nmid \delta_B, p^j || \widetilde{\Delta}_B \\ 4p^{2j_1+j_2}P_p(10)(1+p^{-10})^{-1}, & \text{if } p^{j_1} || \delta_B, p^{j_1+j_2} || \Delta_B, \\ & j_2 > j_1. \end{cases}$$

Over $\mathbb{Z}_2$ the decomposition is given by:

- $\Lambda_B \otimes \mathbb{Z}_2$ is a unimodular lattice of rank 22, if $2 \nmid \Delta_B$;
- $\Lambda_B \otimes \mathbb{Z}_2$ is a direct sum of a unimodular lattice of rank 20 and a 2^j-modular lattice Λ_j of rank 2, whose unimodular scaling N_j is an even lattice, if $2^j || \delta_B, 2 \nmid \widetilde{\Delta}_B, \frac{\gcd(2b, 2n-2)}{2^j}$ is even;
- $\Lambda_B \otimes \mathbb{Z}_2$ is a direct sum of a unimodular lattice of rank 20 and a 2^j-modular lattice Λ_j of rank 2, whose unimodular scaling N_j is an odd lattice, if $2^j || \delta_B, 2 \nmid \widetilde{\Delta}_B, \frac{\gcd(2b, 2n-2)}{2^j}$ is odd;
- $\Lambda_B \otimes \mathbb{Z}_2$ is a direct sum of a unimodular lattice of rank 20 a 2^{j_1}-modular lattice of rank 1 and a 2^{j_2}-modular lattice of rank 1, if $2^{j_1} || \delta_B, 2^{j_1+j_2} || \Delta_B, j_2 > j_1$;

Then, by applying formula (4.3), the local density $\alpha_2(\Lambda_B)$ is given by

$$\alpha_2(\Lambda_B) = \begin{cases} 2^{22}P_2(11)(1+(\frac{\Delta_B}{8})2^{-11})^{-1}, & \text{if } 2 \nmid \Delta_B; \\ 2^{3j+23}P_2(10)P_2(1)(1+2^{-10})^{-1}(1+(\frac{\widetilde{\Delta}_B}{8})2^{-1})^{-1}, & \text{if } 2^j || \delta_B, 2 \nmid \widetilde{\Delta}_B, N_j \text{ is even;} \\ 2^{3j+23}P_2(10)P_2(1)(1+2^{-10})^{-1}, & \text{if } 2^j || \delta_B, 2 \nmid \widetilde{\Delta}_B, N_j \text{ is odd,} \\ & (\frac{\widetilde{\Delta}_B}{4}) = 1; \\ 2^{3j+22}P_2(10)P_2(1)(1+2^{-10})^{-1}, & \text{if } 2^j || \delta_B, 2 \nmid \widetilde{\Delta}_B, N_j \text{ is odd,} \\ & (\frac{\widetilde{\Delta}_B}{4}) = -1; \\ 2^{3j+22}P_2(10)(1+2^{-10})^{-1}, & \text{if } 2^j || \delta_B, 2^{2j+1} || \Delta_B; \\ 2^{2j_1+j_2+20}P_2(10)(1+2^{-10})^{-1}, & \text{if } 2^{j_1} || \delta_B, 2^{j_1+j_2} || \Delta_B, \\ & j_2 > j_1 + 1; \end{cases}$$

Hence the HM volume of $\mathcal{F}_{h_d}$ equals

$$\begin{aligned}
\mathrm{vol}_{HM}(\widehat{O}^+(\Lambda_{K3,n},h_d)) &= 2[PO(\Lambda_B):P\widehat{O}^+(\Lambda_{K3,n},h_d)]|\Delta_B|^{23/2}\prod_{k=1}^{22}\pi^{-k/2}\Gamma(k/2)\\
&\cdot 2^{-(22+\rho(\delta_B)+\rho(\widetilde{\Delta}_B))}|\Delta_B|^{-1}\delta_B^{-1}\zeta(2)\zeta(4)...\zeta(20)L(11,\left(\frac{\Delta}{*}\right))\\
&\cdot C_2(\Lambda_B)\prod_{p|\delta_B}(1+p^{-10})\cdot\prod_{p|\delta_B,p\nmid\widetilde{\Delta}_B}(1+\left(\frac{\widetilde{\Delta}_B}{p}\right)p^{-1})P_p(1)^{-1}\\
&= 2^{-(21+\rho(\delta_B)+\rho(\widetilde{\Delta}_B))}[PO(\Lambda_B):P\widehat{O}^+(\Lambda_{K3,n},h_d)]|\Delta_B|^{21/2}\delta_B^{-1}\\
&\cdot\prod_{k=1}^{22}\pi^{-k/2}\Gamma(k/2)\zeta(2)\zeta(4)...\zeta(20)L(11,\left(\frac{\Delta_B}{*}\right))C_2(\Lambda_B)\\
&\cdot\prod_{p|\delta_B}(1+p^{-10})\cdot\prod_{p|\delta_B,p\nmid\widetilde{\Delta}_B}(1+\left(\frac{\widetilde{\Delta}_B}{p}\right)p^{-1})P_p(1)^{-1}
\end{aligned}$$

where $C_2(\Lambda_B)$ is defined in the statement of the proposition.

As in [GHS3, Ch. 3.3], we simplify the above formula with the help of the ζ-identity below:

$$\pi^{-\frac{1}{2}-2k}\Gamma(k)\Gamma(k+\frac{1}{2})\zeta(2k)=(-1)^{k+1}\frac{B_{2k}}{2k}.$$

We obtain

$$\begin{aligned}
\mathrm{vol}_{HM}(\widehat{O}^+(\Lambda_{K3,n},h_d)) &= 2^{-(21+\rho(\delta_B)+\rho(\widetilde{\Delta}_B))}[PO(\Lambda_B):P\widehat{O}^+(\Lambda_{K3,n},h_d)]|\Delta_B|^{21/2}\delta_B^{-1}\\
&\cdot\pi^{-11}\Gamma(11)L(11,\left(\frac{\Delta_B}{*}\right))\frac{|B_2B_4...B_{20}|}{20!!}C_2(\Lambda_B)\\
&\cdot\prod_{p|\delta_B}(1+p^{-10})\cdot\prod_{p|\delta_B,p\nmid\widetilde{\Delta}_B}(1+\left(\frac{\widetilde{\Delta}_B}{p}\right)p^{-1})P_p(1)^{-1}. \qquad (4.5)
\end{aligned}$$

■

Now, the index $[PO(\Lambda_B):P\widehat{O}^+(\Lambda_{K3,n},h_d)]$ can be computed in the case $\left(f,\left(\frac{2d}{f},\frac{2n-2}{f}\right)\right)=1$. Note that $f\mid d$, whenever f is odd.

Proposition 4.2
The index $[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)]$ *equals*

- $2^{\rho(2d/f)+\delta_1}$, if $f = n-1$ or $f = 2n-2$, where

$$\delta_1 := \begin{cases} 0, & \text{if } f \text{ is even and } (2d/f) \equiv 1 \pmod 2, \text{ or } (2d/f) \equiv 4 \pmod 8; \\ & \text{or if } f \text{ is odd and } d/f \text{ is even;} \\ -1, & \text{if } f \text{ is even and } (2d/f) \equiv 2 \pmod 4; \text{ or if } f \text{ and } d/f \text{ are odd;} \\ 1, & \text{if } f \text{ is even and } (2d/f) \equiv 0 \pmod 8 \end{cases}$$

- $2^{\rho((2n-2)/f)+\delta_2}$, if $f = d$ or $f = 2d$, where

$$\delta_2 := \begin{cases} 0, & \text{if } f \text{ is even and } ((2n-2)/f) \equiv 1 \pmod 2, \text{ or } ((2n-2)/f) \equiv 4 \pmod 8; \\ & \text{or if } f \text{ is odd and } (n-1)/f \text{ is even;} \\ -1, & \text{if } f \text{ is even and } ((2n-2)/f) \equiv 2 \pmod 4; \text{ or if } f \text{ and } (n-1)/f \text{ are odd;} \\ 1, & \text{if } f \text{ is even and } ((2n-2)/f) \equiv 0 \pmod 8 \end{cases}$$

- $2^{\rho(2d/f)+\rho((2n-2)/f)+\delta_1+\delta_2-1}$, if $f \neq d, 2d, n-1, 2n-2$, and $f = 1$ or $f = 2$; here δ_1 and δ_2 are defined as above.
- $2^{\rho(2d/f)+\rho((2n-2)/f)+\delta_1+\delta_2}$, otherwise.

Proof.

- if $\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1$ and $f = n-1$ or $f = 2n-2$, then

$$\widehat{O}(\Lambda_{K3,n}, h_d) = \widetilde{O}(\Lambda_{K3,n}, h_d) = \widetilde{O}(\Lambda_B),$$

by [GHS1, Prop. 3.12]; hence, by [GHS3, Lemma 3.2],

$$[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)] = [O(\Lambda_B) : \widetilde{O}(\Lambda_B)] = |O(D(\Lambda_B))|.$$

Recall that $D(\Lambda_B)$ is the direct sum of the cyclic groups $p(H)$ and T of orders $2d/f$, resp. $(2n-2)/f$ (cf. (3.8) in Ch. 3). Since $f = n-1$

or $f = 2n-2$, the group $O(T)$ is trivial. The order $|O(p(H))|$ equals $2^{\rho(2d/f)+\delta_1}$ – cf. [GHS1, Prop. 3.12]. Hence

$$|O(D(\Lambda_B))| = |O(p(H))| \cdot |O(T)| = 2^{\rho(2d/f)+\delta_1}.$$

- if $\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1$ and $f = d$ or $f = 2d$, then

$$P\widehat{O}(\Lambda_{K3,n}, h_d) = P\widetilde{O}(\Lambda_B)$$

(cf. (3.15) in Ch. 3); hence, by [GHS3, Lemma 3.2],

$$[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)] = [O(\Lambda_B) : \widetilde{O}(\Lambda_B)] = |O(D(\Lambda_B))|.$$

Since $f = d$ or $f = 2d$, the group $O(p(H))$ is trivial. The order $|O(T)|$ equals $2^{\rho((2n-2)/f)+\delta_2}$ – cf. [GHS1, Prop. 3.12]. Hence

$$|O(D(\Lambda_B))| = |O(p(H))| \cdot |O(T)| = 2^{\rho((2n-2)/f)+\delta_2}.$$

- if $f = 1$ or $f = 2$, and $f \neq d, 2d, n-1, 2n-2$, then $\widetilde{O}(\Lambda_B)$ is an index 2 subgroup of $\widehat{O}(\Lambda_{K3,n}, h_d)$ – cf. Thm. 3.1. Hence

$$\begin{aligned}[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)] = |O(D(\Lambda_B))|/2 &= |O(\langle p(H)\rangle)| \cdot |O(T)|/2 \\ &= 2^{\rho(2d/f)+\rho((2n-2)/f)+\delta_1+\delta_2-1}.\end{aligned}$$

- otherwise, $\widetilde{O}(\Lambda_B)$ coincides with $\widehat{O}(\Lambda_{K3,n}, h_d)$ – cf. Thm. 3.1. Hence

$$\begin{aligned}[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)] = |O(D(\Lambda_B))| &= |O(\langle p(H)\rangle)| \cdot |O(T)| \\ &= 2^{\rho(2d/f)+\rho((2n-2)/f)+\delta_1+\delta_2}.\end{aligned}$$

■

Note that if $\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) = 1$, then

$$\delta_B = \left(\frac{2d}{f}, \frac{2n-2}{f}\right);$$

moreover,

$$\Delta_B = \widetilde{\Delta}_B \delta_B^2 = \frac{4d(n-1)}{f^2}.$$

This implies the following elementary identity on numbers of prime divisors:

$$\rho(2d/f) + \rho((2n-2)/f) = \rho(\delta_B) + \rho(\widetilde{\Delta}_B) + \rho(\delta_B / \gcd(\delta_B, \widetilde{\Delta}_B)). \quad (4.6)$$

Hence, the expression (4.5) simplifies to

Corollary 4.3

$$\begin{aligned} \mathrm{vol}_{HM}(\widehat{O}^+(\Lambda_{K3,n}, h_d)) &= 2^{\rho(\delta_B/\gcd(\delta_B,\widetilde{\Delta}_B))+\delta_1+\delta_2-22} |\Delta_B|^{21/2} \delta_B^{-1} \pi^{-11} \Gamma(11) \\ &\quad \cdot L(11, \left(\frac{\Delta_B}{*}\right)) \frac{|B_2 B_4 ... B_{20}|}{20!!} C_2(\Lambda_B) \cdot \prod_{p|\delta_B} (1 + p^{-10}) \\ &\quad \cdot \prod_{p|\delta_B, p\nmid\widetilde{\Delta}_B} (1 + \left(\frac{\widetilde{\Delta}_B}{p}\right) p^{-1}) P_p(1)^{-1}. \end{aligned} \quad (4.7)$$

If $\left(f, \left(\frac{2d}{f}, \frac{2n-2}{f}\right)\right) > 1$, then $\widetilde{O}(\Lambda_B)$ is a finite index subgroup of $\widehat{O}(\Lambda_{K3,n}, h_d)$ and the index $[PO(\Lambda_B) : P\widehat{O}^+(\Lambda_{K3,n}, h_d)]$ is a divisor of

$$[PO(\Lambda_B) : P\widetilde{O}^+(\Lambda_B)] = |O(D(\Lambda_B))|.$$

Chapter 5

Hodge Classes

The failure of 'naive' global Torelli for higher-dimensional IS manifolds of $K3^{[n]}$-type gives rise to certain Hodge classes in the cohomology of a product of two IS manifolds. These classes do not come from parallel transport, and thus are not induced by a bimeromorphic map between the two manifolds – in this chapter we show that, nonetheless, these classes are algebraic in some cases.

5.1 Preliminaries

Let S be a K3 surface. Let $v \in \widetilde{H}(S, \mathbb{Z})$ be a primitive, effective class of degree $(v, v) = 2n - 2$.

In the following, $\mathcal{M}$ will denote either a moduli space of H-stable sheaves $\mathcal{M}_H(v)$ for a v-generic polarization H on S, or a moduli space of σ-stable objects $\mathcal{M}_\sigma(v)$ for some v-generic stability condition σ on S. Denote the projection maps from $S \times \mathcal{M}$ to S, resp. $\mathcal{M}$ by p, resp. q. Let

$$\widetilde{\varphi} : \widetilde{H}(S, \mathbb{Q}) \to H^2(\mathcal{M}, \mathbb{Q})$$

denote the Mukai homomorphism (1.10).

Let $O^+(\widetilde{H}(S, \mathbb{Z}), v)$ denote the subgroup of orientation-preserving isometries, fixing v. We would like to normalize the Mukai kernel in a way, which is compatible with a certain group operation, coming from the monodromy representation constructed by E. Markman. First we recall the theory:

Theorem 5.1.1 ([Mar5, Thm. 1.6])
There exists a natural homomorphism

$$mon : O^+(\widetilde{H}(S,\mathbb{Z}), v) \to GL(H^*(\mathcal{M},\mathbb{Z})).$$

Its image is a finite index subgroup of $Mon(\mathcal{M})$*.*

Remark. In fact, the image of mon is an index 2 subgroup of the monodromy group, for $n > 2$, by [Mar4, Thm. 1.2; Prop. 1.9] and Verbitsky's Global Torelli Theorem; the homomorphism mon is surjective for $n = 2$, by the paragraph preceding [Mar5, Thm. 1.6].

Put $ch(e_v) := ch(\mathcal{E})$, if a universal family exists on $S \times \mathcal{M}$. Otherwise, let $ch(e_v)$ denote the universal class, constructed in [Mar7]. Denote the subgroup of $\mathrm{Mon}(\mathcal{M})$ which is the image of mon by N. Now put

$$\eta := \widetilde{\varphi}(\mathcal{D}_S(v))/(v,v) \in H^2(\mathcal{M},\mathbb{Q})$$

and

$$\kappa := ch(e_v) \cup p^* \sqrt{\mathrm{td}(S)} \cup q^* \exp(\eta),$$

and define $\varphi : \widetilde{H}(S,\mathbb{Q}) \to H^2(\mathcal{M},\mathbb{Q})$ by

$$\varphi : \alpha \mapsto \pi_{H^2}\{q_*(\kappa \cup p^*\mathcal{D}_S(\alpha))\} \tag{5.1}$$

The map φ has the following properties:

Proposition 5.1.2
i) ([Mar5, Thm. 3.10])*Let* $O^+(K(S), v)$ *act on*

$$H^*(S \times \mathcal{M},\mathbb{Q}) \cong \widetilde{H}(S,\mathbb{Q}) \otimes H^*(\mathcal{M},\mathbb{Q})$$

by $g \in O^+(\widetilde{H}(S,\mathbb{Z}), v)$ *on the first factor, and by* $mon(g)$ *on the second factor. Then the class* κ *is* $O^+(\widetilde{H}(S,\mathbb{Z}), v)$*-invariant;*

ii) (cf. [Mar5, Ch. 3]) *The restriction of* φ *to the sub-Hodge structure* $v^\perp \subset \widetilde{H}(S,\mathbb{Z})$ *is a Hodge isometry*

$$\varphi|_{v^\perp} : v^\perp \to H^2(\mathcal{M},\mathbb{Z}) \subset H^2(\mathcal{M},\mathbb{Q})$$

which coincides with the Mukai isometry (1.11) from Thm. 1.3.4.

By composing the adjoint map to φ with the (Mukai) orthogonal projection $\widetilde{H}(S,\mathbb{Q}) \to v_{\mathbb{Q}}^\perp$ (whose kernel is given by the algebraic class $[\Delta_S] + (\mathcal{D}_S(v))/(v,v)) \otimes v$ on $S \times S$), we obtain a map

$$\varphi^\dagger : H^2(\mathcal{M},\mathbb{Q}) \to v_{\mathbb{Q}}^\perp \subset \widetilde{H}(S,\mathbb{Q}) \tag{5.2}$$

which is right inverse to φ and is given by an algebraic correspondence (cf. [Schl, Ch. 2]).

5.2 Main Statements

Suppose that X is isomorphic to $\mathcal{M}_1 := \mathcal{M}_{\sigma_1}(v_1)$ and Y is isomorphic to $\mathcal{M}_2 := \mathcal{M}_{\sigma_2}(v_2)$. Here $\mathcal{M}_{\sigma_i}(v_i), i = 1,2$ denote moduli spaces of σ_i-stable objects in $\mathcal{D}^b(S_i)$, with Mukai vectors $v_i \in \widetilde{H}(S_i,\mathbb{Z})$ such that $(v_i, v_i) > 0$. Furthermore, suppose that we are given a Hodge isometry

$$\psi : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z}). \tag{5.3}$$

We aim to show that ψ is induced by an algebraic class. Of course, this is a trivial consequence of Thm. 1.2.5, if ψ is a parallel-transport operator, because then ψ will be induced by a bimeromorphic map between X and Y. However, there are examples of Hodge isometries, which are not parallel-transport operators – cf. e.g. [Mar4, Ch. 4]. Examples also arise by considering points over the same period, but lying in different components of the moduli space of polarized IS manifolds of fixed polarization type – cf. Ch. 2.

The first case we consider is the case in which the underlying surfaces are *elliptic*. Recall the following definition:

Definition 5.2.1

A K3 surface S is called **elliptic***, if there exists a fibration $S \to \mathbb{P}^1$, whose generic fiber is a smooth curve of genus one.*

We do not require the existence of a section in the above definition. Now we can prove the following proposition:

Proposition 5.2.2

Assume that S_1 is elliptic. Then ψ is induced by an algebraic correspondence which is a composition of rational Hodge isometries.

PROOF.

Since $\widetilde{H}(S_i, \mathbb{Q}) \cong v_i^{\perp} \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Q}v_i$ (recall that $(v_i, v_i) > 0$), there is a unique extension of ψ to a map of rational vector spaces

$$\theta : \widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_2, \mathbb{Q}),$$

such that $\theta(v_1) = v_2$ and such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{H}(S_1,\mathbb{Q}) \arrow[r, "\theta"] & \widetilde{H}(S_2,\mathbb{Q}) \arrow[d, "\varphi_2"] \\ H^2(\mathcal{M}_1,\mathbb{Q}) \arrow[u, "\varphi_1^{\dagger}"] \arrow[r, "\psi"] & H^2(\mathcal{M}_2,\mathbb{Q}) \end{tikzcd}$$

The map θ is a rational Hodge isometry, because $\psi = \theta|_{v_1^{\perp}}$ is, and because

$$(v_1, v_1) = (\theta(v_1), \theta(v_1)) = (v_2, v_2) = 2n - 2.$$

Since S_1 admits the structure of an elliptic fibration, there is an effective isotropic vector (given by the class of an elliptic fiber) in $H^{1,1}(S_1)$ by [Huy4, Prop. 11.1.3]. Then [Nik2, Thm. 3] implies that the isometry ψ is induced by an algebraic class $\alpha \in H^{2,2}(S_1 \times S_2, \mathbb{Q})$. Hence ψ is also algebraic, since its kernel is the convolution of the algebraic correspondences inducing $\varphi_1^{\dagger}$, θ, and φ_2. ■

REMARK. In particular, the above proposition holds for all K3 surfaces of

Picard number greater than, or equal to five, as these always admit an effective isotropic divisor – cf. [Huy4, Prop. 11.1.3]. In his Beijing ICM address, Mukai claimed to have a proof of Shafarevich's conjecture for any projective $K3$ surface, but it has not been published ([Muk3]). The conjecture states that any rational Hodge isometry $\widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_2, \mathbb{Q})$ is induced by an algebraic class. Thus it could be possible to remove the hypothesis on S_1 being an elliptic fibration in the above theorem.

In the following statements we drop the assumption that the underlying $K3$ surfaces are elliptic. This can be done at the expense of considering special Mukai vectors v_i. So we fix two K3 surfaces S_1 and S_2 and let $\mathcal{M}_1$, $\mathcal{M}_2$ and ψ be defined as in the beggining of this section – cf. (5.3). We consider the following special case first:

Theorem 5.2.3
Suppose that $c_1(v_i) \in H^{1,1}(S_i, \mathbb{Z})$ vanish for $i = 1, 2$. Then the isometry ψ is induced by an algebraic correspondence, i.e. it is given by a cohomological correspondence, whose kernel is an algebraic class.

Proof. In the following we sometimes abuse notation and write the same symbol for a map of abelian groups and for its scalar extension to a homomorphism of rational vector spaces. Let

$$\iota_1 : H^2(S_1, \mathbb{Z}) \hookrightarrow \widetilde{H}(S_1, \mathbb{Z})$$

be the natural embedding into the full cohomology ring. Then we can consider morphisms

$$\varphi_i : \widetilde{H}(S_i, \mathbb{Q}) \to H^2(\mathcal{M}_i, \mathbb{Q}), i = 1, 2,$$

given by normalized kernels, as in (5.1). By Thm. 5.1.2, φ_i are Hodge isometries, when restricted to the sub-Hodge structures $v_i^\perp \subset \widetilde{H}(S_i, \mathbb{Z}), i = 1, 2$. Denote by

$$\varphi_i^\dagger : H^2(\mathcal{M}_i, \mathbb{Q}) \to \widetilde{H}(S_i, \mathbb{Q})$$

the respective right inverses – cf. (5.2).

The condition $c_1(v_1) = 0$ implies that $\iota_1(H^2(S_1,\mathbb{Z}))$ and v_1 are orthogonal with respect to the Mukai pairing on $\widetilde{H}(S_1,\mathbb{Z})$, i.e. ι_1 embeds the lattice $H^2(S_1,\mathbb{Z})$ into the sublattice $v_1^\perp \subset \widetilde{H}(S_1,\mathbb{Z})$. Since the restriction of φ_1 to $v_1^\perp$ is a Hodge isometry onto $H^2(\mathcal{M}_1,\mathbb{Z})$, we obtain a Hodge isometric embedding

$$\varphi_1 \circ \iota_1 : H^2(S_1,\mathbb{Z}) \hookrightarrow H^2(\mathcal{M}_1,\mathbb{Z}).$$

Since there is only one $OH^2(\mathcal{M}_1,\mathbb{Z})$-orbit of primitive isometric embeddings of the $K3$-lattice in $H^2(\mathcal{M}_1,\mathbb{Z})$, the embedding splits $H^2(\mathcal{M}_1,\mathbb{Z})$ into an orthogonal direct sum

$$\varphi_1 \circ \iota_1(H^2(S_1,\mathbb{Z})) \oplus \langle w_1 \rangle,$$

where $w_1 \in H^2(\mathcal{M}_1,\mathbb{Z})$ is a primitive element of length $(w_1,w_1) = 2-2n$.

Consider the Hodge isometry ψ (5.3). We obtain a Hodge isometric embedding

$$\widetilde{\iota}_1 := \varphi_2^\dagger \circ \psi \circ \varphi_1 \circ \iota_1 : H^2(S_1,\mathbb{Z}) \to \widetilde{H}(S_2,\mathbb{Z}).$$

Now the K3 lattice has only one $O(\widetilde{H}(S_2,\mathbb{Z}))$-orbit of primitive isometric embeddings into the Mukai lattice. Hence $\widetilde{H}(S_2,\mathbb{Z})$ splits as a direct sum

$$\widetilde{\iota}_1(H^2(S_1,\mathbb{Z})) \oplus U_2,$$

where U_2 is isometric to the hyperbolic lattice U. Furthermore, we have a splitting

$$\widetilde{H}(S_1,\mathbb{Z}) = \iota_1(H^2(S_1,\mathbb{Z})) \oplus U_1,$$

induced by ι_1; here U_1 is also isometric to U. Hence we can fix an isometric embedding

$$\beta : U_1 \to U_2 \subset \widetilde{H}(S_2,\mathbb{Z})$$

and extend $\widetilde{\iota}_1$ to a Hodge isometry

$$\theta := \widetilde{\iota}_1 \oplus \beta : \widetilde{H}(S_1,\mathbb{Z}) \to \widetilde{H}(S_2,\mathbb{Z}).$$

In particular, this shows that the surfaces S_1 and S_2 are derived-equivalent, by Orlov's theorem ([Or]). The situation is summarized in the following diagram:

$$\begin{tikzcd}
H^2(S_1,\mathbb{Z}) \arrow[r, hook, "\iota_1"] & \widetilde{H}(S_1,\mathbb{Z}) \arrow[r, "\theta"] & \widetilde{H}(S_2,\mathbb{Z}) \\
& & \cup \\
& & v_2^{\perp} \arrow[d, "\varphi_2"] \\
& H^2(\mathcal{M}_1,\mathbb{Z}) \arrow[uuu, "\varphi_1^{\dagger}"] \arrow[r, "\psi"] & H^2(\mathcal{M}_2,\mathbb{Z})
\end{tikzcd}$$

Now consider the homomorphism

$$\widetilde{\psi} := \varphi_2 \circ \theta \circ \varphi_1^{\dagger} : H^2(\mathcal{M}_1,\mathbb{Z}) \to H^2(\mathcal{M}_2,\mathbb{Z}).$$

Let us show that

$$\psi \circ \varphi_1 \circ \iota_1 = \widetilde{\psi} \circ \varphi_1 \circ \iota_1;$$

in other words the two maps coincide, when restricted to the sub-Hodge structure $\varphi_1 \circ \iota_1(H^2(S_1,\mathbb{Z})) \subset H^2(\mathcal{M}_1,\mathbb{Z})$:

$$\begin{aligned}
\widetilde{\psi} \circ \varphi_1 \circ \iota_1 &= \varphi_2 \circ \theta \circ (\varphi_1^{\dagger} \circ \varphi_1 \circ \iota_1) \\
&\quad \text{(by the definition of } \widetilde{\psi}) \\
&= \varphi_2 \circ (\theta \circ \iota_1) \\
&\quad \text{(because } \varphi_1^{\dagger} \circ \varphi_1 \text{ restricts to the identity on } v_1^{\perp}) \\
&= \varphi_2 \circ \widetilde{\iota}_1 \\
&\quad \text{(by the definition of } \theta) \\
&= (\varphi_2 \circ \varphi_2^{\dagger}) \circ \psi \circ \varphi_1 \circ \iota_1 \\
&\quad \text{(by the definition of } \widetilde{\iota}_1) \\
&= \psi \circ \varphi_1 \circ \iota_1 \\
&\quad \text{(because } \varphi_2^{\dagger} \text{ is a right inverse to } \varphi_2).
\end{aligned} \tag{5.4}$$

The sublattice $U_1 \subset \widetilde{H}(S_1,\mathbb{Z})$ has a basis e_1, e_2 given by the standard generators of $H^0(S_1,\mathbb{Z}) \cong \mathbb{Z}$ and $H^4(S_1,\mathbb{Z}) \cong \mathbb{Z}$. Then $\theta(e_1), \theta(e_2)$ is a basis for $U_2 \subset \widetilde{H}(S_2,\mathbb{Z})$ and we can write $v_2 \in U_2$ as

$$v_2 = r_1\theta(e_1) - s_1\theta(e_2),$$

where r_1 and s_1 are coprime integers satisfying $|r_1 s_1| = n - 1$. Put

$$\widetilde{w}_2 = r_1\theta(e_1) + s_1\theta(e_2).$$

The vector $\widetilde{w}_2$ generates the sublattice $v_2^\perp \subset U_2$. Since

$$\widetilde{w}_1 := \theta \circ \varphi_1^\dagger(w_1) \in U_2$$

is a primitive vector of length $2 - 2n$, we can write it as

$$\widetilde{w}_1 = r_2\theta(e_1) + s_2\theta(e_2),$$

where r_2 and s_2 are coprime integers satisfying $|r_2 s_2| = n - 1$. Now the map

$$\varphi_2 : \widetilde{H}(S_2,\mathbb{Q}) \to H^2(\mathcal{M}_2,\mathbb{Q})$$

has kernel $\mathbb{Q}v_2$.

Hence

$$\varphi_2(\theta(e_1)) = \varphi_2(\frac{v_2 + \widetilde{w}_2}{2r_1}) = \frac{1}{2r_1}\varphi_2(\widetilde{w}_2)$$

and

$$\varphi_2(\theta(e_2)) = \varphi_2(\frac{\widetilde{w}_2 - v_2}{2s_1}) = \frac{1}{2s_1}\varphi_2(\widetilde{w}_2).$$

In particular,

$$\widetilde{\psi}(w_1) = \varphi_2(\widetilde{w}_1) = (\frac{r_2}{2r_1} + \frac{s_2}{2s_1})\varphi_2(\widetilde{w}_2).$$

Now since $\varphi_2(\widetilde{w}_2)$ generates $\widetilde{\psi} \circ \varphi_1 \circ \iota_1(H^2(S_1,\mathbb{Z}))^\perp$ and $\psi(w_1)$ generates $\psi \circ \varphi_1 \circ \iota_1(H^2(S_1,\mathbb{Z}))^\perp$, we obtain that $\varphi_2(\widetilde{w}_2) = \pm\psi(w_1)$. In the following, we modify the homomorphism $\widetilde{\psi}$ to a homomorphism which coincides with ψ. Let

$$[\Delta_{S_1}] = \pi_0 + \pi_2 + \pi_4$$

be the Künneth decomposition of the cohomology class, Poincaré dual to the diagonal in $S_1 \times S_1$. By the Standard Conjectures for surfaces, the classes $\pi_i, i = 0, 2, 4$ are algebraic. Given $a, b \in \mathbb{Q}$, define the algebraic class

$$[\Delta_{S_1}(a,b)] := a\pi_0 + \pi_2 + b\pi_4, \tag{5.5}$$

and let

$$\Phi_{[\Delta_{S_1}(a,b)]} : \widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_1, \mathbb{Q})$$

be the corresponding homomorphism, induced by the algebraic correspondence $[\Delta_{S_1}(a,b)]$. The map $\Phi_{[\Delta_{S_1}(a,b)]}$ sends a class $(p, l, q) \in \widetilde{H}(S_1, \mathbb{Q})$ to the class (ap, l, bq). Since r_2 and s_2 are coprime, we can choose u, v satisfying $r_2 u + s_2 v = 1$ and put $a = 2r_1 u, b = 2s_1 v$. Now consider the morphism

$$\psi' := \varphi_2 \circ \theta \circ \Phi_{[\Delta_{S_1}(a,b)]} \circ \varphi_1^{\dagger}.$$

We obtain the following equality:

$$\begin{aligned}
\psi' \circ \varphi_1 \circ \iota_1 &= \varphi_2 \circ \theta \circ \Phi_{[\Delta_{S_1}(a,b)]} \circ (\varphi_1^{\dagger} \circ \varphi_1 \circ \iota_1) \\
&\quad \text{(by the definition of } \psi') \\
&= \varphi_2 \circ \theta \circ (\Phi_{[\Delta_{S_1}(a,b)]} \circ \iota_1) \\
&\quad \text{(because } \varphi_1^{\dagger} \circ \varphi_1 \text{ restricts} \\
&\quad \text{to the identity on } v_1^{\perp}) \\
&= \varphi_2 \circ \theta \circ \iota_1 \\
&\quad \text{(because } \Phi_{[\Delta_{S_1}(a,b)]} \text{ restricts} \\
&\quad \text{to the identity on } \iota_1(H^2(S_1, \mathbb{Z}))) \\
&= \widetilde{\psi} \circ \varphi_1 \circ \iota_1 \text{(by (5.4))}.
\end{aligned} \tag{5.6}$$

In particular, the restrictions of $\widetilde{\psi}$ and ψ' to the sub-Hodge structure $\varphi_1 \circ \iota_1(H^2(S_1, \mathbb{Z})) \subset H^2(\mathcal{M}_1, \mathbb{Z})$ coincide. Furthermore:

$$
\begin{aligned}
\psi'(w_1) &= \varphi_2 \circ \theta \circ \Phi_{[\Delta_{S_1}(a,b)]} \circ \varphi_1^\dagger(w_1) \\
&= \varphi_2 \circ \theta \circ \Phi_{[\Delta_{S_1}(a,b)]}(r_2e_1 + s_2e_2) \\
&= \varphi_2 \circ \theta(ar_2e_1 + bs_2e_2) \\
&= ar_2\varphi_2(\theta(e_1)) + bs_2\varphi_2(\theta(e_2)) \\
&= (\frac{ar_2}{2r_1} + \frac{bs_2}{2s_1})\varphi_2(\widetilde{w}_2) \\
&= (\frac{2r_1ur_2}{2r_1} + \frac{2s_1vs_2}{2s_1})\varphi_2(\widetilde{w}_2) \\
&= (ur_2 + vs_2)\varphi_2(\widetilde{w}_2) = \varphi_2(\widetilde{w}_2) = \psi(w_1).
\end{aligned} \tag{5.7}
$$

Note that the map θ is a Hodge isometry between $\widetilde{H}(S_1, \mathbb{Z}) \to \widetilde{H}(S_2, \mathbb{Z})$; it follows from a theorem of Ploog ([Pl]) that either

$$\theta = \Phi, \text{ or } \theta = \mathcal{D}_{S_1} \circ \Phi,$$

where Φ is an isometry induced by a derived equivalence between S_1 and S_2 and $\mathcal{D}_{S_1}$ is the duality operator on $\widetilde{H}(S_1, \mathbb{Z})$. This implies that θ also has an algebraic kernel. Furthermore, the map ψ' is given by an algebraic correspondence, since it is the convolution of the correspondences $\varphi_1^\dagger$, $\Phi_{[\Delta_{S_1}(a,b)]}$, θ and φ_2. Finally, (5.4)-(5.7) imply that the maps ψ' and ψ coincide and we are done. ■

Remark. Examples of a Hodge isometry ψ, which is not a parallel-transport operator (and hence is NOT induced by a birational map between X and Y) are constructed in [Mar4, Ch. 4]. There $S_1 = S_2 = S$, $X = \mathcal{M}_H(v_1)$, $Y = \mathcal{M}_H(v_2)$, where $v_1 = (r_1, 0, s_1) \in \widetilde{H}(S, \mathbb{Z})$, $v_2 = (r_2, 0, s_2) \in \widetilde{H}(S, \mathbb{Z})$ and $n - 1 = r_1s_1 = r_2s_2$; in addition, the numbers r_i, s_i satisfy $\gcd(r_1, s_1) = \gcd r_2, s_2 = 1$, $r_1 < s_1, r_2 < s_2$, and $r_1 \neq r_2$. The above theorem shows that such isometries are induced by algebraic classes.
As pointed to us by D. Ploog, it is an interesting open question whether moduli spaces, which parametrize objects with compatible Mukai vectors on

two derived equivalent surfaces, are also derived equivalent. If this were true, then Thm. 5.2.3 would be a trivial consequence of it.

Next we consider examples of Mukai vectors with $c_1(v_i) \neq 0$. We need to fix some data first. Start with an integral Hodge isometry between the cohomology lattices of two derived equivalent K3 surfaces S_1 and S_2

$$\theta : \widetilde{H}(S_1, \mathbb{Z}) \to \widetilde{H}(S_2, \mathbb{Z}). \tag{5.8}$$

Let $H \in H^{1,1}(S_1, \mathbb{Z})$ be a primitive element, satisfying $(H, H) \neq 0$ and put $2l := (H, H)$. Fix integers r, L, S, satisfying the following conditions:

$$r > 1, (L, r) = (S, r) = 1, rl - LS > 0.$$

Put

$$m := rl - LS, P := L - mr$$

and note that

$$(P, r) = (L - mr, r) = (L, r) = 1, m \equiv -LS \pmod{r}.$$

In particular P is invertible modulo r and we can fix an integer s, satisfying

$$s \equiv -P^{-1} \pmod{r},$$

which is equivalent to $r \mid Ps + 1$. Note also that

$$S \equiv -mL^{-1} \equiv -mP^{-1} \equiv ms \pmod{r}.$$

Hence $r \mid S - ms$ and we can put $Q := \frac{S-ms}{r}$. Let Λ_T be a rank two lattice with Gram matrix

$$T := \begin{pmatrix} 2s & rQ + \frac{Ps+1}{r} \\ rQ + \frac{Ps+1}{r} & 2PQ \end{pmatrix} \tag{5.9}$$

with respect to some fixed basis $\{h_1, h_2\}$. A simple computation shows that the vector $h := mh_1 + h_2$ has length $(h, h)_T = 2l$. Note that, by [Nik1, Thm.

1.2], there always exists a primitive isometric embedding

$$\kappa : \Lambda_T \hookrightarrow H^2(S_1, \mathbb{Z})$$

(independently of the signature of Λ_T). Moreover, since $OH^2(S_1, \mathbb{Z})$ acts transitively on the primitive vectors of length $2l$, we may assume w.l.o.g. that $\kappa(h) = H$. Now put

$$H_1 := \kappa(h_1), H_2 := \kappa(h_2).$$

Let U be a hyperbolic lattice with basis $\{e_1, e_2\}$. Then it follows immediately from the definition of r, s, P, Q, H_1, H_2 that the map

$$j : U \longrightarrow \widetilde{H}(S_1, \mathbb{Z}), e_1 \mapsto (r^2, rH_1, s), e_2 \mapsto (P, H_2, Q)$$

defines a primitive isometric embedding. Moreover the vector

$$v_1 := (rL, rH, S) = mj(e_1) + rj(e_2)$$

is effective (in the sense of Def. 1.3.1), of square $(v_1, v_1) = 2mr$. Then the moduli space $\mathcal{M}_1 := \mathcal{M}_\sigma(v_1)$ is a smooth IS manifold of dimension $2mr+2$, for some stability condition σ in $\mathrm{Stab}(S_1)$. Now put

$$v_2 = \theta \circ \Phi_{[\Delta_{S_1}(\frac{1}{r}, r)]}(v).$$

Now v_2 is integral, because

$$\begin{aligned}
v_2 &= \theta(\Phi_{[\Delta(\frac{1}{r},r)]}(mj(e_1) + rj(e_2))) \\
&= m\theta(\Phi_{[\Delta(\frac{1}{r},r)]}((r^2, rH_1, s))) + r\theta(\Phi_{[\Delta(\frac{1}{r},r)]}((P, H_2, Q))) \\
&= m\theta((r, rH_1, rs)) + r\theta((\frac{P}{r}, H_2, rQ)) \\
&= mr\theta((1, H_1, s)) + \theta((P, rH_2, r^2Q)).
\end{aligned}$$

It is also primitive in the hyperbolic sublattice of $\widetilde{H}(S_2, \mathbb{Z})$, spanned by $\theta((1, H_1, s))$ and $\theta((P, rH_2, r^2Q))$. Since the latter sublattice is a direct summand in $\widetilde{H}(S_2, \mathbb{Z})$, v_2 is also primitive in $\widetilde{H}(S_2, \mathbb{Z})$. Furthermore, v_2 is effective of square $(v_2, v_2) = 2mr$ and the moduli space $\mathcal{M}_2 := \mathcal{M}_\tau(v_2)$ is a smooth IS manifold of dimension $2mr + 2$, for some stability condition τ in $\mathrm{Stab}(S_2)$. Now let ψ denote the map

$$\psi := \varphi_2 \circ \theta \circ \Phi_{[\Delta_{S_1}(\frac{1}{r}, r)]} \circ \varphi_1^\dagger : H^2(\mathcal{M}_1, \mathbb{Z}) \to H^2(\mathcal{M}_2, \mathbb{Z}).$$

The map ψ is induced by an algebraic class, by definition. Furthermore, we have the following proposition:

Proposition 5.2.4

The map ψ is an integral isometry, which is not a parallel transport operator.

PROOF. Put $\theta_1 := \theta \circ \Phi_{[\Delta_{S_1}(r, \frac{1}{r})]}$. To show that ψ is integral, it is enough to show that the rational isometry

$$\theta_1 : \widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_2, \mathbb{Q})$$

restricts to an integral primitive isometric embedding

$$v_1 \oplus v_1^\perp \hookrightarrow \widetilde{H}(S_2, \mathbb{Z}).$$

That $v_2 = \theta_1(v_1)$ is integral and primitive was already shown above. (*) Furthermore, we have that $v_1^\perp = j(U)^\perp \oplus \langle w \rangle$, where $w := mj(e_1) - rj(e_2)$. Analogously to (*), θ_1 restricts to an integral primitive embedding on the rank one sublattice $\langle w \rangle$. (**)

Now let (a, G, b) be an element of $j(U)^\perp$. Since (a, G, b) is orthogonal to $j(e_1)$, we obtain the relation

$$r(G, H_1) = as + br^2.$$

This immediately implies that $a = ra_1$ for some integer a_1, because s and r are coprime. Hence

$$\begin{aligned} \theta_1((a, G, b)) &= \theta(\Phi_{[\Delta(\frac{1}{r}, r)]}((ra_1, G, b))) \\ &= \theta(a_1, G, rb), \end{aligned}$$

i.e. the restriction of θ_1 to $j(U)^\perp$ is integral. It is also primitive by the unimodularity of $j(U)^\perp \cong \Lambda_{K3}$. (***)

(*)-(***) imply that the restriction of θ_1 is to $v_1 \oplus v_1^\perp$ is a primitive isometric embedding into $\widetilde{H}(S_2, \mathbb{Z})$.

For the second part, assume that ψ is a parallel transport operator. Then, by [Mar1, Cor 7.4] there exists an integral isometry

$$\Psi : \widetilde{H}(S_1, \mathbb{Z}) \to \widetilde{H}(S_2, \mathbb{Z})$$

such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H}(S_1, \mathbb{Z}) \arrow[r, "\Psi"] & \widetilde{H}(S_2, \mathbb{Z}) \\
H^2(\mathcal{M}_1, \mathbb{Z}) \arrow[u, "\varphi_1^\dagger"] \arrow[r, "\psi"] & H^2(\mathcal{M}_2, \mathbb{Z}) \arrow[u, "\varphi_2^\dagger"']
\end{tikzcd}$$

Note that

$$\Psi(v_1) = \pm v_2, \tag{5.10}$$

because they both generate $\varphi_2^\dagger(H^2(\mathcal{M}_2, \mathbb{Z}))^\perp$. But

$$\Psi(v_1) = r\Psi(j(e_1)) + m\Psi(j(e_2)),$$

whereas

$$v_2 = mr\theta((1, H_1, s)) + \theta((P, rH_2, r^2Q)).$$

This contradicts (5.10), because both pairs $\{\Psi(j(e_1)), \Psi(j(e_2))\}$ and $\{\theta((1, H_1, s)), \theta((P, rH_2, r^2Q))\}$ are standard integral bases of the hyperbolic sublattice $\Psi(j(U)) \subset \widetilde{H}(S_2, \mathbb{Z})$ and, up to sign and permutation, the vector $\Psi(v_1)$ has the same coordinates with respect to any standard integral basis of $\Psi(j(U))$.

■

In the following proposition we express the maps $\Phi_{\Delta_{S(a,b)}}$ in terms of Todd classes.

Proposition 5.2.5
We have that

$$\Phi_{\Delta_S(a,b)} = \Phi_\Gamma,$$

where $\Gamma := [\Delta_S] + p^* \sqrt[a+1]{td_S} - q^* \sqrt[1-b]{td_S}$.

PROOF. Since

$$\sqrt{td_S} = (1,0,1) \in \widetilde{H}(S,\mathbb{Z}),$$

we have that $\sqrt[m]{td_S} = (1,0,m)$ for $m \in \mathbb{Q}$. It follows that

$$\begin{aligned}
\Phi_{p^* \sqrt[a+1]{td_S}}((r,H,s)) &= (s+(a+1)r,0,0),\\
\Phi_{q^* \sqrt[1-b]{td_S}}((r,H,s)) &= (s,0,(1-b)s).
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi_\Gamma((r,H,s)) &= (\mathrm{id} + \Phi_{p^* \sqrt[a+1]{td_S}} - \Phi_{q^* \sqrt[1-b]{td_S}})((r,H,s))\\
&= (r,H,s) + ((a-1)r+s,0,0) - (s,0,(1-b)s)\\
&= (ar,H,bs) = \Phi_{\Delta_S(a,b)}((r,H,s))
\end{aligned}$$

for all $(r,H,s) \in \widetilde{H}(S,\mathbb{Z})$. ■

The advantage of the cycle Γ is that it has an obvious lifting to a global algebraic cycle in any family $\mathcal{S} \times_B \mathcal{S} \longrightarrow B$ of pairs of K3 surfaces – in the above expression simply replace Δ_S by $\Delta_{\mathcal{S}}$, and $p^*\sqrt{td_S}, q^*\sqrt{td_S}$ by the pullbacks of the relative Todd cycles, associated to the family.

5.3 The case of twisted K3 surfaces

We would like to expand the previous discussion by considering pairs of derived equivalent twisted K3 surfaces (S_1,α_1) and (S_2,α_2) instead of the derived equivalent surfaces S_1 and S_2 in (5.8).

Let (X,α) be a twisted symplectic manifold and let $\mathcal{E}$ be an α-twisted sheaf of rank $r > 0$ on X. We make use of the normalized characteristic class $\kappa(\mathcal{E}) \in H^*(X,\mathbb{Q})$, defined in [Mar6, Ch. 2.2] as a formal r-th root of the chern character of the object $\mathcal{E}^r \otimes \det(\mathcal{E})^{-1}$ in $\mathcal{D}^b(X)$ – here $\mathcal{E}^r$ denotes the r-th derived tensor power of $\mathcal{E}$ as an object in $\mathcal{D}^b(X, \alpha^r)$. We start by a comparison statement between $\kappa(\mathcal{E})$ and the character $ch^B(\mathcal{E})$, associated to a B-field lift of α (cf. Eq. (1.12)).

Proposition 5.3.1

For every α-twisted sheaf $\mathcal{E}$ of rank r we have the equality

$$\kappa(\mathcal{E}) = ch^B(\mathcal{E})\exp(-c_1^B(\mathcal{E})/r).$$

Proof. This can be easily seen by using Heinloth's note on the computation of gerbe cohomology – cf. [Hei]. To each α-twisted sheaf, the author associates chern classes taking values in the cohomology group $H^*(X^\alpha, \mathbb{Q})$ of the gerbe X^α. Furthermore, if α is torsion, then, by [Hei, L. 1],

$$H^*(X^\alpha, \mathbb{Q}) \cong H^*(X, \mathbb{Q})[[z]],$$

where z is the first chern class of a (differentiable) bundle on X^α, or, equivalently, an α-twisted bundle on X of weight one. As shown in [Hei, Ch. 4], whenever the Dixmier-Douady class of α in $H^3(X, \mathbb{Z})$ vanishes (i.e. whenever α is topologically trivial), there exists a (differentiable) line bundle L_B on X^α, associated to a B-field lift of α and if we put $z := c_1(L_B)$, the $H^*(X, \mathbb{Q})$-valued chern character ch^B, defined by Huybrechts and Stellari satisfies

$$ch(F) = ch^B(F)\exp(z) = ch(F \otimes L_B^{-1})\exp(z)$$

for every α-twisted sheaf F on X. Now put $\tau := \alpha^r$; note that $\det(\mathcal{E})$ is an (algebraic) line bundle on X^τ, which means that $\tau = 1 \in H^2_{an}(X, \mathcal{O}_X^*)$, i.e. the cocycle, associated to X^τ is a coboundary and the gerbe X^τ is, in fact, in the class of trivial gerbes. Let us put

$$w := c_1(\det(\mathcal{E})) \in H^*(X^\tau, \mathbb{Q}), \widetilde{z} := c_1(L_B^r) \in H^*(X^\tau, \mathbb{Q}).$$

We have

$$\begin{aligned} ch(F) &= ch(F \otimes \det(\mathcal{E})^{-1}) \exp(w) \\ &= ch(F \otimes L_B^{-r}) \exp(\widetilde{z}) \\ &= ch^{rB}(F) \exp(\widetilde{z}) \end{aligned} \tag{5.11}$$

$$\begin{aligned} \exp(w) &= \exp(L_B^{-r} \otimes \det(\mathcal{E})) \exp(\widetilde{z}) \\ &= \exp(L_{rB}^{-1} \otimes \det(\mathcal{E})) \exp(\widetilde{z}) \\ &= \exp(c_1^{rB}(\det(\mathcal{E}))) \exp(\widetilde{z}) \\ &= \exp(c_1^B(\mathcal{E})) \exp(\widetilde{z}) \end{aligned} \tag{5.12}$$

Hence, if we substitute (5.12) in (5.11) and compare the $H^*(X, \mathbb{Q})$-valued coefficients of $\exp(w)$, we obtain

$$ch^{rB}(F) = ch(F \otimes \det(\mathcal{E})^{-1}) \exp(-c_1^B(\mathcal{E})),$$

for every τ-twisted sheaf F. In particular, if we consider the multi-Tor sheaves, associated to $\mathcal{E}$ and defined in [Mar6, Ch. 2.2], we obtain

$$\begin{aligned} \kappa^r(\mathcal{E}) &= \sum_{i=1}^{n} (-1)^i ch(Tor_i(\mathcal{E}, r) \otimes \det(\mathcal{E})^{-1}) \text{ (by def.)} \\ &= \sum_{i=1}^{n} (-1)^i ch^{rB}(Tor_i(\mathcal{E}, r)) \exp(-c_1^B(\mathcal{E})) \\ &= ch^{rB}(\mathcal{E}^r) \exp(-c_1^B(\mathcal{E})) \\ &= (ch_B(\mathcal{E}))^r \exp(-c_1^B(\mathcal{E})) \text{ (by [HSt1, Prop. 1.2.iii)]).} \end{aligned}$$

By using a suitable Taylor expansion, we can take r-th root on both sides of the above identity and obtain the desired result – note that units in the torsion-free ring $H^*(X, \mathbb{Q})$ have unique r-th roots. ∎

Now let (S_1, α_1) and (S_2, α_2) be two derived equivalent twisted K3 surfaces.

Choose two B-fields B_1 and B_2 and let

$$\theta : \widetilde{H}(S_1, B_1, \mathbb{Z}) \to \widetilde{H}(S_2, B_2, \mathbb{Z})$$

be an isometry induced by the class $ch^B(\mathcal{E})$ of a twisted universal sheaf of rank $r > 0$; here $B := B_1 \boxplus B_2$. Let

$$c_1^B(\mathcal{E}) = p^*g + q^*h$$

be the Künneth decomposition of $c_1^B(\mathcal{E}) \in H^2(S_1 \times S_2, B, \mathbb{Z})$. Furthermore, let

$$\theta_\kappa : \widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_2, \mathbb{Q})$$

be the rational isometry induced by the class $\kappa(\mathcal{E})p^*\sqrt{td_{S_1}}q^*\sqrt{td_{S_2}}$. Suppose that $v_1 \in \widetilde{H}(S_1, B_1, \mathbb{Z})$ is an effective Mukai vector of the form $v_1 = (rL, rH, S)$ – here L and S are fixed integers, satisfying the conditions from the previous section. Assume further that $r = \widetilde{r}^2$ and h have the form $r = \widetilde{r}^2$, $h = \widetilde{r}\widetilde{h}$, for some integer $\widetilde{r}$. Then the following holds:

Proposition 5.3.2

The rational isometry θ_κ restricts to an integral primitive isometric embedding

$$v_1 \oplus v_1^\perp \hookrightarrow \widetilde{H}(S_2, \mathbb{Z}).$$

.

Proof. By Prop. 5.3.1, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{H}(S_1, \mathbb{Q}) & \xrightarrow{\theta_\kappa} & \widetilde{H}(S_2, \mathbb{Q}) \\ \uparrow{\scriptstyle \exp(\frac{g}{r})} & & \uparrow{\scriptstyle \exp(\frac{-h}{r})} \\ H^2(S_1, B_1, \mathbb{Z}) & \xrightarrow{\theta} & H^2(S_2, B_2, \mathbb{Z}) \end{array}$$

Recall the following identity, which follows from the projection formula:

$$\begin{aligned} q_*(p^*\alpha \cup (p^*\beta \cup q^*\gamma)) = q_*(p^*(\alpha \cup \beta) \cup q^*\gamma)) &= \chi(\alpha \cup \beta).\gamma && (5.13) \\ &= -(\alpha, \mathcal{D}_{S_1}(\beta)).\gamma \text{ (cf. (1.5))} \end{aligned}$$

for any $\alpha, \beta \in H^*(S_1, \mathbb{Q}), \gamma \in H^*(S_2, \mathbb{Q})$.

Now, choose a basis $\{e_i\}, i = 1, ..., 22$ of $H^2(S_1, \mathbb{Z})$ and put

$$\begin{aligned}
\theta((0,0,1)) &=: (r_0, l_1, s);\\
\theta(e_i) &=: (r_i, h_i, s_i), i = 1, ..., 22;\\
\theta((1,0,0)) &=: (P, l_2, Q).
\end{aligned}$$

Denote by $[\beta]_j$ the j-th graded piece of a cohomology class β.

We obtain the following identities from (5.13)

$$\begin{aligned}
r_0 &= [q_*(p^*(0,0,1) \cup v^B(\mathcal{E}))]_0 =\\
&= [q_*(p^*(0,0,1) \cup q^*(\mathrm{rk}(\mathcal{E}), 0, 0))]_0 = r;\\
r_i &= [q_*(p^* e_i \cup v^B(\mathcal{E}))]_0\\
&= [q_*(p^* e_i \cup p^* g \cup q^*(1,0,0))]_0 = (e_i, g)\\
l_1 &= [q_*(p^*(0,0,1) \cup v^B(\mathcal{E}))]_2\\
&= [q_*(p^*(0,0,1) \cup q^* h)]_2 = h.
\end{aligned}$$

Note that $h^2 = 2rs$ because $\theta((0,0,1)) = (r,h,s)$ is isotropic. We can now express explicitly the map θ_κ in terms of the integral map θ:

$$\begin{aligned}
\theta_\kappa((0,0,1)) &= \exp(\frac{-h}{r}) \cup \theta(\exp(-\frac{g}{r}) \cup (0,0,1))\\
&= \exp(\frac{-h}{r}) \cup \theta((0,0,1)) = \exp(\frac{-h}{r}) \cup (r_0, l_1, s))\\
&= \exp(\frac{-h}{r}) \cup (r,h,s) = (r, h - r \cdot \frac{h}{r}, s + r \cdot \frac{h^2}{2r^2} - \frac{h^2}{r}) = (r,0,0)\\
\theta_\kappa(e_i) &= \exp\left(\frac{-h}{r}\right) \cup \theta(\exp(-\frac{g}{r}) \cup e_i)\\
&= \exp(\frac{-h}{r}) \cup \theta(e_i - (0,0,\frac{(e_i,g)}{r}))\\
&= \exp(\frac{-h}{r}) \cup (r_i, h_i, s_i) - \exp(\frac{-h}{r}) \cup \frac{r_i}{r}(r,h,s)\\
&= (r_i, h_i - r_i \cdot \frac{h}{r}, s_i + r_i \frac{h^2}{2r^2} - \frac{(h,h_i)}{r}) - (r_i, 0, 0)\\
&= (0, h_i - r_i \cdot \frac{h}{r}, s_i + r_i \frac{s}{r} - \frac{r_i s + r s_i}{r}) = (0, h_i - r_i \cdot \frac{h}{r}, 0)
\end{aligned}$$

Since θ_κ is a rational isometry and $\theta_\kappa((0,0,1)) = r(1,0,0)$, from the above expression we can deduce that $\theta_\kappa((1,0,0)) = \frac{1}{r}\widetilde{e}$, where $\widetilde{e}$ is a primitive integral isotropic vector. By assumption, r and h have the form $r = \widetilde{r}^2$, $h = \widetilde{r}\widetilde{h}$. Then the orthogonality relation

$$(\theta((0,0,1)), \theta(e_i)) = 0$$

implies that $\widetilde{r}(\widetilde{h}, h_i) = \widetilde{r}^2 s_i + r_i s$, which means that $\widetilde{r} \mid r_i$ (note that $(\widetilde{r}, s) = 1$ by the primitivity of $(\widetilde{r}^2, \widetilde{r}\widetilde{h}, s)$), i.e. the vectors $\theta_\kappa(e_i) = (0, h_i - \frac{r_i}{\widetilde{r}}\widetilde{h}, 0)$ are integral. From this we can conclude, as in the proof of Prop. 5.2.4, that θ_κ restricts to a primitive isometric embedding on $v_1 \oplus v_1^\perp$. ■

Now put $\mathcal{M}_1 := \mathcal{M}_\sigma(v_1)$ for some stability condition σ in $\mathrm{Stab}(S_1, \alpha_1)$, and let $v_2 \in \widetilde{H}(S_2, \mathbb{Q})$ denote the integral vector $\theta_\kappa(v_1)$. Put $\mathcal{M}_2 := \mathcal{M}_\tau(v_2)$ for some stability condition τ in $\mathrm{Stab}(S_2, \alpha_2)$. Now let ψ denote the map

$$\psi := \varphi_2 \circ \theta_\kappa \circ \varphi_1^\dagger : H^2(\mathcal{M}_1, \mathbb{Z}) \to H^2(\mathcal{M}_2, \mathbb{Z}).$$

As a corollary of Prop.5.3.2 we obtain the following:

Corollary 5.3.3

The map ψ is an integral isometry, which is not a parallel transport operator.

Remark. In this chapter we assumed throughout that the Hodge partners X and Y can be represented as moduli spaces of objects on a (twisted) K3 surface. It is reasonable to expect that this assumption could be dropped if one is willing to work with noncommutative spaces. This is closely related to the problem of associating to a general IS manifold X of $K3^{[n]}$-type a so called *K3 category* $\mathcal{D}^b(\mathcal{A})$ ([HMSt1, Def. 2.1]), and representing X as a moduli space of objects in this category. The general K3 category can be thought of as a noncommutative deformation of $\mathcal{D}^b(S)$ (for a K3 surface S) without point-like objects. If we understand correctly, one aim of the forthcoming work [MM] is to associate to a global commutative deformation of a moduli space of sheaves on a K3 surface a (generally noncommutative) deformation of the surface. At least locally to first order, there is a natural way to do this – cf. the example following Def. 6.2 in the next chapter.

Chapter 6

Deformations of twisted FM kernels

In the previous chapter, we represented certain Hodge isometries as a convolution of algebraic correspondences, given by characteristic classes of (quasi)-universal sheaves (and their adjoints) on the product of a K3 surface and an IS manifold, tangent sheaves and FM kernels on the product of two K3 surfaces. It is natural to study these objects in families. Global deformations of the convolution of a universal sheaf with its adjoint on the product of a moduli space of sheaves on a K3 surface with itself are studied in [MM]. In this chapter we give a small glimpse of the local (noncommutative) deformation theory of FM kernels on the product of two (twisted) varieties, such as K3 surfaces.

Let X be a smooth projective variety, ω_X its dualizing sheaf and

$$\delta_X : X \hookrightarrow X \times X$$

the diagonal immersion. Let us recall the basic facts about the Hochschild structure on X, following [CW] and [Cal3]. First we define the *Hochschild cohomology and homology groups* of X:

Definition 6.1

$$\begin{aligned}
\mathrm{HH}^i(X) &:= \mathrm{Hom}_{\mathcal{D}^b(X\times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[i]),\\
\mathrm{HH}_i(X) &:= \mathrm{Hom}_{\mathcal{D}^b(X\times X)}(\delta_{X*}(\omega_X^\vee[i-\dim X]), \mathcal{O}_{\Delta_X}).
\end{aligned}$$

The group $\mathrm{HH}^*(X) := \bigoplus_i \mathrm{HH}^i(X)$ has a graded ring structure and $\mathrm{HH}_*(X) := \bigoplus_i \mathrm{HH}_i(X)$ has a left module structure over it. There is an isomorphism

$$\mathrm{L}\delta_X^*\mathcal{O}_{\Delta_X} \longrightarrow \bigoplus_i \Omega_X^i[i]$$

called the *Hochschild-Kostant-Rosenberg* isomorphism. It induces the graded identifications

$$\mathrm{I}_{\mathrm{HKR}} : \mathrm{HH}_i(X) \longrightarrow H\Omega_X^i := \bigoplus_{q-p=i} H^p(X, \Omega_X^q),$$

$$\mathrm{I}^{\mathrm{HKR}} : \mathrm{HH}^i(X) \longrightarrow H\mathcal{T}_X^i := \bigoplus_{q+p=i} H^p(X, \mathcal{T}_X^q).$$

These are usually modified with the square root of td_X, which makes them compatible with the product and module structures on both sides in some cases (cf. [NWMS]):

$$\widetilde{\mathrm{I}}^{\mathrm{HKR}} := ((-)\lrcorner\sqrt{td_X}) \circ \mathrm{I}^{\mathrm{HKR}} \text{ and } \widetilde{\mathrm{I}}_{\mathrm{HKR}} := ((-) \wedge \sqrt{td_X}) \circ \mathrm{I}_{\mathrm{HKR}}.$$

The group $\mathrm{HH}^2(X)$ parametrizes the so-called *noncommutative first-order deformations* of X. By this we mean the deformations of the category of $\mathcal{O}_X$-modules $\underline{\mathrm{mod}} - \mathcal{O}_X$ (and its attendant subcategory $\mathrm{Coh}(X)$) as an abelian category, in the sense of [LvdB]. These include, but do not exhaust the first-order deformations of $\mathcal{O}_X$ as a sheaf of associative algebras – to such a sheaf $\mathcal{A}_X$, we can associate the abelian category of left $\mathcal{A}_X$-modules $\underline{\mathrm{mod}} - \mathcal{A}_X$. However, each deformation $\mathcal{A}_c, c \in \mathrm{HH}^2(X)$ does admit a geometric description as a category of $\mathbb{X}_c$-modules $\underline{\mathrm{mod}} - \mathbb{X}_c$, where $\mathbb{X}_c$ is a $\mathbb{C}(\varepsilon)$-*linear stack of algebroids* in the sense of Kontsevich ([Kon]) – cf. [Low1]. In the context of deformation quantization, algebroid stacks over the power series ring $\mathbb{C}[[t]]$ (also known as *DQ-algebroids*) have been extensively studied in

[KS].

In fact, by using the HKR isomorphism to identify $\mathrm{HH}^2(X)$ with

$$H^0(X, \bigwedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X),$$

the first-order deformations can be described rather explicitly – cf. [Tod, Ch. 4]. The elements of $H^1(X, \mathcal{T}_X)$ are the Kodaira-Spencer classes, which parametrize the usual deformations of X as a scheme, the elements of $H^0(X, \bigwedge^2 \mathcal{T}_X)$ can be considered as bidifferential operators $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ that determine an associative product (a so-called $*$-*product*) on the module $\mathcal{O}_X \oplus \mathcal{O}_X$ to give a deformation of $\mathcal{O}_X$ as a sheaf of associative algebras (a $*$-*algebra*), whereas the elements of $H^2(X, \mathcal{O}_X)$ correspond to the deformations of X as a $\mathbb{G}_m$-gerbe.

Deformations over more general local rings $(R, \mathfrak{m}_R)$ than $\mathbb{C}(\varepsilon)$ are parametrized by the so-called Maurer-Cartan elements of $\Gamma(X, \mathfrak{g}(X) \otimes \mathfrak{m}_R)$, where $\mathfrak{g}(X)$ is a differential graded Lie algebra related to the Hochschild cochain complex of $\mathcal{O}_X$ – cf. [BGNT1] and [BGNT2].

Now to each FM transform

$$\Phi_{\mathcal{E}} : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(X')$$

there is an induced functorial map in Hochschild homology

$$(\Phi_{\mathcal{E}})_{\mathrm{HH}} : \mathrm{HH}_*(X) \longrightarrow \mathrm{HH}_*(X').$$

If $\Phi_{\mathcal{E}}$ is an equivalence, there is also an induced map in cohomology

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} : \mathrm{HH}^*(X) \longrightarrow \mathrm{HH}^*(X')$$

– cf. [CW, Thm. 8.1].

However, when X and X' are IS manifolds, we can still associate a natural map $\Phi_{\mathcal{E}}^{\mathrm{HH}}$ in Hochschild cohomology to a certain type of functors, other than equivalences. In the following, we use $H^*(X, \mathbb{C})$ and $H\Omega_X^*$ interchangeably – we identify them via the natural isomorphism, coming from the Hodge decomposition. Let us define

$$\Phi_{\mathcal{E}}^{H\Omega} := \widetilde{\mathrm{I}}_{\mathrm{HKR}} \circ (\Phi_{\mathcal{E}})_{\mathrm{HH}} \circ \widetilde{\mathrm{I}}_{\mathrm{HKR}}^{-1}.$$

Note that the isomorphisms $H^*(X,\mathbb{C}) \cong H\Omega_X^*$ and $H^*(X',\mathbb{C}) \cong H\Omega_{X'}^*$ conjugate $\Phi_{\mathcal{E}}^H$ to $\Phi_{\mathcal{E}}^{H\Omega}$ – cf. [MSt, Thm. 1.2]. Here $\Phi_{\mathcal{E}}^H$ denotes the cohomological transform $\Phi_{\mathcal{E}}^H : H^*(X,\mathbb{C}) \to H^*(X',\mathbb{C})$, induced by the class $v(\mathcal{E})$.

As is well-known, contraction against a symplectic form

$$\sigma_X \in H^0(X, \Omega_X^2) \cong H^{2,0}(X)$$

yields isomorphisms $\bigwedge^j \mathcal{T}_X \cong \Omega_X^{2-j}$ for $j \geq 0$ (cf. e.g. [AdT, Ch. 6.3]). Passing to cohomology and conjugating with the modified HKR isomorphisms, we obtain an isomorphism

$$\psi_{\sigma_X} : \mathrm{HH}^*(X) \cong \mathrm{HH}_{2-*}(X).$$

Definition 6.2
Let us call $\Phi_{\mathcal{E}}$ *a functor of Hodge type*, if $\Phi_{\mathcal{E}}^{H\Omega}$ restricts to an isomorphism

$$H^{2,0}(X) \xrightarrow{\sim} H^{2,0}(X') \subset H\Omega_{X'}^{-2}.$$

Now given a functor of Hodge type $\Phi_{\mathcal{E}}$, put

$$\sigma_{X'} := \Phi_{\mathcal{E}}^{H\Omega}(\sigma_X).$$

Then $\sigma_{X'}$ generates $H^{2,0}(X')$ and we can define

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} := \psi_{\sigma_{X'}}^{-1} \circ (\Phi_{\mathcal{E}})_{\mathrm{HH}} \circ \psi_{\sigma_X}.$$

Note that, although the isomorphism ψ_{σ_X} depends on the choice of a symplectic form σ_X (by scaling), the map $\Phi_{\mathcal{E}}^{\mathrm{HH}}$ doesn't depend on this choice. Also $\Phi_{\mathcal{E}}^{\mathrm{HH}}$ coincides with the usual definition, whenever $\Phi_{\mathcal{E}}$ is, in addition, an equivalence – this follows from the compatibility of the modified HKR isomorphisms with the module structures (HH^*, HH_*) and $(H\mathcal{T}^*, H\Omega^*)$ for varieties with trivial canonical bundle – cf. [NWMS, Thm. 1.2].

Now let S be a K3 surface and $\mathcal{M}$ – a moduli space of sheaves on S, of dimension greater than two. Suppose there is a universal sheaf $\mathcal{E}$ on $S \times \mathcal{M}$. An example of a functor of Hodge type that is not an equivalence is furnished by the right adjoint to $\Phi_{\mathcal{E}}$:

$$\Phi_{\mathcal{E}_R} : \mathcal{D}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(S).$$

This is because $H^{2,0}(\mathcal{M}) \cong \mathbb{C}\sigma_{\mathcal{M}}$ is not in the kernel of the restriction of $\Phi_{\mathcal{E}}^{H\Omega}$ to $H\Omega_{\mathcal{M}}^{-2}$:

$$\Phi_{\mathcal{E}_R}^{H\Omega^{-2}} : H\Omega_{\mathcal{M}}^{-2} \longrightarrow H\Omega_S^{-2} \cong H^{2,0}(S).$$

This follows from

$$\begin{aligned}(\overline{\sigma}_S, \Phi_{\mathcal{E}_R}^H(\sigma_{\mathcal{M}})) = (\Phi_{\mathcal{E}}^H(\overline{\sigma}_S), \sigma_{\mathcal{M}}) = ([\Phi_{\mathcal{E}}^H(\overline{\sigma}_S)]_2, \sigma_{\mathcal{M}}) &= (\varphi_{\mathcal{E}}(\overline{\sigma}_S), \sigma_{\mathcal{M}}) \\ &= (\lambda\overline{\sigma}_{\mathcal{M}}, \sigma_{\mathcal{M}}) \neq 0.\end{aligned}$$

Above $\overline{\sigma}_S$ denotes a generator of $H^{0,2}(S)$, $\lambda \in \mathbb{C}^*$ is a non-zero scalar, and $\varphi_{\mathcal{E}}$ denotes the Mukai homomorphism. In particular we obtain a well-defined map

$$\Phi_{\mathcal{E}}^{\mathrm{HH}^2} : \mathrm{HH}^2(\mathcal{M}) \longrightarrow \mathrm{HH}^2(S),$$

which associates to any first-order deformation of $\mathcal{M}$ a (possibly noncommutative) deformation of S.

Next we recall a couple of statements about small deformations of FM equivalences in the untwisted case. For an algebroid stack $\mathbb{X}$ on a smooth projective variety X, $\mathcal{D}^b(\mathbb{X})$ denotes the bounded derived category of $\mathrm{Coh}(\mathbb{X})$, $\mathcal{D}^b(\underline{\mathrm{mod}} - \mathbb{X})$ is the bounded derived category of $\underline{\mathrm{mod}} - \mathbb{X}$, whereas $\mathcal{D}^b_{coh}(\mathbb{X})$ denotes the full subcategory of $\mathcal{D}^b(\underline{\mathrm{mod}} - \mathbb{X})$, consisting of objects with coherent cohomology.

Theorem 6.3 ([Tod, Thm. 4.7])

Let X and Y be smooth projective varieties and let

$$\Phi_{\mathcal{E}} : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$$

be an FM equivalence with kernel $\mathcal{E} \in \mathcal{D}^b(X \times Y)$. Then for any first-order deformation $\mathbb{X}_\alpha, \alpha \in \mathrm{HH}^2(X)$, there is a deformation $\mathbb{Y}_\beta$, where

$$\beta = \Phi_{\mathcal{E}}^{\mathrm{HH}}(\alpha) \in \mathrm{HH}^2(Y)$$

and a deformation of $\mathcal{E}$ to an object $\widetilde{\mathcal{E}}$ of $\mathcal{D}^b(\mathbb{X}_{-\alpha} \times \mathbb{Y}_\beta)$, such that the induced FM transform

$$\Phi_{\widetilde{\mathcal{E}}} : \mathcal{D}^b(\mathbb{X}_\alpha) \to \mathcal{D}^b(\mathbb{Y}_\beta)$$

is an equivalence.

In the analytic setting, there is also a general statement for certain higher-order nilpotent thickenings of a complex manifold, called $*$-quantizations; these are a special type of R-linear algebroid stacks, locally isomorphic to stacks of right invertible modules on a $*$-algebra – cf. [ABP, Def. 2.1.13]; here R denotes the standard Artinian ring $\mathbb{C}(t)/(t^n)$ for some $n > 0$. However, note that, unlike Thm 6.3, the Fourier-Mukai kernel in the following statement is assumed to be concentrated in degree zero, i.e. it is assumed to be a sheaf:

Theorem 6.4 ([ABP, Thms. 2.2.1-2])
Let X and Y be complex manifolds and let

$$\Phi_{\mathcal{E}} : \mathcal{D}^b_{coh}(X) \to \mathcal{D}^b_{coh}(Y)$$

be an FM equivalence with kernel $\mathcal{E} \in \mathrm{Coh}(X \times Y)$ and suppose further that the support of $\mathcal{E}$ is proper. Then for any $*$-quantization $\mathbb{X}$ of X, there is a $*$-quantization $\mathbb{Y}$ of Y and a deformation of $\mathcal{E}$ to $\widetilde{\mathcal{E}} \in \mathrm{Coh}(\mathbb{X}^{op} \times \mathbb{Y})$ such that the induced FM transform

$$\Phi_{\widetilde{\mathcal{E}}} : \mathcal{D}^b_{coh}(\mathbb{X}) \to \mathcal{D}^b_{coh}(\mathbb{Y})$$

is an equivalence.

The proof of the above statement is rather involved – the authors' strategy is to reduce the problem to a problem of deforming a coalgebra in the category of $\mathcal{D}$-modules on quantizations. In [ABP, Ch. 1.4], the authors assert that their methods can be adapted to prove a stronger statement on deformations of FM-kernels on $\mathbb{G}_m$-gerbes.

Given a twisted variety (X, α), let $\mathcal{A}$ denote the endomorphism algebra sheaf $\mathcal{E}nd(\mathcal{F})$ of a locally free α-twisted sheaf $\mathcal{F}$ of rank r, where r is the order of α in $Br(X)$. The sheaf $\mathcal{A}$ is also known as an *Azumaya algebra* (i.e. an étale locally matrix algebra) over $\mathcal{O}_X$. Denote its opposite algebra by $\mathcal{A}^{op}$ and its external enveloping algebra sheaf (on $X \times X$) by $\mathcal{A}^e := \mathcal{A} \boxtimes \mathcal{A}^{op}$. The Azumaya algebra $\mathcal{A}$ is very 'close' to $\mathcal{O}_X$ in a precise sense – many of their additive invariants, such as algebraic K-theory, Hochschild homology, cyclic homology, etc. are isomorphic (modulo torsion) – cf. [TvdB, Cor. 3.1]. By [Lie, Prop. 2.2.2.3], applying the functor $\mathcal{H}om(\mathcal{F}, -)$ produces an equivalence between of α-twisted sheaves on X and the category of right modules $\mathcal{A} - \underline{\text{mod}}$, that we prefer to write as $\underline{\text{mod}} - \mathcal{A}^{op}$. The Hochschild cohomology of $\mathcal{A}$ is defined as

$$\mathrm{HH}^n(\mathcal{A}) := \mathrm{Hom}_{\mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{A}^e)}(\delta_{X*}\mathcal{A}, \delta_{X*}\mathcal{A}[n]).$$

More conceptually, $\mathrm{HH}^n(\mathcal{A})$ should be interpreted as a group of natural transformations $\mathrm{Hom}(1_{\mathfrak{a}}, 1_{\mathfrak{a}}[n])$, where $1_{\mathfrak{a}}$ is the identity quasi-endofunctor of a dg enhancement $\mathfrak{a}$ of the category $\mathcal{D}^b(\underline{\text{mod}} - \mathcal{A}^e)$ – cf. [Ke].

The following statement is probably well-known to experts, but we couldn't find a reference and so would like to give an argument anyway.

Theorem 6.5
Let X and Y be smooth projective varieties and let $\mathcal{A}, \mathcal{B}$ be Azumaya algebras over X, resp. Y. Let

$$\Phi_{\mathcal{E}} : \mathcal{D}^b_{coh}(\mathcal{A}) \to \mathcal{D}^b_{coh}(\mathcal{B})$$

be a Fourier-Mukai equivalence with kernel $\mathcal{E} \in \mathcal{D}^b_{coh}(\mathcal{A}^{op} \boxtimes \mathcal{B})$. Then for any first-order deformation $\mathbb{A}_\alpha, \alpha \in \mathrm{HH}^2(\mathcal{A})$, there is a deformation $\mathbb{B}_\beta, \beta \in \mathrm{HH}^2(\mathcal{B})$ and a deformation of $\mathcal{E}$ to an object of $\mathcal{D}^b_{coh}(\mathbb{A}^{op}_\alpha \times \mathbb{B}_\beta)$ such that the induced Fourier-Mukai transform

$$\Phi_{\widetilde{\mathcal{E}}} : \mathcal{D}^b_{coh}(\mathbb{A}_\alpha) \to \mathcal{D}^b_{coh}(\mathbb{B}_\beta)$$

is an equivalence.

PROOF. Put $\mathcal{C} := \mathcal{A}^{op} \boxtimes \mathcal{B}$. Consider $\mathcal{E}$ as an object in $\mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C})$ via the natural inclusion $\mathcal{D}^b_{coh}(\mathcal{C}) \subset \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C})$. An element of $c \in \mathrm{HH}^2(\mathcal{C})$ is by definition a morphism between the objects $(\delta_{X\times Y})_*\mathcal{C}$ and $(\delta_{X\times Y})_*\mathcal{C}[2]$ in $\mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C}^e)$. Considering them as FM kernels, we can evaluate this map at an object $\mathcal{F} \in \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C})$ to obtain an element $\chi_{\mathcal{C},\mathcal{F}}(c) \in \mathrm{Ext}^2_{\mathcal{D}^b(\underline{\mathrm{mod}}-\mathcal{C})}(\mathcal{F},\mathcal{F})$. We obtain a *characteristic morphism:*

$$\chi_{\mathcal{C},\mathcal{F}} : \mathrm{HH}^2(\mathcal{C}) \longrightarrow \mathrm{Ext}^2_{\mathcal{D}^b(\underline{\mathrm{mod}}-\mathcal{C})}(\mathcal{F},\mathcal{F}). \tag{6.1}$$

Since $\underline{\mathrm{mod}} - \mathcal{C}$ has enough injectives (cf. e.g. [Lie, L. 2.2.3.2]) we can use the obstruction theory, developed in [Low2]. In particular, according to [Low2, Cor. 4.9] the element $\chi_{\mathcal{C},\mathcal{F}}(c)$ is an obstruction to the lifting of $\mathcal{F}$ to an object $\widetilde{\mathcal{F}}$ of the deformation $\mathcal{D}^b(\mathbb{C}_c)$.

Now we can use the product structure on $\mathcal{C}$ and the standard method of Toda of representing the zero element in $\mathrm{Ext}^2_{\mathcal{D}^b(\underline{\mathrm{mod}}-\mathcal{C})}(\mathcal{E},\mathcal{E})$ as a sum of two "opposite" obstruction classes, coming from deforming one of the factors, while keeping the other one fixed.

Consider the products $X \times X \times Y$ and $X \times Y \times X \times Y$, and let p_{ij} and q_{ij} denote the respective pairwise projections. Let $\sigma : X \times X \to X \times X$ denote the standard involution of the two factors. Let

$$\mathcal{E} * (-) : \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{A}^e) \longrightarrow \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C})$$

be the functor induced by left convolution with $\mathcal{E}$ – it is an equivalence of categories, since $\Phi_{\mathcal{E}}$ is. Furthermore, define the functor

$$\widetilde{\Phi} : \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C}^e) \longrightarrow \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C}), \mathcal{F} \mapsto Rq_{34*}(\mathcal{F} \otimes^L q_{12}^* \mathcal{E})$$

(recall that we can tensor left $\mathcal{C}^e$ modules by considering them as $(\mathcal{C}, \mathcal{C})$-bimodules).

Now let us consider the natural map

$$p^* : \mathrm{HH}^2(\mathcal{A}) \longrightarrow \mathrm{HH}^2(\mathcal{C}), \alpha \mapsto \alpha \boxtimes 1.$$

It is induced by evaluating the functor

$$(\mathrm{id} \times \delta_Y)_* p_{12}^*(-) : \mathcal{D}^b(\mathcal{A}^e) \longrightarrow \mathcal{D}^b(\mathcal{C}^e)$$

at $\mathrm{Hom}(\delta_{X*}(\mathcal{A}), \delta_{X*}(\mathcal{A})[2])$. Let us show that the map

$$\chi_{\mathcal{C},\mathcal{E}} \circ p^* : \mathrm{HH}^2(\mathcal{A}) \longrightarrow \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$$

is an isomorphism. Note that, by definition, the map $\chi_{\mathcal{C},\mathcal{E}} \circ p^*$ is induced by evaluating the functor $\widetilde{\Phi} \circ (\mathrm{id} \times \delta_Y)_* \circ p_{12}^*(-)$ at $\mathrm{Hom}(\delta_{X*}(\mathcal{A}), \delta_{X*}(\mathcal{A})[2])$. But the functor $\widetilde{\Phi} \circ (\mathrm{id} \times \delta_Y)_* \circ p_{12}^*(-)$ is naturally isomorphic to the equivalence $\mathcal{E} * (\sigma_*(-))$ – cf. e.g. the proof of [Tod, L. 5.7]. This implies that the map $\chi_{\mathcal{C},\mathcal{E}} \circ p^*$ is an isomorphism. Analogously, we obtain an isomorphism

$$\chi_{\mathcal{C},\mathcal{E}} \circ q^* : \mathrm{HH}^2(\mathcal{B}) \longrightarrow \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}).$$

Now put

$$\widetilde{\alpha} := \chi_{\mathcal{C},\mathcal{E}} \circ p^*(\alpha) \text{ and } \beta := (\chi_{\mathcal{C},\mathcal{E}} \circ q^*)^{-1}(-\widetilde{\alpha}).$$

We obtain

$$\chi_{\mathcal{C},\mathcal{E}}(\alpha \boxplus \beta) = \chi_{\mathcal{C},\mathcal{E}} \circ p^*(\alpha) + \chi_{\mathcal{C},\mathcal{E}} \circ q^*(\beta) = \widetilde{\alpha} - \widetilde{\alpha} = 0.$$

In particular, the obstruction vanishes and the object $\mathcal{E}$ lifts to an object $\widetilde{\mathcal{E}} \in \mathcal{D}^b(\underline{\mathrm{mod}} - \mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta)$. This means that $Li^* \widetilde{\mathcal{E}} \cong \mathcal{E}$ – here

$$i_* : \mathcal{D}^b(\underline{\mathrm{mod}} - \mathcal{C}) \longrightarrow \mathcal{D}^b(\underline{\mathrm{mod}} - \mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta)$$

is the exact inclusion functor, and

$$Li^* : \mathcal{D}^b(\underline{\text{mod}} - \mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta) \longrightarrow \mathcal{D}^-(\underline{\text{mod}} - \mathcal{C})$$

is the left adjoint restriction functor. Furthermore, $\widetilde{\mathcal{E}}$ fits into a distinguished triangle

$$i_*\mathcal{E} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow i_*\mathcal{E}.$$

This implies that $\widetilde{\mathcal{E}} \in \mathcal{D}^b_{coh}(\mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta)$, since i_* preserves coherence. The inclusion $\Phi_{\widetilde{\mathcal{E}}}(\mathcal{D}^b_{coh}(\mathbb{A}_\alpha)) \subset \mathcal{D}^b_{coh}(\mathbb{B}_\beta)$ and the statement that $\Phi_{\widetilde{\mathcal{E}}}$ is an equivalence are proven as in [ABP, L. 5.4.8] and [ABP, Prop. 5.4.9]. There is an alternative approach to showing that $\Phi_{\widetilde{\mathcal{E}}}$ is an equivalence, similar to [HMSt1, Prop. 2.12] – we can use dualizing complexes on algebroid stacks (cf. e.g. [KS, Ch. 2,3], where Serre duality for DQ algebroids is treated) to obtain a left adjoint $\Phi_{\widetilde{\mathcal{E}}_L}$ to $\Phi_{\widetilde{\mathcal{E}}}$. Since $\Phi_{\mathcal{E}}$ is an equivalence, the adjunction counit $\Phi_{\widetilde{\mathcal{E}}_L} \circ \Phi_{\widetilde{\mathcal{E}}} \longrightarrow id$ restricts to an equivalence on the subcategory $i_*\mathcal{D}^b_{coh}(\mathcal{C})$; since the latter generates $\mathcal{D}^b_{coh}(\mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta)$, the adjunction counit is an equivalence on the whole category $\mathcal{D}^b_{coh}(\mathbb{A}_\alpha^{op} \times \mathbb{B}_\beta)$, hence $\Phi_{\widetilde{\mathcal{E}}}$ is fully faithful. A similar consideration for the adjunction unit shows that $\Phi_{\widetilde{\mathcal{E}}_L}$ is also fully faithful. Hence $\Phi_{\widetilde{\mathcal{E}}_L}$ is a quasi-inverse and $\Phi_{\widetilde{\mathcal{E}}}$ is an equivalence. ■

Remark. It could be possible to extend the above result to higher-order thickenings (and even formally) without resorting to $\mathcal{D}$-modules, by developing an obstruction theory, based on relative Hochschild-Atiyah classes, mimicking the approach of [HTh]. For many geometric applications, it is important to represent the obstruction class as a composition of a Hochschild-Atiyah class and a deformation class – cf. e.g. [AdT, Ch. 7], [BFl, Ch. 5], [HMSt3, Ch. 3], [HTh, Ch. 3]. In the notation above, the *Hochschild-Atiyah* class $AH(\mathcal{E})$ is the element of $\mathrm{Hom}(\mathcal{E}, L\delta^*_{X\times Y}\delta_{X\times Y*}\mathcal{C} \otimes^L \mathcal{E})$, obtained by applying the adjunction unit

$$\eta : \delta_{X\times Y*}\mathcal{C} \longrightarrow \delta_{X\times Y*}L\delta^*_{X\times Y}\delta_{X\times Y*}\mathcal{C}$$

(viewed as a morphism of FM kernels) to the object $\mathcal{E}$. Let $c \in HH^2(X)$ and consider c as an element of $\mathrm{Hom}(L\delta^*_{X\times Y}\delta_{X\times Y*}\mathcal{C}, \mathcal{C}[2])$ by adjunction. The composition $c \otimes id_{\mathcal{E}} \circ AH(\mathcal{E})$ yields the class $\chi_{\mathcal{C},\mathcal{E}}(c) \in \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$. In the untwisted case, the class $AH(\mathcal{E})$ is mapped to the exponent of the classical Atiyah class $A(\mathcal{E})$ via the HKR isomorphism – cf. [HMSt3, Rmk. 3.4].

Although we do not use a HKR-type isomorphism between the Hochschild cohomology of an Azumaya algebra and the cohomology of its polyvector fields, it is still available in this generality as a special case of [CvdB, Cor. 1.4], where such statement is shown for Lie algebroids over a ringed site. In fact, it was shown in [BGNT2, Cor. 35], in the analytic setting, that the Hochschild cochain complex of an Azumaya algebra is quasiisomorphic to the Hochschild complex of its underlying complex manifold.

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Lebenslauf

Apostol D. Apostolov wurde am 21.04.1984 in der Stadt Varna, Bulgarien, geboren. Von 1997 bis 2002 besuchte er das Mathematische Gymnasium Varna. Von 2002 bis 2005 studierte er Mathematik und EECS (Elektrotechnik und Informatik) an der Jacobs University Bremen. Die Titel seiner Bachelorarbeiten lauten 'Cohn Localization in Semifirs', betreut von Dr. D. Sheiham, und 'Design and Implementation of a viewpoint-changing system for mathematical texts based on theory morphisms in **OMDoc**', betreut von Prof. Dr. M. Kohlhase. Von 2005 bis 2008 studierte er Mathematik auf Master an der Georg-August-Universität Göttingen. Der Titel seiner Abschlußarbeit lautet 'Rationally Connected Varieties', betreut von Prof. Dr. Y. Tschinkel. Im November 2009 begann er ein Promotionsstudium am Institut für Algebraische Geometrie an der Leibniz Universität Hannover. Seit November 2009 arbeitet er als wissenschaftlicher Mitarbeiter am selbigen Institut.