Martin Wolf

On Supertwistor Geometry And Integrability In Super Gauge Theory

Institut für Theoretische Physik
Fakultät für Mathematik und Physik
Leibniz Universität Hannover

ITP–UH–18/06
Institut für Theoretische Physik
Gottfried Wilhelm Leibniz Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
wolf@itp.uni-hannover.de

To My Family

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Referent:	Prof. Dr. Olaf Lechtenfeld
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Zusammenfassung

In der vorliegenden Arbeit werden im Rahmen des Twistorzugangs verschiedene Aspekte der Integrabilität supersymmetrischer Eichtheorien betrachtet. Nach einer kurzen Darstellung der Grundlagen der Twistortheorie im ersten Kapitel, untersuchen wir zunächst selbstduale supersymmetrische Yang-Mills-Theorie (SYM). Insbesondere wird deren Twistorformulierung erläutert und ausgehend von dieser, werden weitere selbstduale Modelle vorgestellt. Jene folgen durch geeignete Reduktion aus selbstdualer SYM-Theorie und können zum Teil im Rahmen topologischer Feldtheorien verstanden werden. Für deren Twistorformulierung schlagen wir bestimmte gewichtete projektive Superräume als Twistorräume vor. Interessanterweise sind diese Calabi-Yau-Supermannigfaltigkeiten, so daß es ebenfalls möglich ist, geeignete Wirkungsprinzipien zu formulieren. Im dritten Kapitel dieser Arbeit befassen wir uns mit der Twistorformulierung eines supersymmetrischen Bogomolny-Modells in drei Raumzeitdimensionen. Die nicht-supersymmetrische Variante dieses Modells beschreibt statische Yang-Mills-Higgs-Monopole im Prasad-Sommerfield-Limes. Insbesondere wird eine supersymmetrische Erweiterung des Minitwistorraumes betrachtet und, folgend aus einem der Grundprinzipien der Twistortheorie, betten wir diesen in eine konkrete Doppelfaserung ein. Diese Methode hat den Vorteil der Möglichkeit der Formulierung eines Chern-Simons ähnlichen Wirkungsfunktionals, welches fast holomorphe Vektorbündel über dem entsprechenden Korrespondenzraum beschreibt. Letzterer kann mit einer Cauchy-Riemann-Struktur ausgestattet werden. Weiterhin formulieren wir holomorphe BF-Theorie auf dem Minisupertwistorraum und beweisen, daß die Modulräume dieser drei Theorien bijektiv zueinander sind. Im Anschluß werden bestimmte Deformationen der komplexen Struktur des Minisupertwistorraumes betrachtet und ein daraus resultierendes supersymmetrisches Bogomolny-Modell mit massiven Feldern konstruiert. Im vierten Kapitel wird dann die Twistorformulierung von SYM-Theorien vorgestellt. Im letzten Kapitel vertiefen wir die Untersuchung selbstdualer SYM-Theorien bezüglich ihrer (klassischen) Integrabilität. Insbesondere wird die Twistorkonstruktion unendlich dimensionaler Algebren nichtlokaler Symmetrien behandelt. Diese sind zum einen affine Erweiterungen globaler und zum anderen affine Erweiterungen superkonformer Symmetrien. Im weiteren betrachten wir die Konstruktion von self-dualer SYM-Hierarchien. Jene beschreiben unendlich viele Flüsse auf den entsprechenden Lösungsräumen, wobei die niedrigsten Raumzeittranslationen darstellen. Dies bedeutet, daß eine gegebene Lösung zu den Feldgleichungen in eine unendlich dimensionale Familie neuer Lösungen eingebettet werden kann. Weiterhin werden unendlich viele erhaltene nichtlokale Ströme konstruiert.

Schlagworte: Supertwistorgeometrie, Supersymmetrische Eichtheorien, Integrabilität

Abstract

In this thesis, we report on different aspects of integrability in supersymmetric gauge theories. Our main tool of investigation is supertwistor geometry. In the first chapter, we briefly review the basics of twistor geometry. Afterwards, we discuss self-dual super Yang-Mills (SYM) theory and some of its relatives. In particular, a detailed twistor description of self-dual SYM theory is presented. Furthermore, we introduce certain self-dual models which are, in fact, obtainable from self-dual SYM theory by a suitable reduction. Some of them can be interpreted within the context of topological field theories. To provide a twistor description of these models, we propose weighted projective superspaces as twistor spaces. These spaces turn out to be Calabi-Yau supermanifolds. Therefore, it is possible to write down appropriate action principles, as well. In chapter three, we then deal with the twistor formulation of a certain supersymmetric Bogomolny model in three space-time dimensions. The nonsupersymmetric version of this model describes static Yang-Mills-Higgs monopoles in the Prasad-Sommerfield limit. In particular, we consider a supersymmetric extension of mini-twistor space. This space is in turn a part of a certain double fibration. It is then possible to formulate a Chern-Simons type theory on the correspondence space of this fibration. As we explain, this theory describes partially holomorphic vector bundles. It should be noticed that the correspondence space can be equipped with a Cauchy-Riemann structure. Moreover, we formulate holomorphic BF theory on mini-supertwistor space. We then prove that the moduli spaces of all three theories are bijective. In addition, complex structure deformations on mini-supertwistor space are investigated eventually resulting in a twistor correspondence involving a supersymmetric Bogomolny model with massive fields. In chapter four, we review the twistor formulation of non-self-dual SYM theories. The remaining chapter is devoted to a more detailed investigation of (classical) integrability of self-dual SYM theories. In particular, we explain the twistor construction of infinite-dimensional algebras of hidden symmetries. Our discussion is exemplified by deriving affine extensions of internal and space-time symmetries. Furthermore, we construct self-dual SYM hierarchies and their truncated versions. These hierarchies describe an infinite number of flows on the respective solution space. The lowest level flows are space-time translations. The existence of such hierarchies allows us to embed a given solution to the equations of motion of self-dual SYM theory into an infinite-parameter family of new solutions. The dependence of the self-dual SYM fields on the additional moduli can be recovered by solving the equations of the hierarchy. We in addition derive infinitely many nonlocal conservation laws.

Keywords: Supertwistor Geometry, Supersymmetric Gauge Theories, Integrability

Motivation and introduction

Twistor geometry

In 1967, Penrose [187] introduced the foundations of twistor geometry. The corner stone of twistor geometry is the substitution of space-time as background for physical processes by some new background manifold – called twistor space, and furthermore the reinterpretation of physical theories on this new space. As originally proposed, twistor space, or rather the projectivization thereof, which is associated with complexified four-dimensional Minkowski space, is the complex projective space $\mathbb{C}P^3$. Geometrically, this space parametrizes all isotropic two-planes in complexified Minkowski space. Upon this correspondence, it is possible to reinterpret solutions to zero rest mass free field equations on Minkowski space in terms of certain cohomology groups on twistor space [187]–[190]. For instance, if U is some open subset in complexified Minkowski space and if $\mathcal{Z}_{\pm h}$ denotes the sheaf of solutions to the helicity $\pm h$ zero rest mass field equations, then there is an isomorphism between the two cohomology groups $H^1(U', \mathcal{O}_{U'}(\mp 2h-2))$ and $H^0(U, \mathcal{Z}_{\pm h})$, where U' is an appropriate region on twistor space related to U. Here, $\mathcal{O}_{U'}(\mp 2h-2)$ is the sheaf of sections of a certain holomorphic line bundle (over U') having first Chern class $\mp 2h - 2$. The map between representatives of these cohomology groups has been termed Penrose transform. Putting it differently, any solution to zero rest mass free field equations can be represented by certain holomorphic "functions" on twistor space, which are "free" of differential constraints. For a detailed discussion of this correspondence in terms of Čech cohomology, we refer the reader to [90]. Let us mention in passing that, as was shown by Eastwood in [91], it is also possible to generalize this description to the case of massive free fields.

Besides giving insights into the geometric nature of solutions to linear field equations, the ideas of twistor geometry turned out to be extremely powerful for studying various nonlinear classical field equations. Twistor methods have successfully been applied to subclasses of the Einstein's equations of General Relativity and of the (super) Yang-Mills equations, to name just the most prominent ones. Indeed, it is possible to associate with any self-dual oriented Riemannian four-dimensional manifold X, that is, a manifold with self-dual Weyl tensor, a complex three-dimensional twistor space

[191, 21]. All the information about the conformal structure of X is encoded in the complex structure of this "curved" twistor space. Some additional data then allows for the construction of self-dual metrics and conformal structures on X. Among such metrics, there are interesting subclasses as, e.g., self-dual Einstein metrics, Kähler metrics with vanishing scalar curvature and Ricci-flat self-dual metrics. For constructions, see Refs. [249, 238, 250, 193, 116, 117, 131, 118, 264, 97, 147, 239, 119, 175, 150], for instance. Furthermore, the question when such a twistor space is in addition Kähler was answered by Hitchin [114]. He showed that given a compact oriented self-dual Riemannian four-dimensional manifold X, the corresponding twistor space admits a Kähler structure if and only if X is conformally equivalent to either S^4 or $\mathbb{C}P^2$. Besides questions related to Einstein's equations, twistor theory has given striking advances in our understanding of the properties of gauge theories. In an important paper [248], Ward has proven a one-to-one correspondence between gauge equivalence classes of self-dual Yang-Mills fields on complexified four-dimensional space-time and equivalence classes of holomorphic vector bundles over twistor space which obey certain triviality conditions. Upon imposing additional conditions on the bundles over twistor space, one may also reduce the structure group of the bundles on space-time to some real form of the general linear group and in addition, the discussion to, for instance, an Euclidean setting. Furthermore, replacing twistor space by ambitwistor space – the space of complex light rays (null lines) in complexified four-dimensional Minkowski space – one can discuss general Yang-Mills fields, as well. In fact, Isenberg et al. [126] and Witten [259] showed that it is possible to represent gauge equivalence classes of solutions to the full Yang-Mills equations in terms of equivalence classes of certain holomorphic vector bundles on formal neighborhoods of ambitwistor space. Khenkin et al. [135] discussed a generalization thereof by giving an interpretation in terms of certain cohomology groups. A detailed discussion of the cohomology interpretation of solutions to the full Yang-Mills equations can be found in [64, 199]. Of course, the discussion can also be reduced to real Yang-Mills fields and to, e.g., Minkowski space-time. A map between gauge equivalence classes of solutions to Yang-Mills equations on some space-time and equivalence classes of holomorphic vector bundles over a twistor space associated with the space-time under consideration is called a Penrose-Ward transform [174].

Furthermore, in this respect it is also worth mentioning that, by considering the space of all complex null geodesics of some general four-dimensional complex space-time, one can generalize ambitwistor space to a curved setting. As was shown by LeBrun [145], this "curved" ambitwistor space is a five-dimensional complex manifold. Moreover, this construction then forms the basis for a twistorial description of four-dimensional conformal gravity [26, 148].

Besides Penrose-Ward transforms involving the already mentioned twistor and ambitwistor spaces, one may establish such correspondences on quite generic ground. This was considered, e.g., by Eastwood [92]. In fact, suppose we are given three complex manifolds X, Y and Z, where Y is simultaneously fibered over X and Z, that is, we consider a double fibration of the form

$$\begin{tikzcd} & Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Z & & X \end{tikzcd}$$

together with two suitable holomorphic projections $\pi_{1,2}$. Here and in the following, we shall refer to X as space-time, to Y as correspondence space and to Z as twistor space, respectively. Clearly, such a double fibration induces a correspondence between X and Z, that is, there is a relation between points and subsets in either manifold. In particular, a point $x \in X$ gives a submanifold $\pi_1(\pi_2^{-1}(x)) \hookrightarrow Z$, and conversely a point in twistor space, $z \in Z$, yields a submanifold $\pi_2(\pi_1^{-1}(z))$ embedded into space-time X. In addition, this correspondence can be used to transfer data given on Z to data on X and vice versa. Indeed, given some analytic object, Ob_Z, on twistor space (e.g., certain cohomology classes or holomorphic vector bundles), one may relate it to some object Ob_X on space-time. The latter will be constrained by certain partial differential equation since by definition, the pull-back $\pi_1^*\mathrm{Ob}_Z$ must be constant along the fibers of $\pi_1 : Y \to Z$. Under suitable topological conditions, the maps

$$\mathrm{Ob}_Z \mapsto \mathrm{Ob}_X \qquad \text{and} \qquad \mathrm{Ob}_X \mapsto \mathrm{Ob}_Z$$

define a bijection between equivalence classes $[\mathrm{Ob}_Z]$ and $[\mathrm{Ob}_X]$ (in general, the objects under consideration will only be defined up to equivalence). The correspondence thus established gives a way of studying objects on space-time obeying differential equations in terms of objects on twistor space which are "free" of such differential constraints. As before, we shall refer to such a map as Penrose-Ward transform.

Suppose now that the objects in question are holomorphic vector bundles. If we let $\Omega^1_{\pi_1}(Y) := \Omega^1(Y)/\pi_1^*\Omega^1(Z)$ be the sheaf of relative differential one-forms on the correspondence space, i.e., the dual of the vertical tangent vectors spanning the tangent spaces of $\pi_1 : Y \to Z$, we have the following theorem [92]:

Theorem. *Suppose the fibers of* $\pi_1 : Y \to Z$ *are simply connected and* $\pi_1(\pi_2^{-1}(x)) \hookrightarrow Z$ *is compact and connected for all* $x \in X$. *If in addition* $\Omega^1(X) \cong \pi_{2*}\Omega^1_{\pi_1}(Y)$, *then there is a one-to-one correspondence between equivalence classes of holomorphic vector bundles* $E \to Z$ *being holomorphically trivial on any* $\pi_1(\pi_2^{-1}(x)) \hookrightarrow Z$, *equivalence classes of holomorphic vector bundles on* Y *equipped with flat relative connection and equivalence classes of holomorphic vector bundles* $E' \to X$ *equipped with a connection flat on each* $\pi_2(\pi_1^{-1}(z)) \hookrightarrow X$ *for* $z \in Z$.

In particular, if Z is twistor space then $\pi_1(\pi_2^{-1}(x)) \cong \mathbb{C}P^1_x$ and the submanifolds $\pi_2(\pi_1^{-1}(z))$ represent isotropic two-planes in complexified Minkowski space. In case of ambitwistor space, one has $\pi_1(\pi_2^{-1}(x)) \cong (\mathbb{C}P^1 \times \mathbb{C}P^1)_x$ and $\pi_2(\pi_1^{-1}(z))$ are complex light rays. To jump ahead of our story a bit, such double fibrations and correspondences between vector bundles will play the basis of all the discussion presented in this thesis.

The twistor approach to field theories also provides a suitable framework for studying certain supersymmetric field theories. One simply needs to replace the manifolds appearing in double fibrations like the one discussed above, by so-called supermanifolds. Originally, the concept of supermanifolds goes back to the work by Berezin, Kostant and Leites [45, 139, 159]. The idea is basically to define a supermanifold as a particular ringed space $(X, \mathcal{O}_X)$, that is, a topological space X together with a sheaf of supercommutative rings $\mathcal{O}_X$ (the structure sheaf) obeying certain local properties. A thorough description of supermanifolds, as proposed by the above authors, has been developed by Manin [169]. In our subsequent discussion, we shall adopt this approach to supermanifolds. Let us also mention that there are alternative ways to introduce supermanifolds. For instance, a different one has been given by DeWitt [78].[1] It is based on the concept of supernumbers which are elements of a certain Graßmann algebra. We refer also to the book by Bartocci et al. [25] (and references therein), where all existing approaches are described and compared among each other.

As indicated above, the twistor approach facilitates the studies of supersymmetric field theories. For instance, by combining the ideas of formal neighborhoods with those of supermanifolds, Witten has shown [259] that it is possible to recover the full $\mathcal{N} = 3$ super Yang-Mills theory in four dimensions by discussing holomorphic vector bundles

[1] There is another, third approach due to Rogers [214].

over a certain superambitwistor space, which we shall denote by $\mathbb{L}^{5|6}$. In fact, $\mathbb{L}^{5|6}$ is the space of super light rays $\mathbb{C}^{1|6}$ in complexified $\mathcal{N} = 3$ Minkowski superspace $\mathbb{C}^{4|12}$. Furthermore, it can be viewed as a certain degree two hypersurface in the direct product $\mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$. In this respect, it is also important to stress that the $\mathcal{N} = 3$ and $\mathcal{N} = 4$ theories are physically equivalent. The only thing appearing to be different is that the $\mathcal{N} = 3$ formulation makes only the $U(1) \times SU(3)$ subgroup of the R-symmetry group $SU(4)$ manifest. In this sense, Witten's approach can be understood as a twistorial formulation of $\mathcal{N} = 4$ super Yang-Mills theory. Furthermore, Manin [169] generalized Witten's discussion to theories with less supersymmetry. In particular, he established a Penrose-Ward transform between solutions to the $\mathcal{N}$-extended super Yang-Mills equations and holomorphic vector bundles over $\mathcal{N}$-extended superambitwistor space – the space parametrizing super light rays $\mathbb{C}^{1|2\mathcal{N}}$ in $\mathbb{C}^{4|4\mathcal{N}}$ – which admit an extension to a $(3-\mathcal{N})$-th formal neighborhood in superambitwistor space.

The twistor string

Within the last three years, the twistorial studies of supersymmetric field theories received a somewhat unexpected renaissance due to string theory. In 2003, Witten [263] proposed a string theory whose target manifold is the supertwistor space $\mathbb{C}P^{3|4}$. His idea is based on three facts: i) holomorphic vector bundles over twistor space and similarly over supertwistor space are related to gauge and, respectively, super gauge theories on four-dimensional space-time, ii) supertwistor space $\mathbb{C}P^{3|4}$ is a Calabi-Yau supermanifold, that is, its first Chern class vanishes and in addition it admits a Ricci-flat metric and iii) the existence of a string theory – the open topological B model – whose space-time effective action is holomorphic Chern-Simons theory [261]. More specifically, Witten showed that the open topological B model on supertwistor space $\mathbb{C}P^{3|4}$ with a stack of r D5-branes[2] is equivalent to holomorphic Chern-Simons theory on the same space. This theory describes holomorphic structures on a rank r complex vector bundle $\mathcal{E}$ over $\mathbb{C}P^{3|4}$ which are given by the (0,1) part $\mathcal{A}^{0,1}$ of a connection one-form $\mathcal{A}$ on $\mathcal{E}$. The components of $\mathcal{A}^{0,1}$ appear as the excitations of open strings ending on the D5-branes. Furthermore, the spectrum of physical states contained in $\mathcal{A}^{0,1}$ is the same as that of $\mathcal{N} = 4$ super Yang-Mills theory, but the interactions of both theories differ. In fact, by analyzing the linearized [263] and the

[2]These D5-branes are not quite space-filling and defined by the condition that all open string vertex operators do not depend on antiholomorphic Graßmann coordinates on $\mathbb{C}P^{3|4}$.

full [205] field equations, it was shown that holomorphic Chern-Simons theory on $\mathbb{C}P^{3|4}$ is equivalent to $\mathcal{N}=4$ self-dual super Yang-Mills theory in four dimensions as introduced by Siegel [229]. Note that this theory can be considered as a truncation of the full $\mathcal{N}=4$ super Yang-Mills theory.

It was conjectured by Witten that perturbative amplitudes of full $\mathcal{N}=4$ super Yang-Mills theory are recovered by including into the B model so-called D-instantons which wrap certain holomorphic curves in supertwistor space. The presence of these D1-branes leads to additional fermionic states from the strings stretching between the D5- and D1-branes. Scattering amplitudes are then computed in terms of currents constructed from these additional fields, which localize on the D1-branes, by integrating certain correlation functions over the moduli space of these D1-branes in $\mathbb{C}P^{3|4}$. This proposal generalizes an earlier construction of maximally helicity-violating amplitudes by Nair [183]. Thus, by incorporating D1-branes into the topological B model, one can complement $\mathcal{N}=4$ self-dual super Yang-Mills theory to the full theory. However, soon after this proposal it was realized that this construction works only at tree-level, as already at one-loop level amplitudes of super Yang-Mills theory mix with those of conformal supergravity [50]. Recently, Boels et al. [58] proposed a twistor approach to $\mathcal{N}=4$ super Yang-Mills theory which reproduces perturbative Yang-Mills theory without conformal supergravity. For related papers regarding twistor actions, see also Refs. [176, 177]. After all, twistor string theory resulted in striking advances in computing gauge theory scattering amplitudes – even beyond tree-level, as powerful methods inspired by the twistor string have been developed; see, e.g., Ref. [66, 269] for reviews and also [162] for more references.

In addition to twistor string theory on supertwistor space, Witten [263] also mentioned the possibility of formulating a twistor string theory, i.e., the open topological B model on the superambitwistor space which is associated with $\mathcal{N}=3$ complexified Minkowski superspace. Within such a formulation, no D-instantons would be needed as already at classical level, holomorphic Chern-Simons theory on superambitwistor space gives the full $\mathcal{N}=3$ (respectively, $\mathcal{N}=4$) super Yang-Mills theory in four dimensions [259]. Therefore, the mechanism for reproducing perturbative super Yang-Mills theory would be completely different compared to the supertwistor space approach. However, even though this particular superambitwistor space is a Calabi-Yau supermanifold, it is not entirely clear how to formulate the B model on it. This is basically because of the difficulty in making sense of an appropriate measure on this space. Only recently, some progress in this direction has been made by Mason et al. [177]. Therein, the authors formulated a Chern-Simons type theory (whose interpretation on four-dimensional space-time is super Yang-Mills theory) on a certain codimension $2|0$ Cauchy-Riemann submanifold in superambitwistor space. Their description is based on Euclidean signature (instead of Minkowski signature) but as far as perturbation theory is concerned, this would not be of importance.

Since Witten's proposal, a variety of questions associated with supertwistor theory and twistor string theory has been investigated. For instance, an interesting property of Calabi-Yau supermanifolds is that Yau's theorem does not apply, that is, a Kähler supermanifold with vanishing first Chern class does not automatically admit a Ricci-flat metric. This fact was already observed by Sethi [226] and after Witten's paper explored in more detail by the authors of [212, 271, 213, 219, 160]. Another direction triggered by twistor string theory is the discussion of mirror symmetry between Calabi-Yau supermanifolds. For instance, it has been argued that for a certain limit of the Kähler moduli, supertwistor space $\mathbb{C}P^{3|4}$ and superambitwistor space $\mathbb{L}^{5|6} \hookrightarrow \mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$ are mirror manifolds [184, 7]. For further surveys on this material, we refer to [142, 8, 35, 197, 211, 143]. Furthermore, various other target spaces for twistor string theory have been discussed by the authors of [206, 219, 186, 105, 74, 220, 75, 208, 158] leading to, e.g., certain dimensional reductions of self-dual super Yang-Mills and full super Yang-Mills theories. Moreover, alternative formulations of twistor string theories have been proposed in Refs. [48, 49, 230, 157, 23]. In addition, issues related to gravity were investigated, as well. See, e.g., [104, 9, 10, 29, 57, 2, 5, 6]. For other aspects, see also Refs. [1, 231, 141, 14, 224, 222, 225, 144].

Integrability

Besides twistor string theory, $\mathcal{N} = 4$ super Yang-Mills theory appears to be connected with another string theory. Approximately ten years ago, a correspondence – nowadays known as AdS/CFT correspondence – was discovered. Maldacena conjectured an equivalence between $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions and type IIB superstring theory on the curved background $\mathrm{AdS}_5 \times S^5$ [167, 108, 262]. Besides matching of global symmetries, this conjecture claims full dynamical agreement of both theories. For instance, one of the statements is that the spectrum of the scaling dimensions of the gauge theory should coincide with the energy spectrum of the string states. However, due to the weak-strong nature of this correspondence, that is, the weak coupling regime of either theory gets mapped to the strong coupling regime of the other, it is extremely difficult to test or rather prove this conjecture. Therefore, one is searching for appropriate tools which may help to clarify this relation on general grounds.

One such tool which emerged in recent years is integrability. As already indicated, $\mathcal{N}=4$ super Yang-Mills theory appears to be integrable at classical level in the sense of admitting a certain twistor formulation (thus yielding a Lax pair formulation). Quantum integrable structures in $\mathcal{N}=4$ super Yang-Mills theory have first been discovered by Minahan et al. [180] being inspired by the work of Berenstein et al. [43].[3] They found that the planar dilatation operator, which measures the planar scaling dimension of local operators, in a certain sector of the theory can be interpreted at one-loop level as Hamiltonian of an integrable quantum spin chain. Based on this observation, it has then been shown that it is indeed possible to interpret the complete one-loop dilatation operator as Hamiltonian of an integrable quantum spin chain; see, e.g., [31, 39] and references therein. For discussions beyond leading order, see also [30, 40, 232, 33, 178, 195, 44, 34, 272, 101, 94, 41]. Another development pointing towards integrable structures was initiated by Bena et al. [42]. Their investigation is based on the observation that the Green-Schwarz formulation of the superstring on $\mathrm{AdS}_5 \times S^5$ can be interpreted as a coset theory, where the fields take values in the supercoset space

$$PSU(2,2|4)/(SO(4,1) \times SO(5))$$

[179, 134, 215]. Although this is not a symmetric space (the denominator group is too small) [168], this coset theory admits an infinite set of conserved nonlocal charges, quite similar to those that exist in two-dimensional field theories.[4] For the construction of nonlocal conserved charges in two-dimensional sigma models, we refer to [164, 165]. Such charges are in turn related to Kac-Moody algebras [81, 83, 227] and generate Yangian algebras [79, 80] as has been discussed, e.g., in [52]. For reviews of Yangian algebras see Refs. [53, 166]. Some time later, the construction of an analogous set of nonlocal conserved charges using the pure spinor formulation of the superstring [46, 47, 243] on $\mathrm{AdS}_5 \times S^5$ was given in [244]. For further developments, we refer the reader to Refs. [11, 19, 121, 20, 112, 51, 234, 32, 12, 270, 73, 99, 100, 13, 170, 133, 113, 216, 56], for example.[5] Clearly, the question that arises of what could be the charges in super Yang-

[3] For an ealier account of integrable structures in QCD, see, e.g., Refs. [161, 95, 36, 62, 37, 38].

[4] These charges were also independently found by Polyakov [198].

[5] See also Refs. [102, 61, 138, 233].

Mills theory. Within the spin chain approach, Dolan et al. [84, 85, 86] related these nonlocal charges for the superstring to a corresponding set of nonlocal charges in the superconformal gauge theory in the extreme weak coupling limit (see also [18, 16, 181, 17]).

Outline and main results

This thesis is devoted to studying different questions related to supersymmetric gauge theories. The main tool we shall be using is twistor theory.

In the first chapter, we begin by reviewing some of the basic aspects and properties of twistor spaces which correspond to flat space-times. No prior knowledge of twistor geometry is assumed. The unifying framework for our discussion is complex flag manifolds, which are natural generalizations of complex projective spaces. Furthermore, after having presented the concepts of supermanifolds, supervector bundles, etc., we explain the extension of twistor geometry to supertwistor geometry. As otherwise it would carry us too far afield from the main thread of development, we present the material only to that extent which is of need in later applications. For detailed expositions on the subject, we refer to the books [123, 169, 174, 194, 256].

In the second chapter, we give a first application by discussing self-dual super Yang-Mills theory and some related self-dual models. In particular, we start by describing the equivalence of the Čech and Dolbeault approaches to holomorphic vector bundles over supermanifolds. This lies somehow in the heart of the twistor approach to gauge theories. After that, we give a detailed explanation of the Penrose-Ward transform for $\mathcal{N}$-extended self-dual super Yang-Mills theory including appropriate superfield expansions, etc. By replacing supertwistor space by certain weighted projective superspaces, we develop Penrose-Ward transforms for truncations of self-dual super Yang-Mills theories. As the spaces under consideration will always be Calabi-Yau supermanifolds, we are also able to write down appropriate action principles. The discussion of the second part of this chapter is based on the work done together with Alexander Popov [206].

The third chapter is devoted to the twistor construction of certain supersymmetric Bogomolny models in three space-time dimensions. In [74], it was shown that scattering amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory which are localized on holomorphic curves in supertwistor space can be reduced to amplitudes of $\mathcal{N} = 8$ super Yang-Mills theory in three dimensions which are localized on holomorphic curves in a supersymmetric extension of mini-twistor space. Note that the simplest of such curves in mini-twistor space is the Riemann sphere which coincides with the spectral curve of the BPS $SU(2)$ monopole. Note also that every static $SU(2)$ monopole of charge k may be constructed from an algebraic curve in mini-twistor space [115], and an $SU(r)$ monopole is defined by $r-1$ such holomorphic curves [182]. The corresponding string theory after this reduction is the topological B model on mini-supertwistor space with r not quite space-filling D3-branes (defined analogously to the D5-branes in the six-dimensional case) and additional D1-branes wrapping holomorphic cycles in mini-supertwistor space. It is reasonable to assume that the latter correspond to monopoles and substitute the D-instantons in the case of supertwistor space. In [74], also a twistor string theory corresponding to a certain massive super Yang-Mills theory in three dimensions was described. The target space of the underlying B model is a Calabi-Yau supermanifold obtained from mini-supertwistor space by a deformation of its complex structure. The goal of this chapter is to complement the discussion given in [74]. In particular, we show that the B model with only the D3-branes included corresponds to a field theory on mini-supertwistor space obtained by a reduction of holomorphic Chern-Simons theory on supertwistor space. We show that this field theory is a holomorphic BF-type theory[6] which in turn is equivalent to a supersymmetric Bogomolny model. This model can be understood as the BPS equations of $\mathcal{N}=8$ super Yang-Mills theory in three dimensions. The action functional of holomorphic BF theory on mini-supertwistor space is not of Chern-Simons type, but one can introduce a Chern-Simons type action on the correspondence space of mini-supertwistor space. This space admits a so-called Cauchy-Riemann structure. After enlarging the integrable distribution defining this Cauchy-Riemann structure by one real direction to a distribution $\mathscr{T}$, one is led to the notion of $\mathscr{T}$-flat vector bundles over the correspondence space. These bundles take over the role of holomorphic vector bundles, and they can be defined by a $\mathscr{T}$-flat connection one-form $\mathcal{A}_{\mathscr{T}}$ [210]. The condition of $\mathscr{T}$-flatness of $\mathcal{A}_{\mathscr{T}}$ can be derived as the equations of motion of a theory we shall call partially holomorphic Chern-Simons theory. This theory can be obtained by a dimensional reduction of holomorphic Chern-Simons theory on supertwistor space. We prove that there are one-to-one correspondences between equivalence classes of holomorphic vector bundles subject to certain triviality conditions over mini-supertwistor space, equivalence classes of $\mathscr{T}$-flat vector bundles over the correspondence space and gauge equivalence classes of solutions to supersymmetric Bogomolny equations in three dimensions. In other words, the moduli

[6]These theories were introduced in [204] and considered, e.g., in [128, 28].

spaces of all three theories are bijective. By deforming the complex structure on mini-supertwistor space, which in turn induces a deformation of the Cauchy-Riemann structure on the correspondence space, we obtain a similar correspondence but with additional mass terms for fermions and scalars in the supersymmetric Bogomolny equations. The twistorial description of the supersymmetric Bogomolny equations has the nice feature of yielding novel methods for constructing explicit solutions. For simplicity, we restrict our discussion to solutions where only fields with helicity ± 1 and a Higgs field are nontrivial. The corresponding Abelian configurations give rise to the Dirac monopole-antimonopole systems. For the non-Abelian case, we present two ways of constructing solutions: first, by using a dressed version of the Penrose transform and second, by considering a nilpotent deformation of the holomorphic vector bundle corresponding to an arbitrary seed solution of the ordinary Bogomolny equations. The third chapter is based on the work done together with Alexander Popov and Christian Sämann [208].

The fourth chapter deals with a discussion of full super Yang-Mills theories on four-dimensional space-time. In particular, we review Witten's [259] and Manin's [169] constructions of Penrose-Ward transforms relating gauge equivalence classes of solutions to the $\mathcal{N}$-extended super Yang-Mills equations to certain equivalence classes of holomorphic vector bundles over superambitwistor space which admit an extension to a $(3-\mathcal{N})$-th order formal neighborhood.

As indicated above, there exist infinitely many nonlocal conserved charges in $\mathcal{N}=4$ super Yang-Mills theory. So far, they have been formulated within the spin chain approach [84, 85, 86] which is perturbative in nature. Note that field theoretic expressions for these charges at zero 't Hooft coupling are known and were also given in [84, 85]. It is therefore reasonable to consider the problem of constructing such charges from first principles. As a modest step, one may first study a simplification of the theory – namely its self-dual truncation. It will be the purpose of the fifth chapter to use twistor theory for studying infinite-dimensional algebras of hidden symmetries in $\mathcal{N}$-extended self-dual super Yang-Mills theories. Here, we generalize the results known for the self-dual Yang-Mills equations [196, 70, 71, 72, 241, 82, 76, 200, 202, 203, 127, 174]. In particular, we will first consider deformation theory of holomorphic vector bundles over supertwistor space. Using the Penrose-Ward transform, we relate these infinitesimal deformations to symmetries of the self-dual super Yang-Mills equations. After some general words on Riemann-Hilbert problems, hidden symmetries and related algebras and using also a cohomological classification, we exemplify our discussion by constructing Kac-Moody symmetries which come from affine extensions of the gauge algebra. In addition, we consider affine extensions of the superconformal algebra to obtain super Kac-Moody-Virasoro-type symmetries. The existence of such algebras originates from the fact that the full group of continuous transformations acting on the space of holomorphic vector bundles over supertwistor space is a semi-direct product of the group of local holomorphic automorphisms of the supertwistor space and of the group of one-cochains with respect to a certain covering with values in the sheaf of holomorphic maps of supertwistor space into the gauge group. See [202] for a discussion in the purely bosonic setting. By focusing on a certain Abelian subalgebra of the affinely extended superconformal algebra, we introduce a family of generalized supertwistor spaces. These manifolds, being parametrized by certain integers, then allow us to introduce truncated self-dual super Yang-Mills hierarchies. Such a hierarchy consists of a finite system of partial differential equations, where the self-dual super Yang-Mills equations are embedded in. The lowest level flows of such a hierarchy represent space-time translations. Furthermore, one may also consider the asymptotic limit to obtain the full hierarchy. Indeed, the existence of such a hierarchy allows us to embed a given solution into an infinite-parameter family of new solutions. This generalizes the results known for the self-dual Yang-Mills equations [172, 237, 173, 4, 129, 174]. We remark that such symmetries of the latter equations are intimately connected with one-loop maximally helicity violating amplitudes [24, 67, 68, 217, 106]. As for certain values of the parameters the generalized supertwistor spaces become Calabi-Yau, we are also able to give action principles for the truncated hierarchies. In addition, we also construct infinitely many nonlocal conservation laws in self-dual super Yang-Mills theory which are associated with the symmetries in question (see [4] in the context of the self-dual Yang-Mills hierarchy). The discussion of the fifth chapter is based on [266, 267].

Finally, we give a short summary of the results derived in this thesis and close with an outlook.

Conventions

In the sequel, we shall be using the standard abbreviations YM theory and SYM theory for Yang-Mills theory and super Yang-Mills theory, respectively. We also abbreviate holomorphic Chern-Simons theory by hCS theory. In addition, the reader may find a list with symbols which are most frequently used throughout this work at the end of this thesis.

Disclaimer

I apologize in advance to those people whose names were either mentioned incorrectly or not mentioned at all. I have tried to track down all the literature related to the topics discussed below but nevertheless because of the broad fields of twistor theory, integrable systems, etc., it might have happened that some of the works slipped through my hands. If so then simply because of the fact that I was not aware of them.

Hannover, Summer 2006

Martin Wolf

Contents

Chapter I

Supertwistor geometry

This first chapter is intended to give some of the main ideas of supertwistor geometry. The material is presented only to that extent which is needed in later applications. For pedagogical reasons, we begin with ordinary twistor geometry. As we will be discussing the twistorial approach to supersymmetric field theories, we need to generalize this framework to also include so-called supermanifolds – manifolds parametrized by $\mathbb{Z}_2$-graded coordinates. We will shortly realize, however, that for developing these generalizations, it is necessary to work in a bit more abstract mathematical setting. In due course, we therefore also present some definitions and notions to simplify understanding. The reader unfamiliar with the underlying mathematics may consult, e.g., Refs. [25, 107, 169, 257]. A thorough introduction into twistor geometry can be found in the books [123, 169, 174, 194, 256].

I.1 Twistor spaces

§I.1 Flag manifolds. Let us begin by recalling the definition of a flag manifold. Consider the complex space $\mathbb{C}^n$. Then we define its flag manifolds by

$$F_{d_1\cdots d_m}(\mathbb{C}^n) \ := \ \{(S_1,\ldots,S_m)\,|\,S_i\subset\mathbb{C}^n,\ \dim_{\mathbb{C}}S_i=d_i,\ S_1\subset S_2\subset\cdots\subset S_m\}, \qquad \text{(I.1)}$$

where the S_is are subspaces of $\mathbb{C}^n$. Typical examples of such flag manifolds are the projective space $F_1=\mathbb{C}P^{n-1}$ and the Graßmannian $F_k=G_{k,n}(\mathbb{C})$. Note that all of these manifolds are compact complex manifolds. Moreover, they have an equivalent representation as homogeneous spaces. That is, consider the decomposition

$$\mathbb{C}^{d_1}\oplus\mathbb{C}^{d_2-d_1}\oplus\mathbb{C}^{d_3-d_2}\oplus\cdots\oplus\mathbb{C}^{d_m-d_{m-1}}\oplus\mathbb{C}^{n-d_m}$$

of $\mathbb{C}^n$. The subgroup of $U(n)$ which preserves this decomposition is

$$U(d_1) \times U(d_2 - d_1) \times \cdots \times U(d_m - d_{m-1}) \times U(n - d_m) \ \subset \ U(n)$$

and hence we may give an equivalent definition by the quotient

$$F_{d_1 \cdots d_m}(\mathbb{C}^n) \ := \ \frac{U(n)}{U(d_1) \times U(d_2 - d_1) \times \cdots \times U(d_m - d_{m-1}) \times U(n - d_m)}. \tag{I.2}$$

From this it rather straightforwardly follows that their dimensionality is given by the formula

$$\dim_{\mathbb{C}} F_{d_1 \cdots d_m}(\mathbb{C}^n) \ = \ d_1(n - d_1) + (d_2 - d_1)(n - d_2) + \cdots + (d_m - d_{m-1})(n - d_m).$$

§I.2 Twistor space. We now want to use the notion of flag manifolds to introduce fundamental complex manifolds considered in twistor geometry. In order to describe field theories in four dimensions, we restrict ourselves to four dimensions. So, let $\mathbb{T}$ be a fixed complex four-dimensional vector space which we call twistor space. In due course, we shall also endow $\mathbb{T}$ with various real structures. Furthermore, we have a natural fibration in terms of flag manifolds

$$\begin{tikzcd} & F_{12}(\mathbb{T}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_1(\mathbb{T}) & & F_2(\mathbb{T}) \end{tikzcd} \tag{I.3}$$

together with the canonical projections $\pi_i(S_1, S_2) = S_i$ for $i = 1, 2$. In fact, the double fibration (I.3) is the first in a row of important double fibrations in twistor geometry we will encounter throughout this thesis. We shall now develop some of its basic properties. We follow the literature and define

$$\begin{aligned} \mathbb{P}^3 &:= F_1(\mathbb{T}) \ = \ \mathbb{C}P^3, \\ \mathbb{M}^4 &:= F_2(\mathbb{T}) \ = \ G_{2,4}(\mathbb{C}), \\ \mathbb{F}^5 &:= F_{12}(\mathbb{T}), \end{aligned} \tag{I.4}$$

and call them projective twistor space, compactified complexified four-dimensional space-time and correspondence space, respectively. At this point we stress that Minkowski and Euclidean spaces can naturally be realized as four-dimensional subsets of the complex four-dimensional manifold $\mathbb{M}^4$ (see Ref. [256] for a detailed discussion). By virtue of the double fibration (I.3), geometric data is transferred from $\mathbb{M}^4$ to $\mathbb{P}^3$ and vice versa according to the subsequent proposition (see, e.g., [256]):

Proposition I.1. *There is the following correspondence between points and subsets:*

$$\begin{array}{llcl} \text{(i)} & \textit{point in } \mathbb{P}^3 & \longleftrightarrow & \mathbb{C}P^2 \subset \mathbb{M}^4 \\ \text{(ii)} & \mathbb{C}P^1 \subset \mathbb{P}^3 & \longleftrightarrow & \textit{point in } \mathbb{M}^4 \end{array}$$

§I.3 Local coordinates. Next we introduce local coordinates. For this, we let

$$x \;=\; (x^{\alpha\dot\alpha}) \;\in\; \mathrm{Mat}(2,\mathbb{C}) \;\overset{\varphi}{\mapsto}\; \begin{bmatrix} x^{\alpha\dot\alpha} \\ \mathbb{1}_2 \end{bmatrix} \tag{I.5}$$

be a coordinate mapping for $\mathbb{M}^4$. The brackets denote, as usual, the span. Then we define a coordinate chart on $\mathbb{M}^4$ by

$$\mathcal{M}^4 \;:=\; \varphi(\mathrm{Mat}(2,\mathbb{C})) \;\cong\; \mathbb{C}^4. \tag{I.6}$$

We call $\mathcal{M}^4$ affine complexified four-dimensional space-time. Note that it is simply one of six possible choices of standard coordinate charts for $\mathbb{M}^4$. Using the projections $\pi_{1,2}$, we may naturally define the affine parts of $\mathbb{F}^5$ and $\mathbb{P}^3$ according to

$$\mathcal{F}^5 \;:=\; \pi_2^{-1}(\mathcal{M}^4) \qquad \text{and} \qquad \mathcal{P}^3 \;:=\; \pi_1(\pi_2^{-1}(\mathcal{M}^4)). \tag{I.7}$$

Then we have

$$\mathcal{F}^5 \;\cong\; \mathcal{M}^4 \times \mathbb{C}P^1. \tag{I.8}$$

To see this, let $[\lambda_{\dot\alpha}] = [\lambda_{\dot 1}, \lambda_{\dot 2}]$ be homogeneous coordinates on $\mathbb{C}P^1$ and x as above. Then consider the mapping

$$\begin{aligned} (x^{\alpha\dot\alpha}, [\lambda_{\dot\alpha}]) \;\mapsto\; & \left(\begin{bmatrix} x^{\alpha\dot\alpha} \\ \mathbb{1}_2 \end{bmatrix} \lambda_{\dot\alpha}, \begin{bmatrix} x^{\alpha\dot\alpha} \\ \mathbb{1}_2 \end{bmatrix} \right) \;=\; \left(\begin{bmatrix} x^{\alpha\dot\alpha}\lambda_{\dot\alpha} \\ \lambda_{\dot\alpha} \end{bmatrix}, \begin{bmatrix} x^{\alpha\dot\alpha} \\ \mathbb{1}_2 \end{bmatrix} \right) \\ =\; & \left(S_1^{x,\lambda}, S_2^{x,\lambda} \right) \;\in\; \mathbb{F}^5, \end{aligned}$$

which in fact proves (I.8).

It then follows by our above discussion that the projection $\pi_1 \,:\, \mathcal{F}^5 \to \mathcal{P}^3$ is given by

$$(x^{\alpha\dot\alpha}, [\lambda_{\dot\alpha}]) \;\overset{\pi_1}{\mapsto}\; \begin{bmatrix} x^{\alpha\dot\alpha}\lambda_{\dot\alpha} \\ \lambda_{\dot\alpha} \end{bmatrix} \in\; \mathbb{P}^3. \tag{I.9}$$

Therefore, in terms of these coordinates our double fibration (I.3) takes the following form:

$$\begin{array}{ccc} & (x^{\alpha\dot\alpha},[\lambda_{\dot\alpha}])\in\mathbb{F}^5 & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \begin{bmatrix} x^{\alpha\dot\alpha}\lambda_{\dot\alpha} \\ \lambda_{\dot\alpha}\end{bmatrix}\in\mathbb{P}^3 & & \begin{bmatrix} x^{\alpha\dot\alpha} \\ \mathbb{1}_2\end{bmatrix}\in\mathbb{M}^4 \end{array} \qquad \begin{array}{ccc} & \mathcal{F}^5 & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathcal{P}^3 & & \mathcal{M}^4 \end{array} \tag{I.10}$$

Altogether, these considerations allow us to introduce affine coordinates according to:

- $\mathcal{M}^4$: $x^{\alpha\dot\alpha}$,
- $\mathcal{F}^5$: $x^{\alpha\dot\alpha}$ and $\lambda_\pm$ with $\lambda_+ = \lambda_-^{-1}$ on $U_+ \cap U_-$, where $\{U_+, U_-\}$ denotes the canonical cover of $\mathbb{C}P^1$,
- $\mathcal{P}^3$: $z^\alpha_\pm$ and $z^3_\pm$; $\pi_1 : (x^{\alpha\dot\alpha}, \lambda_\pm) \mapsto (z^\alpha_\pm = x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha}, z^3_\pm = \lambda_\pm)$, with $\lambda^+_{\dot\alpha} = \lambda_+\lambda^-_{\dot\alpha}$ and $(\lambda^+_{\dot\alpha}) := {}^t(1, \lambda_+)$.

The last point deserves a little more attention. Let us denote the covering of $\mathcal{P}^3$ by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ the coordinates $(z^\alpha_\pm, z^3_\pm)$ satisfy

$$z^\alpha_+ \;=\; \frac{1}{z^3_-} z^\alpha_- \qquad \text{and} \qquad z^3_+ \;=\; \frac{1}{z^3_-}. \tag{I.11}$$

Hence, the twistor space[1] $\mathcal{P}^3$ is a rank two holomorphic vector bundle over the Riemann sphere. In more details, it is the total space of

$$\mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1) \;\to\; \mathbb{C}P^1, \tag{I.12}$$

where $\mathcal{O}_{\mathbb{C}P^1}(1)$ denotes the hyperplane line bundle (the dual of the tautological line bundle) over $\mathbb{C}P^1$. Moreover, the equations

$$z^\alpha_\pm \;=\; x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha}, \tag{I.13}$$

are also called incidence relations. In a given trivialization, holomorphic sections of the bundle $\mathcal{P}^3 \to \mathbb{C}P^1$ are of the form (I.13) and are parametrized by the moduli $x^{\alpha\dot\alpha}$. In other words, Eqs. (I.13) make Prop. I.1. transparent: a fixed point $p = (z^\alpha_\pm, z^3_\pm) \in \mathcal{P}^3$ corresponds to a null two-plane $\mathbb{C}^2_p \subset \mathcal{M}^4$ and furthermore, a fixed point $x = (x^{\alpha\dot\alpha}) \in \mathcal{M}^4$ corresponds to a holomorphic embedding of a rational degree one curve $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$. To see that these two-planes are indeed null, we solve (I.13) for a generic $p = (z^\alpha_\pm, z^3_\pm = \lambda_\pm)$,

$$x^{\alpha\dot\alpha} \;=\; \hat{x}^{\alpha\dot\alpha} + \mu^\alpha_\pm\lambda^{\dot\alpha}_\pm, \tag{I.14}$$

[1]By a slight abuse of terminology, we shall also refer to $\mathcal{P}^3$ as the twistor space.

where $\mu_\pm^\alpha$ is arbitrary. Here, $\hat{x}^{\alpha\dot{\alpha}}$ denotes a particular solution to (I.13). By recalling that any null four-vector $k^{\alpha\dot{\alpha}}$ can be decomposed as $k^{\alpha\dot{\alpha}} = v^\alpha w^{\dot{\alpha}}$ for two generic commuting spinors v^α and $w^{\dot{\alpha}}$, we have thus shown that $\mathbb{C}_p^2$ is totally null. Hence, $\mathcal{P}^3$ is the space of all null two-planes in $\mathcal{M}^4 \cong \mathbb{C}^4$.

§I.4 Remark. Two remarks are in order. By definition, the projective twistor space $\mathbb{P}^3$ is the same as $\mathbb{C}P^3$. On this space we may introduce homogeneous coordinates $[z^\alpha, \pi_{\dot{\alpha}}]$. Then consider $\mathbb{C}P^3 \setminus \mathbb{C}P^1$, where $\mathbb{C}P^1 \subset \mathbb{C}P^3$ is defined by setting $\pi_{\dot{\alpha}} = 0$ and $z^\alpha \neq 0$. We can cover $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ by two coordinate patches, say $\mathcal{U}_+$ and $\mathcal{U}_-$, for which $\pi_{\dot{1}} \neq 0$ and $\pi_{\dot{2}} \neq 0$, respectively, and introduce the coordinates

$$\begin{aligned} z_+^\alpha &:= \frac{z^\alpha}{\pi_{\dot{1}}} \quad \text{and} \quad z_+^3 := \frac{\pi_{\dot{2}}}{\pi_{\dot{1}}} \quad \text{on} \quad \mathcal{U}_+, \\ z_-^\alpha &:= \frac{z^\alpha}{\pi_{\dot{2}}} \quad \text{and} \quad z_-^3 := \frac{\pi_{\dot{1}}}{\pi_{\dot{2}}} \quad \text{on} \quad \mathcal{U}_-, \end{aligned} \tag{I.15}$$

which are related on $\mathcal{U}_+ \cap \mathcal{U}_-$ by (I.11). This shows that $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ is biholomorphic to $\mathcal{P}^3 = \mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1)$.

There is yet another interpretation of $\mathcal{P}^3 \to \mathbb{C}P^1$. The Riemann sphere $\mathbb{C}P^1$ can be emdedded into $\mathbb{C}P^3$. The normal bundle of $\mathbb{C}P^1$ inside $\mathbb{C}P^3$ is $\mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1)$. Kodaira's theorem on relative deformation theory [136, 137] states that if Y is a compact complex submanifold of a complex manifold – not necessarily compact, and if $H^1(Y, N_{Y|X}) \cong 0$, where $N_{Y|X}$ is the normal sheaf of Y in X, then there exists a d-parameter family of deformations of Y inside X, where $d = \dim_{\mathbb{C}} H^0(Y, N_{Y|X})$. In our example,

$$H^1(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1)) \;\cong\; 0$$

and $d = 4$.[2]

§I.5 Two other twistor spaces. Above, we have introduced (projective) twistor space in terms of a particular double fibration. However, this is not the only possibility as there is a variety of other twistor spaces which turn out to be extremely important in later applications. In the remainder of this section, we pick two examples which again are based on flag manifolds. Let us consider the following two correspondences:

$$\begin{tikzcd}[column sep=small] & F_{23}(\mathbb{T}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_3(\mathbb{T}) & & F_2(\mathbb{T}) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & F_{123}(\mathbb{T}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_{13}(\mathbb{T}) & & F_2(\mathbb{T}) \end{tikzcd} \tag{I.16}$$

[2] We shall often denote a vector bundle and the corresponding sheaf of sections by the same letter; cf. also footnote 5. In addition, we shall sometimes write N_Y instead of $N_{Y|X}$.

The space $F_2(\mathbb{T}) = \mathbb{M}^4$ is, of course, compactified complexified four-dimensional space-time. Note that the space $F_3(\mathbb{T})$, that is, the space of all three-dimensional subspaces in $\mathbb{T}$, is naturally dual to $F_1(\mathbb{T}^*)$ as we have a natural duality between lines and hyperplanes in a vector space. For that reason we call $F_3(\mathbb{T})$ dual projective twistor space and denote it by $\mathbb{P}^3_*$, in the following. The twistor manifold $F_{13}(\mathbb{T})$ is called projective ambitwistor space as it inherits aspects of the projective and the dual projective twistor spaces. We shall denote it by $\mathbb{L}^5$. Then one can prove the following proposition:

Proposition I.2. *There are the following correspondences:*

$$\begin{array}{llccc}
\text{(a)} & \text{(i)} & \textit{point in } \mathbb{P}^3_* & \longleftrightarrow & \mathbb{C}P^2 \subset \mathbb{M}^4 \\
 & \text{(ii)} & \mathbb{C}P^1_* \subset \mathbb{P}^3_* & \longleftrightarrow & \textit{point in } \mathbb{M}^4 \\
\text{(b)} & \text{(i)} & \textit{point in } \mathbb{L}^5 & \longleftrightarrow & \mathbb{C}P^1 \subset \mathbb{M}^4 \\
 & \text{(ii)} & \mathbb{C}P^1 \times \mathbb{C}P^1_* \subset \mathbb{L}^5 & \longleftrightarrow & \textit{point in } \mathbb{M}^4
\end{array}$$

Note that for all these manifolds we have coordinate representations similar to (I.5), and we shall develop a suitable notation for these coordinates as they are needed. We want to conclude this section by stressing that ambitwistor space $\mathbb{L}^5$ can be regarded as a submanifold in $\mathbb{P}^3 \times \mathbb{P}^3_*$ as there is an embedding

$$\begin{aligned}
\mathbb{L}^5 \ &\hookrightarrow\ \mathbb{P}^3 \times \mathbb{P}^3_*, \\
(S_1 \subset S_3) \ &\mapsto\ (S_1, S_3).
\end{aligned} \tag{I.17}$$

In fact, $\mathbb{L}^5$ is a degree two hypersurface in $\mathbb{P}^3 \times \mathbb{P}^3_*$ since this embedding is given by the zero locus

$$z^\alpha \rho_\alpha - w^{\dot\alpha} \pi_{\dot\alpha} \ =\ 0, \tag{I.18}$$

where $[z^\alpha, \pi_{\dot\alpha}]$ are homogeneous coordinates on $\mathbb{P}^3$ and $[\rho_\alpha, w^{\dot\alpha}]$ on $\mathbb{P}^3_*$, respectively. Eq. (I.18) is just the orthogonality relation between the vectors characterizing S_1 and the ones characterizing S_3, the latters being normal vectors to the hyperplanes. The minus sign in (I.18) has been chosen for convenience.

I.2 Supermanifolds

So far, we have discussed twistor spaces in the purely even (bosonic) setting. In the sequel, we extend the discussion to supertwistor spaces. To do this, let us first present some preliminaries.

§I.6 Graded rings and modules. The first notions we need are $\mathbb{Z}_2$-rings and modules and some relations among them. So, let $R \cong R_0 \oplus R_1$ be a $\mathbb{Z}_2$-graded ring, that is, $R_0R_0 \subset R_0$, $R_1R_0 \subset R_1$, $R_0R_1 \subset R_1$ and $R_1R_1 \subset R_0$. We call elements of R_0 even and elements of R_1 odd. An element of R is said to be homogeneous if it is either even or odd. The degree (or parity) of a homogeneous element is defined to be zero if it is even and one if it is odd, respectively. We denote the degree of a homogeneous element $r \in R$ by p_r (p for parity). We define the supercommutator, $[\cdot,\cdot\} : R \times R \to R$, by

$$[r_1, r_2\} \;:=\; r_1r_2 - (-)^{p_{r_1}p_{r_2}} r_2r_1, \tag{I.19}$$

for all homogeneous elements $r_{1,2} \in R$. The ring R is called supercommutative if the supercommutator vanishes for all of the ring's elements. An important example of such $\mathbb{Z}_2$-graded rings is the Graßmann algebra over $\mathbb{C}^n$,

$$R \;=\; \Lambda^\bullet\mathbb{C}^n \;:=\; \bigoplus_p \Lambda^p\mathbb{C}^n, \tag{I.20}$$

with the $\mathbb{Z}_2$-grading being

$$R \;=\; \bigoplus_p \Lambda^{2p}\mathbb{C}^n \oplus \bigoplus_p \Lambda^{2p+1}\mathbb{C}^n. \tag{I.21}$$

An R-module M is a $\mathbb{Z}_2$-graded bimodule which satisfies

$$rm \;=\; (-)^{p_rp_m} mr, \tag{I.22}$$

for all homogeneous $r \in R$, $m \in M$, with $M \cong M_0 \oplus M_1$. An additive mapping of R-modules, $\varphi : M \to N$, is called an even morphism if it preserves the grading and is R-linear. We denote the group of such morphisms by $\mathrm{Hom}_0(M,N)$. On the other hand, we call an additive mapping of R-modules odd if it reverses the grading, $p_{\varphi(m)} = p_m + 1$, and is R-linear, that is, $\varphi(rm) = (-)^{p_r} r\varphi(m)$ and $\varphi(mr) = \varphi(m)r$. The group of such morphisms is denoted by $\mathrm{Hom}_1(M,N)$. Then we set

$$\mathrm{Hom}(M,N) \;:=\; \mathrm{Hom}_0(M,N) \oplus \mathrm{Hom}_1(M,N)$$

and it can be given an R-module structure.

Furthermore, there is a natural mapping Π – called the parity map – defined by

$$(\Pi M)_0 \;:=\; M_1 \qquad \text{and} \qquad (\Pi M)_1 \;:=\; M_0 \tag{I.23}$$

and by requiring that i) addition in ΠM is the same as in M, ii) right multiplication by R is the same as in M and iii) left multiplication differs by a sign, i.e., $r\Pi(m) = (-)^{p_r}\Pi(rm)$ for $r \in R$, $m \in M$ and $\Pi(m) \in \Pi M$. Corresponding to the morphism $\varphi : M \to N$, we let $\varphi^\Pi : \Pi M \to \Pi N$ be the morphism which agrees with φ as a mapping of sets. Moreover, corresponding to the morphism $\varphi : M \to N$, we can find morphisms

$$\begin{aligned} \Pi\varphi : M &\to \Pi N, \qquad \text{with} \qquad (\Pi\varphi)(m) := \Pi(\varphi(m)), \\ \varphi\Pi : \Pi M &\to N, \qquad \text{with} \qquad (\varphi\Pi)(\Pi m) := \varphi(m), \end{aligned} \tag{I.24}$$

and hence $\varphi^\Pi = \Pi\varphi\Pi$. We stress that R is an R-module itself, and as such ΠR is, as well. However, ΠR is no longer a $\mathbb{Z}_2$-graded ring since $(\Pi R)_1(\Pi R)_1 \subset (\Pi R)_1$, for instance.

Next we need the notion of free R-modules. A free R-module of rank $m|n$ is defined to be an R-module isomorphic to

$$R^{m|n} := R^m \oplus (\Pi R)^n. \tag{I.25}$$

This has a free system of generators, m of which are even and n of which are odd, respectively. We stress that the decomposition of $R^{m|n}$ into $R^{m|0}$ and $R^{0|n}$ has, in general, no invariant meaning and does not coincide with the decomposition into even and odd parts,

$$[R_0^m \oplus (\Pi R_1)^n] \oplus [R_1^m \oplus (\Pi R_0)^n].$$

Only when $R_1 = 0$, these decompositions are the same.

§I.7 Supermatrices, supertranspose, supertrace and superdeterminant. Let R be a supercommutative ring and $R^{m|n}$ be a freely generated R-module. Just as in the commutative case, morphisms between free R-modules can be given by matrices. The standard matrix format is

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{I.26}$$

where A is said to be even (respectively, odd) if A_1 and A_4 are filled with even (respectively, odd) elements of the ring while A_2 and A_3 are filled with odd (respectively, even) elements. Furthermore, A_1 is a $p \times m$-, A_2 a $q \times m$-, A_3 a $p \times n$- and A_4 a $q \times n$-matrix. The set of matrices in standard format with elements in R is denoted by $\mathrm{Mat}(m|n, p|q, R)$. It forms a $\mathbb{Z}_2$-graded module which, with the usual matrix multiplication, is naturally isomorphic to $\mathrm{Hom}(R^{m|n}, R^{p|q})$. We denote by $GL(m|n, R)$ the even invertible automorphisms of $R^{m|n}$.

The supertranspose of $A \in \mathrm{Mat}(m|n, p|q, R)$ is defined according to

$$ {}^{\mathrm{st}}\!A \ := \ \begin{pmatrix} {}^{\mathrm{t}}\!A_1 & (-)^{p_A}\, {}^{\mathrm{t}}\!A_3 \\ -(-)^{p_A}\, {}^{\mathrm{t}}\!A_2 & {}^{\mathrm{t}}\!A_4 \end{pmatrix}, \tag{I.27} $$

where the superscript "t" denotes the usual transpose. The supertransposition satisfies ${}^{\mathrm{st}}(A+B) = {}^{\mathrm{st}}\!A + {}^{\mathrm{st}}\!B$ and ${}^{\mathrm{st}}(AB) = (-)^{p_A p_B}\, {}^{\mathrm{st}}\!B\, {}^{\mathrm{st}}\!A$.

We shall use the following definition of the supertrace of $A \in \mathrm{Mat}(m|n, p|q, R)$:

$$ \mathrm{str} A \ := \ \mathrm{tr} A_1 - (-)^{p_A} \mathrm{tr} A_4. \tag{I.28} $$

Clearly, the supertrace of the supercommutator vanishes and the supertrace of a supertransposed matrix is the same as the supertrace of the matrix one has started with.

Finally, let $A \in GL(m|n, R)$. The superdeterminant is given by

$$ \mathrm{sdet} A \ := \ \det(A_1 - A_2 A_4^{-1} A_3) \det A_4^{-1}, \tag{I.29} $$

where the right-hand side is well-defined for $A_1 \in GL(m|0, R_0)$ and $A_4 \in GL(n|0, R_0)$. Furthermore, it belongs to $GL(1|0, R_0)$, that is, to R_0. The superdeterminant satisfies also the usual rules,

$$ \mathrm{sdet}(AB) \ = \ \mathrm{sdet} A\, \mathrm{sdetB} \qquad \text{and} \qquad \mathrm{sdet}\, {}^{\mathrm{st}}\!A \ = \ \mathrm{sdet} A $$

for $A, B \in GL(m|n, R)$.

§I.8 Supermanifolds. We have now all necessary ingredients to give the definition of a supermanifold. As it will be most convenient for us, we shall follow Manin [169] and consider the graded space approach to supermanifolds.[3] The idea is roughly to extend the structure sheaf of a manifold to a sheaf of supercommutative rings. However, note that there are other approaches as, e.g., the one by DeWitt [78, 25] (see also [69] for a review). Furthermore, we stress in advance that the subsequent definition will not be the most generic one. For a more general discussion, we refer the interested reader to Ref. [169].

[3] We note that the systematic theory of smooth supermanifolds goes back to the work by Kostant [139] and Leites [159].

Definition I.1. *A complex supermanifold of complex dimension $m|n$ is a ringed space $X^{m|n} := (X, \mathcal{O}_X)$, where X is a topological space of real dimension $2m$, and $\mathcal{O}_X$ is a sheaf of supercommutative rings on X such that, if we let $\mathcal{N}$ be the ideal subsheaf in $\mathcal{O}_X$ of all nilpotent elements in $\mathcal{O}_X$, the following is fulfilled:*

(i) $X_{\rm red} = (X, \mathcal{O}_{\rm red} := \mathcal{O}_X/\mathcal{N})$ *is a complex manifold.*[4]

(ii) *For each point $x \in X$ there is a neighborhood $U \ni x$ such that*

$$\mathcal{O}_X|_U \;\cong\; \mathcal{O}_{\rm red}(\Lambda^\bullet \mathcal{E})|_U,$$

where $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_{\rm red}$-modules of purely even rank $n|0$ on X and $\Lambda^\bullet$ denotes the exterior algebra, i.e., the tensor algebra modulo the ideal generated by the superanticommutator.

We shall call $X_{\rm red}$ the body of $(X, \mathcal{O}_X)$. If no confusion arises, we will not mention the sheaf $\mathcal{O}_X$ explicitly. The latter is also called the sheaf of holomorphic superfunctions on X or simply structure sheaf of the supermanifold. Note that a similar definition can also be given for differentiable supermanifolds. Furthermore, as for purely even complex manifolds, it will turn out to be useful to consider the sheaf $\mathcal{S}_X$ of smooth complex-valued superfunctions on X, which is defined in a similar fashion. Clearly, $\mathcal{O}_X$ is a subsheaf of $\mathcal{S}_X$. In the sequel, we shall loosely speak of functions but always mean superfunctions. Moreover, we will often suppress the explicit appearance of the dimensionality and simply write X instead of $X^{m|n}$.

Let $(z^1, \ldots, z^m)$ be local coordinates on $U \subset X$ and $(\eta_1, \ldots, \eta_n)$ be basis sections of $\mathcal{E}$. Then $(z^1, \ldots, z^m, \eta_1, \ldots, \eta_n)$ is an even-odd system of coordinates for the space $(U, \mathcal{O}_{\rm red}(\Lambda^\bullet \mathcal{E})|_U)$. Any $f \in \Gamma(U, \mathcal{O}_{\rm red}(\Lambda^\bullet \mathcal{E}))$ can thus be expressed as

$$f(z, \eta) \;=\; \sum_I \eta^I f_I(z), \tag{I.30}$$

where I is a multiindex and the f_Is are local functions on $X_{\rm red}$.

Let us give a first example of a supermanifold: let $E \to X$ be a rank r holomorphic vector bundle over an ordinary complex manifold X and $\mathcal{E}$ the sheaf of sections of E.[5]

[4] Recall that a ringed space $(X, \mathcal{O}_X)$ with the property that for each $x \in X$ there is a neighborhood $U \ni x$ such that there is a ringed space isomorphism $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_V)$, where $V \subset \mathbb{C}^m$ and $\mathcal{O}_V$ is the sheaf of holomorphic functions on V can be given the structure of a complex manifold. Moreover, any complex manifold arises in this manner.

Then $(X, \mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}))$ is a supermanifold by the very definition, since $\mathcal{E}$ is locally free. Such a supermanifold is called globally split. A particular example is $\mathbb{C}^{m|n}$ defined by $\mathbb{C}^{m|n} = (\mathbb{C}^m, \mathcal{O}_{\rm red}(\Lambda^\bullet\mathbb{C}^n))$. Note that due to a theorem by Batchelor [27], any differentiable supermanifold is globally split. This is basically because of the existence of a partition of unity. The reader should be warned that, in general, complex supermanifolds are not globally split.

§I.9 Vector bundles. As for ordinary manifolds, we need the notions of tangent and cotangent bundles. Since these are just special vector bundles and in view of our later discussion it is certainly worth stating some more general words on vector bundles. The idea to define a holomorphic supervector bundle over a complex supermanifold $(X, \mathcal{O}_X)$ is to use the notion of a locally free sheaf of rank $r|s$, which is defined to be a sheaf of $\mathcal{O}_X$-modules locally isomorphic to $\mathcal{O}_X^{r|s} := \mathcal{O}_X^{\oplus r} \oplus (\Pi\mathcal{O}_X)^{\oplus s}$.[5] In the following, we shall suppress the prefix super: thus, vector bundle means supervector bundle. Next let $\varphi : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a holomorphic mapping of complex supermanifolds, that is, φ is defined to be the pair $(\phi, \tilde{\phi})$, where $\phi : Y \to X$ is a continuous mapping of topological spaces and $\tilde{\phi} : \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ is a morphism of sheaves of rings: for any open subset U in X there is a morphism $\tilde{\phi}_U : \mathcal{O}_X|_U \to \mathcal{O}_Y|_{\phi^{-1}(U)}$. Note that $\varphi_*\mathcal{O}_Y$ is also called the zeroth direct image sheaf of the sheaf $\mathcal{O}_Y$. By the pull-back via φ of a vector bundle $\mathcal{E}$ over X, we mean the locally free sheaf of $\mathcal{O}_Y$-modules

$$\varphi^*\mathcal{E} \ := \ \mathcal{O}_Y \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{E}, \tag{I.31}$$

where $\varphi^{-1}\mathcal{O}_X$ (respectively, $\varphi^{-1}\mathcal{E}$) denotes the topological inverse sheaf of $\mathcal{O}_X$ (respectively, of $\mathcal{E}$). For instance, $\varphi^{-1}\mathcal{O}_X$ is the sheaf on Y defined by the presheaf

$$Y \supset V \text{ open } \mapsto \ \Gamma(\phi(V), \mathcal{O}_X)$$

with the obvious restriction mappings. It is characterized by the property that the stalk $(\varphi^{-1}\mathcal{O}_X)_y = \mathcal{O}_X|_{\phi(y)}$ for all $y \in Y$. For a complex vector bundle, replace $\mathcal{O}_X$ by $\mathcal{S}_X$.

[5] Remember the following fact: Let X be a purely even complex manifold. Then there is a one-to-one correspondence between rank r locally free sheaves $\mathcal{E}$ of $\mathcal{O}_X$-modules on X and rank r holomorphic vector bundles E over X. In fact, if $\{U_i\}$ is an open covering of X and if $\{U_i\}$ trivializes $E \to X$, then the group of holomorphic sections of E over U_i is $\mathcal{O}_X^{\oplus r}|_{U_i}$. Conversely, if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules and if $\phi_i : \mathcal{E}|_{U_i} \to \mathcal{O}_X^{\oplus r}|_{U_i}$ is the explicit isomorphism, then on any $U_i \cap U_j$ we may define $r \times r$ matrices of holomorphic functions according to $\phi_j \circ \phi_i^{-1} : \mathcal{O}_X^{\oplus r}|_{U_i \cap U_j} \to \mathcal{O}_X^{\oplus r}|_{U_i \cap U_j}$ which give the transition functions of a holomorphic vector bundle $E \to X$.

The tangent sheaf TX of a complex supermanifold X is then the sheaf of local vector fields, that is, the superderivations of the ring of functions. It is locally free – in fact, it is a locally free sheaf of $\mathcal{O}_X$-modules – and of rank equal to the dimension of X. Let (z^i, η_j) be local coordinates on X. Then TX is freely generated by its sections $(\partial/\partial z^i, \partial/\partial\eta_j)$. The cotangent sheaf $\Omega^1(X)$ is defined to be

$$\Omega^1(X) \ := \ T^*X \ = \ \mathscr{H}om_{\mathcal{O}_X}(TX, \mathcal{O}_X), \tag{I.32}$$

where $\mathscr{H}om_{\mathcal{O}_X}(TX, \mathcal{O}_X)$ is the sheaf of local morphisms from $TX \to \mathcal{O}_X$. There is also an obvious differential $\mathrm{d} : \mathcal{O}_X \to \Omega^1(X)$ with the property $X \lrcorner\, \mathrm{d}f = Xf$ for $X \in TX$.

§I.10 Remark. In the sequel, we shall only be dealing with locally free sheaves. Therefore, we will often not make any notational distinction between such sheaves and the vector bundles corresponding to them, and we also use the two objects interchangeably.

I.3 Supertwistor spaces

Now we want to use the above machinery for generalizing the twistor spaces of Sec. I.1 to supertwistor spaces.

§I.11 Flag supermanifolds. For the sake of concreteness, let us consider $\mathbb{C}^{m|n}$. Then we define the flag superspace[6] $F_{d_1\cdots d_k}(\mathbb{C}^{m|n})$ to be the set of all k-tuples $(S_1, \ldots, S_k)$ of free submodules of $\mathbb{C}^{m|n}$ satisfying $S_1 \subset \cdots \subset S_k \subset \mathbb{C}^{m|n}$ and $d_i := \operatorname{rank} S_i = p_i|q_i$. This naturally generalizes our definition (I.1). In fact, one can also introduce suitable coordinate systems and thus a suitable structure sheaf which makes this set into a complex supermanifold. As in the purely even setting, we can give an equivalent definition by considering the decomposition

$$\mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2-d_2} \oplus \mathbb{C}^{d_3-d_2} \oplus \cdots \oplus \mathbb{C}^{d_k-d_{k-1}} \oplus \mathbb{C}^{m|n-d_k}$$

of $\mathbb{C}^{m|n}$, where $d_i - d_j = (p_i - p_j)|(q_i - q_j)$. Let now $U(m|n) \subset GL(m|n, \mathbb{C})$ be the unitary automorphisms of $\mathbb{C}^{m|n}$.[7] The subgroup of $U(m|n)$ which preserves this decomposition is

$$U(d_1) \times U(d_2 - d_1) \times \cdots \times U(d_k - d_{k-1}) \times U(m|n - d_k) \ \subset \ U(m|n)$$

[6] See also Ref. [122].

[7] Hermitian conjugation is defined by composing supertransposition and complex conjugation.

and hence we may write the quotient

$$F_{d_1\cdots d_k}(\mathbb{C}^{m|n}) \ := \ \frac{U(m|n)}{U(d_1)\times U(d_2-d_1)\times\cdots\times U(d_k-d_{k-1})\times U(m|n-d_k)}. \tag{I.33}$$

One can actually show that $U(m|n)$ has real dimension $m^2+n^2|2mn$. Then it is a rather straightforward exercise to determine the dimension of a flag supermanifold. So, we leave it to the interested reader. Note that flag supermanifolds are, in general, not globally split [169].

§I.12 Supertwistor space. Similarly to our discussion in Sec. I.1, we let $\mathbb{T}$ be of the form $\mathbb{C}^{4|\mathcal{N}}$. We call this space $\mathcal{N}$-extended supertwistor space [96]. Then we introduce

$$\begin{aligned} \mathbb{P}^{3|\mathcal{N}} &:= F_{1|0}(\mathbb{T}),\\ \mathbb{M}_R^{4|2\mathcal{N}} &:= F_{2|0}(\mathbb{T}),\\ \mathbb{F}_R^{5|2\mathcal{N}} &:= F_{1|0,2|0}(\mathbb{T}). \end{aligned} \tag{I.34}$$

This time we call them projective supertwistor space, compactified complexified four-dimensional antichiral superspace-time and correspondence space, respectively. Therefore, the double fibration (I.3) becomes:

$$\begin{tikzcd} & \mathbb{F}_R^{5|2\mathcal{N}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^{3|\mathcal{N}} & & \mathbb{M}_R^{4|2\mathcal{N}} \end{tikzcd} \tag{I.35}$$

Note that $\mathbb{P}^{3|\mathcal{N}} = \mathbb{C}P^{3|\mathcal{N}} = (\mathbb{P}^3, \mathcal{O}_{\mathrm{red}}(\Lambda^\bullet(\mathbb{C}^{\mathcal{N}}\otimes\mathcal{O}_{\mathbb{P}^3}(-1))))$, where $\mathcal{O}_{\mathbb{P}^3}(-1)$ is the tautological sheaf on $\mathbb{P}^3$. Furthermore, Prop. I.1. generalizes accordingly.

Proposition I.3. *There is the following geometric correspondence:*

$$\begin{array}{llcl} \text{(i)} & \textit{point in } \mathbb{P}^{3|\mathcal{N}} & \longleftrightarrow & \mathbb{C}P^{2|\mathcal{N}} \subset \mathbb{M}_R^{4|2\mathcal{N}} \\ \text{(ii)} & \mathbb{C}P^{1|0} \subset \mathbb{P}^{3|\mathcal{N}} & \longleftrightarrow & \textit{point in } \mathbb{M}_R^{4|2\mathcal{N}} \end{array}$$

In the following, we shall abbreviate $X^{m|0} \equiv X^m$ for any ordinary manifold X. As before, we may introduce local coordinates. Eventually, one finds

$$\begin{tikzcd} & \mathcal{F}_R^{5|2\mathcal{N}} \cong \mathcal{M}_R^{4|2\mathcal{N}} \times \mathbb{C}P^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^{3|\mathcal{N}} & & \mathcal{M}_R^{4|2\mathcal{N}} \end{tikzcd} \tag{I.36}$$

together with

- $\mathcal{M}_R^{4|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}}$: $x_R^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$, where $i = 1, \ldots, \mathcal{N}$,
- $\mathcal{F}_R^{5|2\mathcal{N}}$: $x_R^{\alpha\dot\alpha}$, $\eta_i^{\dot\alpha}$ and $\lambda_\pm$ with $\lambda_+ = \lambda_-^{-1}$ on $U_+ \cap U_-$, where $\{U_+, U_-\}$ denotes again the canonical cover of $\mathbb{C}P^1$,
- $\mathcal{P}^{3|\mathcal{N}}$: $z_\pm^\alpha$, $z_\pm^3$ and $\eta_i^\pm$; $\pi_1 : (x_R^{\alpha\dot\alpha}, \lambda_\pm, \eta_i^{\dot\alpha}) \mapsto (z_\pm^\alpha = x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm, z_\pm^3 = \lambda_\pm, \eta_i^\pm = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm)$, with $\lambda_{\dot\alpha}^+ = \lambda_+\lambda_{\dot\alpha}^-$ and $(\lambda_{\dot\alpha}^+) := {}^t(1, \lambda_+)$.

Again, the last point shows that $\mathcal{P}^{3|\mathcal{N}}$ is a rank $2|\mathcal{N}$ holomorphic vector bundle

$$\mathcal{P}^{3|\mathcal{N}} \;=\; \mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^{\mathcal{N}}. \tag{I.37}$$

Moreover, the relations

$$z_\pm^\alpha \;=\; x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm \qquad \text{and} \qquad \eta_i^\pm \;=\; \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm \tag{I.38}$$

explicitly say that a fixed point $p = (z_\pm^\alpha, z_\pm^3, \eta_i^\pm) \in \mathcal{P}^{3|\mathcal{N}}$ corresponds to a null $2|\mathcal{N}$-dimensional subspace of $\mathcal{M}_R^{4|2\mathcal{N}}$. By analogy with the purely even setting, we shall refer to those as super null planes of dimension $2|\mathcal{N}$ in $\mathcal{M}_R^{4|2\mathcal{N}}$. Hence, $\mathcal{P}^{3|\mathcal{N}}$ is the space of super null planes of $\mathcal{M}_R^{4|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}}$. On the other hand, a fixed point $(x_R, \eta) = (x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathcal{M}_R^{4|2\mathcal{N}}$ corresponds to a holomorphic embedding of a rational curve $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. Before closing this paragraph, we would like to stress that the argumentation of §I.4 also applies here.

§I.13 Two other supertwistor spaces. Above, we have introduced the (projective) supertwistor space. Let us now also generalize the other two twistor spaces given in §I.5: the dual projective supertwistor space and the projective superambitwistor space, respectively. So, let us consider the following two double fibrations:

$$\begin{tikzcd}[column sep=small] & F_{2|\mathcal{N},3|\mathcal{N}}(\mathbb{T}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_{3|\mathcal{N}}(\mathbb{T}) & & F_{2|\mathcal{N}}(\mathbb{T}) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & F_{1|0,2|0,2|\mathcal{N},3|\mathcal{N}}(\mathbb{T}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_{1|0,3|\mathcal{N}}(\mathbb{T}) & & F_{2|0,2|\mathcal{N}}(\mathbb{T}) \end{tikzcd} \tag{I.39}$$

The space $\mathbb{M}_L^{4|2\mathcal{N}} := F_{2|\mathcal{N}}(\mathbb{T})$ is compactified complexified four-dimensional chiral super space-time, which is naturally dual to $F_{2|0}(\mathbb{T}^*)$. Note also that the space $F_{3|\mathcal{N}}(\mathbb{T})$ is naturally dual to $F_{1|0}(\mathbb{T}^*)$. For that reason we call $F_{3|\mathcal{N}}(\mathbb{T})$ dual projective supertwistor space and denote it by $\mathbb{P}_*^{3|\mathcal{N}}$. Furthermore, $\mathbb{M}^{4|4\mathcal{N}} := F_{2|0,2|\mathcal{N}}(\mathbb{T})$ is compactified complexified superspace-time and obviously, we have the natural fibration:

$$\begin{tikzcd}[column sep=small] & \mathbb{M}^{4|4\mathcal{N}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{M}_R^{4|2\mathcal{N}} & & \mathbb{M}_L^{4|2\mathcal{N}} \end{tikzcd} \tag{I.40}$$

The manifold $F_{1|0,3|\mathcal{N}}(\mathbb{T})$ is called projective superambitwistor space and we shall denote it by $\mathbb{L}^{5|2\mathcal{N}}$. Then one can show:

Proposition I.4. *There are the following correspondences:*

(a)	(i)	*point in* $\mathbb{P}^{3|\mathcal{N}}_*$	$\longleftrightarrow$	$\mathbb{C}P^{2|\mathcal{N}} \subset \mathbb{M}^{4|2\mathcal{N}}_L$
	(ii)	$\mathbb{C}P^1_* \subset \mathbb{P}^{3|\mathcal{N}}_*$	$\longleftrightarrow$	*point in* $\mathbb{M}^{4|2\mathcal{N}}_L$
(b)	(i)	*point in* $\mathbb{L}^{5|2\mathcal{N}}$	$\longleftrightarrow$	$\mathbb{C}P^{1|2\mathcal{N}} \subset \mathbb{M}^{4|4\mathcal{N}}$
	(ii)	$\mathbb{C}P^1 \times \mathbb{C}P^1_* \subset \mathbb{L}^{5|2\mathcal{N}}$	$\longleftrightarrow$	*point in* $\mathbb{M}^{4|4\mathcal{N}}$

Note that for all these manifolds we can construct coordinate representations similar to those given in §I.12. As before, we shall develop suitable notation for these coordinates as we need them.

Finally, we want to stress that also the projective superambitwistor space $\mathbb{L}^{5|2\mathcal{N}}$ can be viewed as a degree two hypersurface in $\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*$. In fact, if we let $[z^\alpha, \pi_{\dot\alpha}, \eta_i]$ be homogeneous coordinates on $\mathbb{P}^{3|\mathcal{N}}$ and $[\rho_\alpha, w^{\dot\alpha}, \theta^i]$ on $\mathbb{P}^{3|\mathcal{N}}_*$, respectively, then $\mathbb{L}^{5|2\mathcal{N}}$ is given by $\mathbb{L}^{5|2\mathcal{N}} = (\mathbb{L}^5, \mathcal{O}_{\mathbb{L}^{5|2\mathcal{N}}} = \mathcal{O}_{\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*}/\mathcal{I})$, with

$$\mathcal{O}_{\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*} \ = \ \mathcal{O}_{\text{red}}(\Lambda^\bullet(\mathbb{C}^{\mathcal{N}} \otimes \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3_*}(-1,0) \oplus \mathbb{C}^{\mathcal{N}*} \otimes \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3_*}(0,-1))) \tag{I.41}$$

and $\mathcal{I}$ is the ideal subsheaf in $\mathcal{O}_{\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*}$ given by

$$\mathcal{I} \ = \ \langle z^\alpha \rho_\alpha - w^{\dot\alpha}\pi_{\dot\alpha} + 2\theta^i \eta_i \rangle. \tag{I.42}$$

Furthermore, we have abbreviated

$$\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3_*}(m,n) \ := \ \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^3}(m) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^3_*}(n), \tag{I.43}$$

where $\mathrm{pr}_1 : \mathbb{P}^3 \times \mathbb{P}^3_* \to \mathbb{P}^3$ and $\mathrm{pr}_2 : \mathbb{P}^3 \times \mathbb{P}^3_* \to \mathbb{P}^3_*$, respectively.

I.4 Connections and curvature

So far, we have been dealing with just supermanifolds and vector bundles. Next we want to introduce some additional geometric structure, such as connections. This will then allow us to talk about curvature and characteristic classes.

§I.14 Connections and curvature. As we have already mentioned in §I.9, the notion of a holomorphic vector bundle over a complex supermanifold $(X, \mathcal{O}_X)$ is equivalent to the notion of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules. Then a connection ∇ is an even morphism of sheaves

$$\nabla : \mathcal{E} \;\to\; \Omega^1(X) \otimes \mathcal{E} \tag{I.44}$$

satisfying the Leibniz formula

$$\nabla(f\sigma) \;=\; \mathrm{d}f \otimes \sigma + f\nabla\sigma, \tag{I.45}$$

where f is a local holomorphic function on X and σ is a local section of $\mathcal{E}$. Let $(Z^I) = (z^i, \eta_j)$ be local coordinates on X and TX be generated by $(\partial/\partial Z^I) = (\partial/\partial z^i, \partial/\partial\eta_j)$. Therefore,

$$\mathrm{d} \;=\; \mathrm{d}Z^I\partial_I \;=\; \mathrm{d}z^i\frac{\partial}{\partial z^i} + \mathrm{d}\eta_j\frac{\partial}{\partial\eta_j} \tag{I.46}$$

and (I.45) reads as

$$\nabla_I(f\sigma) \;=\; (\partial_I f)\sigma + (-)^{p_I p_f} f\nabla_I\sigma. \tag{I.47}$$

Locally, ∇ has the form

$$\nabla \;=\; \mathrm{d} + \mathcal{A}, \tag{I.48}$$

where $\mathcal{A} \in \Gamma(X, \Omega^1(X) \otimes \operatorname{End}\mathcal{E})$.

As usual, ∇^2 induces the curvature

$$\mathcal{F} \;\in\; \Gamma(X, \Lambda^2\Omega^1(X) \otimes \operatorname{End}\mathcal{E}), \tag{I.49}$$

where $\Lambda^\bullet$ denotes the exterior algebra, i.e., it is the tensor algebra modulo the ideal generated by the superanticommutator. Here, we have just introduced one version of a holomorphic de Rham complex on supermanifolds. There are other possible ways; see, e.g., Ref. [169] for details. Note that the above definitions carry naturally over to complex vector bundles.

§I.15 Integral forms and Berezin integral. In the purely even setting, differential forms are objects which can be integrated over. However, in the case of supermanifolds, the situation is more subtle. First, let us introduce the holomorphic Berezinian. Let X be a complex supermanifold with tangent sheaf TX. The holomorphic Berezinian line bundle, $\operatorname{Ber} X$, or holomorphic Berezinian for short, is defined to be the line bundle over X having holomorphic super Jacobians as transition functions. Thus, it can be considered as the natural extension of the canonical sheaf (sheaf of sections of the canonicle bundle) on a purely even complex manifold. Then integral forms are defined to be sections of the sheaf

$$\Sigma^\bullet(X) \;:=\; \operatorname{Ber} X \otimes \Lambda^\bullet(TX) \;=\; \mathscr{H}om_{\mathcal{O}_X}(\Lambda^\bullet\Omega^1(X), \operatorname{Ber} X). \tag{I.50}$$

Thus, $\operatorname{Ber} X = \Sigma^0(X)$ and if X is $m|n$ dimensional with local coordinates (z^i, η_j), then $\omega \in \operatorname{Ber} X$ is locally of the form

$$\omega \;=\; \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^m \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n \otimes f, \tag{I.51}$$

where f is some local section of $\mathcal{O}_X$. Sections of $\operatorname{Ber} X$ are also called holomorphic volume forms. A complexification $T_{\mathbb{C}}X$ of TX splits into a direct sum $T_{\mathbb{C}}X \cong T^{1,0}_{\mathbb{C}}X \oplus T^{0,1}_{\mathbb{C}}X$. Clearly, this then implies that

$$\begin{aligned}
\Sigma^k_{\mathbb{C}}(X) &= \operatorname{Ber}_{\mathbb{C}} X \otimes \Lambda^k(T_{\mathbb{C}}X) \\
&= \bigoplus_{p+q=k} \operatorname{Ber}^{1,0}_{\mathbb{C}} X \otimes \Lambda^p(T^{1,0}_{\mathbb{C}}X) \otimes \operatorname{Ber}^{0,1}_{\mathbb{C}} X \otimes \Lambda^q(T^{0,1}_{\mathbb{C}}X) \\
&=: \bigoplus_{p+q=k} \Sigma^{p,q}_{\mathbb{C}}(X).
\end{aligned} \tag{I.52}$$

Let now X be a differentiable supermanifold with local coordinates (x^i, η_j). Furthermore, suppose that X_{red} is connected and given an orientation. Let $\operatorname{Ber}_0 X = \Gamma_0(X, \operatorname{Ber} X)$, that is, the volume forms with compact support. Then $\omega \in \operatorname{Ber}_0 X$ is locally

$$\begin{aligned}
\omega &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n \otimes \sum_I \eta^I f_I(x) \\
&= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n \otimes \sum_{\{|I_i|\leq 1\}} \eta_1^{I_1} \cdots \eta_n^{I_n} f_{I_1 \cdots I_n}(x),
\end{aligned} \tag{I.53}$$

where the f_Is are local functions on X_{red}. We define the Berezin integral as

$$\int_X \omega \;:=\; (-)^{\frac{n}{2}(n-1)} \int_{X_{\text{red}}} \mathrm{d}x^1 \cdots \mathrm{d}x^m f_{1\cdots 1}(x). \tag{I.54}$$

For a more general volume form with compact support, we define the integral by using a partition of unity and the additivity property of the integral. Furthermore, if we let $\Sigma^k_0(X)$ be the integral k-forms with compact support, then such a form can be hooked into a differential k-form with compact support, i.e., an element of $\Lambda^k\Omega^1_0(X)$, to give an element of $\operatorname{Ber}_0 X$ which, by virtue of the above formula, can be integrated over X.

§I.16 Remark. Two things are worth mentioning. First of all, integral forms can be given the structure of a complex, since one may introduce a mapping $\delta : \Sigma^\bullet(X) \to \Sigma^\bullet(X)$ with $\delta^2 = 0$. Therefore, one can discuss cohomology of integral forms [169]. Another issue concerns the integration theory on supermanifolds. In general, one is interested in objects which can be integrated over immersed sub(super)manifolds Y of some supermanifold X. One way of doing this is to introduce so-called k-densities. They are sections of $(\mathrm{Ber}\,\Pi\mathcal{T}^*)^*$, where $\mathcal{T}$ is the tautological sheaf on the relative Graßmannian $G_X(k, TX) \to X$. Then if $\varphi : Y \to X$ is an immersion of supermanifolds and ω a density on X, one can canonically define a volume form $\varphi^*(\omega)$ on Y. We shall not go into further details at this point and refer the reader to the book by Manin [169] for a nice discussion of these aspects. We also refer to the work by Bernstein and Leites [54, 55] who generalized the integration theory of volume forms for the first time.

§I.17 Formal Calabi-Yau supermanifolds. Given a rank $r|s$ complex vector bundle $(\mathcal{E}, \nabla)$ over a complex supermanifold X, we define the k-th Chern class of $\mathcal{E}$ to be

$$c_k(\mathcal{E}) \ := \ \frac{1}{k!}\, \frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_{t=0} \mathrm{sdet}\,(\mathbb{1} + t\,\omega) \qquad \text{for} \qquad k \ \leq \ r+s, \tag{I.55}$$

where $\omega := \frac{1}{2\pi\mathrm{i}}\mathcal{F}$ is (up to an overall factor) the curvature of ∇. The first few Chern classes are given by:

$$\begin{aligned} c_0(\mathcal{E}) \ &= \ 1, \\ c_1(\mathcal{E}) \ &= \ \mathrm{str}\,\omega, \\ c_2(\mathcal{E}) \ &= \ \tfrac{1}{2}((\mathrm{str}\,\omega)^2 - \mathrm{str}\,\omega^2), \\ &\ \ \vdots \end{aligned} \tag{I.56}$$

The total Chern class is then $c(\mathcal{E}) = \sum_{k=0}^{r+s} c_k(\mathcal{E})$. Note that in a similar fashion, one may also introduce the k-th Chern character according to

$$ch_k(\mathcal{E}) \ := \ \frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_{t=0} \mathrm{str}\,\exp(t\,\omega) \qquad \text{for} \qquad k \ \leq \ r+s. \tag{I.57}$$

As in the purely even setting, one may prove the following useful result: for a short exact sequence of complex vector bundles over X,

$$0 \ \to \ \mathcal{F} \ \to \ \mathcal{E} \ \to \ \mathcal{G} \ \to \ 0, \tag{I.58}$$

we have

$$c(\mathcal{E}) \ = \ c(\mathcal{F})c(\mathcal{G}). \tag{I.59}$$

In particular, this formula yields

$$c_1(\mathcal{E}) \;=\; c_1(\mathcal{F}) + c_1(\mathcal{G}). \tag{I.60}$$

More details about Chern classes, Chern characters, etc. can be found, e.g., in the book by Bartocci et al. [25].

Furthermore, the total Chern class of a complex supermanifold X is defined to be the total Chern class of TX. In this case, we shall simply write $c(X)$. In the purely even case, we have a relation between the first Chern class of a vector bundle and its determinant line bundle. In fact, both agree (up to a sign). The question is whether we have an extension of this relation to our present setting. The analog of the determinant line bundle is, as we have already seen above, the superdeterminant line bundle – the Berezinian line bundle. Using splitting principle arguments, one may indeed deduce that the first Chern class of $\operatorname{sdet}\mathcal{E}$ for some vector bundle $\mathcal{E}$ coincides, again up to a sign, with the first Chern class of $\mathcal{E}$,

$$c_1(\operatorname{sdet}\mathcal{E}) \;=\; \mp c_1(\mathcal{E}), \tag{I.61}$$

where the sign depends on whether $\operatorname{sdet}\mathcal{E}$ is of rank $1|0$ or $0|1$, respectively. When talking about TX, we shall use our old notation $\operatorname{Ber} X$.

Definition I.2. *Let X be a complex supermanifold. Then X is called a formal Calabi-Yau supermanifold if it fulfills the following equivalent conditions:*

(i) *The first Chern class of X vanishes.*

(ii) *The holomorphic Berezinian of X is trivial.*

(iii) *There exists a globally defined and nowhere vanishing holomorphic volume form.*

Before we will discuss some examples, let us point out an important issue: in contrast to ordinary Calabi-Yau manifolds, formal Calabi-Yau supermanifolds do not necessarily admit Ricci-flat metrics – even if one assumes compactness. For some expositions on this issue, see Refs. [226] and [212, 271, 213, 219, 160].

Let us now discuss some examples. First, consider the projective superspace $\mathbb{C}P^{m|n} = (\mathbb{C}P^m, \mathcal{O}_{\mathbb{C}P^{m|n}})$ with $\mathcal{O}_{\mathbb{C}P^{m|n}} = \mathcal{O}_{\text{red}}(\Lambda^\bullet(\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{C}P^m}(-1)))$. To compute the total Chern class $c(\mathbb{C}P^{m|n})$, we use the short exact sequence

$$0 \;\to\; \mathcal{O}_{\mathbb{C}P^{m|n}} \;\to\; \mathcal{O}_{\mathbb{C}P^{m|n}}(1) \otimes \mathbb{C}^{m+1} \oplus \Pi\mathcal{O}_{\mathbb{C}P^{m|n}}(1) \otimes \mathbb{C}^n \;\to\; T\mathbb{C}P^{m|n} \;\to\; 0, \tag{I.62}$$

as we have to take equivalence classes with respect to overall rescalings. Here, $\mathcal{O}_{\mathbb{C}P^{m|n}}(1)$ denotes the dual tautological sheaf on $\mathbb{C}P^{m|n}$. Note that the above sequence is nothing but a $\mathbb{Z}_2$-graded extension of the Euler sequence.[8] Using our relations given in §I.17, we find

$$c(\mathbb{C}P^{m|n}) \;=\; c(\mathcal{O}_{\mathbb{C}P^{m|n}}(1)\otimes\mathbb{C}^{m+1}\oplus\Pi\mathcal{O}_{\mathbb{C}P^{m|n}}(1)\otimes\mathbb{C}^{n})$$

which immediately implies

$$c_1(\mathbb{C}P^{m|n}) \;=\; c_1(\mathcal{O}_{\mathbb{C}P^{m|n}}(1)\otimes\mathbb{C}^{m+1}) - c_1(\mathcal{O}_{\mathbb{C}P^{m|n}}(1)\otimes\mathbb{C}^{n})$$

and hence

$$c_1(\mathbb{C}P^{m|n}) \;=\; (m+1-n)x, \tag{I.63}$$

where $x := c_1(\mathcal{O}_{\mathbb{C}P^{m|n}}(1))$. Thus, we may conclude that our supertwistor space $\mathbb{P}^{3|\mathcal{N}} = \mathbb{C}P^{3|\mathcal{N}}$ becomes a formal Calabi-Yau supermanifold if and only if $\mathcal{N}=4$.

A similar argumentation can be given for superambitwistor space $\mathbb{L}^{5|2\mathcal{N}}$ as introduced in §I.13. There, it was shown that $\mathbb{L}^{5|2\mathcal{N}}$ can be realized as a hypersurface in $\mathbb{P}^{3|\mathcal{N}}\times\mathbb{P}^{3|\mathcal{N}}_*$. Thus, we have a short exact sequence of sheaves

$$0 \;\to\; T\mathbb{L}^{5|2\mathcal{N}} \;\to\; \mathcal{O}_{\mathbb{L}^{5|2\mathcal{N}}}\otimes T(\mathbb{P}^{3|\mathcal{N}}\times\mathbb{P}^{3|\mathcal{N}}_*) \;\to\; N_{\mathbb{L}^{5|2\mathcal{N}}} \;\to\; 0, \tag{I.64}$$

where $N_{\mathbb{L}^{5|2\mathcal{N}}}$ is the normal sheaf of $\mathbb{L}^{5|2\mathcal{N}}$ in $\mathbb{P}^{3|\mathcal{N}}\times\mathbb{P}^{3|\mathcal{N}}_*$. In fact, $N_{\mathbb{L}^{5|2\mathcal{N}}}\cong\mathcal{O}_{\mathbb{L}^{5|2\mathcal{N}}}(1,1) = \mathcal{O}_{\mathbb{P}^{3|\mathcal{N}}\times\mathbb{P}^{3|\mathcal{N}}_*}(1,1)/\mathcal{I}$, where $\mathcal{I}$ is the ideal subsheaf (I.42).[9] Then a short calculation reveals that (I.64) implies

$$c_1(\mathbb{L}^{5|2\mathcal{N}}) \;=\; (3-\mathcal{N})x + (3-\mathcal{N})y, \tag{I.65}$$

where $x := c_1(\mathcal{O}_{\mathbb{L}^{5|2\mathcal{N}}}(1,0))$ and $y := c_1(\mathcal{O}_{\mathbb{L}^{5|2\mathcal{N}}}(0,1))$, respectively. Hence, $\mathbb{L}^{5|6}$ is a formal Calabi-Yau supermanifold. Furthermore, it has been shown [184, 7] that $\mathbb{L}^{5|6}$ and $\mathbb{P}^{3|4}$ are related in some sense by mirror symmetry. For related aspects of mirror symmetry see also Refs. [142, 8, 35, 197, 211, 143].

Before coming to our next topic, let us illustrate a final example given by LeBrun [149]. Let X be some ordinary complex manifold. As we have seen in §I.8, $(X,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}^*))$ is a complex supermanifold, where $\mathcal{E}$ is a rank $r|0$ locally free sheaf of $\mathcal{O}_{\rm red}$-modules. The holomorphic Berezinian of $(X,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}^*))$ is then given by $\mathrm{Ber}\,(X,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}^*)) =$

[8] The proof of (I.62) is a straightforward extension of the one given in the purely even setting. The proof of the Euler sequence for $\mathbb{C}P^m$ can be found in, e.g., [107].

[9] See also our discussion given in §IV.9.

$\mathcal{K}\otimes\Lambda^r\mathcal{E}$, where $\mathcal{K}$ is the canonical sheaf (sheaf of sections of the canonical bundle) on X. Next one can show that there is a short exact sequence of $\mathcal{O}_{\rm red}$-modules on X

$$0 \ \to \ \Omega^1(X)\otimes\mathcal{F} \ \to \ {\rm Jet}^1\mathcal{F} \ \to \ \mathcal{F} \ \to \ 0, \tag{I.66}$$

where ${\rm Jet}^1\mathcal{F}$ is the sheaf of first-order jets for $\mathcal{F}$.[10] Let now X be some complex three-dimensional manifold which admits a spin structure. Then X can be extended to a formal Calabi-Yau supermanifold of dimension $3|4$ by setting $\mathcal{E} = {\rm Jet}^1\mathcal{K}^{-1/2}$ since from the above sequence we obtain

$$\begin{aligned}\Lambda^4\mathcal{E} \ &\cong \ \mathcal{K}^{-1/2}\otimes\Lambda^3(\Omega^1(X)\otimes\mathcal{K}^{-1/2})\\ &\cong \ \mathcal{K}^{-1/2}\otimes\Lambda^3\Omega^1(X)\otimes\mathcal{K}^{-3/2}\\ &\cong \ \mathcal{K}^{-1/2}\otimes\mathcal{K}\otimes\mathcal{K}^{-3/2} \ \cong \ \mathcal{K}^{-1}\end{aligned}$$

and hence ${\rm Ber}\,(X,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}^*)) = \mathcal{K}\otimes\Lambda^4\mathcal{E} \ \cong \ \mathcal{K}\otimes\mathcal{K}^{-1}$ is trivial. Therefore, we can conclude that $(X,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathcal{E}^*)$ is a formal Calabi-Yau supermanifold. For instance, the supertwistor space $\mathbb{P}^{3|4}$ fits into this construction scheme, since $\mathcal{K} = \mathcal{O}_{\mathbb{P}^3}(-4)$ and hence ${\rm Jet}^1\mathcal{K}^{-1/2} \cong \mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathbb{C}^4$, as can most easily be deduced from the Euler sequence (I.62).

I.5 Real structures

Up to now, we have only dealt with complex (super)twistor spaces. In order to discuss real gauge theories, that is, gauge theories living on either Euclidean or Minkowski spaces (or conformal compactifications thereof) with some unitary group as gauge group, we need to put certain real structures on the supertwistor space $\mathbb{T}$ and its dual $\mathbb{T}^*$. This in turn induces real structures on all the supermanifolds appearing in our double fibrations. Therefore, our first goal is to choose proper coordinates on $\mathbb{T}$ and $\mathbb{T}^*$ as well as on the supermanifolds participating in (I.40). Recall that in §I.12 and §I.13 we have already given partial results on this matter.

§I.18 Local coordinates. In §I.13, we have denoted the coordinates on $\mathbb{T}$ by $(z^\alpha,\pi_{\dot\alpha},\eta_i)$ and on $\mathbb{T}^*$ by $(\rho_\alpha, w^{\dot\alpha},\theta^i)$, respectively.[11] Furthermore, in the same paragraph we also discussed the canonical bilinear form

$$\langle(z^\alpha,\pi_{\dot\alpha},\eta_i),(\rho_\alpha,w^{\dot\alpha},\theta^i)\rangle \ = \ z^\alpha\rho_\alpha - w^{\dot\alpha}\pi_{\dot\alpha} + 2\theta^i\eta_i \tag{I.67}$$

[10] For a proof, see, e.g., Ref. [169].

[11] In fact, they are just homogeneous coordinates for the projectivized versions.

as an element of $\mathbb{T}\otimes\mathbb{T}^*$. Consider the double fibration (I.40),

$$\begin{tikzcd} & \mathbb{M}^{4|4\mathcal{N}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{M}_R^{4|2\mathcal{N}} & & \mathbb{M}_L^{4|2\mathcal{N}} \end{tikzcd} \tag{I.68}$$

and recall that $\mathbb{M}_R^{4|2\mathcal{N}} = F_{2|0}(\mathbb{T})$, $\mathbb{M}^{4|4\mathcal{N}} = F_{2|0,2|\mathcal{N}}(\mathbb{T})$ and $\mathbb{M}_L^{4|2\mathcal{N}} = F_{2|\mathcal{N}}(\mathbb{T}) \cong F_{2|0}(\mathbb{T}^*)$. According to §I.12, local coordinates on $\mathbb{M}_R^{4|2\mathcal{N}}$ are $(x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha})$. In a similar fashion, one may take $(x_L^{\alpha\dot\alpha},\theta^{i\alpha})$ as local coordinates on $\mathbb{M}_L^{4|2\mathcal{N}}$. A $(2|0,2|\mathcal{N})$-flag in $\mathbb{T}$ is the same thing as a pair of $2|0$-dimensional subspaces in $\mathbb{T}$ and $\mathbb{T}^*$, orthogonal with respect to the bilinear form (I.67). Making use of the identifications

$$(z^\alpha = x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \pi_{\dot\alpha}=\lambda_{\dot\alpha}, \eta_i = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}) \quad \text{and} \quad (w^{\dot\alpha} = x_L^{\alpha\dot\alpha}\mu_{\dot\alpha}, \rho_\alpha=\mu_\alpha, \theta^i=\theta^{i\alpha}\mu_\alpha), \tag{I.69}$$

as discussed in §I.12 in the case of the supertwistor space together with the orthogonality relation induced by (I.67), we find

$$x_R^{\alpha\dot\alpha} - x_L^{\alpha\dot\alpha} + 2\theta^{i\alpha}\eta_i^{\dot\alpha} \ = \ 0, \tag{I.70}$$

where summation over repeated indices is implied. This equation can generically be solved by, e.g., putting

$$x_{R,L}^{\alpha\dot\alpha} \ = \ x^{\alpha\dot\alpha} \mp \theta^{i\alpha}\eta_i^{\dot\alpha}. \tag{I.71}$$

We may thus take $(x^{\alpha\dot\alpha},\theta^{i\alpha},\eta_i^{\dot\alpha})$ as local coordinates on $\mathbb{M}^{4|4\mathcal{N}}$, with the obvious projections $\pi_{1,2}$

$$\begin{aligned} \pi_1 \ &: \ (x^{\alpha\dot\alpha},\theta^{i\alpha},\eta_i^{\dot\alpha}) \in \mathbb{M}^{4|4\mathcal{N}} \ \mapsto \ (x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha}) \in \mathbb{M}_R^{4|2\mathcal{N}}, \\ \pi_2 \ &: \ (x^{\alpha\dot\alpha},\theta^{i\alpha},\eta_i^{\dot\alpha}) \in \mathbb{M}^{4|4\mathcal{N}} \ \mapsto \ (x_L^{\alpha\dot\alpha},\theta^{i\alpha}) \in \mathbb{M}_L^{4|2\mathcal{N}}. \end{aligned} \tag{I.72}$$

Clearly, by virtue of the discussion given in §I.12 and of the double fibration (I.68), $(x^{\alpha\dot\alpha},\theta^{i\alpha},\eta_i^{\dot\alpha})$ are defined on $\pi_1^{-1}(\mathcal{M}_R^{4|2\mathcal{N}})\cap\pi_2^{-1}(\mathcal{M}_L^{4|2\mathcal{N}}) =: \mathcal{M}^{4|4\mathcal{N}} \cong \mathbb{C}^{4|4\mathcal{N}}$.

§I.19 Euclidean signature. As we shall see in the next chapter, the supertwistor space will play a key role in discussing self-dual SYM theories. As the corresponding equations of motion are natural extensions of the self-dual YM equations, we are interested in their formulation in an Euclidean setting. Though not being subject of the present discussion, it is also possible to formulate them in the case of split signature. For details see, e.g., [205].

Let us introduce the ϵ-tensors according to

$$(\epsilon^{\alpha\beta}) \ = \ (\epsilon^{\dot\alpha\dot\beta}) \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad (\epsilon_{\alpha\beta}) \ = \ (\epsilon_{\dot\alpha\dot\beta}) \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{I.73}$$

which satisfy $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}$. Furthermore, define

$$(T_i^{\ j}) \ := \ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{I.74}$$

For the choice $\mathcal{N} = 4$ (the most interesting one for our purposes), Euclidean signature will be induced by the antiholomorphic involution

$$\tau_E(z^\alpha, \pi_{\dot\alpha}, \eta_i) \ := \ (\epsilon_{\alpha\beta}\bar z^\beta, \epsilon_{\dot\alpha\dot\beta}\bar\pi_{\dot\beta}, -T_i^{\ j}\bar\eta_j) \tag{I.75}$$

for the coordinates $(z^\alpha, \pi_{\dot\alpha}, \eta_i)$ on $\mathbb{T}$. Here, summation over repeated indices is implied and bar denotes complex conjugation. Note that reality of the odd coordinates for Euclidean signature can only be imposed if $\mathcal{N}$ is even [140, 163]; the $\mathcal{N} = 0$ and $\mathcal{N} = 2$ cases are obtained by suitable truncations of the $\mathcal{N} = 4$ case. Furthermore, we adopt the convention

$$\tau_E(ab) \ = \ \tau_E(a)\tau_E(b), \tag{I.76}$$

where a, b are any of the coordinates $(z^\alpha, \pi_{\dot\alpha}, \eta_i)$. The extension of the involution τ_E to any holomorphic function f is defined to be

$$\tau_E(f(\cdots)) \ := \ \overline{f(\tau_E(\cdots))}. \tag{I.77}$$

Using (I.75) together with the incidence relations (I.69), we find

$$\tau_E(x_R^{\alpha\dot\alpha}) \ = \ \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\bar x_R^{\beta\dot\beta} \qquad \text{and} \qquad \tau_E(\eta_i^{\dot\alpha}) \ = \ \epsilon^{\dot\alpha\dot\beta}T_i^{\ j}\bar\eta_j^{\dot\beta} \tag{I.78}$$

for the local coordinates $(x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha})$ on $\mathcal{M}_R^{4|8} \subset \mathbb{M}_R^{4|8}$. The fixed point set of these involutions then defines antichiral Euclidean superspace $\mathcal{M}_{R,\tau_E}^{4|8} \subset \mathcal{M}_R^{4|8}$ with $\mathcal{M}_{R,\tau_E}^{4|8} \cong \mathbb{R}^{4|8}$. One may choose the following parametrization:

$$x_R^{2\dot2} \ = \ \bar x_R^{1\dot1} \ =: \ x^4 - \mathrm{i}x^3 \qquad \text{and} \qquad x_R^{2\dot1} \ = \ -\bar x_R^{1\dot2} \ =: \ -x^2 + \mathrm{i}x^1, \tag{I.79}$$

with real x^μ. Hence, the metric is of Euclidean type. Note that later on, we shall also choose parametrizations different from (I.79).

§I.20 Minkowski signature. Besides self-dual SYM theories, we are also interested in full SYM theories which are most interesting when considered on Minkowski space-time. In this situation, the superambitwistor space plays the central role in the discussion.

Minkowski signature is induced by the antiholomorphic involution

$$\tau_M(z^\alpha,\pi_{\dot\alpha},\eta_i,\rho_\alpha,w^{\dot\alpha},\theta^i) \;:=\; (-\bar w^{\dot\alpha},\bar\rho_\alpha,\bar\theta^i,\bar\pi_{\dot\alpha},-\bar z^\alpha,\bar\eta_i) \tag{I.80}$$

on $\mathbb{T}\times\mathbb{T}^*$, where bar denotes, as before, complex conjugation. This time, however, we choose

$$\tau_M(ab) \;=\; \tau_M(b)\tau_M(a), \tag{I.81}$$

where a,b are any of the coordinates on $\mathbb{T}\times\mathbb{T}^*$. The extension to holomorphic functions is then given analogously as for τ_E.

Next using Eqs. (I.69) together with the conditions (I.71), we immediately arrive at

$$\tau_M(x^{\alpha\dot\beta}) \;=\; -\bar x^{\beta\dot\alpha}, \qquad \tau_M(\eta_i^{\dot\alpha}) \;=\; \bar\theta^{i\alpha} \qquad \text{and} \qquad \tau_M(\theta^{i\alpha}) \;=\; \bar\eta_i^{\dot\alpha}. \tag{I.82}$$

The fixed point set

$$\tau_M(x^{\alpha\dot\beta}) \;=\; -\bar x^{\beta\dot\alpha} \;=\; x^{\alpha\dot\beta} \qquad \text{and} \qquad \tau_M(\eta_i^{\dot\alpha}) \;=\; \bar\theta^{i\alpha} \;=\; \eta_i^{\dot\alpha} \tag{I.83}$$

together with the parametrization

$$(x^{\alpha\dot\alpha}) \;=:\; \begin{pmatrix} -\mathrm{i}(x^0-x^3) & x^2+\mathrm{i}x^1 \\ -x^2+\mathrm{i}x^1 & -\mathrm{i}(x^0+x^3) \end{pmatrix} \tag{I.84}$$

for real x^μ, yields a metric of Minkowski signature, that is, $(-+++)$.

Chapter II

Self-dual super gauge theory

After these introductory words on (super)twistor spaces, we shall now discuss a first application. In the introduction we have already seen that the twistor approach to gauge theory involves certain holomorphic vector bundles over spaces appearing in a double fibration like (I.35). As is well known, holomorphic vector bundles can be described within two different approaches: the Čech and the Dolbeault pictures. Both pictures, however, turn out to be equivalent – and each of it has its own advantages and disadvantages. In the sequel, we shall be using both on equal footing. Therefore, we first describe the equivalence of both pictures in a more general setting and then discuss as a first example self-dual SYM theory[1] and some related models.

II.1 Čech-Dolbeault correspondence

§II.1 Čech cochains, cocycles and cohomology. Let us recall some basic definitions already adopted to complex supermanifolds. Consider a complex supermanifold X with an open covering $\mathfrak{U} = \{\mathcal{U}_a\}$. Furthermore, we are interested in smooth maps from open subsets of X into some Lie (super)group G as well as in a sheaf $\mathfrak{S}$ of such G-valued functions. A q-cochain of the covering $\mathfrak{U}$ with values in $\mathfrak{S}$ is a collection $\psi = \{\psi_{a_0 \cdots a_q}\}$ of sections of the sheaf $\mathfrak{S}$ over nonempty intersections $\mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_q}$. We will denote the set of such q-cochains by $C^q(\mathfrak{U}, \mathfrak{S})$. We stress that it has a group structure, where the multiplication is just pointwise multiplication.

We may define the subsets of cocycles $Z^q(\mathfrak{U}, \mathfrak{S}) \subset C^q(\mathfrak{U}, \mathfrak{S})$. For example, for $q = 0, 1$

[1] See Ref. [228, 245, 246, 247] for an earlier account on self-dual SYM theory.

they are given by

$$\begin{aligned} Z^0(\mathfrak{U},\mathfrak{S}) &:= \{\psi\in C^0(\mathfrak{U},\mathfrak{S})\mid \psi_a=\psi_b \ \text{ on } \ \mathcal{U}_a\cap\mathcal{U}_b\neq\emptyset\},\\ Z^1(\mathfrak{U},\mathfrak{S}) &:= \{\psi\in C^1(\mathfrak{U},\mathfrak{S})\mid \psi_{ab}=\psi_{ba}^{-1} \ \text{ on } \ \mathcal{U}_a\cap\mathcal{U}_b\neq\emptyset \\ &\qquad\qquad \text{and } \ \psi_{ab}\psi_{bc}\psi_{ca}=1 \ \text{ on } \ \mathcal{U}_a\cap\mathcal{U}_b\cap\mathcal{U}_c\neq\emptyset\}. \end{aligned} \tag{II.1}$$

These sets will be of particular interest. We remark that from the first of these two definitions it follows that $Z^0(\mathfrak{U},\mathfrak{S})$ coincides with the group

$$H^0(X,\mathfrak{S}) \ \equiv \ \mathfrak{S}(X) \ = \ \Gamma(X,\mathfrak{S}),$$

which is the group of global sections of the sheaf $\mathfrak{S}$. Note that in general the subset $Z^1(\mathfrak{U},\mathfrak{S})\subset C^1(\mathfrak{U},\mathfrak{S})$ is not a subgroup of the group $C^1(\mathfrak{U},\mathfrak{S})$.

We say that two cocycles $f,\tilde f\in Z^1(\mathfrak{U},\mathfrak{S})$ are equivalent if $\tilde f_{ab}=\psi_a^{-1}f_{ab}\psi_b$ for some $\psi\in C^0(\mathfrak{U},\mathfrak{S})$. The set of equivalence classes induced by this equivalence relation is the first (pointed) Čech cohomology set and denoted by $H^1(\mathfrak{U},\mathfrak{S})$. If the $\mathcal{U}_a$s are all Stein – in case of supermanifolds X we need $X_{\rm red}$ to be covered by Stein manifolds – we have the bijection

$$H^1(\mathfrak{U},\mathfrak{S}) \ \cong \ H^1(X,\mathfrak{S}), \tag{II.2}$$

otherwise one takes the inductive limit. To sum up, we see that, for instance, the elements of $H^1(X,\mathfrak{H})$ with $\mathfrak{H}:=GL(r|s,\mathcal{O}_X)$ classify rank $r|s$ locally free sheaves of $\mathcal{O}_X$-modules up to isomorphism. Hence, $H^1(X,\mathfrak{H})$ is the moduli space of holomorphic vector bundles over X with complex rank $r|s$.

§II.2 Dolbeault cohomology. Let X be a complex supermanifold and consider a rank $r|s$ complex vector bundle $\mathcal{E}\to X$. Furthermore, we let

$$\Omega^{p,q}(X) \ := \ \mathscr{H}om_{\mathcal{S}_X}(\Lambda^p(T^{1,0}_{\mathbb{C}}X)\otimes\Lambda^q(T^{0,1}_{\mathbb{C}}X),\mathcal{S}_X) \tag{II.3}$$

be the differential (p,q)-forms on X. In spirit of our discussion given in §I.14, we have a natural antiholomorphic exterior derivative $\bar\partial\,:\,\Omega^{p,q}(X)\to\Omega^{p,q+1}(X)$. A $(0,1)$-connection on $\mathcal{E}$ is defined by a covariant differential

$$\nabla^{0,1}\,:\,\mathcal{E} \ \to \ \Omega^{0,1}(X)\otimes\mathcal{E} \tag{II.4}$$

satisfying the Leibniz formula; see also Eq. (I.45). Locally, it is of the form

$$\nabla^{0,1} \ = \ \bar\partial+\mathcal{A}^{0,1}, \tag{II.5}$$

where $\mathcal{A}^{0,1} \in \Gamma(X, \Omega^{0,1}(X) \otimes \operatorname{End} \mathcal{E})$. The complex vector bundle $\mathcal{E}$ is said to be holomorphic if the $(0,1)$-connection is flat, that is, if the corresponding curvature vanishes,

$$\mathcal{F}^{0,2} \;=\; (\nabla^{0,1})^2 \;=\; \bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \;=\; 0. \tag{II.6}$$

In other words, $\nabla^{0,1}$ defines a holomorphic structure on $\mathcal{E}$. Note that (II.6) is also called equation of motion of holomorphic Chern-Simons (hCS) theory.

Let $\mathfrak{A}^{0,1}$ be the sheaf of solutions to (II.6). The group $\Gamma(X, \mathfrak{S})$, where $\mathfrak{S} := GL(r|s, \mathcal{S}_X)$, is acting on $\Gamma(X, \mathfrak{A}^{0,1})$ by

$$\mathcal{A}^{0,1} \;\mapsto\; g^{-1}\mathcal{A}^{0,1} g + g^{-1}\bar{\partial} g, \tag{II.7}$$

where $g \in \Gamma(X, \mathfrak{S})$, and without changing the holomorphic structure on $\mathcal{E}$. Therefore, the Dolbeault cohomology set

$$H^1_{\nabla^{0,1}}(X, \mathcal{E}) \;:=\; \Gamma(X, \mathfrak{A}^{0,1})/\Gamma(X, \mathfrak{S}) \tag{II.8}$$

parametrizes all different holomorphic structures on the complex vector bundle $\mathcal{E}$.

§II.3 Equivalence of Čech and Dolbeault pictures. Let us now show that the above approaches to holomorphic vector bundles are actually equivalent. This fact may be understood as a non-Abelian generalization of Dolbeault's theorem.

Theorem II.1. *Let X be a complex supermanifold with an open Stein covering $\mathfrak{U} = \{\mathcal{U}_a\}$ and $\mathcal{E} \to X$ be a rank $r|s$ complex vector bundle over X. Then there is a one-to-one correspondence between $H^1_{\nabla^{0,1}}(X, \mathcal{E})$ and the subset of $H^1(X, \mathfrak{H})$ consisting of those elements of $H^1(X, \mathfrak{H})$ representing vector bundles which are smoothly equivalent to $\mathcal{E}$, i.e.,*

$$(\mathcal{E}, f = \{f_{ab}\}, \nabla^{0,1}) \;\sim\; (\tilde{\mathcal{E}}, \tilde{f} = \{\tilde{f}_{ab}\}, \bar{\partial}),$$

where $\tilde{f}_{ab} = \psi_a^{-1} f_{ab} \psi_b$ for some $\psi = \{\psi_a\} \in C^0(\mathfrak{U}, \mathfrak{S})$.

Proof: Let $\mathcal{E} \to X$ be a rank $r|s$ complex vector bundle represented by $f = \{f_{ab}\} \in H^1(X, \mathfrak{S})$. Furthermore, consider the subset of $C^0(\mathfrak{U}, \mathfrak{S})$ consisting of those elements $\psi = \{\psi_a\}$ obeying

$$\psi_b \bar{\partial} \psi_b^{-1} \;=\; f_{ab}^{-1} \psi_a \bar{\partial} \psi_a^{-1} f_{ab} + f_{ab}^{-1} \bar{\partial} f_{ab}$$

on $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$. Due to Eq. (II.6), elements $\mathcal{A}^{0,1} = \{\mathcal{A}_a^{0,1}\}$ of $H^1_{\nabla^{0,1}}(X, \mathcal{E})$ are locally of the form $\mathcal{A}_a^{0,1} = \psi_a \bar{\partial} \psi_a^{-1}$ and glued together according to the above formula. Hence, $\mathcal{A}^{0,1} \in H^1_{\nabla^{0,1}}(X, \mathcal{E})$

determines a zero-cochain $\psi = \{\psi_a\}$ with the above property. This ψ can in turn be used to define the transition functions of a rank $r|s$ holomorphic vector bundle $\tilde{\mathcal{E}} \to X$ by setting

$$\tilde{f}_{ab} \;:=\; \psi_a^{-1} f_{ab} \psi_b,$$

that is, $\bar{\partial}\tilde{f}_{ab} = 0$. Clearly, the bundle $\tilde{\mathcal{E}}$ defined by this $\tilde{f} = \{\tilde{f}_{ab}\} \in H^1(X, \mathfrak{H})$ is smoothly equivalent to $\mathcal{E}$. Conversely, given $\tilde{f} = \{\tilde{f}_{ab}\} \in H^1(X, \mathfrak{H})$ as transition functions of a holomorphic vector bundle $\tilde{\mathcal{E}}$ which is smoothly equivalent to $\mathcal{E}$, the $\tilde{f}_{ab}$s can always be written in the above form and hence, one can reconstruct a differential $(0,1)$-form $\mathcal{A}^{0,1}$ such that $\mathcal{F}^{0,2} = 0$.

Bijectivity is shown by virtue of a short exact sequence of sheaves

$$0 \;\to\; \mathfrak{H} \;\hookrightarrow\; \mathfrak{S} \;\overset{\delta^0}{\to}\; \mathfrak{A}^{0,1} \;\overset{\delta^1}{\to}\; 0,$$

where $\delta^0 : \mathfrak{S} \to \mathfrak{A}^{0,1}$ is defined on any open subset $\mathcal{U}$ on X by $\delta^0 : \psi_{\mathcal{U}} \mapsto \psi_{\mathcal{U}}\bar{\partial}\psi_{\mathcal{U}}^{-1}$, with $\psi_{\mathcal{U}} \in \Gamma(\mathcal{U}, \mathfrak{S})$. The map δ^1 sends $\mathcal{A}^{0,1} \in \mathfrak{A}^{0,1}$ to $\mathcal{F}^{0,2}$ which by construction vanishes. The above sequence induces an exact sequence of cohomology sets

$$0 \;\to\; H^0(X, \mathfrak{H}) \;\to\; H^0(X, \mathfrak{S}) \;\to\; H^0(X, \mathfrak{A}^{0,1}) \;\to\; H^1(X, \mathfrak{H}) \;\overset{\rho}{\to}\; H^1(X, \mathfrak{S}).$$

By definition, $H^1(X, \mathfrak{H})$ (respectively, $H^1(X, \mathfrak{S})$) parametrizes holomorphic (respectively, smooth) vector bundles over X. The kernel of ρ coincides with the subset of $H^1(X, \mathfrak{H})$ whose elements are mapped into the class of $H^1(X, \mathfrak{S})$ representing holomorphic vector bundles which are smoothly equivalent to $\mathcal{E}$. By virtue of the exactness of the cohomology sequence, we find

$$H^1_{\nabla^{0,1}}(X, \mathcal{E}) \;=\; H^0(X, \mathfrak{A}^{0,1})/H^0(X, \mathfrak{S}) \;\cong\; \ker \rho.$$

■

§II.4 Remark. In the following, we shall mostly be interested in complex vector bundles which are trivial as smooth bundles. Furthermore, we also restrict our discussion to rank $r|0 \equiv r$ complex vector bundles, although it straightforwardly generalizes to rank $r|s$.

II.2 Self-dual super Yang-Mills theory

Subject of this section is the discussion of $\mathcal{N}$-extended self-dual SYM theory. We first present the Čech approach to holomorphic vector bundles over supertwistor space and derive in this setting the field equations of self-dual SYM theory on four-dimensional space-time. Second, we reconsider the whole discussion in the Dolbeault picture. The latter then also allows us to formulate appropriate action principles for both, hCS theory on supertwistor space and self-dual SYM theory on Euclidean four-dimensional space in the case of maximal supersymmetry, that is, for $\mathcal{N} = 4$.

§II.5 Penrose-Ward transform. Let us consider the double fibration (I.36) and recall that $\mathcal{M}_R^{4|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}}$, i.e.,

$$\begin{tikzcd} & \mathcal{F}_R^{5|2\mathcal{N}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}} \end{tikzcd} \tag{II.9}$$

where $\mathcal{P}^{3|\mathcal{N}}$ is the supertwistor space given by (I.37) and $\mathcal{F}_R^{5|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1$. Recall also the form of the two projections $\pi_{1,2}$,

$$\begin{aligned} \pi_1 \,:\, (x_R^{\alpha\dot\alpha}, \lambda_{\dot\alpha}^{\pm}, \eta_i^{\dot\alpha}) \ &\mapsto\ (z_\pm^\alpha = x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^{\pm}, \pi_{\dot\alpha}^{\pm} = \lambda_{\dot\alpha}^{\pm}, \eta_i^{\pm} = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^{\pm}),\\ \pi_2 \,:\, (x_R^{\alpha\dot\alpha}, \lambda_{\dot\alpha}^{\pm}, \eta_i^{\dot\alpha}) \ &\mapsto\ (x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}). \end{aligned} \tag{II.10}$$

We denote the coverings of $\mathcal{P}^{3|\mathcal{N}}$ and $\mathcal{F}^{5|2\mathcal{N}}$ by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ and $\hat{\mathfrak{U}} = \{\hat{\mathcal{U}}_+, \hat{\mathcal{U}}_-\}$, respectively. Consider a rank r holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and its pull-pack $\pi_1^*\mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$ as defined in §I.9. These bundles are characterized by the transition functions $f = \{f_{+-}\}$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ and $\pi_1^* f$ on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. For notational simplicity, we shall use the same letter, f, for the transition functions of both bundles in the following course of discussion. By definition of a pull-back, f is constant along $\pi_1 : \mathcal{F}_R^{5|2\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}$. The relative tangent sheaf[2] $T\mathcal{F}_R/\mathcal{P} := (\Omega^1(\mathcal{F}_R^{5|2\mathcal{N}})/\pi_1^*\Omega^1(T\mathcal{P}^{3|\mathcal{N}}))^*$ is of rank $2|\mathcal{N}$ and freely generated by

$$D_\alpha^{\pm} \ =\ \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha}^R \qquad \text{and} \qquad D_\pm^i \ =\ \lambda_\pm^{\dot\alpha}\partial_{\dot\alpha}^i. \tag{II.11}$$

Here, we have again abbreviated $\partial_{\alpha\dot\alpha}^R := \partial/\partial x_R^{\alpha\dot\alpha}$ and $\partial_{\dot\alpha}^i := \partial/\partial\eta_i^{\dot\alpha}$. In the sequel, we shall write $\mathscr{T} := T\mathcal{F}_R/\mathcal{P}$.[3] Therefore, the transition functions of $\pi_1^*\mathcal{E}$ are annihilated by the vector fields (II.11). Letting $\bar\partial_{\mathcal{P}}$ and $\bar\partial_{\mathcal{F}}$ be the anti-holomorphic parts of the exterior derivatives on the supertwistor space and its correspondence space, respectively, we have $\pi_1^*\bar\partial_{\mathcal{P}} = \bar\partial_{\mathcal{F}} \circ \pi_1^*$. Hence, the transition functions of $\pi_1^*\mathcal{E}$ are also annihilated by $\bar\partial_{\mathcal{F}}$.

As indicated, we assume that the bundle $\mathcal{E}$ is smoothly trivial and moreover $\mathbb{C}^{4|2\mathcal{N}}$-trivial, that is, holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. These conditions imply that there exists some smooth $GL(r,\mathbb{C})$-valued functions $\psi = \{\psi_+, \psi_-\} \in C^0(\hat{\mathfrak{U}}, \mathfrak{S})$, which define trivializations of $\pi_1^*\mathcal{E}$ over $\hat{\mathcal{U}}_\pm$, such that f_{+-} can

[2]If one has a fibration $\pi : Z \to X$, the relative tangent sheaf TZ/X (sheaf of vertical vector fields) is defined by the following short exact sequence: $0 \to TZ/X \to TZ \overset{\pi_*}{\to} \pi^*TX \to 0$.

[3]Here, we have just introduced the letter $\mathscr{T}$ as abbreviation for the relative tangent sheaf. However, it can be put in a general context of integrable distributions as subsheaves of the tangent sheaf. See §III.6 for details.

be decomposed as

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \tag{II.12}$$

and

$$\bar{\partial}_{\mathcal{F}}\psi_\pm \;=\; 0. \tag{II.13}$$

Note that in particular this formula implies that the $\psi_\pm$s depend holomorphically on $\lambda_\pm$. Applying the vector fields (II.11) to (II.12), we realize that

$$\psi_+ D_\alpha^+ \psi_+^{-1} \;=\; \psi_- D_\alpha^+ \psi_-^{-1} \qquad \text{and} \qquad \psi_+ D_+^i \psi_+^{-1} \;=\; \psi_- D_+^i \psi_-^{-1}$$

must be – by an extension of Liouville's theorem – at most linear in λ_+. Therefore, we may introduce a Lie-algebra valued one-form $\mathcal{A}_{\mathscr{T}}$ which has components only along $\mathscr{T}$, such that

$$\begin{aligned} D_\alpha \lrcorner \mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{U}}_\pm} \;&:=\; \mathcal{A}_\alpha^\pm \;=\; \psi_\pm D_\alpha^\pm \psi_\pm^{-1} \;=\; \lambda_\pm^{\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} \\ D^i \lrcorner \mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{U}}_\pm} \;&:=\; \mathcal{A}_\pm^i \;=\; \psi_\pm D_\pm^i \psi_\pm^{-1} \;=\; \lambda_\pm^{\dot\alpha}\mathcal{A}_{\dot\alpha}^i. \end{aligned} \tag{II.14}$$

In fact, $\mathcal{A}_{\mathscr{T}}$ defines a relative connection

$$\nabla_{\mathscr{T}} \,:\, \pi_1^*\mathcal{E} \;\to\; \Omega^1_{\mathscr{T}}(\mathcal{F}_R^{5|2\mathcal{N}}) \otimes \pi_1^*\mathcal{E}, \tag{II.15}$$

which is flat. Here, $\Omega^1_{\mathscr{T}}(\mathcal{F}_R^{5|2\mathcal{N}}) \;:=\; \mathscr{T}^*$ are the relative differential one-forms on the correspondence space.

Eqs. (II.14) can be rewritten as

$$\begin{aligned} (D_\alpha^\pm + \mathcal{A}_\alpha^\pm)\psi_\pm \;&=\; 0, \\ (D_\pm^i + \mathcal{A}_\pm^i)\psi_\pm \;&=\; 0, \\ \bar{\partial}_{\mathcal{F}}\psi_\pm \;&=\; 0. \end{aligned} \tag{II.16}$$

The compatibility conditions for this linear system read as

$$\begin{gathered} [\nabla^R_{\alpha\dot\alpha}, \nabla^R_{\beta\dot\beta}] + [\nabla^R_{\alpha\dot\beta}, \nabla^R_{\beta\dot\alpha}] \;=\; 0, \qquad [\nabla^i_{\dot\alpha}, \nabla^R_{\beta\dot\beta}] + [\nabla^i_{\dot\beta}, \nabla^R_{\beta\dot\alpha}] \;=\; 0, \\ \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} \;=\; 0, \end{gathered} \tag{II.17}$$

where we have defined the covariant derivatives

$$\nabla^R_{\alpha\dot\alpha} \;:=\; \partial^R_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \nabla^i_{\dot\alpha} \;:=\; \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}. \tag{II.18}$$

Eqs. (II.17) are the constraint equations of $\mathcal{N}$-extended self-dual SYM theory. Note that the first of those equations represents the self-duality equation for the gauge potential $\mathcal{A}_{\alpha\dot\alpha}$, since

$$[\nabla^R_{\alpha\dot\alpha}, \nabla^R_{\beta\dot\beta}] \;=\; \epsilon_{\alpha\beta} f_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta}, \tag{II.19}$$

where $f_{\dot\alpha\dot\beta}$ (respectively, $f_{\alpha\beta}$) is symmetric in its indices. Furthermore, $f_{\dot\alpha\dot\beta}$ (respectively, $f_{\alpha\beta}$) represents the anti-self-dual (respectively, the self-dual) part of the field strength. By virtue of (II.17), the anti-self-dual part is put to zero.

Next let us briefly discuss how to obtain the functions $\psi_\pm$ in (II.16) from a given gauge potential. Formally, a solution is given by

$$\psi_\pm \;=\; P\exp\left(-\int_{\mathscr{C}}\mathcal{A}\right). \tag{II.20}$$

Here, "P" denotes the path-ordering symbol and

$$\mathcal{A} \;=\; \mathrm{d}x^{\alpha\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} + \mathrm{d}\eta_i^{\dot\alpha}\mathcal{A}^i_{\dot\alpha}. \tag{II.21}$$

The contour $\mathscr{C}$ is any real curve within an isotropic two-plane $\mathbb{C}^{2|\mathcal{N}}$ from a point $(\hat x,\hat\eta)$ to a point (x,η), with

$$\begin{aligned} x^{\alpha\dot\alpha}(s) \;&=\; \hat x^{\alpha\dot\alpha} + s\varepsilon^\alpha\lambda_\pm^{\dot\alpha},\\ \eta_i^{\dot\alpha}(s) \;&=\; \hat\eta_i^{\dot\alpha} + s\varepsilon_i\lambda_\pm^{\dot\alpha}, \end{aligned} \tag{II.22}$$

for $s\in[0,1]$; the choice of the contour plays no role, since the curvature is zero when restricted to the two-plane. Furthermore, $(\varepsilon^\alpha,\varepsilon_i)$ are some free parameters.

In §I.19, we introduced an antiholomorphic involution τ_E corresponding to Euclidean signature. Upon extending this involution to $\pi_1^*\mathcal{E}\to\mathcal{F}^{5|2\mathcal{N}}$, that is, upon requiring

$$f_{+-}(\cdots) \;=\; [f_{+-}(\tau_E(\cdots))]^\dagger, \tag{II.23}$$

where dagger denotes Hermitian conjugation, one ends up with real self-dual SYM fields. In particular, one finds

$$\tau_E(\mathcal{A}_{\alpha\dot\alpha}) \;=\; -\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\mathcal{A}^\dagger_{\beta\dot\beta} \;=\; \mathcal{A}_{\alpha\dot\alpha} \quad\text{and}\quad \tau_E(\mathcal{A}^i_{\dot\alpha}) \;=\; -\epsilon_{\dot\alpha\dot\beta}T_j{}^i(\mathcal{A}^j_{\dot\beta})^\dagger \;=\; \mathcal{A}^i_{\dot\alpha}, \tag{II.24}$$

where $T_i{}^j$ has been defined in (I.74). This reduces the gauge group $GL(r,\mathbb{C})$ to the unitary group $U(r)$. Unless otherwise stated, we shall implicitly assume that (II.23) has been imposed. If one in addition requires that $\det(f_{+-})=1$, the structure group is reduced further to $SU(r)$.

Before we are discussing the field expansions of the superfields $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$, let us give some integral formulas

$$\mathcal{A}_{\alpha\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}}\mathrm{d}\lambda_+\,\frac{\mathcal{A}^+_\alpha}{\lambda_+\lambda_+^{\dot\alpha}} \qquad\text{and}\qquad \mathcal{A}^i_{\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}}\mathrm{d}\lambda_+\,\frac{\mathcal{A}^i_+}{\lambda_+\lambda_+^{\dot\alpha}} \tag{II.25}$$

where the contour $\mathscr{C}=\{\lambda_+\in\mathbb{C}P^1\,|\,|\lambda_+|=1\}$ encircles $\lambda_+=0$. In fact, these contour integrals give the explicit form of the Penrose-Ward transform.

§II.6 Field expansions, field equations and action functional. Let us stick to the $\mathcal{N}=4$ case. The others are then obtained by suitable truncations. Recall that the field content of $\mathcal{N}=4$ self-dual SYM theory consists of a (self-dual) gauge potential $\mathring{\mathcal{A}}_{\alpha\dot\alpha}$, four positive chirality spinors $\mathring{\chi}^i_\alpha$, six scalars $\mathring{W}^{ij}=-\mathring{W}^{ji}$, four negative chirality spinors $\mathring{\chi}_{i\dot\alpha}$ and an anti-self-dual two-form $\mathring{G}_{\dot\alpha\dot\beta}$, all in the adjoint representation of $SU(r)$. The circle refers to the lowest component in the superfield expansions of the corresponding superfields $\mathcal{A}_{\alpha\dot\alpha}$, χ^i_α, W^{ij}, $\chi_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$, respectively. The constraint equations (II.17) can formally be solved by setting

$$[\nabla^R_{\alpha\dot\alpha},\nabla^R_{\beta\dot\beta}] \;=\; \epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta},\quad [\nabla^i_{\dot\alpha},\nabla^R_{\beta\dot\beta}] \;=\; \epsilon_{\dot\alpha\dot\beta}\chi^i_\beta \quad\text{and}\quad \{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\} \;=\; \epsilon_{\dot\alpha\dot\beta}W^{ij}. \tag{II.26}$$

Using Bianchi identities, we find the remaining fields

$$\chi_{i\dot\alpha} \;:=\; \tfrac{2}{3}\nabla^j_{\dot\alpha}W_{ij} \qquad\text{and}\qquad G_{\dot\alpha\dot\beta} \;:=\; -\tfrac{1}{4}\nabla^i_{(\dot\alpha}\chi_{i\dot\beta)}, \tag{II.27}$$

respectively. Here, we introduced the common abbreviation $W_{ij}:=\frac{1}{2!}\epsilon_{ijkl}W^{kl}$ and parentheses mean normalized symmetrization. Next we follow the literature [109, 110, 77] and impose the transversal gauge,

$$\eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha} \;=\; 0, \tag{II.28}$$

in order to remove the superfluous gauge degrees of freedom associated with the odd coordinates $\eta^{\dot\alpha}_i$.[4] Putting it differently, this reduces the allowed gauge transformations to ordinary gauge transformations. Furthermore, this leads to the recursion operator $\mathscr{D}$ given by

$$\mathscr{D} \;:=\; \eta^{\dot\alpha}_i\nabla^i_{\dot\alpha} \;=\; \eta^{\dot\alpha}_i\partial^i_{\dot\alpha}. \tag{II.29}$$

Again by virtue of Bianchi identities, one arrives after a somewhat lengthy calculation at the following set of recursion relations:

$$\begin{aligned}
\mathscr{D}\mathcal{A}_{\alpha\dot\alpha} &= -\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\chi^i_\alpha,\\
(1+\mathscr{D})\mathcal{A}^i_{\dot\alpha} &= \epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_jW^{ij},\\
\mathscr{D}W_{ij} &= -\eta^{\dot\alpha}_{[i}\chi_{j]\dot\alpha},\\
\mathscr{D}\chi^i_\alpha &= -\eta^{\dot\alpha}_j\nabla^R_{\alpha\dot\alpha}W^{ij},\\
\mathscr{D}\chi_{i\dot\alpha} &= \eta^{\dot\beta}_iG_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j[W^{jk},W_{ki}],\\
\mathscr{D}G_{\dot\alpha\dot\beta} &= \eta^{\dot\gamma}_i\epsilon_{\dot\gamma(\dot\alpha}[\chi_{j\dot\beta)},W^{ij}],
\end{aligned} \tag{II.30}$$

[4] See also Ref. [15] for a more general setting.

where, as before, parentheses mean normalized symmetrization while the brackets denote normalized antisymmetrization of the enclosed indices.[5] These equations determine all superfields to $(n+1)$-st order, provided one knows them to n-th order in the odd coordinates. At this point, it is helpful to present some formulas which simplify this argumentation a lot. Consider some generic superfield f. Its explicit η-expansion looks as

$$f \;=\; \overset{\circ}{f} + \sum_{k\geq 1} \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, f^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}. \tag{II.31}$$

Furthermore, we have $\mathscr{D} f = \eta^{\dot\gamma_1}_{j_1}[\ \ ]^{j_1}_{\dot\gamma_1}$, where the bracket $[\ \ ]^{j_1}_{\dot\gamma_1}$ is a composite expression of some superfields. For example, we have $\mathscr{D}\mathcal{A}_{\alpha\dot\alpha} = \eta^{\dot\gamma_1}_{j_1}[\alpha\dot\alpha]^{j_1}_{\dot\gamma_1}$, with $[\alpha\dot\alpha]^{j_1}_{\dot\gamma_1} = -\epsilon_{\dot\alpha\dot\gamma_1}\chi^{j_1}_{\alpha}$. Let now

$$\mathscr{D}[\ \ ]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k} \;=\; \eta^{\dot\gamma_{k+1}}_{j_{k+1}}[\ \ ]^{j_1\cdots j_{k+1}}_{\dot\gamma_1\cdots\dot\gamma_{k+1}}. \tag{II.32}$$

Then we find by induction

$$f \;=\; \overset{\circ}{f} + \sum_{k\geq 1} \frac{1}{k!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[\ \ ]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}. \tag{II.33}$$

In case the recursion relation of f was given by $(1+\mathscr{D})f = \eta^{\dot\gamma_1}_{j_1}[\ \ ]^{j_1}_{\dot\gamma_1}$, as it happens to be for $\mathcal{A}^i_{\dot\alpha}$, then $\overset{\circ}{f} = 0$ and the superfield expansion is of the form

$$f \;=\; \sum_{k\geq 1} \frac{k}{(k+1)!}\, \eta^{\dot\gamma_1}_{j_1} \cdots \eta^{\dot\gamma_k}_{j_k}\, \overset{\circ}{[\ \ ]}{}^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}. \tag{II.34}$$

Using these expressions, one obtains the following results for the superfields $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$:

$$\begin{aligned}
\mathcal{A}_{\alpha\dot\alpha} \;&=\; \overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \epsilon_{\dot\alpha\dot\beta}\overset{\circ}{\chi}{}^i_{\alpha}\eta^{\dot\beta}_i + \cdots\,, \\
\mathcal{A}^i_{\dot\alpha} \;&=\; \tfrac{1}{2!}\epsilon_{\dot\alpha\dot\beta}\overset{\circ}{W}{}^{ij}\eta^{\dot\beta}_j - \tfrac{1}{3!}\epsilon^{ijkl}\epsilon_{\dot\alpha\dot\beta}\overset{\circ}{\chi}_{k\dot\gamma}\eta^{\dot\gamma}_l\eta^{\dot\beta}_j + \\
&\qquad + \tfrac{3}{2\cdot 4!}\epsilon^{ijkl}\epsilon_{\dot\alpha\dot\beta}(\overset{\circ}{G}_{\dot\gamma\dot\delta}\delta^m_l + \epsilon_{\dot\gamma\dot\delta}[\overset{\circ}{W}{}^{mn}, \overset{\circ}{W}_{nl}])\eta^{\dot\gamma}_k\eta^{\dot\delta}_m\eta^{\dot\beta}_j + \cdots\,.
\end{aligned} \tag{II.35}$$

Upon substituting the superfield expansions (II.35) into the constraint equations

[5] Note that these equations resemble the supersymmetry transformations, but nevertheless they should not be confused with them.

(II.17), we obtain

$$
\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} &= 0,\\
\epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^i_\beta &= 0,\\
\overset{\circ}{\Box}{}^R\overset{\circ}{W}{}^{ij} &= -\epsilon^{\alpha\beta}\{\overset{\circ}{\chi}{}^i_\alpha,\overset{\circ}{\chi}{}^j_\beta\},\\
\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}_{i\dot\beta} &= 2[\overset{\circ}{W}_{ij},\overset{\circ}{\chi}{}^j_\alpha],\\
\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma} &= \{\overset{\circ}{\chi}{}^i_\alpha,\overset{\circ}{\chi}_{i\dot\gamma}\}-\tfrac12[\overset{\circ}{W}_{ij},\overset{\circ}{\nabla}{}^R_{\alpha\dot\gamma}\overset{\circ}{W}{}^{ij}],
\end{aligned}
\tag{II.36}
$$

which are the $\mathcal{N}=4$ self-dual SYM equations. The equations for less supersymmetry are obtained from those by suitable truncations. We have also introduced the abbreviation

$$
\overset{\circ}{\Box}{}^R := \tfrac12\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\nabla}{}^R_{\beta\dot\beta}. \tag{II.37}
$$

We stress that Eqs. (II.36) represent the field equations to lowest order in the superfield expansions. With the help of the recursion operator (II.29), one may verify that they are in one-to-one correspondence with the constraint equations (II.17); see also Ref. [77] for a detailed discussion.

Furthermore, one easily checks that the above field equations can be derived from the following action functional:

$$
S = \int \mathrm{d}^4x_R\,\mathrm{tr}\left\{\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\overset{\circ}{f}_{\dot\alpha\dot\beta}+\overset{\circ}{\chi}{}^{i\alpha}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^{\dot\alpha}_i+\tfrac12\overset{\circ}{W}_{ij}\overset{\circ}{\Box}{}^R\overset{\circ}{W}{}^{ij}+\overset{\circ}{W}_{ij}\{\overset{\circ}{\chi}{}^i_\alpha,\overset{\circ}{\chi}{}^{j\alpha}\}\right\}, \tag{II.38}
$$

which has first been introduced by Siegel [229].

§II.7 HCS theory on $\mathcal{P}^{3|4}$. Let us now try to understand the twistor analog of the action functional (II.38). So far, we have worked in the Čech approach to holomorphic vector bundles. In particular, we started with a smoothly trivial holomorphic vector bundle $(\mathcal{E}, f=\{f_{+-}\},\bar\partial_{\mathcal{P}})$ over supertwistor space and additionally required holomorphic triviality along any $\mathbb{C}P^1_{x_R,\eta}\hookrightarrow\mathcal{P}^{3|4}$; for the sake of concreteness, we again stick to the $\mathcal{N}=4$ case. As we have learned in the previous section, there is another equivalent approach – the Dolbeault approach. To switch to this picture, we need to find $(\tilde{\mathcal{E}},\tilde f=\{\mathbb{1}_r\},\bar\partial_{\mathcal{P}}+\mathcal{A}^{0,1})$, since $\mathcal{E}$ is assumed to be smoothly trivial. Moreover, within this approach we shall be able to write down an action functional on supertwistor space yielding the functional (II.38) after performing suitable integrations. Up to now, it is not known how to formulate an appropriate action principle within the Čech approach. This is mainly due to the fact that the Čech approach makes the construction manifestly on-shell: certain holomorphic functions (the transition functions) on supertwistor space yield solutions to the equations of motion of $\mathcal{N}=4$ SYM theory via contour integrals of the form (II.25).

First, let us make a more careful analysis of the real structure τ_E as introduced in §I.19. There we have seen that the fixed point set $\tau_E(x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha}) = (x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha})$ – cf. also Eqs. (I.78) – is a real slice $\mathbb{R}^{4|8}\subset\mathbb{C}^{4|8}$ corresponding to Euclidean superspace. When restricting to $\mathbb{R}^{4|8}$, we have the diffeomorphisms

$$\mathbb{R}^{4|8}\times S^2 \;\cong\; \mathbb{C}^{2|4}\times\mathbb{C}P^1 \;\cong\; \mathcal{P}^{3|4}. \tag{II.39}$$

In fact, the map from $\mathcal{P}^{3|4}$ with coordinates $(z_\pm^\alpha, z_\pm^3, \eta_i^\pm)$ to $\mathbb{C}^{2|4}\times\mathbb{C}P^1$ with coordinates $(x^{\alpha\dot 1},\eta_i^{\dot 1},\lambda_\pm)$ is explicitly given by

$$\begin{gathered} x^{1\dot 1} \;=\; \frac{z_+^1+z_+^3\bar z_+^2}{1+z_+^3\bar z_+^3} \;=\; \frac{\bar z_-^3 z_-^1+\bar z_-^2}{1+z_-^3\bar z_-^3},\quad x^{2\dot 1} \;=\; \frac{z_+^2-z_+^3\bar z_+^1}{1+z_+^3\bar z_+^3} \;=\; \frac{\bar z_-^3 z_-^2-\bar z_-^1}{1+z_-^3\bar z_-^3}\,, \\ \lambda_\pm \;=\; z_\pm^3 \end{gathered} \tag{II.40}$$

and

$$\begin{aligned} \eta_1^{\dot 1} &= \frac{\eta_1^+ + z_+^3\bar\eta_2^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_1^- + \bar\eta_2^-}{1+z_-^3\bar z_-^3},\quad \eta_2^{\dot 1} = \frac{\eta_2^+ - z_+^3\bar\eta_1^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_2^- - \bar\eta_1^-}{1+z_-^3\bar z_-^3},\\ \eta_3^{\dot 1} &= \frac{\eta_3^+ + z_+^3\bar\eta_4^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_3^- + \bar\eta_4^-}{1+z_-^3\bar z_-^3},\quad \eta_4^{\dot 1} = \frac{\eta_4^+ - z_+^3\bar\eta_3^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_4^- - \bar\eta_3^-}{1+z_-^3\bar z_-^3}. \end{aligned} \tag{II.41}$$

These relations define a (smooth) projection

$$\mathcal{P}^{3|4} \;\to\; \mathbb{R}^{4|8}. \tag{II.42}$$

Therefore, we may conclude that in the Euclidean setting no double fibration like (II.9) is needed. It is rather enough to restrict the discussion to the nonholomorphic fibration (II.42). This, however, is a very special feature of the present setting and we shall find other examples where double fibrations – even if reality conditions are imposed – are inevitable.

Let us continue with the fibration (II.42). The vector fields (II.11) which generate the relative tangent sheaf $\mathscr{T} = T\mathcal{F}_R/\mathcal{P}$ do now, together with $\partial_{\bar\lambda_\pm}$, generate the anti-holomorphic tangent sheaf $T_{\mathbb{C}}^{0,1}\mathcal{P}^{3|4}$, since $\partial_{\bar z_\pm^\alpha}$, $\partial_{\bar z_\pm^3}$ and $\partial_{\bar\eta_i^\pm}$ can be rewritten according to

$$\begin{aligned} \frac{\partial}{\partial\bar z_\pm^\alpha} &= \epsilon_{\alpha\beta}\gamma_\pm\bar V_\beta^\pm,\\ \frac{\partial}{\partial\bar z_+^3} &= \bar V_3^+ - \gamma_+ x^{\alpha\dot 1}\bar V_\alpha^+ - \gamma_+\eta_i^{\dot 1}\bar V_+^i,\quad \frac{\partial}{\partial\bar z_-^3} = \bar V_3^- - \gamma_- x^{\alpha\dot 2}\bar V_\alpha^- + \gamma_-\eta_i^{\dot 2}\bar V_-^i,\\ \frac{\partial}{\partial\bar\eta_i^\pm} &= \gamma_\pm T_j{}^i\bar V_\pm^j, \end{aligned} \tag{II.43}$$

where we have defined

$$\bar{V}^{\pm}_{\alpha} \;:=\; \lambda^{\dot\alpha}_{\pm}\partial^R_{\alpha\dot\alpha}, \qquad \bar{V}^{\pm}_{3} \;:= \partial_{\bar\lambda_{\pm}} \qquad \text{and} \qquad \bar{V}^{i}_{\pm} \;:=\; \lambda^{\dot\alpha}_{\pm}\partial^i_{\dot\alpha} \tag{II.44}$$

and

$$\gamma_{\pm} \;:=\; \frac{1}{1+\lambda_{\pm}\bar\lambda_{\pm}}. \tag{II.45}$$

This makes it obvious that $T^{0,1}_{\mathbb{C}}\mathcal{P}^{3|4} = \langle \bar V^{\pm}_{\alpha}, \bar V^{\pm}_3, \bar V^i_{\pm}\rangle$. The matrix $(T_i{}^j)$ has been defined in (I.74). A short calculation reveals that the sheaf of differential $(0,1)$-forms on $\mathcal{P}^{3|4}$ is, for instance, freely generated by the sections

$$\bar E^{\alpha}_{\pm} \;=\; -\gamma_{\pm}\hat\lambda^{\pm}_{\dot\alpha}\mathrm{d}x^{\alpha\dot\alpha}_R, \qquad \bar E^3_{\pm} \;=\; \mathrm{d}\bar\lambda_{\pm}, \qquad \text{and} \qquad \bar E^{\pm}_i \;=\; -\gamma_{\pm}\hat\lambda^{\pm}_{\dot\alpha}\mathrm{d}\eta^{\dot\alpha}_i, \tag{II.46}$$

which, in fact, are dual to $(\bar V^{\pm}_{\alpha}, \bar V^{\pm}_3, \bar V^i_{\pm})$. Here, we have introduced

$$\begin{aligned}(\hat\lambda^+_{\dot\alpha}) \;&:=\; \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1\\ \bar\lambda_+\end{pmatrix} \;=\; \begin{pmatrix} -\bar\lambda_+\\ 1\end{pmatrix},\\ (\hat\lambda^-_{\dot\alpha}) \;&:=\; \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} \bar\lambda_-\\ 1\end{pmatrix} \;=\; \begin{pmatrix} -1\\ \bar\lambda_-\end{pmatrix}.\end{aligned} \tag{II.47}$$

Thus, $\bar\partial_{\mathcal{P}}$ is given by

$$\bar\partial_{\mathcal{P}}|_{\mathcal{U}_{\pm}} \;=\; \mathrm{d}\bar z^{\alpha}_{\pm}\frac{\partial}{\partial\bar z^{\alpha}_{\pm}} + \mathrm{d}\bar z^3_{\pm}\frac{\partial}{\partial\bar z^3_{\pm}} + \mathrm{d}\bar\eta^{\pm}_i\frac{\partial}{\partial\bar\eta^{\pm}_i} \;=\; \bar E^{\alpha}_{\pm}\bar V^{\pm}_{\alpha} + \bar E^3_{\pm}\bar V^{\pm}_3 + \bar E^i_{\pm}\bar V^{\pm}_i, \tag{II.48}$$

where, as before, $\mathcal{U}_{\pm}$ are the two sets covering the supertwistor space.

After this digression, we can now proceed as in §II.5 and discuss holomorphic vector bundles $(\mathcal{E}, f = \{f_{+-}\}, \bar\partial_{\mathcal{P}})$ over $\mathcal{P}^{3|4}$ which are smoothly trivial and in addition $\mathbb{R}^{4|8}$-trivial. Eventually, one again finds a linear system of the form (II.16), that is,

$$\begin{aligned}(\bar V^{\pm}_{\alpha} + \mathcal{A}^{\pm}_{\alpha})\psi_{\pm} \;&=\; 0,\\ \bar V^{\pm}_3\psi_{\pm} \;&=\; 0,\\ (\bar V^i_{\pm} + \mathcal{A}^i_{\pm})\psi_{\pm} \;&=\; 0,\end{aligned} \tag{II.49}$$

where $f_{+-} = \psi^{-1}_+\psi_-$. But this time, $\mathcal{A}^{\pm}_{\alpha}$ and $\mathcal{A}^i_{\pm}$ (and, of course, $\mathcal{A}^{\pm}_3$, which is absent in the present gauge) are interpreted as components of a differential $(0,1)$-form $\mathcal{A}^{0,1} \in \Gamma(\mathcal{P}^{3|4}, \Omega^{0,1}(\mathcal{P}^{3|4})\otimes \mathrm{End}\,\tilde{\mathcal{E}})$, where the bundle $\tilde{\mathcal{E}} \to \mathcal{P}^{3|4}$ is smoothly equivalent to $\mathcal{E} \to \mathcal{P}^{3|4}$, i.e.,

$$(\mathcal{E}, f = \{f_{+-}\}, \bar\partial_{\mathcal{P}}) \;\sim\; (\tilde{\mathcal{E}}, \tilde f = \{\mathbb{1}_r\}, \bar\partial_{\mathcal{P}} + \mathcal{A}^{0,1}).$$

This is nothing but a special case of Thm. II.1. Note that

$$\mathcal{A}^{0,1}|_{\mathcal{U}_\pm} \;=\; \psi_\pm \bar\partial_{\mathcal{P}} \psi_\pm^{-1}, \qquad \text{with} \qquad \mathcal{A}^{0,1}|_{\mathcal{U}_+} \;=\; \mathcal{A}^{0,1}|_{\mathcal{U}_-} \tag{II.50}$$

by smooth triviality of $\tilde{\mathcal{E}}$. By following the analysis of §II.6, one again reproduces the equations of motion of $\mathcal{N}=4$ self-dual SYM theory. So, we do not repeat the argumentation at this point.

Instead, we shall now change the trivialization of $\mathcal{E}$. In fact, there exist matrix-valued functions $\hat\psi = \{\hat\psi_+, \hat\psi_-\} \in C^0(\mathfrak{U}, \mathfrak{S})$ such that[6]

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \;=\; \hat\psi_+^{-1}\hat\psi_-, \qquad \text{with} \qquad \bar V^i_\pm \hat\psi_\pm \;=\; 0. \tag{II.51}$$

From (II.51) it then follows that

$$g \;:=\; \psi_+\hat\psi_+^{-1} \;=\; \psi_-\hat\psi_-^{-1} \tag{II.52}$$

is a globally well-defined matrix-valued function generating a gauge transformation

$$\begin{aligned} \psi_\pm \;&\mapsto\; \hat\psi_\pm \;=\; g^{-1}\psi_\pm, \\ \mathcal{A}^\pm_\alpha \;&\mapsto\; \hat{\mathcal{A}}^\pm_\alpha \;=\; g^{-1}\mathcal{A}^\pm_\alpha g + g^{-1}\bar V^\pm_\alpha g \;=\; \hat\psi_\pm \bar V^\pm_\alpha \hat\psi_\pm^{-1}, \\ 0 \;=\; \mathcal{A}^\pm_3 \;&\mapsto\; \hat{\mathcal{A}}^\pm_3 \;=\; g^{-1}\bar V^\pm_3 g \;=\; \hat\psi_\pm \bar V^\pm_3 \hat\psi_\pm^{-1}, \\ \mathcal{A}^i_\pm \;&\mapsto\; \hat{\mathcal{A}}^i_\pm \;=\; g^{-1}\mathcal{A}^i_\pm g + g^{-1}\bar V^i_\pm g \;=\; \hat\psi_\pm \bar V^i_\pm \hat\psi_\pm^{-1} \;=\; 0. \end{aligned} \tag{II.53}$$

Thus, we end up with

$$\begin{aligned} (\bar V^\pm_\alpha + \hat{\mathcal{A}}^\pm_\alpha)\hat\psi_\pm \;&=\; 0, \\ (\bar V^\pm_3 + \hat{\mathcal{A}}^\pm_3)\hat\psi_\pm \;&=\; 0, \\ \bar V^i_\pm \hat\psi_\pm \;&=\; 0, \end{aligned} \tag{II.54}$$

which is gauge equivalent to (II.49).

The compatibility conditions of the linear system (II.54) are, of course, the field equations of hCS theory on the supertwistor space $\mathcal{P}^{3|4}$. On $\mathcal{U}_\pm$, they read as

$$\begin{aligned} \bar V^\pm_\alpha \hat{\mathcal{A}}^\pm_\beta - \bar V^\pm_\beta \hat{\mathcal{A}}^\pm_\alpha + [\hat{\mathcal{A}}^\pm_\alpha, \hat{\mathcal{A}}^\pm_\beta] \;&=\; 0, \\ \bar V^\pm_3 \hat{\mathcal{A}}^\pm_\alpha - \bar V^\pm_\alpha \hat{\mathcal{A}}^\pm_3 + [\hat{\mathcal{A}}^\pm_3, \hat{\mathcal{A}}^\pm_\alpha] \;&=\; 0. \end{aligned} \tag{II.55}$$

As in §II.6, we now have to find the explicit superfield expansions of the components $\hat{\mathcal{A}}^\pm_\alpha$ and $\hat{\mathcal{A}}^\pm_3$, respectively. However, their form is fixed by the geometry of supertwistor space.

[6] Recall that $\mathcal{P}^{3|4} = \mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^3$.

Recall that $\hat{\mathcal{A}}^\pm_\alpha$ and $\hat{\mathcal{A}}^\pm_3$ are $\mathcal{O}_{\mathbb{C}P^1}(1)$- and $\overline{\mathcal{O}}_{\mathbb{C}P^1}(-2)$-valued. Together with the fact that the $\eta^\pm_i$s take values in the bundle $\Pi\mathcal{O}_{\mathbb{C}P^1}(1)$, this determines the dependence of $\hat{\mathcal{A}}^\pm_\alpha$ and $\hat{\mathcal{A}}^\pm_3$ on $\lambda_\pm$ and $\bar{\lambda}_\pm$ [205],

$$
\begin{aligned}
\hat{\mathcal{A}}^\pm_\alpha &= \lambda^{\dot\alpha}_\pm \overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \eta^\pm_i \overset{\circ}{\chi}{}^i_\alpha + \gamma_\pm \tfrac{1}{2!}\, \eta^\pm_i \eta^\pm_j\, \hat{\lambda}^{\dot\alpha}_\pm \overset{\circ}{W}{}^{ij}_{\alpha\dot\alpha} + \gamma^2_\pm\, \tfrac{1}{3!}\, \eta^\pm_i \eta^\pm_j \eta^\pm_k\, \hat{\lambda}^{\dot\alpha}_\pm \hat{\lambda}^{\dot\beta}_\pm\, \overset{\circ}{\chi}{}^{ijk}_{\alpha\dot\alpha\dot\beta} + \\
&\qquad + \gamma^3_\pm\, \tfrac{1}{4!}\, \eta^\pm_i \eta^\pm_j \eta^\pm_k \eta^\pm_l\, \hat{\lambda}^{\dot\alpha}_\pm \hat{\lambda}^{\dot\beta}_\pm \hat{\lambda}^{\dot\gamma}_\pm\, \overset{\circ}{G}{}^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma}, \\
\hat{\mathcal{A}}^\pm_3 &= \pm\gamma^2_\pm\, \tfrac{1}{2!}\, \eta^\pm_i \eta^\pm_j\, \overset{\circ}{W}{}^{ij} \pm \gamma^3_\pm\, \tfrac{1}{3!}\, \eta^\pm_i \eta^\pm_j \eta^\pm_k\, \hat{\lambda}^{\dot\alpha}_\pm\, \overset{\circ}{\chi}{}^{ijk}_{\dot\alpha} \pm \\
&\qquad \pm \gamma^4_\pm\, \tfrac{1}{4!}\, \eta^\pm_i \eta^\pm_j \eta^\pm_k \eta^\pm_l\, \hat{\lambda}^{\dot\alpha}_+ \hat{\lambda}^{\dot\beta}_\pm \overset{\circ}{G}{}^{ijkl}_{\dot\alpha\dot\beta}.
\end{aligned}
\tag{II.56}
$$

Here, $\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha}$, $\overset{\circ}{\chi}{}^i_\alpha$, $\overset{\circ}{W}{}^{ij}$, $\overset{\circ}{\chi}_{i\dot\alpha} := \frac{1}{3!}\epsilon_{ijkl}\overset{\circ}{\chi}{}^{jkl}_{\dot\alpha}$ and $\overset{\circ}{G}_{\dot\alpha\dot\beta} := \frac{1}{4!}\epsilon_{ijkl}\overset{\circ}{G}{}^{ijkl}_{\dot\alpha\dot\beta}$ is again the field content of $\mathcal{N}=4$ self-dual SYM theory. Note that, of course, the above expansions are unique only up to gauge transformations generated by group-valued functions which may depend on $\lambda_\pm$ and $\bar{\lambda}_\pm$. Also the absence of terms to zeroth and first order in $\eta^\pm_i$ in the expansion of $\hat{\mathcal{A}}^\pm_3$ is due to the existence of a gauge in which $\hat{\mathcal{A}}^\pm_3$ vanishes identically. Moreover, not all coefficient fields are independent degrees of freedom. Some of them are composite expressions,

$$
\overset{\circ}{W}{}^{ij}_{\alpha\dot\alpha} = -\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{W}{}^{ij}, \quad \overset{\circ}{\chi}{}^{ijk}_{\alpha\dot\alpha\dot\beta} = -\tfrac{1}{2}\overset{\circ}{\nabla}{}^R_{\alpha(\dot\alpha}\overset{\circ}{\chi}{}^{ijk}_{\dot\beta)} \quad \text{and} \quad \overset{\circ}{G}{}^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma} = -\tfrac{1}{3}\overset{\circ}{\nabla}{}^R_{\alpha(\dot\alpha}\overset{\circ}{G}{}^{ijkl}_{\dot\beta\dot\gamma)}, \tag{II.57}
$$

which follow upon substituting the field expansions (II.56) into the second equation of (II.55). Again we have abbreviated $\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha} := \partial^R_{\alpha\dot\alpha} + \overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha}$. The field expansions (II.56) together with the first equation of (II.55) eventually reproduce (II.36).

Now we have all ingredients to give the twistor analog of the action functional (II.38). In fact, we have just seen that the equations of motion of hCS theory,

$$
\bar{\partial}_{\mathcal{P}}\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1} \;=\; 0,
$$

reproduce the equations of motion of $\mathcal{N}=4$ self-dual SYM theory. Luckily, as was discussed in §I.17 of the previous chapter, the supertwistor space $\mathcal{P}^{3|4}$ is a formal Calabi-Yau supermanifold. In particular, this means that it admits a globally defined nowhere vanishing holomorphic volume form Ω. On the patches $\mathcal{U}_\pm$ of $\mathcal{P}^{3|4}$, it is given by

$$
\Omega|_{\mathcal{U}_\pm} \;=\; \pm \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \mathrm{d}z^3_\pm \mathrm{d}\eta^\pm_1 \mathrm{d}\eta^\pm_2 \mathrm{d}\eta^\pm_3 \mathrm{d}\eta^\pm_4. \tag{II.58}
$$

Assuming that $\mathcal{A}^{0,1}$ contains no antiholomorphic odd components and does not depend on $\bar{\eta}^\pm_i$ (see our above discussion), we may write down [261, 263]

$$
S \;=\; \int_{\mathcal{Y}} \Omega\wedge\mathrm{tr}\left\{\mathcal{A}^{0,1}\wedge\bar{\partial}_{\mathcal{P}}\mathcal{A}^{0,1} + \tfrac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right\}, \tag{II.59}
$$

where $\mathcal{Y}$ is the submanifold of $\mathcal{P}^{3|4}$ constrained by $\bar{\eta}_i^{\pm} = 0$. The action functional (II.59) of hCS theory represents the twistor analog of (II.38) we were looking for. Upon substituting the field expansions (II.56) into (II.59), integrating over the odd coordinates and over the Riemann sphere, we eventually end up with (II.38).

§II.8 Summary. Even though we have restricted our above discussion to the $\mathcal{N} = 4$ case, one may equally well talk about the cases with less supersymmetry, i.e., the cases with $\mathcal{N} < 4$. Of course, then the whole story is restricted to the level of the equations of motion, as there are no appropriate action principles. Nevertheless, we may collect all the things said above and summarize as follows:

Theorem II.2. *There is a one-to-one correspondence between gauge equivalence classes of local solutions to the $\mathcal{N}$-extended self-dual SYM equations on four-dimensional space-time and equivalence classes of holomorphic vector bundles $\mathcal{E}$ over supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ which are smoothly trivial and holomorphically trivial on any projective line $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$.*

Putting it differently, by Thm. II.1. we let $H^1_{\nabla^{0,1}}(\mathcal{P}^{3|\mathcal{N}}, \tilde{\mathcal{E}})$ be the moduli space of hCS theory on $\mathcal{P}^{3|\mathcal{N}}$ for vector bundles $\tilde{\mathcal{E}}$ smoothly equivalent to $\mathcal{E}$. Furthermore, we denote by $\mathcal{M}^{\mathcal{N}}_{\text{SDYM}}$ the moduli space of $\mathcal{N}$-extended self-dual SYM theory obtained from the solution space by quotiening with respect to the group of gauge transformations. Then we have a bijection

$$H^1_{\nabla^{0,1}}(\mathcal{P}^{3|\mathcal{N}}, \tilde{\mathcal{E}}) \;\cong\; \mathcal{M}^{\mathcal{N}}_{\text{SDYM}}. \tag{II.60}$$

II.3 Other self-dual models in four dimensions

Above we have related $\mathcal{N}$-extended self-dual SYM theory to hCS theory on supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. The $\mathcal{N} = 4$ case turned out to be very special in the sense of allowing to write down action functionals for self-dual SYM and hCS theories. Clearly, the reason is the formal Calabi-Yau property of $\mathcal{P}^{3|4}$. The natural question one may now pose is that of extending the above approach to other geometries but at the same time keeping the formal Calabi-Yau property. One such class of geometries is weighted projective superspaces [263, 206]. In fact, they are natural extensions of the projective superspace $\mathbb{C}P^{m|n}$. The following discussion is based on the work done together with Alexander Popov [206].

§II.9 Weighted projective superspaces. First, consider ordinary weighted projective spaces. They are defined by some $\mathbb{C}^*$-action on the complex space $\mathbb{C}^{m+1}$. By letting $(z^1, \dots, z^{m+1})$ be coordinates on $\mathbb{C}^{m+1}$, we define the weighted projective space $W\mathbb{C}P^m[k_1, \dots, k_{m+1}]$ for $k_i \in \mathbb{Z}$ according to

$$W\mathbb{C}P^m[k_1, \dots, k_{m+1}] \;:=\; (\mathbb{C}^{m+1} \setminus \{0\})/\mathbb{C}^*, \tag{II.61}$$

where the $\mathbb{C}^*$-action is given by

$$(z^1, \dots, z^{m+1}) \;\mapsto\; (t^{k_1} z^1, \dots, t^{k_{m+1}} z^{m+1}) \qquad \text{for} \qquad t \in \mathbb{C}^*. \tag{II.62}$$

Clearly, what we have just defined is a toric variety and as such it need not be a manifold. In general, there may be nontrivial fixed points under coordinate identifications leading to singularities. However, we shall ignore this subtlety at this point and assume that the generic expression (II.61) is a complex manifold. Anyhow, our later discussions will be unaffected by this issue since, in spirit of our above discussion, we shall be considering only certain subsets which are always manifolds.

In analogy to $\mathbb{C}P^{m|n} = (\mathcal{O}_{\mathbb{C}P^m}, \mathcal{O}_{\text{red}}(\Lambda^\bullet(\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{C}P^m}(-1)))$ from Chap. I, we define the weighted projective superspace $W\mathbb{C}P^{m|n}[k_1, \dots, k_{m+1}|l_1, \dots, k_n]$ for $k_i, l_i \in \mathbb{Z}$ by

$$\begin{aligned} W\mathbb{C}P^{m|n}[k_1, \dots, k_{m+1}|l_1, \dots, k_n] \;&:=\; (W\mathbb{C}P^m[k_1, \dots, k_{m+1}], \mathcal{O}_{W\mathbb{C}P^{m|n}}), \\ \mathcal{O}_{W\mathbb{C}P^{m|n}} \;&:=\; \mathcal{O}_{\text{red}}(\Lambda^\bullet(\mathcal{O}_{W\mathbb{C}P^m}(-l_1) \oplus \dots \oplus \mathcal{O}_{W\mathbb{C}P^m}(-l_n))). \end{aligned} \tag{II.63}$$

Furthermore, by extending the Euler sequence (I.62) to the present setting (see, e.g., [124] for the purely even case), one may readily deduce that the first Chern class is given by

$$c_1(W\mathbb{C}P^{m|n}) \;=\; \left(\sum_{i=1}^{m+1} k_i - \sum_{i=1}^{n} l_i\right) x, \tag{II.64}$$

where $x := c_1(\mathcal{O}_{W\mathbb{C}P^{m|n}}(1))$. Hence, for appropriate numbers k_i and l_i, the weighted projective superspace $W\mathbb{C}P^{m|n}$ becomes a formal Calabi-Yau supermanifold. Note also that with this definition, we have

$$W\mathbb{C}P^{m|n}[1, \dots, 1|1, \dots, 1] \;\equiv\; \mathbb{C}P^{m|n}.$$

§II.10 HCS theory on $\mathcal{P}^{3|2}_{p,q}$. For the sake of concreteness, let us now consider an open subset of $W\mathbb{C}P^{3|2}[1,1,1,1|p,q]$ defined by

$$\mathcal{P}^{3|2}_{p,q} \;:=\; W\mathbb{C}P^{3|2}[1,1,1,1|p,q] \setminus W\mathbb{C}P^{1|2}[1,1|p,q]. \tag{II.65}$$

This space can be identified with

$$\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(p) \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(q) \ \to \ \mathbb{C}P^1 \tag{II.66}$$

and as such, it can be covered by two patches, which we denote by $\mathcal{U}_\pm$. Obviously, for the particular combination $p+q=4$, it becomes a formal Calabi-Yau supermanifold. In the following, we shall only be interested in this case. Let $[z^\alpha, \pi_{\dot\alpha}, \eta_i]$ be homogeneous coordinates on $W\mathbb{C}P^{3|2}[1,1,1,1|p,q]$. Since the body of $\mathcal{P}^{3|2}_{p,q}$ is the twistor space $\mathcal{P}^3$, we may take as local coordinates (I.15) together with

$$\begin{aligned} \eta_1^+ &:= \frac{\eta_1}{\pi_{\dot 1}^p} \quad \text{and} \quad \eta_2^+ := \frac{\eta_2}{\pi_{\dot 1}^q} \quad \text{on} \quad \mathcal{U}_+, \\ \eta_1^- &:= \frac{\eta_1}{\pi_{\dot 2}^p} \quad \text{and} \quad \eta_2^- := \frac{\eta_2}{\pi_{\dot 2}^q} \quad \text{on} \quad \mathcal{U}_-, \end{aligned} \tag{II.67}$$

which are related by $\eta_1^+ = (z_+^3)^p \eta_1^-$ and $\eta_2^+ = (z_+^3)^q \eta_2^-$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$. Note that as even coordinates on $\mathcal{P}^{3|2}_{p,q}$ either $(z_\pm^\alpha, z_\pm^3)$ or $(x_R^{\alpha\dot\alpha}, \lambda_\pm)$ can be used if proper reality conditions as those discussed in §I.19 and in §II.7, respectively, have been imposed. Therefore, we can again take $\bar V_\alpha^\pm$ and $\bar V_3^\pm$ of (II.44) as even generators of the antiholomorphic tangent sheaf $T^{0,1}_{\mathbb{C}}\mathcal{P}^{3|2}_{p,q}$.

Having given all the ingredients, we may now consider hCS theory on $\mathcal{P}^{3|2}_{p,q}$. Let $\mathcal{E}$ be a smoothly trivial rank r complex vector bundle over $\mathcal{P}^{3|2}_{p,q}$ equipped with a holomorphic structure $\mathcal{A}^{0,1} \in H^1_{\nabla^{0,1}}(\mathcal{P}^{3|2}_{p,q}, \mathcal{E})$, that is, $\mathcal{A}^{0,1}$ is subject to (II.6). Furthermore, by virtue of the twistor approach, we shall assume that there exists a gauge in which the component $\mathcal{A}_3^\pm$ is zero. This corresponds to the holomorphic triviality on any $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|2}_{p,q}$ of the holomorphic vector bundle which is associated to any solution of hCS theory. The equations of motion (II.6) of hCS theory on the patches $\mathcal{U}_\pm$ of $\mathcal{P}^{3|2}_{p,q}$ are again given by (II.55), since there exists a gauge in which $\mathcal{A}^{0,1}$ does neither contain antiholomorphic odd components nor depend on $\bar\eta_i$. Let us now discuss particular examples.

§II.11 HCS theory on $\mathcal{P}^{3|2}_{1,3}$. Consider the case $p=1$ and $q=3$, where the fermionic coordinates $\eta_1^\pm$ and $\eta_2^\pm$ are $\Pi\mathcal{O}_{\mathbb{C}P^1}(1)$- and $\Pi\mathcal{O}_{\mathbb{C}P^1}(3)$-valued, respectively. Therefore, the components $\mathcal{A}_\alpha^\pm$ and $\mathcal{A}_3^\pm$ of the $(0,1)$-form $\mathcal{A}^{0,1}$ are again $\mathcal{O}_{\mathbb{C}P^1}(1)$- and $\overline{\mathcal{O}}_{\mathbb{C}P^1}(-2)$-valued. As before, this fixes the dependence of $\mathcal{A}_\alpha^\pm$ and $\mathcal{A}_3^\pm$ on $\lambda_\pm$ and $\bar\lambda_\pm$ up to gauge transformations. In particular, we obtain

$$\begin{aligned} \mathcal{A}_\alpha^\pm &= \lambda_\pm^{\dot\alpha} \mathring{\mathcal{A}}_{\alpha\dot\alpha} + \eta_1^\pm \mathring{\chi}_\alpha + \tfrac{1}{2!\sqrt{3}} \eta_2^\pm \gamma_\pm^2 \hat\lambda_\pm^{\dot\alpha} \hat\lambda_\pm^{\dot\beta} \mathring{\psi}_{\alpha\dot\alpha\dot\beta} + \tfrac{1}{3!} \eta_1^\pm \eta_2^\pm \gamma_\pm^3 \hat\lambda_\pm^{\dot\alpha} \hat\lambda_\pm^{\dot\beta} \hat\lambda_\pm^{\dot\gamma} \mathring{G}_{\alpha\dot\alpha\dot\beta\dot\gamma}, \\ \mathcal{A}_3^\pm &= \pm \tfrac{1}{\sqrt{3}} \eta_2^\pm \gamma_\pm^3 \hat\lambda_\pm^{\dot\alpha} \mathring{\psi}_{\dot\alpha} \pm \tfrac{1}{2!} \eta_1^\pm \eta_2^\pm \gamma_\pm^4 \hat\lambda_\pm^{\dot\alpha} \hat\lambda_\pm^{\dot\beta} \mathring{G}_{\dot\alpha\dot\beta}, \end{aligned} \tag{II.68}$$

where $\gamma_\pm$ has been defined in (II.45). As before, not all component fields are independent degrees of freedom. Upon substituting the above expansions into (II.55), we find

$$\overset{\circ}{\psi}_{\alpha\dot\alpha\dot\beta} \;=\; -\overset{\circ}{\nabla}{}^R_{\alpha(\dot\alpha}\overset{\circ}{\psi}_{\dot\beta)} \qquad \text{and} \qquad \overset{\circ}{G}_{\alpha\dot\alpha\dot\beta\dot\gamma} \;=\; -\overset{\circ}{\nabla}{}^R_{\alpha(\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma)}. \tag{II.69}$$

The field equations resulting from (II.55) are then easily obtained to be

$$\begin{aligned} \overset{\circ}{f}_{\dot\alpha\dot\beta} &= 0,\\ \epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}_\beta &= 0,\\ \epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\psi}_{\dot\beta} &= 0,\\ \epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma} &= \{\overset{\circ}{\chi}_\alpha, \overset{\circ}{\psi}_{\dot\gamma}\}. \end{aligned} \tag{II.70}$$

As one may check, these equations follow also from the action functional

$$S \;=\; \int \mathrm{d}^4x_R\ \mathrm{tr}\left\{\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\overset{\circ}{f}_{\dot\alpha\dot\beta} + \overset{\circ}{\psi}{}^{\dot\alpha}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^\alpha\right\}, \tag{II.71}$$

which can be obtained from (II.59) by integration over the odd coordinates and over the sphere $\mathbb{C}P^1$. Note that this action has an obvious supersymmetry the transformation laws being

$$\begin{aligned} \delta_\xi\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} &= \xi_{\dot\alpha}\overset{\circ}{\chi}_\alpha \qquad \text{and} \qquad \delta_\xi\overset{\circ}{G}_{\dot\alpha\dot\beta} = -\epsilon^{\alpha\beta}\xi_{(\dot\alpha}\overset{\circ}{\nabla}{}^R_{\alpha\dot\beta)}\overset{\circ}{\chi}_\beta,\\ \delta_\xi\overset{\circ}{\chi}_\alpha &= 0 \qquad \text{and} \qquad \delta_\xi\overset{\circ}{\psi}_{\dot\alpha} = -\xi^{\dot\beta}(\overset{\circ}{G}_{\dot\alpha\dot\beta} + \overset{\circ}{f}_{\dot\alpha\dot\beta}), \end{aligned} \tag{II.72}$$

where $\xi_{\dot\alpha}$ is a constant (anticommuting) spinor. The action describes a truncation of $\mathcal{N}=4$ self-dual SYM theory for which all the scalars and three of the dotted and three of the undotted fermions are put to zero.

§II.12 HCS theory on $\mathcal{P}^{3|2}_{2,2}$. Now we consider the case $p=q=2$, i.e., the odd coordinates $\eta^\pm_i$ take values in $\Pi\mathcal{O}_{\mathbb{C}P^1}(2)$. The equations of motion of hCS theory on $\mathcal{P}^{3|2}_{2,2}$ have the same form (II.55). Again, the functional dependence on $\lambda_\pm$ and $\bar\lambda_\pm$ is fixed up to gauge transformations by the geometry of $\mathcal{P}^{3|2}_{2,2}$. That is, this dependence has the form

$$\begin{aligned} \mathcal{A}^\pm_\alpha &= \lambda^{\dot\alpha}_\pm\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \tfrac{1}{\sqrt{3}}\eta^\pm_i\gamma_\pm\hat\lambda^{\dot\alpha}_\pm\overset{\circ}{\phi}{}^i_{\alpha\dot\alpha} + \tfrac{1}{3!}\eta^\pm_1\eta^\pm_2\gamma^3_\pm\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\hat\lambda^{\dot\gamma}_\pm\overset{\circ}{G}_{\alpha\dot\alpha\dot\beta\dot\gamma},\\ \mathcal{A}^\pm_3 &= \pm\tfrac{1}{\sqrt{3}}\eta^\pm_i\gamma^2_\pm\overset{\circ}{\phi}{}^i \pm \tfrac{1}{2!}\eta^\pm_1\eta^\pm_2\gamma^4_\pm\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\overset{\circ}{G}_{\dot\alpha\dot\beta} \end{aligned} \tag{II.73}$$

together with

$$\overset{\circ}{\phi}{}^i_{\alpha\dot\alpha} \;=\; -\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\phi}{}^i \qquad \text{and} \qquad \overset{\circ}{G}_{\alpha\dot\alpha\dot\beta\dot\gamma} \;=\; -\overset{\circ}{\nabla}{}^R_{\alpha(\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma)}. \tag{II.74}$$

The remaining nontrivial equations read

$$\begin{aligned} \overset{\circ}{f}_{\dot\alpha\dot\beta} &= 0, \\ \overset{\circ}{\Box}{}^R\overset{\circ}{\phi}{}^i &= 0, \\ \epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma} &= \epsilon_{ij}\{\overset{\circ}{\phi}{}^i, \overset{\circ}{\nabla}{}^R_{\alpha\dot\gamma}\overset{\circ}{\phi}{}^j\}, \end{aligned} \tag{II.75}$$

where $\overset{\circ}{\Box}{}^R$ has been introduced in (II.37). The associated action functional is given by

$$S = \int \mathrm{d}^4 x_R \ \mathrm{tr}\left\{\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\overset{\circ}{f}_{\dot\alpha\dot\beta} + \epsilon_{ij}\overset{\circ}{\phi}{}^i\overset{\circ}{\Box}{}^R\overset{\circ}{\phi}{}^j\right\}, \tag{II.76}$$

and can be obtained from (II.59). Note that formally (II.76) looks as the bosonic truncation of the self-dual $\mathcal{N}=4$ SYM theory, i.e., all the spinors and four of the six scalars of self-dual $\mathcal{N}=4$ SYM theory are put to zero. However, in (II.76) the parity of the scalars $\overset{\circ}{\phi}{}^i$ is different, as they are Graßmann odd. To understand their nature, note that in the expansions (II.73) we have two Graßmann odd vectors $\overset{\circ}{\phi}{}^i_{\alpha\dot\alpha}$ which satisfy the equations

$$\begin{aligned} \epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\phi}{}^i_{\beta\dot\beta} &= 0, \\ \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\phi}{}^i_{\beta\dot\beta} &= 0 \end{aligned} \tag{II.77}$$

following from (II.55). Solutions to these equations describe tangent vectors $\delta\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha}$ (with assigned odd parity) to the solution space of the self-duality equations $\overset{\circ}{f}_{\dot\alpha\dot\beta}=0$ [260, 242]. However, due to the first equation of (II.74) (which solves the first equation of (II.77) and reduces the second to $\overset{\circ}{\Box}{}^R\overset{\circ}{\phi}{}^i=0$), the $\overset{\circ}{\phi}{}^i_{\alpha\dot\alpha}$s are projected to zero in the moduli space of solutions to the equations $\overset{\circ}{f}_{\dot\alpha\dot\beta}=0$. By choosing $\overset{\circ}{\phi}{}^i=0$, we remain with the equations

$$\overset{\circ}{f}_{\dot\alpha\dot\beta} = 0 \qquad \text{and} \qquad \epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma} = 0, \tag{II.78}$$

which can be obtained from the Lorentz-invariant Siegel action [229]

$$S = \int \mathrm{d}^4 x_R \ \mathrm{tr}\left\{\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\overset{\circ}{f}_{\dot\alpha\dot\beta}\right\}, \tag{II.79}$$

describing self-dual YM theory.

§II.13 HCS theory on $\mathcal{P}^{3|2}_{4,0}$. Finally, we want to discuss the case in which the odd coordinate $\eta_1^\pm$ has weight four and $\eta_2^\pm$ weight zero, i.e., we consider $\mathcal{P}^{3|2}_{4,0}$. Proceeding as in the previous two paragraphs, we obtain the following field expansions:

$$\begin{aligned} \mathcal{A}^\pm_\alpha &= \lambda^{\dot\alpha}_\pm\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \tfrac{1}{3!}\eta_1^\pm\gamma_\pm^3\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\hat\lambda^{\dot\gamma}_\pm\overset{\circ}{\chi}_{\alpha\dot\alpha\dot\beta\dot\gamma} + \eta_2^\pm\lambda^{\dot\alpha}_\pm\overset{\circ}{\psi}_{\alpha\dot\alpha} + \tfrac{1}{3!}\eta_1^\pm\eta_2^\pm\gamma_\pm^3\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\hat\lambda^{\dot\gamma}_\pm\overset{\circ}{G}_{\alpha\dot\alpha\dot\beta\dot\gamma}, \\ \mathcal{A}^\pm_3 &= \pm\tfrac{1}{2!}\eta_1^\pm\gamma_\pm^4\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\overset{\circ}{\chi}_{\dot\alpha\dot\beta} \pm \tfrac{1}{2!}\eta_1^\pm\eta_2^\pm\gamma_\pm^4\hat\lambda^{\dot\alpha}_\pm\hat\lambda^{\dot\beta}_\pm\overset{\circ}{G}_{\dot\alpha\dot\beta} \end{aligned} \tag{II.80}$$

together with the conditions

$$\mathring{\chi}_{\alpha\dot\alpha\dot\beta\dot\gamma} \ = \ -\mathring{\nabla}^R_{\alpha(\dot\alpha}\mathring{\chi}_{\dot\beta\dot\gamma)} \qquad \text{and} \qquad \mathring{G}_{\alpha\dot\alpha\dot\beta\dot\gamma} \ = \ -\mathring{\nabla}^R_{\alpha(\dot\alpha}\mathring{G}_{\dot\beta\dot\gamma)} - \{\mathring{\psi}_{\alpha(\dot\alpha},\mathring{\chi}_{\dot\beta\dot\gamma)}\}. \tag{II.81}$$

The field equations of this theory read as

$$\begin{aligned} \mathring{f}_{\dot\alpha\dot\beta} \ &= \ 0, \\ \epsilon^{\alpha\beta}\mathring{\nabla}^R_{\alpha(\dot\alpha}\mathring{\psi}_{\beta\dot\beta)} \ &= \ 0, \\ \epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}^R_{\alpha\dot\alpha}\mathring{\chi}_{\dot\beta\dot\gamma} \ &= \ 0, \\ \epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}^R_{\alpha\dot\alpha}\mathring{G}_{\dot\beta\dot\gamma} \ &= \ -\epsilon^{\dot\alpha\dot\beta}\{\mathring{\psi}_{\alpha\dot\alpha},\mathring{\chi}_{\dot\beta\dot\gamma}\}. \end{aligned} \tag{II.82}$$

In this case, the action functional from which these equations arise is

$$S \ = \ \int \mathrm{d}^4x_R \ \mathrm{tr}\left\{\mathring{G}^{\dot\alpha\dot\beta}\mathring{f}_{\dot\alpha\dot\beta} + \epsilon^{\alpha\beta}(\mathring{\nabla}^R_{\alpha\dot\alpha}\mathring{\psi}_{\beta\dot\beta})\mathring{\chi}^{\dot\alpha\dot\beta}\right\}. \tag{II.83}$$

This time, the multiplet contains a space-time vector $\mathring{\psi}_{\alpha\dot\alpha}$ and an anti-self-dual two-form $\mathring{\chi}_{\dot\alpha\dot\beta}$ which are both Graßmann odd. Such fields are well known from topological YM theories [260, 242]. In this respect, the model (II.82), (II.83) can be understood as a truncated self-dual sector of these theories. One may, of course, also consider more than just two fermionic coordinates in order to enlarge the multiplet. This may lead to other truncations of topological YM theories. We will come to related issues when dealing with (truncated) self-dual SYM hierarchies in Chap. V. Note that the above constructions have been formalized in [219] in the context of exotic supermanifolds.

Chapter III

Supersymmetric Bogomolny monopole equations

Approximately two decades ago, it has been conjectured by Ward [253, 254, 255] that all integrable models in less than four space-time dimensions can be obtained from the self-dual YM equations in four dimensions. Typical examples of such systems are the non-linear Schrödinger equation, the Korteweg-de Vries equation, the sine-Gordon model, etc. All of these models follow from the self-dual YM equations by incorporating suitable algebraic ansätze for the self-dual gauge potential followed by a dimensional reduction. Also the Bogomolny monopole equations on $\mathbb{R}^3$, describing static Yang-Mills-Higgs monopoles in the Prasad-Sommerfield limit, may be added to this list. In fact, Hitchin showed [115] that the Bogomolny monopole equations can be described by using twistorial methods. He constructed a twistor space – the so-called mini-twistor space – corresponding to $\mathbb{R}^3$. Geometrically, it is the space of oriented lines in $\mathbb{R}^3$. Furthermore, he then gave the construction of a Penrose-Ward transform relating equivalence classes of certain holomorphic vector bundles over mini-twistor space to gauge equivalence classes of solutions to the Bogomolny monopole equations. Our subsequent discussion is devoted to an extension of Hitchin's approach to a supersymmetric setting. It is based on the work done together with Alexander Popov and Christian Sämann [208]. We will obtain the mini-supertwistor space, which leads us to a twistorial description of the supersymmetrized Bogomolny model. To jump ahead of our story a bit the mini-supertwistor space can be considered as an open subset of the weighted projective superspace $W\mathbb{C}P^{2|4}[2,1,1|1,1,1,1]$ and as such it is a formally Calabi-Yau; cf. also §II.9. Furthermore, on the way of our discussion we will meet with the notion of Cauchy-Riemann structures (see, e.g., Ref. [146] for the purely even case) which naturally generalize the notion of complex structures. This allows us to use tools familiar from complex geometry.

III.1 Cauchy-Riemann supermanifolds

The supersymmetric Bogomolny monopole equations we are interested in are obtained from the four-dimensional $\mathcal{N}=4$ self-dual SYM equations by a dimensional reduction $\mathbb{R}^4 \to \mathbb{R}^3$. In this section, we study in detail the meaning of this reduction on the level of the supertwistor space. Note that we shall always be working in the real setting as discussed in §II.7. In particular, we find that the supertwistor space $\mathcal{P}^{3|4}$, when interpreted as the real manifold $\mathbb{R}^{4|8} \times S^2$, reduces to the space $\mathbb{R}^{3|8} \times S^2$. As a complex manifold, however, $\mathcal{P}^{3|4}$ reduces to the rank $1|4$ holomorphic vector bundle $\mathcal{P}^{2|4} := \mathcal{O}_{\mathbb{C}P^1}(2) \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^4$. Due to this difference, the twistor correspondence gets more involved. For instance, we need a double fibration. We also show that $\mathbb{R}^{3|8} \times S^2$ can be equipped with so-called Cauchy-Riemann structures.

§III.1 Dimensional reduction $\mathbb{R}^{4|8} \times S^2 \to \mathbb{R}^{3|8} \times S^2$. It is well known that the Bogomolny equations on $\mathbb{R}^3$ describing BPS monopoles [59, 209] can be obtained from the self-dual YM equations on $\mathbb{R}^4$ by demanding the components of a gauge potential to be independent of x^4 and by putting the four-component of the gauge potential to be the Higgs field [171, 115, 22]. Obviously, one can similarly reduce the $\mathcal{N}=4$ self-dual SYM equations (II.36) on $\mathbb{R}^4$ by imposing the $\frac{\partial}{\partial x^4}$-invariance condition on all the fields

$$(\overset{\circ}{f}_{\alpha\beta}, \overset{\circ}{\chi}{}^i_\alpha, \overset{\circ}{W}{}^{ij}, \overset{\circ}{\chi}_{i\dot\alpha}, \overset{\circ}{G}_{\dot\alpha\dot\beta})$$

in the supermultiplet and obtain supersymmetric Bogomolny equations on $\mathbb{R}^3$. Recall that both $\mathcal{N}=4$ self-dual SYM theory and $\mathcal{N}=4$ SYM theory have an $SU(4) \cong Spin(6)$ R-symmetry group. In the case of the full $\mathcal{N}=4$ SYM theory, the R-symmetry group and supersymmetry get enlarged to $Spin(7)$ and $\mathcal{N}=8$ supersymmetry by a reduction from four to three dimensions, cf. Ref. [74]. However, the situation in the dimensionally reduced $\mathcal{N}=4$ self-dual SYM theory is more involved since there is no parity symmetry interchanging left-handed and right-handed fields, and only the $SU(4)$ subgroup of $Spin(7)$ is manifest as an R-symmetry of the Bogomolny model.

Recall that on $\mathbb{R}^4 \cong \mathbb{C}^2$ we may use the complex coordinates $x_R^{\alpha\dot\alpha}$ satisfying the reality conditions induced by (I.78) or the real coordinates x^μ defined by

$$x_R^{2\dot2} = \bar{x}_R^{1\dot1} =: -\mathrm{i}(x^1 - \mathrm{i}x^2) \qquad \text{and} \qquad x_R^{2\dot1} = -\bar{x}_R^{1\dot2} =: -\mathrm{i}(x^3 - \mathrm{i}x^4). \tag{III.1}$$

This choice differs from the one given in (I.79) by a change of coordinates. However, it is most convenient for our subsequent discussion. It will also yield a better comparability with related literature.

Translations generated by the vector field $\mathscr{T}_4 := \frac{\partial}{\partial x^4}$ are isometries of $\mathbb{R}^{4|8}$ and by taking the quotient with respect to the action of the Abelian group

$$\mathscr{G} \;:=\; \{\exp(a\mathscr{T}_4)\,|\,x^4 \mapsto x^4 + a, a \in \mathbb{R}\} \tag{III.2}$$

generated by $\mathscr{T}_4$, we obtain the superspace $\mathbb{R}^{3|8} \cong \mathbb{R}^{4|8}/\mathscr{G}$. Recall that the eight odd complex coordinates $\eta_i^{\dot\alpha}$ satisfy certain reality conditions induced by (I.78). The vector field $\mathscr{T}_4$ is trivially lifted to $\mathbb{R}^{4|8} \times S^2$ and therefore the supertwistor space, considered as the smooth supermanifold $\mathbb{R}^{4|8} \times S^2$, is reduced to $\mathbb{R}^{3|8} \times S^2 \cong (\mathbb{R}^{4|8} \times S^2)/\mathscr{G}$. In other words, smooth $\mathscr{T}_4$-invariant functions on $\mathcal{P}^{3|4} \cong \mathbb{R}^{4|8} \times S^2$ can be considered as "free" smooth functions on the supermanifold $\mathbb{R}^{3|8} \times S^2$.

Recall that the rotation group $SO(4)$ of Euclidean four-dimensional space is locally isomorphic to $SU(2)_L \times SU(2)_R \cong Spin(4)$. Upon dimensional reduction to three dimensions, the rotation group $SO(3)$ of $(\mathbb{R}^3, \delta_{rs})$ with $r, s = 1, 2, 3$ is locally $SU(2) \cong Spin(3)$, which is the diagonal group $\mathrm{diag}(SU(2)_L \times SU(2)_R)$. Therefore, the distinction between undotted, i.e., $SU(2)_L$, and dotted, i.e., $SU(2)_R$, indices disappears. This implies that one can relabel the bosonic coordinates $x_R^{\alpha\dot\beta}$ by $x_R^{\dot\alpha\dot\beta}$ and split them as

$$x_R^{\dot\alpha\dot\beta} \;=\; x_R^{(\dot\alpha\dot\beta)} + x_R^{[\dot\alpha\dot\beta]} \;:=\; \tfrac{1}{2}(x_R^{\dot\alpha\dot\beta} + x_R^{\dot\beta\dot\alpha}) + \tfrac{1}{2}(x_R^{\dot\alpha\dot\beta} - x_R^{\dot\beta\dot\alpha}), \tag{III.3}$$

into symmetric

$$y^{\dot\alpha\dot\beta} \;:=\; -\mathrm{i}x_R^{(\dot\alpha\dot\beta)}, \quad y^{\dot1\dot1} \;=\; -\bar y^{\dot2\dot2} \;=\; (x^1 + \mathrm{i}x^2) \;=:\; y, \quad y^{\dot1\dot2} \;=\; \bar y^{\dot1\dot2} \;=\; -x^3 \tag{III.4}$$

and antisymmetric

$$x_R^{[\dot\alpha\dot\beta]} \;=\; \epsilon^{\dot\alpha\dot\beta}x^4 \tag{III.5}$$

parts. More abstractly, this splitting corresponds to the decomposition $\mathbf{4} \cong \mathbf{3} \oplus \mathbf{1}$ of the irreducible real vector representation $\mathbf{4}$ of the group $Spin(4) \cong SU(2)_L \times SU(2)_R$ into two irreducible real representations $\mathbf{3}$ and $\mathbf{1}$ of the group $Spin(3) \cong SU(2) = \mathrm{diag}(SU(2)_L \times SU(2)_R)$. For future use, we also introduce the derivations

$$\partial_{(\dot\alpha\dot\beta)} \;:=\; \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial x_R^{\dot\alpha\dot\beta}} + \frac{\partial}{\partial x_R^{\dot\beta\dot\alpha}}\right), \tag{III.6}$$

which read explicitly as

$$\partial_{(11)} \;=\; \frac{\partial}{\partial y^{11}}, \qquad \partial_{(1\dot{2})} \;=\; \frac{1}{2}\frac{\partial}{\partial y^{1\dot{2}}} \qquad \text{and} \qquad \partial_{(\dot{2}\dot{2})} \;=\; \frac{\partial}{\partial y^{\dot{2}\dot{2}}}. \tag{III.7}$$

Altogether, we thus have

$$\frac{\partial}{\partial x_R^{\dot{\alpha}\dot{\beta}}} \;=\; -\mathrm{i}\partial_{(\dot{\alpha}\dot{\beta})} - \tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^4}. \tag{III.8}$$

§III.2 Holomorphic reduction $\mathcal{P}^{3|4} \to \mathcal{P}^{2|4}$. The vector field $\mathscr{T}_4 = \frac{\partial}{\partial x^4}$ yields a free twistor space action of the Abelian group $\mathscr{G} \cong \mathbb{R}$ which is the real part of the holomorphic action of the complex group $\mathscr{G}_{\mathbb{C}} \cong \mathbb{C}$. In other words, we have

$$\begin{aligned} \mathscr{T}_4 \;&=\; \frac{\partial}{\partial x^4} \;=\; \frac{\partial z_+^a}{\partial x^4}\frac{\partial}{\partial z_+^a} + \frac{\partial \bar{z}_+^a}{\partial x^4}\frac{\partial}{\partial \bar{z}_+^a} \\ &=\; \left(-\frac{\partial}{\partial z_+^2} + z_+^3\frac{\partial}{\partial z_+^1}\right) + \left(-\frac{\partial}{\partial \bar{z}_+^2} + \bar{z}_+^3\frac{\partial}{\partial \bar{z}_+^1}\right) \;=:\; \mathscr{T}_+' + \bar{\mathscr{T}}_+' \end{aligned} \tag{III.9}$$

in the coordinates (z_+^a, η_i^+) for $a = 1,2,3$ on $\mathcal{U}_+$, where

$$\mathscr{T}_+' \;:=\; \mathscr{T}'|_{\mathcal{U}_+} \;=\; -\frac{\partial}{\partial z_+^2} + z_+^3\frac{\partial}{\partial z_+^1} \tag{III.10}$$

is a holomorphic vector field on $\mathcal{U}_+$. Similarly, we obtain

$$\mathscr{T}_4 \;=\; \mathscr{T}_-' + \bar{\mathscr{T}}_-', \qquad \text{with} \qquad \mathscr{T}_-' \;:=\; \mathscr{T}'|_{\mathcal{U}_-} \;=\; -z_-^3\frac{\partial}{\partial z_-^2} + \frac{\partial}{\partial z_-^1} \tag{III.11}$$

on $\mathcal{U}_-$ and $\mathscr{T}_+' = \mathscr{T}_-'$ on $\mathcal{U}_+ \cap \mathcal{U}_-$. For holomorphic functions f on $\mathcal{P}^{3|4}$ we clearly have $\mathscr{T}_4 f = \mathscr{T}' f$, and therefore $\mathscr{T}'$-invariant holomorphic functions on $\mathcal{P}^{3|4}$ can be considered as "free" holomorphic functions on a reduced space $\mathcal{P}^{2|4} \cong \mathcal{P}^{3|4}/\mathscr{G}_{\mathbb{C}}$ obtained as the quotient space of $\mathcal{P}^{3|4}$ by the action of the complex Abelian group $\mathscr{G}_{\mathbb{C}}$ generated by $\mathscr{T}'$.

In the purely even case, the space $\mathcal{P}^2 \cong \mathcal{P}^3/\mathscr{G}_{\mathbb{C}}$ was called mini-twistor space [115] and we shall refer to $\mathcal{P}^{2|4}$ as the mini-supertwistor space. To sum up, the reduction of the supertwistor correspondence induced by the $\mathscr{T}_4$-action is described by the following diagram:

$$\begin{tikzcd} \mathcal{P}^{3|4} \arrow[r, phantom, "\cong"] \arrow[dd] & \mathbb{R}^{4|8} \times S^2 \arrow[r] \arrow[d] & \mathbb{R}^{4|8} \arrow[dd] \\ & \mathbb{R}^{3|8} \times S^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^{2|4} & & \mathbb{R}^{3|8} \end{tikzcd} \tag{III.12}$$

Here, "$\downarrow$" symbolizes projections generated by the action of the groups $\mathscr{G}$ or $\mathscr{G}_{\mathbb{C}}$ and π_2 is the canonical projection. The projection π_1 will be described momentarily.

§III.3 Geometry of mini-supertwistor space. It is not difficult to see that the functions

$$\begin{aligned} w_+^1 &:= -\mathrm{i}(z_+^1 + z_+^3 z_+^2), \quad w_+^2 := z_+^3 \quad \text{and} \quad \eta_i^+ \quad \text{on} \quad \mathcal{U}_+, \\ w_-^1 &:= -\mathrm{i}(z_-^2 + z_-^3 z_-^1), \quad w_-^2 := z_-^3 \quad \text{and} \quad \eta_i^- \quad \text{on} \quad \mathcal{U}_- \end{aligned} \tag{III.13}$$

are constant along the $\mathscr{G}_\mathbb{C}$-orbits in $\mathcal{P}^{3|4}$ and thus descend to the patches $\mathcal{V}_\pm := \mathcal{U}_\pm \cap \mathcal{P}^{2|4}$ covering the (orbit) space $\mathcal{P}^{2|4} \cong \mathcal{P}^{3|4}/\mathscr{G}_\mathbb{C}$. On the overlap $\mathcal{V}_+ \cap \mathcal{V}_-$, we have

$$w_+^1 = \frac{1}{(w_-^2)^2} w_-^1, \qquad w_+^2 = \frac{1}{w_-^2} \qquad \text{and} \qquad \eta_i^+ = \frac{1}{w_-^2}\eta_i^- \tag{III.14}$$

which coincides with the transformation laws of canonical coordinates on the total space

$$\mathcal{O}_{\mathbb{C}P^1}(2) \oplus \Pi\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \mathbb{C}^4 = \mathcal{P}^{2|4} \tag{III.15}$$

of the holomorphic vector bundle

$$\mathcal{P}^{2|4} \rightarrow \mathbb{C}P^1. \tag{III.16}$$

Clearly, this space is a formal Calabi-Yau supermanifold. Hence, it comes with a globally well-defined nowhere vanishing holomorphic volume form

$$\Omega|_{\mathcal{V}_\pm} = \pm \mathrm{d}w_\pm^1 \wedge \mathrm{d}w_\pm^2 \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm. \tag{III.17}$$

The body of this supermanifold is the mini-twistor space [115]

$$\mathcal{P}^2 \cong \mathcal{O}_{\mathbb{C}P^1}(2).$$

Note that the space $\mathcal{P}^{2|4}$ can be considered as an open subset of the weighted projective superspace $W\mathbb{C}P^{2|4}[2,1,1|1,1,1,1]$.

The real structure τ_E (cf. our discussion given in §I.19) on $\mathcal{P}^{3|4}$ induces a real structure on $\mathcal{P}^{2|4}$ acting on local coordinates by the formula

$$\tau_E(w_\pm^1, w_\pm^2, \eta_i^\pm) = \left(-\frac{\bar{w}_\pm^1}{(\bar{w}_\pm^2)^2}, -\frac{1}{\bar{w}_\pm^2}, \pm\frac{1}{\bar{w}_\pm^2} T_i{}^j \bar{\eta}_j^\pm\right), \tag{III.18}$$

where the matrix $(T_i{}^j)$ has been defined in (I.74). From (III.18), one sees that τ_E has no fixed points in $\mathcal{P}^{2|4}$ but leaves invariant projective lines $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{2|4}$ defined by the equations

$$\begin{aligned} w_+^1 &= y - 2\lambda_+ x^3 - \lambda_+^2 \bar{y}, \quad \eta_i^+ = \eta_i^{\dot{1}} + \lambda_+ \eta_i^{\dot{2}}, \quad \text{with} \quad \lambda_+ = w_+^2 \in U_+, \\ w_-^1 &= \lambda_-^2 y - 2\lambda_- x^3 - \bar{y}\ , \quad \eta_i^- = \lambda_- \eta_i^{\dot{1}} + \eta_i^{\dot{2}}, \quad \text{with} \quad \lambda_- = w_-^2 \in U_- \end{aligned} \tag{III.19}$$

for fixed $(x,\eta)\in\mathbb{R}^{3|8}$. Here, $y=x^1+\mathrm{i}x^2$, $\bar{y}=x^1-\mathrm{i}x^2$ and x^3 are coordinates on $\mathbb{R}^3$ and $U_\pm$ denote again the canonical patches covering $\mathbb{C}P^1$.

By using the coordinates (III.4), we can rewrite (III.19) as

$$w^1_\pm \;=\; \lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}y^{\dot\alpha\dot\beta}, \quad w^2_\pm \;=\; \lambda_\pm \qquad \text{and} \qquad \eta^\pm_i \;=\; \eta^{\dot\alpha}_i\lambda^\pm_{\dot\alpha}, \tag{III.20}$$

where the explicit form of $\lambda^\pm_{\dot\alpha}$ has been given in §I.3. In fact, Eqs. (III.20) are the incidence relations which lead to the double fibration

$$\begin{tikzcd} & \mathcal{F}^{5|8} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^{2|4} & & \mathbb{R}^{3|8} \end{tikzcd} \tag{III.21}$$

where $\mathcal{F}^{5|8}\cong\mathbb{R}^{3|8}\times S^2$, π_2 is again the canonical projection onto $\mathbb{R}^{3|8}$ and the projection π_1 is defined by the formula

$$\pi_1(x^r,\lambda_\pm,\eta^{\dot\alpha}_i) \;=\; \pi_1(y^{\dot\alpha\dot\beta},\lambda^\pm_{\dot\alpha},\eta^{\dot\alpha}_i) \;=\; (w^1_\pm,w^2_\pm,\eta^\pm_i), \tag{III.22}$$

where $r=1,2,3$, and $w^{1,2}_\pm$ and $\eta^\pm_i$ are given in (III.20). The diagram (III.21), which is a part of (III.12), describes the following proposition:

Proposition III.1. *There exist the following geometric correspondences:*

(i)	*point p in $\mathcal{P}^{2|4}$*	$\longleftrightarrow$	*oriented $\mathbb{R}^{1|0}_p$ in $\mathbb{R}^{3|8}$*
(ii)	*τ_E -invariant $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{P}^{2|4}$*	$\longleftrightarrow$	*point (x,η) in $\mathbb{R}^{3|8}$*

§III.4 Cauchy-Riemann supermanifolds. Consider the double fibration (III.21). The correspondence space $\mathbb{R}^{3|8}\times S^2$ is the smooth $5|8$-dimensional supermanifold. As it is of "wrong" dimensionality, it cannot be a complex supermanifold but it can be understood as a so-called Cauchy-Riemann (CR) supermanifold, i.e., as a partially complex supermanifold. Recall that a CR structure on a smooth supermanifold X of real dimension $m|n$ is a locally direct subsheaf $\overline{\mathscr{D}}$ of rank $r|s$ of the complexified tangent sheaf $T_{\mathbb{C}}X$ such that $\mathscr{D}\cap\overline{\mathscr{D}}=\{0\}$ and $\overline{\mathscr{D}}$ is involutive (integrable), i.e., $\overline{\mathscr{D}}$ is closed with respect to the Lie superbracket. Of course, the distribution $\mathscr{D}$ is integrable if $\overline{\mathscr{D}}$ is integrable. The pair $(X,\overline{\mathscr{D}})$ is called a CR supermanifold of dimension $m|n=\dim_{\mathbb{R}}X$, of rank $r|s=\dim_{\mathbb{C}}\overline{\mathscr{D}}$ and of codimension $m-2r|n-2s$. In particular, a CR structure on X in the special case $m|n=2r|2s$ is a complex structure on X. Thus, the notion of CR supermanifolds generalizes that of complex supermanifolds.

Given a CR supermanifold $(X,\overline{\mathscr{D}})$, we let $\Omega^k(X) := \Lambda^k T^*_{\mathbb{C}}X$ be the sheaf of complex-valued smooth differential k-forms. Then we define locally free subsheaves of $\mathcal{S}_X$-modules by

$$\hat{\Omega}^{p,q}_{\mathrm{CR}}(X) \ := \ \{\omega \in \Omega^{p+q}(X) \,|\, \bar{V}_0 \wedge \bar{V}_1 \wedge \cdots \wedge \bar{V}_q \lrcorner\, \omega = 0, \ \forall\, \bar{V}_i \in \overline{\mathscr{D}}\}. \tag{III.23}$$

Furthermore, we set $\hat{\Omega}^{p,-1}_{\mathrm{CR}}(X) := \{0 \in \Omega^0_{\mathrm{CR}}(X)\}$. Clearly, we have

$$\mathrm{d} \,:\, \hat{\Omega}^{p,q}_{\mathrm{CR}}(X) \ \to \ \hat{\Omega}^{p,q+1}_{\mathrm{CR}}(X).$$

We now define complex-valued differential (p,q)-forms $\Omega^{p,q}_{\mathrm{CR}}(X)$ on X according to

$$\Omega^{p,q}_{\mathrm{CR}}(X) \ := \ \hat{\Omega}^{p,q}_{\mathrm{CR}}(X)/\hat{\Omega}^{p+1,q-1}_{\mathrm{CR}}(X). \tag{III.24}$$

Then we can introduce a natural family of $\bar{\partial}$-operators, i.e., $\bar{\partial} \,:\, \Omega^{p,q}_{\mathrm{CR}}(X) \to \Omega^{p,q+1}_{\mathrm{CR}}(X)$ by requiring that the diagrams

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{\bar{\partial}} & \Omega^{p,q}_{\mathrm{CR}}(X) & \xrightarrow{\bar{\partial}} & \Omega^{p,q+1}_{\mathrm{CR}}(X) & \xrightarrow{\bar{\partial}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{\mathrm{d}} & \hat{\Omega}^{p,q}_{\mathrm{CR}}(X) & \xrightarrow{\mathrm{d}} & \hat{\Omega}^{p+1,q}_{\mathrm{CR}}(X) & \xrightarrow{\mathrm{d}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{\mathrm{d}} & \hat{\Omega}^{p-1,q+1}_{\mathrm{CR}}(X) & \xrightarrow{\mathrm{d}} & \hat{\Omega}^{p,q+1}_{\mathrm{CR}}(X) & \xrightarrow{\mathrm{d}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

should commute.

§III.5 Cauchy-Riemann supertwistors. Let us now come back to our example (III.21). On the manifold $\mathbb{R}^{3|8} \times S^2$, one can introduce several CR structures. For instance, we may choose

$$\mathcal{F}^{5|8}_0 \ := \ (\mathbb{R}^{3|8} \times S^2, \overline{\mathscr{D}}_0) \ \cong \ \mathbb{R}^{1|0} \times \mathbb{C}^{1|4} \times \mathbb{C}P^1 \tag{III.25}$$

for the distribution

$$\overline{\mathscr{D}}_0 \ = \ \left\langle \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{\lambda}_\pm}, \frac{\partial}{\partial \bar{\eta}^{\,i}_i} \right\rangle. \tag{III.26}$$

Another one is obtained by setting

$$\mathcal{F}^{5|8} \;:=\; (\mathbb{R}^{3|8}\times S^2, \overline{\mathscr{D}} = \pi_1^* T^{0,1}_{\mathbb{C}}\mathcal{P}^{2|4}), \tag{III.27}$$

i.e.,

$$\overline{\mathscr{D}} \;=\; \left\langle \pi_1^*\frac{\partial}{\partial \bar{w}^1_\pm}, \pi_1^*\frac{\partial}{\partial \bar{w}^2_\pm}, \pi_1^*\frac{\partial}{\partial \bar{\eta}^\pm_i} \right\rangle . \tag{III.28}$$

In the following, we shall suppress the explicit appearance of π_1^*. In spirit of LeBrun's [146], we call $\mathcal{F}^{5|8}$ the CR supertwistor space.[1] Clearly, all the criteria for a CR structure are satisfied for our above two choices and moreover, in both cases the CR structures have rank $2|4$.

Let us denote the covering of $\mathcal{F}^{5|8}$ by $\hat{\mathfrak{V}} = \{\hat{\mathcal{V}}_+, \hat{\mathcal{V}}_-\}$. Up to now, we have used the coordinates $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot\alpha})$ or $(y^{\dot\alpha\dot\beta}, \lambda^\pm_{\dot\alpha}, \hat{\lambda}^\pm_{\dot\alpha}, \eta^{\dot\alpha}_i)$ on the two patches $\hat{\mathcal{V}}_\pm$. More appropriate for the distribution (III.28) are, however, the coordinates (III.19) together with

$$\begin{aligned} w^3_+ \;&:=\; \tfrac{1}{1+\lambda_+\bar{\lambda}_+}\left[\bar{\lambda}_+ y + (1-\lambda_+\bar{\lambda}_+)x^3 + \lambda_+\bar{y}\right] \qquad \text{on} \qquad \hat{\mathcal{V}}_+,\\ w^3_- \;&:=\; \tfrac{1}{1+\lambda_-\bar{\lambda}_-}\left[\lambda_- y + (\lambda_-\bar{\lambda}_- - 1)x^3 + \bar{\lambda}_-\bar{y}\right] \qquad \text{on} \qquad \hat{\mathcal{V}}_-. \end{aligned} \tag{III.29}$$

In terms of the coordinates (III.4) and $\lambda^\pm_{\dot\alpha}$, we can rewrite (III.19) and (III.29) concisely as

$$w^1_\pm \;=\; \lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}y^{\dot\alpha\dot\beta}, \quad w^2_\pm \;=\; \lambda_\pm, \quad w^3_\pm \;=\; -\gamma_\pm\lambda^\pm_{\dot\alpha}\hat{\lambda}^\pm_{\dot\beta}y^{\dot\alpha\dot\beta} \quad \text{and} \quad \eta^\pm_i \;=\; \eta^{\dot\alpha}_i\lambda^\pm_{\dot\alpha}, \tag{III.30}$$

where the factors $\gamma_\pm$ have been given in (II.45). Note that $w^3_\pm$ is real and all the other coordinates in (III.30) are complex. The relations between the coordinates on $\hat{\mathcal{V}}_+\cap\hat{\mathcal{V}}_-$ follow directly from their definitions (III.30).

The coordinates $w^{1,2}_\pm$ and $\eta^\pm_i$ have already appeared in (III.20) since $\mathcal{P}^{2|4}$ is a complex subsupermanifold of $\mathcal{F}^{5|8}$. Recall that formulas (III.22) together with (III.30) define a projection

$$\pi_1 \;:\; \mathcal{F}^{5|8} \;\to\; \mathcal{P}^{2|4} \tag{III.31}$$

onto mini-supertwistor space $\mathcal{P}^{2|4}$. The fibers over points $p\in\mathcal{P}^{2|4}$ of this projection are real one-dimensional spaces $\ell_p \cong \mathbb{R}$ parametrized by $w^3_\pm$. Note that the pull-back to $\mathcal{F}^{5|8}$ of the real structure τ_E on $\mathcal{P}^{2|4}$ given in (III.18) reverses the orientation of each line ℓ_p, since $\tau_E(w^3_\pm) = -w^3_\pm$.

[1] In the purely even case, a CR five-dimensional manifold can be constructed as a sphere bundle over an arbitrary three-dimensional manifold with conformal metric [146].

In order to clarify the geometry of the fibration (III.31), we note that the body $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$ of the supermanifold $\mathcal{F}^{5|8}$ can be considered as the sphere bundle

$$S(T\mathbb{R}^3) \;=\; \{(x,u) \in T\mathbb{R}^3 \,|\, \delta_{rs}u^r u^s = 1\} \;\cong\; \mathbb{R}^3 \times S^2 \tag{III.32}$$

whose fibers at points $x \in \mathbb{R}^3$ are spheres of unit vectors in $T_x\mathbb{R}^3$ [115]. Since this bundle is trivial, its projection onto $\mathbb{R}^3$ is obviously $\pi_2(x,u) = x$. Moreover, the complex two-dimensional mini-twistor space $\mathcal{P}^2$ can be described as the space of all oriented lines in $\mathbb{R}^3$. That is, any such line ℓ is defined by a unit vector u^r in the direction of ℓ and a shortest vector v^r from the origin in $\mathbb{R}^3$ to ℓ, and one can show [115] that

$$\mathcal{P}^2 \;=\; \{(v,u) \in T\mathbb{R}^3 \,|\, \delta_{rs}u^r v^s = 0\,,\ \delta_{rs}u^r u^s = 1\} \;\cong\; T\mathbb{C}P^1 \;\cong\; \mathcal{O}_{\mathbb{C}P^1}(2). \tag{III.33}$$

The fibers of the projection $\pi_1 : \mathbb{R}^3 \times S^2 \to \mathcal{P}^2$ are the orbits of the action of the group $\mathscr{G}' \cong \mathbb{R}$ on $\mathbb{R}^3 \times S^2$ given by $(v^r, u^s) \mapsto (v^r + tu^r, u^s)$ for $t \in \mathbb{R}$ and

$$\mathcal{P}^2 \;\cong\; \mathbb{R}^3 \times S^2/\mathscr{G}'. \tag{III.34}$$

Recall that $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$ is a (real) hypersurface in the twistor space $\mathcal{P}^3$. On the other hand, $\mathcal{P}^2$ is a complex two-dimensional submanifold of $\mathcal{F}^5$ and therefore

$$\mathcal{P}^2 \;\subset\; \mathcal{F}^5 \;\subset\; \mathcal{P}^3.$$

Similarly, we have in the supertwistor case

$$\mathcal{P}^{2|4} \;\subset\; \mathcal{F}^{5|8} \;\subset\; \mathcal{P}^{3|4}.$$

The formulas given in (III.19) and (III.29), respectively, define a coordinate transformation $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot{\alpha}}) \;\mapsto\; (w_\pm^a, \eta_i^\pm)$ on $\mathcal{F}^{5|8}$. From corresponding inverse formulas defining the transformation $(w_\pm^a, \eta_i^\pm) \mapsto (y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm, \eta_i^{\dot{\alpha}})$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial w_+^1} \;&=\; \gamma_+^2\left(\frac{\partial}{\partial y} - \bar{\lambda}_+\frac{\partial}{\partial x^3} - \bar{\lambda}_+^2\frac{\partial}{\partial \bar{y}}\right) \;=:\; \gamma_+^2 W_1^+,\\
\frac{\partial}{\partial w_+^2} \;&=\; W_2^+ + 2\gamma_+^2(x^3 + \lambda_+\bar{y})W_1^+ - \gamma_+^2(\bar{y} - 2\bar{\lambda}_+x^3 - \bar{\lambda}_+^2 y)W_3^+ - \gamma_+\bar{\eta}_i^{\dot{1}}V_+^i,\\
\frac{\partial}{\partial w_+^3} \;&=\; 2\gamma_+\left(\lambda_+\frac{\partial}{\partial y} + \bar{\lambda}_+\frac{\partial}{\partial \bar{y}} + \frac{1}{2}(1 - \lambda_+\bar{\lambda}_+)\frac{\partial}{\partial x^3}\right) \;=:\; W_3^+,
\end{aligned} \tag{III.35}$$

as well as

$$
\begin{aligned}
\frac{\partial}{\partial w^1_-} &= \gamma_-^2\left(\bar\lambda_-^2\frac{\partial}{\partial y}-\bar\lambda_-\frac{\partial}{\partial x^3}-\frac{\partial}{\partial \bar y}\right) \ =: \ \gamma_-^2 W_1^-,\\
\frac{\partial}{\partial w^2_-} &= W_2^- + 2\gamma_-^2(x^3-\lambda_- y)W_1^- + \gamma_-^2(\bar\lambda_-^2\bar y - 2\bar\lambda_- x^3 - y)W_3^- + \gamma_-\bar\eta_i^{\dot 2}V_-^i,\\
\frac{\partial}{\partial w^3_-} &= 2\gamma_-\left(\bar\lambda_-\frac{\partial}{\partial y}+\lambda_-\frac{\partial}{\partial \bar y}+\frac{1}{2}(\lambda_-\bar\lambda_- - 1)\frac{\partial}{\partial x^3}\right) \ =: \ W_3^-,
\end{aligned}
\tag{III.36}
$$

where $W_2^\pm := \frac{\partial}{\partial\lambda_\pm}$. Thus, when working in the coordinates $(y,\bar y,x^3,\lambda_\pm,\bar\lambda_\pm,\eta_i^{\dot\alpha})$ on $\hat{\mathcal{V}}_\pm\subset\mathcal{F}^{5|8}$, we will use the even vector fields $W_a^\pm$ with $a=1,2,3$ and the odd vector fields $V_\pm^i$ together with their complex conjugates $\bar W_{1,2}^\pm$ and $\bar V_\pm^i$, respectively. Note that the vector field $W_3^\pm$ is real!

Hence, we learn that when using the coordinates $(y^{\dot\alpha\dot\beta},\lambda^\pm_{\dot\alpha},\hat\lambda^\pm_{\dot\alpha},\eta_i^{\dot\alpha})$, the CR tangent sheaf $T^{1,0}_{\mathrm{CR}}\mathcal{F}^{5|8}$ is freely generated by

$$
\begin{aligned}
W_1^\pm \ = \ \hat\lambda_\pm^{\dot\alpha}\hat\lambda_\pm^{\dot\beta}\partial_{(\dot\alpha\dot\beta)},\quad W_2^\pm \ = \ \partial_{\lambda_\pm},\quad W_3^\pm \ = \ 2\gamma_\pm\hat\lambda_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\partial_{(\dot\alpha\dot\beta)},\\
V_\pm^i \ = \ -\hat\lambda_\pm^{\dot\alpha}T_j{}^i\frac{\partial}{\partial\eta_j^{\dot\alpha}}
\end{aligned}
\tag{III.37}
$$

while $T^{0,1}_{\mathrm{CR}}\mathcal{F}^{5|8}$ is generated by

$$
\bar W_1^\pm \ = \ -\lambda_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\partial_{(\dot\alpha\dot\beta)},\quad \bar W_2^\pm \ = \ \partial_{\bar\lambda_\pm},\quad \bar V_\pm^i \ = \ \lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_i^{\dot\alpha}}. \tag{III.38}
$$

It is not too difficult to see that forms dual to the vector fields (III.37) and (III.38) are

$$
\begin{aligned}
\Theta^1_\pm \ := \ \gamma_\pm^2\lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\mathrm{d}y^{\dot\alpha\dot\beta},\quad \Theta^2_\pm \ := \ \mathrm{d}\lambda_\pm,\quad \Theta^3_\pm \ := \ -\gamma_\pm\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta}\mathrm{d}y^{\dot\alpha\dot\beta},\\
E_i^\pm \ := \ \gamma_\pm\lambda^\pm_{\dot\alpha}T_i{}^j\mathrm{d}\eta_j^{\dot\alpha}
\end{aligned}
\tag{III.39}
$$

and

$$
\bar\Theta^1_\pm \ = \ -\gamma_\pm^2\hat\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta}\mathrm{d}y^{\dot\alpha\dot\beta},\quad \bar\Theta^2_\pm \ = \ \mathrm{d}\bar\lambda_\pm,\quad \bar E_i^\pm \ = \ -\gamma_\pm\hat\lambda^\pm_{\dot\alpha}\mathrm{d}\eta_i^{\dot\alpha}, \tag{III.40}
$$

where $T_i{}^j$ has been given in (I.74). The exterior derivative $\bar\partial$ on $\mathcal{F}^{5|8}$ reads as

$$
\bar\partial|_{\hat{\mathcal{V}}_\pm} \ = \ \mathrm{d}\bar w^1_\pm\frac{\partial}{\partial\bar w^1_\pm}+\mathrm{d}\bar w^2_\pm\frac{\partial}{\partial\bar w^2_\pm}+\mathrm{d}\bar\eta_i^\pm\frac{\partial}{\partial\bar\eta_i^\pm} \ = \ \bar\Theta^1_\pm\bar W_1^\pm+\bar\Theta^2_\pm\bar W_2^\pm+\bar E_i^\pm\bar V_\pm^i. \tag{III.41}
$$

Note again that $\Theta^3_\pm$ and $W_3^\pm$ are both real. To homogenize the notation later on, we shall also use $\bar W_3^\pm$ and $\partial_{\bar w_3^\pm}$ instead of $W_3^\pm$ and $\partial_{w_3^\pm}$, respectively.

III.2 Partially holomorphic Chern-Simons theory

We have discussed how mini-supertwistor and CR supertwistor spaces arise via dimensional reductions from supertwistor space of four-dimensional space-time. Subject of this section is the discussion of a generalization of Chern-Simons theory and its relatives to this setup. We call the theory we are about to introduce partially holomorphic Chern-Simons theory or phCS theory for short. Roughly speaking, this theory is a mixture of Chern-Simons and holomorphic Chern-Simons (hCS) theory on the CR supertwistor space $\mathcal{F}^{5|8}$ which has one real and two complex even dimensions. This theory is a reduction of hCS theory on $\mathcal{P}^{3|4}$. As we will show below, there is a one-to-one correspondence between the moduli space of solutions to the equations of motion of phCS theory on $\mathcal{F}^{5|8}$ and the moduli space of solutions to the supersymmetrized Bogomolny equations on $\mathbb{R}^3$, quite similar to Thm. II.2. for $\mathcal{N}$-extended self-dual SYM theory.

§III.6 Partially flat connections. Let X be a smooth supermanifold of real dimension $m|n$ and $T_\mathbb{C}X$ the complexified tangent sheaf of X. A locally direct subsheaf $\mathscr{T} \subset T_\mathbb{C}X$ is integrable if i) $\mathscr{T}\cap\overline{\mathscr{T}}$ has constant rank $r|s$ and ii) $\mathscr{T}$ and $\mathscr{T}\cap\overline{\mathscr{T}}$ are closed under the Lie superbracket. Note that a CR structure is the special case of an integrable distribution $\mathscr{T}$ with $r|s = 0|0$.

For any smooth function f on X, let $\mathrm{d}_\mathscr{T} f$ denote the restriction of $\mathrm{d}f$ to $\mathscr{T}$, i.e., $\mathrm{d}_\mathscr{T}$ is the composition

$$\mathcal{S}_X \ \overset{\mathrm{d}}{\to}\ T^*_\mathbb{C}X \ \to\ \mathscr{T}^*, \tag{III.42}$$

where $\mathscr{T}^*$ denotes the sheaf of smooth complex-valued differential one-forms dual to $\mathscr{T}$; cf., e.g., Rawnsley's discussion for the purely even case [210]. The operator $\mathrm{d}_\mathscr{T}$ can be extended to act on relative differential k-forms denoted by $\Omega^k_\mathscr{T}(X) := \Lambda^k\mathscr{T}^*$,

$$\mathrm{d}_\mathscr{T}\ :\ \Omega^k_\mathscr{T}(X)\ \to\ \Omega^{k+1}_\mathscr{T}(X). \tag{III.43}$$

Let $\mathcal{E}$ be a complex vector bundle over X. A connection on $\mathcal{E}$ along the distribution $\mathscr{T}$ – a $\mathscr{T}$-connection – is an even morphism of sheaves

$$\nabla_\mathscr{T}\ :\ \mathcal{E}\ \to\ \Omega^1_\mathscr{T}(X)\otimes\mathcal{E} \tag{III.44}$$

satisfying the Leibniz formula

$$\nabla_\mathscr{T}(f\sigma)\ =\ f\nabla_\mathscr{T}\sigma + \mathrm{d}_\mathscr{T} f\otimes\sigma, \tag{III.45}$$

where σ is a local section of $\mathcal{E}$ and f is a local smooth function. This $\mathscr{T}$-connection extends to

$$\nabla_{\mathscr{T}} \;:\; \Omega^k_{\mathscr{T}}(X,\mathcal{E}) \;\to\; \Omega^{k+1}_{\mathscr{T}}(X,\mathcal{E}), \tag{III.46}$$

where $\Omega^k_{\mathscr{T}}(X,\mathcal{E}) := \Omega^k_{\mathscr{T}}(X)\otimes\mathcal{E}$. Locally, $\nabla_{\mathscr{T}}$ has the form

$$\nabla_{\mathscr{T}} \;=\; \mathrm{d}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}}, \tag{III.47}$$

where the standard $\mathrm{End}\,\mathcal{E}$-valued $\mathscr{T}$-connection one-form $\mathcal{A}_{\mathscr{T}}$ has components only along the distribution $\mathscr{T}$. As usual, $\nabla^2_{\mathscr{T}}$ naturally induces

$$\mathcal{F}_{\mathscr{T}} \;\in\; \Gamma(X,\Omega^2_{\mathscr{T}}(X)\otimes\mathrm{End}\,\mathcal{E}) \tag{III.48}$$

which is the curvature of $\mathcal{A}_{\mathscr{T}}$. We say that $\nabla_{\mathscr{T}}$ is flat, if $\mathcal{F}_{\mathscr{T}}=0$. For a flat $\nabla_{\mathscr{T}}$, the pair $(\mathcal{E},\nabla_{\mathscr{T}})$ is called a $\mathscr{T}$-flat vector bundle. In particular, if $\mathscr{T}$ is a CR structure then $(\mathcal{E},\nabla_{\mathscr{T}})$ is a CR vector bundle. Moreover, if $\mathscr{T}$ is the integrable sheaf $T^{0,1}_{\mathbb{C}}X$ on some complex supermanifold X then the $\mathscr{T}$-flat complex vector bundle $(\mathcal{E},\nabla_{\mathscr{T}})$ is a holomorphic bundle.

§III.7 Field equations on the CR supertwistor space. Consider the CR supertwistor space $\mathcal{F}^{5|8}$ and a distribution $\mathscr{T}$ generated by the vector fields $\bar W_1^\pm$, $\bar W_2^\pm$, $\bar V^i_\pm$ from the CR structure $\overline{\mathscr{D}}$ and $\bar W_3^\pm$. This distribution is integrable since all conditions described in §III.6 are satisfied, e.g., the only nonzero commutator is

$$[\bar W_2^\pm,\bar W_3^\pm] \;=\; \pm 2\gamma_\pm^2\bar W_1^\pm \tag{III.49}$$

and therefore $\mathscr{T}$ is closed under the Lie superbracket. Also,

$$\mathscr{V} \;:=\; \mathscr{T}\cap\overline{\mathscr{T}} \tag{III.50}$$

is of (real) rank $1|0$ and hence integrable. The vector field $\bar W_3^\pm$ is a basis section for $\mathscr{V}$ over the patches $\hat{\mathcal{V}}_\pm\subset\mathcal{F}^{5|8}$. Note that mini-supertwistor space $\mathcal{P}^{2|4}$ is a subsupermanifold of $\mathcal{F}^{5|8}$ transversal to the leaves of $\mathscr{V}=\mathscr{T}\cap\overline{\mathscr{T}}$ and furthermore, $\mathscr{T}|_{\mathcal{P}^{2|4}}=\overline{\mathscr{D}}$. Thus, we have an integrable distribution $\mathscr{T}$ defined by

$$0 \;\to\; \pi_1^* T^{0,1}_{\mathbb{C}}\mathcal{P}^{2|4} \;\to\; \mathscr{T} \;\to\; (\Omega^{1,0}_{\mathrm{CR}}(\mathcal{F}^{5|8})/\pi_1^*\Omega^{1,0}(\mathcal{P}^{2|4}))^* \;\to\; 0 \tag{III.51}$$

on the CR supertwistor space $\mathcal{F}^{5|8}$ and we will denote by $\mathscr{T}_b$ its part generated by the even vector fields $\bar W_1^\pm$, $\bar W_2^\pm$ and $\bar W_3^\pm$,

$$\mathscr{T}_b \;:=\; \langle \bar W_1^\pm,\bar W_2^\pm,\bar W_3^\pm\rangle. \tag{III.52}$$

Let $\mathcal{E}$ be a trivial rank r complex vector bundle over $\mathcal{F}^{5|8}$ and $\mathcal{A}_{\mathscr{T}}$ a $\mathscr{T}$-connection one-form on $\mathcal{E}$ with $\mathscr{T}$ given by (III.51). Consider now the subspace $\mathcal{X}$ of $\mathcal{F}^{5|8}$ which is parametrized by the same even coordinates but only the holomorphic odd coordinates of $\mathcal{F}^{5|8}$, i.e., on $\mathcal{X}$, all objects are holomorphic in $\eta_i^\pm$. As it was already noted, the mini-supertwistor space is a formal Calabi-Yau supermanifold. In particular, this ensures the existence of a holomorphic volume form on $\mathcal{P}^{2|4}$. Moreover, $\mathcal{P}^{2|4} \subset \mathcal{F}^{5|8}$ and the pull-back $\hat{\Omega}$ of this form is globally defined on $\mathcal{F}^{5|8}$. Locally, on the patches $\hat{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$, one obtains

$$\hat{\Omega}|_{\hat{\mathcal{V}}_\pm} \;=\; \pm \mathrm{d}w_\pm^1 \wedge \mathrm{d}w_\pm^2 \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm . \tag{III.53}$$

This well-defined integral form allows us to integrate on $\mathcal{X}$ by pairing it with elements from $\Omega^3_{\mathscr{T}_b}(\mathcal{X})$. Assume that $\mathcal{A}_{\mathscr{T}}$ neither contains antiholomorphic odd components nor depends on $\bar{\eta}_i^\pm$,

$$\bar{V}_\pm^i \lrcorner \mathcal{A}_{\mathscr{T}} \;=\; 0 \qquad \text{and} \qquad \bar{V}_\pm^i(\mathcal{A}_a^\pm) \;=\; 0, \quad \text{with} \quad \mathcal{A}_a^\pm \;:=\; \bar{W}_a^\pm \lrcorner \mathcal{A}_{\mathscr{T}}, \tag{III.54}$$

that is, $\mathcal{A}_{\mathscr{T}} \in \Omega^1_{\mathscr{T}_b}(\mathcal{X}, \mathrm{End}\,\mathcal{E})$. Now, we introduce a CS-type action functional

$$S \;=\; \int_{\mathcal{X}} \hat{\Omega} \wedge \mathrm{tr}\, \left\{ \mathcal{A}_{\mathscr{T}} \wedge \mathrm{d}_{\mathscr{T}} \mathcal{A}_{\mathscr{T}} + \tfrac{2}{3} \mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}} \right\}, \tag{III.55}$$

where

$$\mathrm{d}_{\mathscr{T}}|_{\hat{\mathcal{V}}_\pm} \;=\; \mathrm{d}\bar{w}_\pm^a \frac{\partial}{\partial \bar{w}_\pm^a} + \mathrm{d}\bar{\eta}_i^\pm \frac{\partial}{\partial \bar{\eta}_i^\pm} \tag{III.56}$$

is the $\mathscr{T}$-part of the exterior derivative d on $\mathcal{F}^{5|8}$.

The action (III.55) leads to the CS-type field equations

$$\mathrm{d}_{\mathscr{T}} \mathcal{A}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}} \;=\; 0, \tag{III.57}$$

which are the equations of motion of partially holomorphic Chern-Simons (phCS) theory. In the nonholonomic basis $(\bar{W}_a^\pm, \bar{V}_\pm^i)$ of the distribution $\mathscr{T}$ over $\hat{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$, these equations read as

$$\begin{aligned} \bar{W}_1^\pm \mathcal{A}_2^\pm - \bar{W}_2^\pm \mathcal{A}_1^\pm + [\mathcal{A}_1^\pm, \mathcal{A}_2^\pm] \;&=\; 0\ , \\ \bar{W}_2^\pm \mathcal{A}_3^\pm - \bar{W}_3^\pm \mathcal{A}_2^\pm + [\mathcal{A}_2^\pm, \mathcal{A}_3^\pm] \mp 2\gamma_\pm^2 \mathcal{A}_1^\pm \;&=\; 0, \\ \bar{W}_1^\pm \mathcal{A}_3^\pm - \bar{W}_3^\pm \mathcal{A}_1^\pm + [\mathcal{A}_1^\pm, \mathcal{A}_3^\pm] \;&=\; 0, \end{aligned} \tag{III.58}$$

where the components $\mathcal{A}_a^\pm$ have been defined in (III.54).

§III.8 Equivalence to supersymmetric Bogomolny equations. Note that from (III.37) and (III.38), it follows that

$$\bar{W}_1^+ \;=\; \lambda_+^2 \bar{W}_1^-\,, \quad \bar{W}_2^+ \;=\; -\bar{\lambda}_+^{-2}\bar{W}_2^- \quad \text{and} \quad \gamma_+^{-1}\bar{W}_3^+ \;=\; \lambda_+\bar{\lambda}_+\left(\gamma_-^{-1}\bar{W}_3^-\right) \tag{III.59}$$

and therefore $\mathcal{A}_1^\pm$, $\mathcal{A}_2^\pm$ and $\gamma_\pm^{-1}\mathcal{A}_3^\pm$ take values in the bundles $\mathcal{O}_{\mathbb{C}P^1}(2)$, $\overline{\mathcal{O}}_{\mathbb{C}P^1}(-2)$ and $\mathcal{O}_{\mathbb{C}P^1}(1)\otimes\overline{\mathcal{O}}_{\mathbb{C}P^1}(1)$, respectively. Together with the definitions (III.54) of $\mathcal{A}_a^\pm$ and the fact that the $\eta_i^\pm$s are $\Pi\mathcal{O}_{\mathbb{C}P^1}(1)$-valued, this determines the dependence of $\mathcal{A}_a^\pm$ on $\eta_i^\pm$, $\lambda_\pm$ and $\bar{\lambda}_\pm$ to be

$$\mathcal{A}_1^\pm \;=\; -\lambda_\pm^{\dot{\alpha}}\mathcal{B}_{\dot{\alpha}}^\pm \qquad \text{and} \qquad \mathcal{A}_3^\pm \;=\; 2\gamma_\pm\hat{\lambda}_\pm^{\dot{\alpha}}\mathcal{B}_{\dot{\alpha}}^\pm \tag{III.60}$$

with the abbreviation

$$\begin{aligned}\mathcal{B}_{\dot{\alpha}}^\pm \;:=&\; \lambda_\pm^{\dot{\beta}}\mathring{\mathcal{B}}_{\dot{\alpha}\dot{\beta}} + \eta_i^\pm\mathring{\chi}_{\dot{\alpha}}^i + \tfrac{1}{2!}\gamma_\pm\eta_i^\pm\eta_j^\pm\hat{\lambda}_\pm^{\dot{\beta}}\mathring{W}^{ij}_{\dot{\alpha}\dot{\beta}} + \tfrac{1}{3!}\gamma_\pm^2\eta_i^\pm\eta_j^\pm\eta_k^\pm\hat{\lambda}_\pm^{\dot{\beta}}\hat{\lambda}_\pm^{\dot{\gamma}}\mathring{\chi}^{ijk}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \\ &+\; \tfrac{1}{4!}\gamma_\pm^3\eta_i^\pm\eta_j^\pm\eta_k^\pm\eta_l^\pm\hat{\lambda}_\pm^{\dot{\beta}}\hat{\lambda}_\pm^{\dot{\gamma}}\hat{\lambda}_\pm^{\dot{\delta}}\mathring{G}^{ijkl}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\end{aligned} \tag{III.61}$$

and

$$\mathcal{A}_2^\pm \;=\; \pm\tfrac{1}{2!}\gamma_\pm^2\eta_i^\pm\eta_j^\pm\mathring{W}^{ij} \pm \tfrac{1}{3!}\gamma_\pm^3\eta_i^\pm\eta_j^\pm\eta_k^\pm\hat{\lambda}_\pm^{\dot{\alpha}}\mathring{\chi}^{ijk}_{\dot{\alpha}} \pm \tfrac{1}{4!}\gamma_\pm^4\eta_i^\pm\eta_j^\pm\eta_k^\pm\eta_l^\pm\hat{\lambda}_\pm^{\dot{\alpha}}\hat{\lambda}_\pm^{\dot{\beta}}\mathring{G}^{ijkl}_{\dot{\alpha}\dot{\beta}}. \tag{III.62}$$

The expansions (III.61) and (III.62) are defined up to gauge transformations generated by group-valued functions which may depend on $\lambda_\pm$ and $\bar{\lambda}_\pm$. In particular, it is assumed in this twistor correspondence that for solutions to (III.58), there exists a gauge in which terms of zeroth and first order in $\eta_i^\pm$ are absent in $\mathcal{A}_2^\pm$. Recall that in the Čech approach, this corresponds to the holomorphic triviality of the bundle defined by such solutions when restricted to projective lines. Putting it differently, we consider a subset in the set of all solutions of phCS theory on $\mathcal{F}^{5|8}$, and we will always mean this subset when speaking about solutions to phCS theory.

Note that in (III.61) and (III.62), all coefficient fields $\mathring{\mathcal{B}}_{\dot{\alpha}\dot{\beta}}, \mathring{\chi}_{\dot{\alpha}}^i, \ldots$ depend only on $y^{\dot{\alpha}\dot{\beta}}$. Furthermore, not all of them represent independent degrees of freedom. Upon substituting the superfield expansions (III.61) and (III.62) into the first two equations of (III.58), we find the relations

$$\begin{aligned}\mathring{W}^{ij}_{\dot{\alpha}\dot{\beta}} \;&=\; -\left(\partial_{(\dot{\alpha}\dot{\beta})}\mathring{W}^{ij} + [\mathring{\mathcal{B}}_{\dot{\alpha}\dot{\beta}}, \mathring{W}^{ij}]\right), \\ \mathring{\chi}^{ijk}_{\dot{\alpha}(\dot{\beta}\dot{\gamma})} \;&=\; -\tfrac{1}{2}\left(\partial_{(\dot{\alpha}(\dot{\beta})}\mathring{\chi}^{ijk}_{\dot{\gamma})} + [\mathring{\mathcal{B}}_{\dot{\alpha}(\dot{\beta}}, \mathring{\chi}^{ijk}_{\dot{\gamma})}]\right), \\ \mathring{G}^{ijkl}_{\dot{\alpha}(\dot{\beta}\dot{\gamma}\dot{\delta})} \;&=\; -\tfrac{1}{3}\left(\partial_{(\dot{\alpha}(\dot{\beta})}\mathring{G}^{ijkl}_{\dot{\gamma}\dot{\delta})} + [\mathring{\mathcal{B}}_{\dot{\alpha}(\dot{\beta}}, \mathring{G}^{ijkl}_{\dot{\gamma}\dot{\delta})}]\right)\end{aligned} \tag{III.63}$$

showing that $\overset{\circ}{W}{}^{ij}_{\dot\alpha\dot\beta}$, $\overset{\circ}{\chi}{}^{ijk}_{\dot\alpha\dot\beta\dot\gamma}$ and $\overset{\circ}{G}{}^{ijkl}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ are composite fields. Furthermore, the field $\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta}$ can be decomposed into its symmetric part, denoted by $\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}=\overset{\circ}{\mathcal{A}}_{(\dot\alpha\dot\beta)}$, and its antisymmetric part, proportional to $\overset{\circ}{\Phi}$, such that

$$\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta} \;=\; \overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}-\tfrac{\mathrm{i}}{2}\epsilon_{\dot\alpha\dot\beta}\overset{\circ}{\Phi}. \tag{III.64}$$

Hence, we have recovered the covariant derivative $\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}=\partial_{(\dot\alpha\dot\beta)}+\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}$ and the (scalar) Higgs field $\overset{\circ}{\Phi}$. Defining

$$\overset{\circ}{\chi}_{i\dot\alpha} \;:=\; \tfrac{1}{3!}\epsilon_{ijkl}\overset{\circ}{\chi}{}^{jkl}_{\dot\alpha} \qquad\text{and}\qquad \overset{\circ}{G}_{\dot\alpha\dot\beta} \;:=\; \tfrac{1}{4!}\epsilon_{ijkl}\overset{\circ}{G}{}^{ijkl}_{\dot\alpha\dot\beta}, \tag{III.65}$$

we have thus obtained the supermultiplet in three dimensions:

$$\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta},\overset{\circ}{\chi}{}^{i}_{\dot\alpha},\overset{\circ}{\Phi},\overset{\circ}{W}{}^{ij},\overset{\circ}{\chi}_{i\dot\alpha},\overset{\circ}{G}_{\dot\alpha\dot\beta}.$$

We shall again abbreviate $\overset{\circ}{W}_{ij}:=\frac{1}{2!}\epsilon_{ijkl}\overset{\circ}{W}{}^{kl}$. Eqs. (III.58) together with the field expansions (III.61) and (III.62), the constraints (III.63) and the definitions (III.64) and (III.65) yield the maximally supersymmetrically extended Bogomolny monopole equations:

$$\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} &= -\tfrac{\mathrm{i}}{2}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\Phi},\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\chi}{}^{i}_{\dot\gamma} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{\Phi},\overset{\circ}{\chi}{}^{i}_{\dot\alpha}],\\
\overset{\circ}{\triangle}\overset{\circ}{W}{}^{ij} &= -\tfrac{1}{4}[\overset{\circ}{\Phi},[\overset{\circ}{W}{}^{ij},\overset{\circ}{\Phi}]]-\epsilon^{\dot\alpha\dot\beta}\{\overset{\circ}{\chi}{}^{i}_{\dot\alpha},\overset{\circ}{\chi}{}^{j}_{\dot\beta}\},\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\chi}_{i\dot\gamma} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{\chi}_{i\dot\alpha},\overset{\circ}{\Phi}]+2[\overset{\circ}{W}_{ij},\overset{\circ}{\chi}{}^{j}_{\dot\alpha}],\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{G}_{\dot\gamma\dot\delta} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{G}_{\dot\alpha\dot\delta},\overset{\circ}{\Phi}]+\{\overset{\circ}{\chi}{}^{i}_{\dot\alpha},\overset{\circ}{\chi}_{i\dot\delta}\}-\tfrac{1}{2}[\overset{\circ}{W}_{ij},\overset{\circ}{\nabla}_{\dot\alpha\dot\delta}\overset{\circ}{W}{}^{ij}]+\tfrac{\mathrm{i}}{4}\epsilon_{\dot\alpha\dot\delta}[\overset{\circ}{W}_{ij},[\overset{\circ}{\Phi},\overset{\circ}{W}{}^{ij}]],
\end{aligned} \tag{III.66}$$

which can also be derived from Eqs. (II.36) by demanding that all the fields in (II.36) are independent of the coordinate x^4. Here, we have introduced the abbreviation

$$\overset{\circ}{\triangle} \;:=\; \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}\overset{\circ}{\nabla}_{\dot\alpha\dot\gamma}\overset{\circ}{\nabla}_{\dot\beta\dot\delta}. \tag{III.67}$$

Note that (III.53) can be rewritten as $\hat\Omega|_{\hat{\mathcal{V}}_\pm}=\pm\Theta^1_\pm\wedge\Theta^2_\pm\mathrm{d}\eta^\pm_1\cdots\mathrm{d}\eta^\pm_4$, where the differential one-forms $\Theta^{1,2}_\pm$ have been given in (III.39). Substituting this expression and the expansions (III.61) and (III.61) into the action (III.55), we arrive after a straightforward calculation at the action

$$\begin{aligned}
S \;=\; \int\mathrm{d}^3x\,\mathrm{tr}\Big\{&\overset{\circ}{G}{}^{\dot\alpha\dot\beta}\left(\overset{\circ}{f}_{\dot\alpha\dot\beta}+\tfrac{\mathrm{i}}{2}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\Phi}\right)+\overset{\circ}{\chi}{}^{i\dot\alpha}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\chi}{}^{\dot\beta}_{i}+\tfrac{1}{2}\overset{\circ}{W}_{ij}\overset{\circ}{\triangle}\overset{\circ}{W}{}^{ij}+\\
&+\;\tfrac{\mathrm{i}}{2}\overset{\circ}{\chi}{}^{i}_{\dot\alpha}[\overset{\circ}{\chi}{}^{\dot\alpha}_{i},\overset{\circ}{\Phi}]+\overset{\circ}{W}_{ij}\{\overset{\circ}{\chi}{}^{i}_{\dot\alpha},\overset{\circ}{\chi}{}^{j\dot\alpha}\}+\tfrac{1}{8}[\overset{\circ}{W}_{ij},\overset{\circ}{\Phi}][\overset{\circ}{W}{}^{ij},\overset{\circ}{\Phi}]\Big\},
\end{aligned} \tag{III.68}$$

producing (III.66).

§III.9 PhCS theory in the Čech description. Our starting point in §III.7 was to consider a trivial complex vector bundle $\mathcal{E}$ over $\mathcal{F}^{5|8}$ endowed with a $\mathscr{T}$-connection. Such a $\mathscr{T}$-connection $\nabla_{\mathscr{T}} = \mathrm{d}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}}$ on $\mathcal{E}$ is flat if $\mathcal{A}_{\mathscr{T}}$ solves Eqs. (III.57), and then $(\mathcal{E}, f = \{\mathbb{1}_r\}, \nabla_{\mathscr{T}})$ is a $\mathscr{T}$-flat bundle in the Dolbeault description. As for holomorphic vector bundles (cf. Sec. II.1), one may turn to the Čech description of $\mathscr{T}$-flat bundles in which the connection one-form $\mathcal{A}_{\mathscr{T}}$ disappears and all the information is hidden in a transition function. In fact, let $\mathcal{E} \to X$ be a rank $r|s$ complex vector bundle over some smooth supermanifold X and denote by

$$H^1_{\nabla_{\mathscr{T}}}(X, \mathcal{E}) \;=\; \Gamma(X, \mathfrak{A}_{\mathscr{T}})/\Gamma(X, \mathfrak{S}) \tag{III.69}$$

the moduli space of phCS theory, where $\mathfrak{A}_{\mathscr{T}}$ is the sheaf of solutions to (III.57) and $\mathfrak{S} = GL(r|s, \mathcal{S}_X)$, as before. Furthermore, we shall need the subsheaf $\mathcal{C}_X \subset \mathcal{S}_X$ of $\mathscr{T}$-functions on X, that is, $\mathrm{d}_{\mathscr{T}} f = 0$ for $f \in \Gamma(U, \mathcal{C}_X)$ as well as the sheaf $\mathfrak{C} := GL(r|s, \mathcal{C}_X)$. Quite generically, we may then state the following theorem:

Theorem III.1. *Let X be a smooth supermanifold with an open Stein covering $\mathfrak{U} = \{\mathcal{U}_a\}$ and $\mathcal{E} \to X$ be a rank $r|s$ complex vector bundle over X. Then there is a one-to-one correspondence between $H^1_{\nabla_{\mathscr{T}}}(X, \mathcal{E})$ and the subset of $H^1(X, \mathfrak{C})$ consisting of those elements of $H^1(X, \mathfrak{C})$ representing vector bundles which are smoothly equivalent to $\mathcal{E}$, i.e.,*

$$(\mathcal{E}, f = \{f_{ab}\}, \nabla_{\mathscr{T}}) \;\sim\; (\tilde{\mathcal{E}}, \tilde{f} = \{\tilde{f}_{ab}\}, \mathrm{d}_{\mathscr{T}}),$$

where $\tilde{f}_{ab} = \psi_a^{-1} f_{ab} \psi_b$ for some $\psi = \{\psi_a\} \in C^0(\mathfrak{U}, \mathfrak{S})$.

The proof is similar to the one of Thm. II.1. For that reason, we shall omit it here and instead continue with our example.

Consider our smoothly trivial vector bundle $\mathcal{E} \to \mathcal{F}^{5|8}$ from above. Since the $\mathscr{T}$-connection one-form $\mathcal{A}_{\mathscr{T}}$ is flat, it is given as a pure gauge configuration on each patch and we have

$$\mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{V}}_\pm} \;=\; \psi_\pm \mathrm{d}_{\mathscr{T}} \psi_\pm^{-1}, \tag{III.70}$$

together with the gluing condition

$$\psi_+ \mathrm{d}_{\mathscr{T}} \psi_+^{-1} \;=\; \psi_- \mathrm{d}_{\mathscr{T}} \psi_-^{-1} \tag{III.71}$$

for the trivial bundle $\mathcal{E}$. Therefore, we can define a $\mathscr{T}$-flat complex vector bundle $\tilde{\mathcal{E}} \to \mathcal{F}^{5|8}$ with the canonical flat $\mathscr{T}$-connection $\mathrm{d}_{\mathscr{T}}$ and the transition function

$$\tilde{f}_{+-} \;:=\; \psi_+^{-1} \psi_-. \tag{III.72}$$

The condition $\mathrm{d}_{\mathscr{T}}\tilde{f}_{+-} = 0$ reads explicitly as

$$\begin{aligned} \bar{W}_1^+ \tilde{f}_{+-} &= 0, \quad \bar{W}_2^+ \tilde{f}_{+-} = 0, \quad \bar{V}_+^i \tilde{f}_{+-} = 0, \\ \bar{W}_3^+ \tilde{f}_{+-} &= 0. \end{aligned} \tag{III.73}$$

Recall that the vector fields appearing in the first line generate the antiholomorphic distribution $\overline{\mathscr{D}}$, which is the chosen CR structure. In other words, the bundle $\tilde{\mathcal{E}}$ is holomorphic along the mini-supertwistor space $\mathcal{P}^{2|4} \subset \mathcal{F}^{5|8}$ and flat along the fibers of the projection $\pi_1 : \mathcal{F}^{5|8} \to \mathcal{P}^{2|4}$ as follows directly from second line of (III.73). Let us now additionally assume that $\tilde{\mathcal{E}}$ is $\mathbb{R}^{3|8}$-trivial, that is, holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}^{5|8}$ given by (III.20). Recall that this assumption was already used in (III.62). As before, this extra ingredient guarantees the existence of a gauge in which the component $\mathcal{A}_2^\pm$ of $\mathcal{A}_{\mathscr{T}}$ vanishes. Hence, there exist $GL(r,\mathbb{C})$-valued functions $\hat{\psi} = \{\hat{\psi}_+, \hat{\psi}_-\} \in C^0(\hat{\mathfrak{V}}, \mathfrak{S})$ such that

$$\tilde{f}_{+-} = \psi_+^{-1}\psi_- = \hat{\psi}_+^{-1}\hat{\psi}_-, \qquad \text{with} \qquad \bar{W}_2^\pm \hat{\psi}_\pm = 0 \tag{III.74}$$

and

$$g := \psi_+ \hat{\psi}_+^{-1} = \psi_- \hat{\psi}_-^{-1} \tag{III.75}$$

is a matrix-valued function generating a gauge transformation

$$\psi_\pm \mapsto \hat{\psi}_\pm = g^{-1}\psi_\pm, \tag{III.76}$$

which acts on the gauge potential according to

$$\begin{aligned} \mathcal{A}_1^\pm &\mapsto \hat{\mathcal{A}}_1^\pm = g^{-1}\mathcal{A}_1^\pm g + g^{-1}\bar{W}_1^\pm g = \hat{\psi}_\pm \bar{W}_1^\pm \hat{\psi}_\pm^{-1}, \\ \mathcal{A}_2^\pm &\mapsto \hat{\mathcal{A}}_2^\pm = g^{-1}\mathcal{A}_2^\pm g + g^{-1}\bar{W}_2^\pm g = \hat{\psi}_\pm \bar{W}_2^\pm \hat{\psi}_\pm^{-1} = 0, \\ \mathcal{A}_3^\pm &\mapsto \hat{\mathcal{A}}_3^\pm = g^{-1}\mathcal{A}_3^\pm g + g^{-1}\bar{W}_3^\pm g = \hat{\psi}_\pm \bar{W}_3^\pm \hat{\psi}_\pm^{-1}, \\ 0 = \mathcal{A}_\pm^i := \psi_\pm \bar{V}_\pm^i \psi_\pm^{-1} &\mapsto \hat{\mathcal{A}}_\pm^i = g^{-1}\bar{V}_\pm^i g = \hat{\psi}_\pm \bar{V}_\pm^i \hat{\psi}_\pm^{-1}. \end{aligned} \tag{III.77}$$

In this new gauge, one generically has $\hat{\mathcal{A}}_\pm^i \neq 0$.

Note that (III.70) can be rewritten as the following linear system of differential equations:

$$\begin{aligned} (\bar{W}_a^\pm + \mathcal{A}_a^\pm)\psi_\pm &= 0, \\ \bar{V}_\pm^i \psi_\pm &= 0. \end{aligned} \tag{III.78}$$

The compatibility conditions of this linear system are Eqs. (III.58). This means that for any solution $\mathcal{A}_a^\pm$ to (III.58), one can construct solutions $\psi_\pm$ to (III.78) and, conversely, for any given $\psi_\pm$ obtained via a splitting (III.72) of a transition function $\tilde{f}_{+-}$, one can construct a solution (III.70) to (III.58).

Similarly, Eqs. (III.77) can be rewritten as the gauge equivalent linear system

$$\begin{aligned}(\bar{W}_1^\pm + \hat{\mathcal{A}}_1^\pm)\hat{\psi}_\pm &= 0,\\ \bar{W}_2^\pm\hat{\psi}_\pm &= 0,\\ (\bar{W}_3^\pm + \hat{\mathcal{A}}_3^\pm)\hat{\psi}_\pm &= 0,\\ (\bar{V}_\pm^i + \hat{\mathcal{A}}_\pm^i)\hat{\psi}_\pm &= 0.\end{aligned} \tag{III.79}$$

Note that due to the holomorphicity of $\hat{\psi}_\pm$ in $\lambda_\pm$ and the condition $\hat{\mathcal{A}}^+_{\mathscr{T}} = \hat{\mathcal{A}}^-_{\mathscr{T}}$ on $\hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-$, the components $\hat{\mathcal{A}}_1^\pm$, $\gamma_\pm^{-1}\hat{\mathcal{A}}_3^\pm$ and $\hat{\mathcal{A}}_\pm^i$ must take the form

$$\hat{\mathcal{A}}_1^\pm \;=\; -\lambda_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\mathcal{B}_{\dot\alpha\dot\beta}, \quad \gamma_\pm^{-1}\hat{\mathcal{A}}_3^\pm \;=\; 2\hat{\lambda}_\pm^{\dot\alpha}\lambda_\pm^{\dot\beta}\mathcal{B}_{\dot\alpha\dot\beta} \qquad \text{and} \qquad \hat{\mathcal{A}}_\pm^i \;=\; \lambda_\pm^{\dot\alpha}\mathcal{A}_{\dot\alpha}^i, \tag{III.80}$$

with λ-independent superfields $\mathcal{B}_{\dot\alpha\dot\beta} := \mathcal{A}_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\epsilon_{\dot\alpha\dot\beta}\Phi$ and $\mathcal{A}^i_{\dot\alpha}$. Introducing the first-order differential operators $D_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}$ and $\nabla^i_{\dot\alpha} = \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}$, we arrive at the following compatibility conditions of the linear system (III.79):

$$\begin{gathered}[D_{\dot\alpha\dot\gamma}, D_{\dot\beta\dot\delta}] + [D_{\dot\alpha\dot\delta}, D_{\dot\beta\dot\gamma}] \;=\; 0, \qquad [\nabla^i_{\dot\alpha}, D_{\dot\beta\dot\gamma}] + [\nabla^i_{\dot\gamma}, D_{\dot\beta\dot\alpha}] \;=\; 0,\\ \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} \;=\; 0.\end{gathered} \tag{III.81}$$

These equations also follow from (III.57) after substituting the expansions (III.80). These equations can be understood as the constraint equations of our supersymmetric Bogomolny model. In fact, proceeding as in §II.6, we can derive all the superfields together with their field expansions and the equations of motion they are subject to. Eventually, one finds (III.66). So, let us only sketch this way.

Eqs. (III.81) can equivalently be rewritten as

$$[D_{\dot\alpha\dot\gamma}, D_{\dot\beta\dot\delta}] \;=\; \epsilon_{\dot\gamma\dot\delta}\Sigma_{\dot\alpha\dot\beta}, \quad [\nabla^i_{\dot\alpha}, D_{\dot\beta\dot\gamma}] \;=\; \epsilon_{\dot\alpha\dot\gamma}\Sigma^i_{\dot\beta} \quad \text{and} \quad \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} \;=\; \epsilon_{\dot\alpha\dot\beta}\Sigma^{ij}, \tag{III.82}$$

where $\Sigma_{\dot\alpha\dot\beta} = \Sigma_{\dot\beta\dot\alpha}$ and $\Sigma^{ij} = -\Sigma^{ji}$. Note that the first equation in (III.81) immediately shows that $f_{\dot\alpha\dot\beta} = -\frac{\mathrm{i}}{2}\nabla_{\dot\alpha\dot\beta}\Phi$ and thus the contraction of the first equation of (III.82) with $\epsilon^{\dot\gamma\dot\delta}$ gives $\Sigma_{\dot\alpha\dot\beta} = f_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\nabla_{\dot\alpha\dot\beta}\Phi = 2f_{\dot\alpha\dot\beta}$. The Bianchi identities for the differential operators $D_{\dot\alpha\dot\beta}$ and $\nabla^i_{\dot\alpha}$ yield in a straightforward manner further field equations, which allow us to identify the superfields $\Sigma^i_{\dot\alpha}$ and Σ^{ij} with the spinors $\chi^i_{\dot\alpha}$ and the scalars W^{ij}, respectively. Moreover, $\chi_{i\dot\alpha}$ is given by $\chi_{i\dot\alpha} := \frac{1}{3}\epsilon_{ijkl}\nabla^j_{\dot\alpha}W^{kl}$ and $G_{\dot\alpha\dot\beta}$ is defined by $G_{\dot\alpha\dot\beta} := -\frac{1}{4}\nabla^i_{(\dot\alpha}\chi_{i\dot\beta)}$.

Collecting the above information, one obtains the equations of motion for the superfields $\mathcal{A}_{\dot\alpha\dot\beta}$, $\chi^i_{\dot\alpha}$, Φ, W^{ij}, $\chi_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$ which take the same form as (III.66) but with all the fields now being superfields. Thus, the projection of the superfields onto the zeroth order components of their η-expansions gives (III.66).

To extract the physical field content from the superfields, we need their explicit expansions in powers of $\eta^{\dot\alpha}_i$. For this, one again imposes the transversal gauge condition (II.28). The constraint equations (III.82) together with the Bianchi identities yield the recursion relations analogously to Eqs. (II.30). In fact, one simply needs to dimensionally reduce (II.30) to obtain the recursion relations for the present setting. Then by iterating the equations one may compute the superfield expansions. Furthermore, as in §II.6 one deduces the one-to-one correspondence between the constraint equations (III.82) and the field equations (III.66). We shall come back to this issue when dealing with explicit solution construction algorithms for Eqs. (III.66) in Sec. III.5.

III.3 Holomorphic BF theory

In the preceding section, we have defined a theory on the CR supertwistor space $\mathcal{F}^{5|8}$ entering into the double fibration (III.21) which we called partially holomorphic Chern-Simons theory. We have shown that this theory is equivalent to a supersymmetric Bogomolny-type Yang-Mills-Higgs theory in three Euclidean dimensions. The purpose of this section is to show, that one can also introduce a theory (including an action functional) on mini-supertwistor space $\mathcal{P}^{2|4}$, which is equivalent to phCS theory on $\mathcal{F}^{5|8}$. Thus, one can define at each level of the double fibration (III.21) a theory accompanied by a proper action functional and, moreover, these three theories are all equivalent.

§III.10 Field equations of hBF theory on $\mathcal{P}^{2|4}$. Consider the mini-supertwistor space $\mathcal{P}^{2|4}$. Let E be a trivial rank r complex vector bundle over $\mathcal{P}^{2|4}$ with a connection one-form $\mathcal{A}$. Assume that its $(0,1)$-part $\mathcal{A}^{0,1}$ contains no antiholomorphic odd components and does not depend on $\bar\eta^{\pm}_i$, that is, $\bar V^i_\pm \lrcorner \mathcal{A}^{0,1} = 0$ and $\bar V^i_\pm(\partial_{\bar w^{1,2}_\pm} \lrcorner \mathcal{A}^{0,1}) = 0$. Recall that on $\mathcal{P}^{2|4}$ we have a holomorphic volume form Ω which is locally given by (III.17). Hence, we can define a holomorphic BF (hBF) type theory – as introduced and discussed in Refs. [204, 128, 130, 28] – on $\mathcal{P}^{2|4}$ with the action

$$S \;=\; \int_{\mathcal{Y}} \Omega\wedge \mathrm{tr}\{B(\bar\partial_{\mathcal{P}}\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1})\} \;=\; \int_{\mathcal{Y}} \Omega\wedge\mathrm{tr}\{B\mathcal{F}^{0,2}\}. \tag{III.83}$$

Here, B is a scalar field in the adjoint representation of the group $GL(r,\mathbb{C})$, $\bar\partial_{\mathcal{P}}$ is the antiholomorphic part of the exterior derivative on $\mathcal{P}^{2|4}$ and $\mathcal{F}^{0,2}$ the $(0,2)$-part of the curvature two-form. The space $\mathcal{Y}$ is the subsupermanifold of $\mathcal{P}^{2|4}$ constrained by $\bar\eta_i^\pm = 0$.

The equations of motion following from the action functional (III.83) are

$$\begin{aligned} \bar\partial_{\mathcal{P}}\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1} &= 0, \\ \bar\partial_{\mathcal{P}}B + [\mathcal{A}^{0,1}, B] &= 0. \end{aligned} \tag{III.84}$$

These equations as well as the Lagrangian in (III.83) can be obtained from Eqs. (III.57) and the Lagrangian in (III.55), respectively, by imposing the condition $\partial_{\bar w^3_\pm}\mathcal{A}_{\bar w^a_\pm} = 0$ and identifying

$$\mathcal{A}^{0,1}|_{\mathcal{V}_\pm} = \mathrm{d}\bar w^1_\pm \mathcal{A}_{\bar w^1_\pm} + \mathrm{d}\bar w^2_\pm \mathcal{A}_{\bar w^2_\pm} \qquad \text{and} \qquad B^\pm := B|_{\mathcal{V}_\pm} = \mathcal{A}_{\bar w^3_\pm}. \tag{III.85}$$

Note that $\mathcal{A}_{\bar w^3_\pm}$ behaves on $\mathcal{P}^{2|4}$ as a scalar. Thus, (III.84) can be obtained from (III.57) by demanding invariance of all fields under the action of the group $\mathscr{G}'$ from §III.5 such that $\mathcal{P}^{2|4} \cong \mathcal{F}^{5|8}/\mathscr{G}'$.

§III.11 Čech description. When restricted to the patches $\mathcal{V}_\pm$, Eqs. (III.84) can be solved by

$$\mathcal{A}^{0,1}|_{\mathcal{V}_\pm} = \tilde\psi_\pm \bar\partial_{\mathcal{P}}\tilde\psi_\pm^{-1} \qquad \text{and} \qquad B^\pm = \tilde\psi_\pm B_0^\pm \tilde\psi_\pm^{-1}, \tag{III.86}$$

where $B_0^\pm$ is a holomorphic $\mathfrak{gl}(r,\mathbb{C})$-valued function on $\mathcal{V}_\pm$,

$$\bar\partial_{\mathcal{P}} B_0^\pm = 0. \tag{III.87}$$

On the intersection $\mathcal{V}_+\cap\mathcal{V}_-$, we have the gluing conditions

$$\tilde\psi_+\bar\partial_{\mathcal{P}}\tilde\psi_+^{-1} = \tilde\psi_-\bar\partial_{\mathcal{P}}\tilde\psi_-^{-1} \qquad \text{and} \qquad \tilde\psi_+ B_0^+\tilde\psi_+^{-1} = \tilde\psi_- B_0^-\tilde\psi_-^{-1} \tag{III.88}$$

as E is a trivial bundle. From (III.88), we learn that

$$\tilde f_{+-} := \tilde\psi_+^{-1}\tilde\psi_- \tag{III.89}$$

can be identified with the holomorphic transition function of a bundle $\tilde E$ with the canonical holomorphic structure $\bar\partial_{\mathcal{P}}$, and

$$B_0^+ = \tilde f_{+-} B_0^- \tilde f_{+-}^{-1}, \tag{III.90}$$

i.e., $B_0 \in H^0(\mathcal{P}^{2|4}, \mathrm{End}\,\tilde{E})$ and $B \in H^0(\mathcal{P}^{2|4}, \mathrm{End}\,E)$. Note that the pull-back $\pi_1^*\tilde{E}$ of the bundle $\tilde{E}$ to the space $\mathcal{F}^{5|8}$ can be identified with the bundle $\tilde{\mathcal{E}}$,

$$\tilde{\mathcal{E}} \;=\; \pi_1^*\tilde{E}, \tag{III.91}$$

with the transition function $\tilde{f}_{+-} = \psi_+^{-1}\psi_- = \tilde{\psi}_+^{-1}\tilde{\psi}_-$ if one additionally assumes that $\tilde{E}$ is $\mathbb{R}^{3|8}$-trivial, that is, holomorphically trivial on any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{2|4}$. Recall that the transition functions of the bundle $\tilde{\mathcal{E}}$ do not depend on $w^3_\pm$ and therefore they can always be considered as the pull-backs of transition functions of a bundle $\tilde{E}$ over $\mathcal{P}^{2|4}$.

§III.12 Moduli space. By construction, $B = \{B^\pm\}$ is a $\mathfrak{gl}(r, \mathbb{C})$-valued function generating trivial infinitesimal gauge transformations of $\mathcal{A}^{0,1}$ and therefore it does not contain any physical degrees of freedom. Remember that solutions to the first equation of (III.84) are defined up to gauge transformations

$$\mathcal{A}^{0,1} \;\mapsto\; \tilde{\mathcal{A}}^{0,1} \;=\; g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}_{\mathcal{P}}g^{-1} \tag{III.92}$$

generated by smooth $GL(r, \mathbb{C})$-valued functions g on $\mathcal{P}^{2|4}$. Clearly, the transformations (III.92) do not change the holomorphic structure $\nabla^{0,1}$ on the bundle E. On infinitesimal level, the transformations (III.92) take the form

$$\delta\mathcal{A}^{0,1} \;=\; \bar{\partial}_{\mathcal{P}}B + [\mathcal{A}^{0,1}, B], \tag{III.93}$$

with $B \in H^0(\mathcal{P}^{2|4}, \mathrm{End}\,E)$ and such a field B, solving the second equation of (III.84), generates holomorphic transformations such that $\delta\mathcal{A}^{0,1} = 0$. Their finite version is

$$\tilde{\mathcal{A}}^{0,1} \;=\; g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}_{\mathcal{P}}g^{-1} \;=\; \mathcal{A}^{0,1}, \tag{III.94}$$

and for a gauge potential $\mathcal{A}^{0,1}$ given by (III.86), such a g takes the form

$$g_\pm \;=\; \tilde{\psi}_\pm \mathrm{e}^{B_0^\pm}\tilde{\psi}_\pm^{-1}, \quad \text{with} \quad g_+ \;=\; g_- \quad \text{on} \quad \mathcal{V}_+ \cap \mathcal{V}_-. \tag{III.95}$$

§III.13 Summary. Collecting all the things derived in the preceding sections, we may now summarize our discussion as follows:

Theorem III.2. *The are one-to-one correspondences between equivalence classes of* $\mathbb{R}^{3|8}$*-trivial holomorphic vector bundles* E *over mini-supertwistor space* $\mathcal{P}^{3|8}$*, equivalence classes of* $\mathbb{R}^{3|8}$*-trivial* $\mathscr{T}$*-flat vector bundles* $\mathcal{E}$ *over CR supertwistor space* $\mathcal{F}^{5|8}$ *– all being smoothly trivial – and gauge equivalence classes of local solutions to the maximally supersymmetrized Bogomolny monopole equations on* $\mathbb{R}^3$*.*

By virtue of Thms. II.1. and III.1., we let i) $H^1_{\nabla^{0,1}}(\mathcal{P}^{2|4},\tilde{E})$ be the moduli space of hBF theory on $\mathcal{P}^{2|4}$ for vector bundles $\tilde{E}$ smoothly equivalent to E and ii) $H^1_{\nabla_{\mathscr{T}}}(\mathcal{F}^{5|8},\tilde{\mathcal{E}})$ be the moduli space of phCS theory on $\mathcal{F}^{5|8}$ for vector bundles $\tilde{\mathcal{E}}$ smoothly equivalent to $\mathcal{E}$, respectively. Then we have the bijections

$$H^1_{\nabla^{0,1}}(\mathcal{P}^{2|4},\tilde{E}) \;\cong\; H^1_{\nabla_{\mathscr{T}}}(\mathcal{F}^{5|8},\tilde{\mathcal{E}}) \;\cong\; \mathcal{M}_{\mathrm{sB}}, \tag{III.96}$$

where $\mathcal{M}_{\mathrm{sB}}$ denotes the moduli space of the supersymmetric Bogomolny monopole equations on $\mathbb{R}^3$ obtained from the solution space by quotiening with respect to the group of gauge transformations.

Pictorially, we have established the following diagram:

$$\begin{tikzcd} & \text{phCS theory on } \mathcal{F}^{5|8} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{hBF theory on } \mathcal{P}^{2|4} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{supersymmetric}\\ \text{Bogomolny model on } \mathbb{R}^3\end{array} \end{tikzcd}$$

III.4 Massive fields

In [74] (see also the subsequent work [75]), Chiou et al. developed a twistor string theory corresponding to a certain massive SYM theory in three dimensions. It was argued, that the mass terms in this theory arise from coupling the R-symmetry current to a constant background field when performing the dimensional reduction. In this section, we want to study the analogous construction for the supersymmetric Bogomolny model which we discussed in the previous sections. We focus on the geometric origin of the additional mass terms by discussing the associated twistor description. More explicitly, we establish a correspondence between holomorphic bundles over the deformed mini-supertwistor space introduced in [74] and solutions to massive supersymmetric Bogomolny equations in three dimensions.

§III.14 Mini-supertwistor and CR supertwistor spaces as vector bundles. We start from the observation that mini-supertwistor space $\mathcal{P}^{2|4}$ can be considered as the total space of a rank $0|4$ holomorphic vector bundle over mini-twistor space $\mathcal{P}^2$, that is,

$$\mathcal{P}^{2|4} \;\to\; \mathcal{P}^2. \tag{III.97}$$

The mini-twistor space $\mathcal{P}^2$ is covered by two patches, say $\mathcal{W}_\pm$, with coordinates $w^1_\pm$ and $w^2_\pm$. The additional fiber coordinates in the vector bundle $\mathcal{P}^{2|4}$ over $\mathcal{P}^2$ are the Graßmann variables $\eta^\pm_i$. For later convenience, we rearrange them into the vector $\eta^\pm = (\eta^\pm_i)$. On $\mathcal{W}_+ \cap \mathcal{W}_-$, we have the relation

$$\eta^+ \;=\; \varphi_{+-}\eta^-, \tag{III.98}$$

with the transition function

$$\varphi_{+-} \;=\; w^2_+(\delta_i{}^j) \;=\; w^2_+\mathbb{1}_4. \tag{III.99}$$

The CR supertwistor space $\mathcal{F}^{5|8}$ is a CR vector bundle, i.e., it has a transition function annihilated by the vector fields $\partial_{\bar{w}^1_\pm}$, $\partial_{\bar{w}^2_\pm}$ from the distribution $\overline{\mathscr{D}}$ on $\mathcal{F}^5$, over the CR twistor space $\mathcal{F}^5 \cong \mathbb{R}^3 \times S^2$,

$$\mathcal{F}^{5|8} \;\to\; \mathcal{F}^5, \tag{III.100}$$

with complex fiber coordinates $\eta^\pm_i$ over the patches $\hat{\mathcal{W}}_\pm$ covering $\mathcal{F}^5$. Recall that we have the double fibration

$$\begin{tikzcd}[column sep=small] & \mathcal{F}^5 \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \mathcal{P}^2 & & \mathbb{R}^3 \end{tikzcd} \tag{III.101}$$

in the purely even case and the transition function of the vector bundle (III.100) can be identified with

$$\nu_1^*\varphi_{+-} \;=\; \lambda_+(\delta_i{}^j) \;=\; \lambda_+\mathbb{1}_4, \tag{III.102}$$

i.e., we have the same transformation (III.98) relating η^+ to η^- on $\hat{\mathcal{W}}_+ \cap \hat{\mathcal{W}}_-$. Note that our notation often does not distinguish between objects on $\mathcal{P}^2$ and their pull-backs to $\mathcal{F}^5$.

Thus, we arrive at the diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{F}^{5|8} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd] & \\ \mathcal{P}^{2|4} \arrow[dd] & & \mathbb{R}^{3|8} \arrow[dd] \\ & \mathcal{F}^5 \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ \mathcal{P}^2 & & \mathbb{R}^3 \end{tikzcd} \tag{III.103}$$

combining the double fibrations (III.21) and (III.101).

§III.15 Deformed mini-supertwistor and CR supertwistor spaces. Let us define a holomorphic vector bundle

$$\mathcal{P}^{2|4}_M \;\to\; \mathcal{P}^2 \tag{III.104}$$

with complex coordinates $\tilde{\eta}^\pm = (\tilde{\eta}^\pm_i)$ on the fibers over $\mathcal{W}_\pm \subset \mathcal{P}^2$ which are related by the transition function

$$\tilde{\varphi}_{+-} \;=\; w^2_+ \mathrm{e}^{\frac{w^1_+}{w^2_+}M} \tag{III.105}$$

on the intersection $\mathcal{W}_+ \cap \mathcal{W}_-$, i.e.,

$$\tilde{\eta}^+ \;=\; \tilde{\varphi}_{+-}\tilde{\eta}^- . \tag{III.106}$$

For reasons which will become more transparent in the later discussion, we demand that M is traceless and Hermitian. It is also assumed that M is constant.

This supermanifold $\mathcal{P}^{2|4}_M$ was introduced in [74] as the target space of twistor string theory[2] which corresponds by our subsequent discussion to a SYM theory in three dimensions with massive spinors and both massive and massless scalar fields. In the following, we provide a twistorial derivation of analogous mass terms in our supersymmetric Bogomolny model and explain their geometric origin.

Consider the rank 0|4 holomorphic vector bundle (III.104) and its pull-back

$$\mathcal{F}^{5|8}_M \;:=\; \nu_1^*\mathcal{P}^{2|4}_M \;\to\; \mathcal{F}^5 \tag{III.107}$$

to the space $\mathcal{F}^5$ from the double fibration (III.101). Note that the supervector bundle $\mathcal{F}^{5|8}_M \to \mathcal{F}^5$ is smoothly equivalent to the supervector bundle $\mathcal{F}^{5|8} \to \mathcal{F}^5$ since in the coordinates $(y^{\dot\alpha\dot\beta}, \lambda_\pm, \bar\lambda_\pm) = (y, \bar y, x^3, \lambda_\pm, \bar\lambda_\pm)$ on $\mathcal{F}^5$, the pulled-back transition function $\nu_1^*\tilde{\varphi}_{+-}$ can be split

$$\nu_1^*\tilde{\varphi}_{+-} \;=\; \lambda_+ \mathrm{e}^{\frac{1}{\lambda_+}y^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta}M} \;=\; \varphi_+(\lambda_+\mathbb{1}_4)\varphi_-^{-1} \;\sim\; \lambda_+\mathbb{1}_4. \tag{III.108}$$

Here,

$$\varphi_+ \;=\; \mathrm{e}^{-(x^3+\lambda_+\bar y)M} \;=\; \mathrm{e}^{\lambda^+_{\dot\alpha}y^{\dot\alpha\dot2}M} \quad\text{and}\quad \varphi_- \;=\; \mathrm{e}^{(x^3-\lambda_- y)M} \;=\; \mathrm{e}^{-\lambda^-_{\dot\alpha}y^{\dot\alpha\dot1}M} \tag{III.109}$$

[2]For this, $\mathcal{P}^{2|4}_M$ has to be a formal Calabi-Yau supermanifold, which is the reason underlying the above restriction to $\mathrm{tr}M = 0$.

are matrix-valued functions well-defined on the patches $\hat{\mathcal{W}}_+$ and $\hat{\mathcal{W}}_-$, respectively. Remember that $\tilde{\eta}_i^+$ and $\tilde{\eta}_i^-$ are related by (III.106) and their pull-backs to $\mathcal{F}^5$ (which we denote again by the same letter) are related by the transition function (III.108). Therefore, we have

$$(\varphi_+^{-1}\tilde{\eta}^+) \;=\; \lambda_+(\varphi_-^{-1}\tilde{\eta}^-). \tag{III.110}$$

From this we conclude that

$$\tilde{\eta}^+ \;=\; \varphi_+\eta^+ \;=\; \mathrm{e}^{\lambda^+_{\dot\alpha}y^{\dot\alpha\dot2}M}\eta^+ \qquad \text{and} \qquad \tilde{\eta}^- \;=\; \varphi_-\eta^- \;=\; \mathrm{e}^{-\lambda^-_{\dot\alpha}y^{\dot\alpha\dot1}M}\eta^-, \tag{III.111}$$

where $\eta^\pm = (\eta_i^\pm)$ are the fiber coordinates of the bundle (III.100) related by (III.98) on the intersection $\hat{\mathcal{W}}_+ \cap \hat{\mathcal{W}}_-$.

The fibration $\mathcal{F}_M^{5|8} \to \mathcal{F}^5$ can also be understood in the Dolbeault picture. It follows from (III.111) that

$$\bar{W}_1^\pm\tilde{\eta}_i^\pm \;=\; 0, \qquad \bar{W}_2^\pm\tilde{\eta}_i^\pm \;=\; 0 \qquad \text{and} \qquad \bar{W}_3^\pm\tilde{\eta}_i^\pm + M_i{}^j\tilde{\eta}_j^\pm \;=\; 0. \tag{III.112}$$

Recall that the vector fields $\bar{W}_a^\pm$ generate an integrable distribution $\mathscr{T}_b = \langle\bar{W}_a^\pm\rangle$ together with the operator

$$\mathrm{d}_{\mathscr{T}_b}|_{\hat{\mathcal{W}}_\pm} \;=\; \mathrm{d}\bar{w}^a_\pm\frac{\partial}{\partial\bar{w}^a_\pm}, \tag{III.113}$$

which annihilates the transition function (III.108) of the bundle (III.107). Due to formulas (III.108) and (III.112), the vector bundle $\mathcal{F}_M^{5|8}$ with canonical $\mathscr{T}_b$-flat connection $\mathrm{d}_{\mathscr{T}_b}$ is diffeomorphic to the vector bundle $\mathcal{F}^{5|8}$ with the $\mathscr{T}_b$-flat connection $\nabla_{\mathscr{T}_b} = \mathrm{d}_{\mathscr{T}_b} + \mathcal{A}_{\mathscr{T}_b}$ the components $\mathcal{A}_a^\pm = \bar{W}_a^\pm \lrcorner \mathcal{A}_{\mathscr{T}_b}$ of which are given by

$$\mathcal{A}_1^\pm \;=\; 0, \qquad \mathcal{A}_2^\pm \;=\; 0 \qquad \text{and} \qquad \mathcal{A}_3^\pm \;=\; M. \tag{III.114}$$

In other words, we have an equivalence of the following data:

$$(\mathcal{F}_M^{5|8}, \tilde{\varphi}, \mathrm{d}_{\mathscr{T}_b}) \;\sim\; (\mathcal{F}^{5|8}, \varphi = \{\lambda_+\mathbb{1}_4\}, \nabla_{\mathscr{T}_b}). \tag{III.115}$$

By construction, the connection one-form $\mathcal{A}_{\mathscr{T}_b}$, given explicitly in (III.114), is a solution to the field equations

$$\mathrm{d}_{\mathscr{T}_b}\mathcal{A}_{\mathscr{T}_b} + \mathcal{A}_{\mathscr{T}_b} \wedge \mathcal{A}_{\mathscr{T}_b} \;=\; 0 \tag{III.116}$$

of phCS theory on $\mathcal{F}^5$, which are equivalent via the arguments of §III.8 to the Bogomolny equations on $\mathbb{R}^3$. Due to this correspondence, (III.114) is equivalent to a solution of the Bogomolny equations with vanishing YM gauge potential $a_{\dot\alpha\dot\beta}$ and constant Higgs field

$$\phi \;=\; (\phi_i{}^j) \;=\; -\mathrm{i}(M_i{}^j), \tag{III.117}$$

which takes values in the Lie algebra $\mathfrak{su}(4)$ of the R-symmetry group $SU(4)$. Thus, the data (III.115) are equivalent to the trivial vector bundle $\mathbb{R}^{3|8} \to \mathbb{R}^3$ together with the differential operator

$$D_{\dot\alpha\dot\beta} \;=\; \partial_{(\dot\alpha\dot\beta)} - \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}M \tag{III.118}$$

encoding the information about the matrix M, i.e.,

$$(\mathcal{F}^{5|8}_M, \tilde\varphi, \mathrm{d}_{\mathscr{T}_b}) \;\sim\; (\mathcal{F}^{5|8}, \varphi, \nabla_{\mathscr{T}_b}) \;\sim\; (\mathbb{R}^{3|8}, D_{\dot\alpha\dot\beta}). \tag{III.119}$$

Note that the gauge potential $A_{\dot\alpha\dot\beta}$ corresponding to[3] $A_s \in \mathfrak{u}(r)$ in a different basis and the Higgs fields $\Phi \in \mathfrak{u}(r)$ considered in Sec. III.2 can be combined with $a_{\dot\alpha\dot\beta}$ and ϕ into the fields

$$A_{\dot\alpha\dot\beta} \otimes \mathbb{1}_4 + \mathbb{1}_r \otimes a_{\dot\alpha\dot\beta} \qquad \text{and} \qquad \Phi \otimes \mathbb{1}_4 + \mathbb{1}_r \otimes \phi \tag{III.120}$$

acting on the tensor product $V_{U(r)} \otimes V_{SU(4)}$ of the (adjoint) representation space $V_{U(r)}$ of the gauge group and the representation space $V_{SU(4)}$ of the R-symmetry group.

For the sake of completeness, we note that the deformed complex vector bundle $\mathcal{P}^{2|4}_M \to \mathcal{P}^2$ with the transition function $\tilde\varphi_{+-}$ from (III.105) and the holomorphic structure

$$\bar\partial_b|_{\mathcal{W}_\pm} \;=\; \mathrm{d}\bar w^1_\pm \frac{\partial}{\partial \bar w^1_\pm} + \mathrm{d}\bar w^2_\pm \frac{\partial}{\partial \bar w^2_\pm} \tag{III.121}$$

is smoothly equivalent to the bundle $\mathcal{P}^{2|4} \to \mathcal{P}^2$ with the transition function φ_{+-} from (III.99) and the holomorphic structure defined by the fields $\mathcal{A}^{0,1}$ and B with the components

$$\mathcal{A}_{\bar w^1_\pm} \;=\; 0, \qquad \mathcal{A}_{\bar w^2_\pm} \;=\; \mp\frac{w^1_\pm}{(1+w^2_\pm \bar w^2_\pm)^2}M \qquad \text{and} \qquad B_\pm \;=\; \mathcal{A}_{\bar w^3_\pm} \;=\; M. \tag{III.122}$$

The fields $\mathcal{A}^{0,1}$ and B obviously satisfy the field equations

$$\bar\partial_b \mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \;=\; 0 \qquad \text{and} \qquad \bar\partial_b B + [\mathcal{A}^{0,1}, B] \;=\; 0 \tag{III.123}$$

of hBF theory on $\mathcal{P}^2$. By repeating the discussion of Sec. III.3, one can now show the equivalence of the data

$$(\mathcal{P}^{2|4}_M, \tilde\varphi, \bar\partial_b) \;\sim\; (\mathcal{P}^{2|4}, \varphi = \{\lambda_+ \mathbb{1}_4\}, \nabla^{0,1}_b) \;\sim\; (\mathcal{F}^{5|8}_M, \tilde\varphi, \mathrm{d}_{\mathscr{T}_b}), \tag{III.124}$$

which extends the equivalences described in (III.119).

[3] Here, A_s with $s = 1, 2, 3$ are the components of the ordinary gauge potential in three dimensions.

§III.16 The deformed CR supertwistor space as a supermanifold. For developing a twistor correspondence involving the deformed CR supertwistor space $\mathcal{F}_M^{5|8}$, the description of $\mathcal{F}_M^{5|8}$ as a rank 0|4 complex vector bundle with a constant gauge potential (III.114) which twists the direct product of even and odd spaces is not sufficient. We rather have to interpret the total space of $\mathcal{F}_M^{5|8}$ as a supermanifold with deformed CR structure and deformed distribution $\mathscr{T}_M$.

Let us begin with the vector fields on $\mathcal{F}_M^{5|8}$. Remember that a covariant derivative along a vector field on the base space of a bundle can be lifted to a vector field on the total space of the bundle. In our case of the bundle (III.107), the lift of (III.112) reads as

$$\bar{W}_1^{\pm}\tilde{\eta}_i^{\pm} \;=\; 0, \qquad \bar{W}_2^{\pm}\tilde{\eta}_i^{\pm} \;=\; 0 \qquad \text{and} \qquad \left(\bar{W}_3^{\pm} + M_k{}^j\tilde{\eta}_j^{\pm}\frac{\partial}{\partial\tilde{\eta}_k^{\pm}}\right)\tilde{\eta}_i^{\pm} \;=\; 0. \tag{III.125}$$

To see the explicit form of the vector fields corresponding to the integrable distribution

$$\mathscr{T}_M \;=\; \left\langle \frac{\partial}{\partial\bar{w}_{\pm}^a}, \frac{\partial}{\partial\bar{\tilde{\eta}}_i^{\pm}}\right\rangle \tag{III.126}$$

on $\mathcal{F}_M^{5|8}$, it is convenient to switch to the coordinates $(y^{\dot\alpha\dot\beta}, \lambda_{\pm}, \bar{\lambda}_{\pm}, \eta_i^{\dot\alpha})$ by the formulas

$$\begin{aligned} w_{\pm}^1 \;&=\; \lambda_{\dot\alpha}^{\pm}\lambda_{\dot\beta}^{\pm}y^{\dot\alpha\dot\beta}, \quad w_{\pm}^2 \;=\; \lambda_{\pm} \quad \text{and} \quad w_{\pm}^3 \;=\; -\gamma_{\pm}\lambda_{\dot\alpha}^{\pm}\hat{\lambda}_{\dot\beta}^{\pm}y^{\dot\alpha\dot\beta}, \\ \tilde{\eta}_i^{+} \;&=\; \left(\mathrm{e}^{\lambda_{\dot\alpha}^{+}y^{\dot\alpha\dot 2}M}\right)_i{}^j\eta_j^{\dot\beta}\lambda_{\dot\beta}^{+} \quad \text{and} \quad \tilde{\eta}_i^{-} \;=\; \left(\mathrm{e}^{-\lambda_{\dot\alpha}^{-}y^{\dot\alpha\dot 1}M}\right)_i{}^j\eta_j^{\dot\beta}\lambda_{\dot\beta}^{-}. \end{aligned} \tag{III.127}$$

By a straightforward calculation, we obtain

$$\begin{aligned} \mathrm{d}_{\mathscr{T}_M}|_{\hat{\mathcal{V}}_{\pm}} \;&=\; \mathrm{d}\bar{w}_{\pm}^a\frac{\partial}{\partial\bar{w}_{\pm}^a} + \mathrm{d}\bar{\tilde{\eta}}_i^{\pm}\frac{\partial}{\partial\bar{\tilde{\eta}}_i^{\pm}} \\ &=\; \bar{\Theta}_{\pm}^1\bar{\mathcal{W}}_1^{\pm} + \bar{\Theta}_{\pm}^2\bar{\mathcal{W}}_2^{\pm} + (\bar{\Theta}_{\pm}^3 \mp \gamma_{\pm}^2\hat{\lambda}_{\dot\alpha}^{\pm}\hat{\lambda}_{\dot\beta}^{\pm}y^{\dot\alpha\dot\beta}\Theta_{\pm}^2)\bar{\mathcal{W}}_3^{\pm} + \bar{\mathcal{E}}_i^{\pm}\bar{V}_{\pm}^i, \end{aligned} \tag{III.128}$$

where

$$\begin{aligned} \bar{\mathcal{W}}_1^{\pm} \;&:=\; \bar{W}_1^{\pm} \mp \lambda_{\pm}(T\bar{M}T)_i{}^j\hat{\lambda}_{\dot\alpha}^{\pm}\eta_j^{\dot\alpha}\bar{V}_{\pm}^i, \qquad \bar{\mathcal{W}}_2^{\pm} \;:=\; \bar{W}_2^{\pm}, \\ \bar{\mathcal{W}}_3^{\pm} \;&:=\; \bar{W}_3^{\pm} + \gamma_{\pm}(TM)_i{}^j\lambda_{\dot\alpha}^{\pm}\eta_j^{\dot\alpha}V_{\pm}^i + \gamma_{\pm}(T\bar{M}T)_i{}^j\hat{\lambda}_{\dot\alpha}^{\pm}\eta_j^{\dot\alpha}\bar{V}_{\pm}^i, \\ \bar{\mathcal{E}}_i^{+} \;&:=\; \bar{E}_i^{+} + \gamma_{+}\hat{\lambda}_{\dot\alpha}^{+}\hat{\lambda}_{\dot\beta}^{+}\eta_j^{\dot\beta}(T\bar{M}T)_i{}^j\mathrm{d}y^{\dot 1\dot\alpha}, \\ \bar{\mathcal{E}}_i^{-} \;&:=\; \bar{E}_i^{-} + \gamma_{-}\hat{\lambda}_{\dot\alpha}^{-}\hat{\lambda}_{\dot\beta}^{-}\eta_j^{\dot\beta}(T\bar{M}T)_i{}^j\mathrm{d}y^{\dot 2\dot\alpha} \end{aligned} \tag{III.129}$$

and $\bar{W}_a^{\pm}, V_{\pm}^i, \bar{V}_{\pm}^i$ and $\bar{\Theta}_{\pm}^a, \Theta_{\pm}^a$ were given in (III.37)–(III.40) and $(AB)_i{}^j := A_i{}^kB_k{}^j$. Actually, the formulas (III.127) and their inverses define a diffeomorphism between the supermanifolds $\mathcal{F}_M^{5|8} = (\mathbb{R}^{3|8}\times S^2, \mathscr{T}_M)$ and $\mathcal{F}^{5|8} = (\mathbb{R}^{3|8}\times S^2, \mathscr{T})$ which have different integrable distributions $\mathscr{T}_M$ and $\mathscr{T}$ (and different CR structures).

Next we need the vector fields on $\mathcal{P}^{2|4}_M$. In the above discussion, we used a transformation from the coordinates $\tilde{\eta}^\pm_i$ to the coordinates $\eta^\pm_i$ on $\mathcal{F}^{5|8}_M$, which are (pulled-back) sections of $\Pi\mathcal{O}_{\mathbb{C}P^1}(1)$. The corresponding splitting of the transition function was given in (III.108)–(III.110). One can find a similar splitting of the transition function (III.105) also on the complex supermanifold $\mathcal{P}^{2|4}_M$ and obtain new coordinates $\hat{\eta}^\pm_i$, which are sections of $\Pi\mathcal{O}_{\mathbb{C}P^1}(1)$, as well. Explicitly, we have

$$\mathrm{e}^{\frac{w^1_+}{w^2_+}M} \;=\; \mathrm{e}^{\left(1-\frac{1}{1+w^2_+\bar{w}^2_+}\right)\frac{w^1_+}{w^2_+}M}\,\mathrm{e}^{\frac{w^1_+}{w^2_+(1+w^2_+\bar{w}^2_+)}M} \;=\; \mathrm{e}^{\frac{\bar{w}^2_+ w^1_+}{1+w^2_+\bar{w}^2_+}M}\,\mathrm{e}^{\frac{\bar{w}^2_- w^1_-}{1+w^2_-\bar{w}^2_-}M}\,,$$

which yields the formulas

$$\tilde{\eta}^+ \;=\; \mathrm{e}^{\frac{\bar{w}^2_+ w^1_+}{1+w^2_+\bar{w}^2_+}M}\,\hat{\eta}^+ \qquad\text{and}\qquad \tilde{\eta}^- \;=\; \mathrm{e}^{-\frac{\bar{w}^2_- w^1_-}{1+w^2_-\bar{w}^2_-}M}\,\hat{\eta}^-. \tag{III.130}$$

From this and (III.106) it follows that

$$\hat{\eta}^+_i \;=\; w^2_+\hat{\eta}^-_i \tag{III.131}$$

and these coordinates have the desired property. Furthermore, in the $(0,1)$-part of the differential

$$\begin{aligned}
\bar{\partial}_{\mathcal{P}_M}|_{\mathcal{V}_\pm} \;&=\; \mathrm{d}\bar{w}^1_\pm\frac{\partial}{\partial\bar{w}^1_\pm}+\mathrm{d}\bar{w}^2_\pm\frac{\partial}{\partial\bar{w}^2_\pm}+\mathrm{d}\bar{\tilde{\eta}}^\pm_i\frac{\partial}{\partial\bar{\tilde{\eta}}^\pm_i}\\
&=\; \mathrm{d}\bar{\hat{w}}^1_\pm\partial_{\bar{\hat{w}}^1_\pm}+\mathrm{d}\bar{\hat{w}}^2_\pm\left(\partial_{\bar{\hat{w}}^2_\pm}\mp\gamma^2_\pm\hat{w}^1_\pm M_i{}^j\hat{\eta}^\pm_j\partial_{\hat{\eta}^\pm_i}\right)\;+\\
&\qquad\qquad +\;\left(\mathrm{d}\bar{\hat{\eta}}^\pm_i\pm\gamma^2_\pm\bar{\hat{w}}^1_\pm\bar{M}_i{}^j\bar{\hat{\eta}}^\pm_j\mathrm{d}\hat{w}^2_\pm\right)\partial_{\bar{\hat{\eta}}^\pm_i}\,,
\end{aligned} \tag{III.132}$$

where we introduced $\hat{w}^{1,2}_\pm \;=\; w^{1,2}_\pm$ for clarity, we see explicitly the deformation of the complex structure from $\mathcal{P}^{2|4}$ to $\mathcal{P}^{2|4}_M$. Note that the coordinates $\hat{\eta}^\pm_i$ can be pulled-back to $\mathcal{F}^{5|8}_M$, and there they are related to the coordinates $\eta^\pm$ by

$$\hat{\eta}^\pm \;=\; \mathrm{e}^{-w^3_\pm M}\eta^\pm. \tag{III.133}$$

§III.17 Mass-deformed Bogomolny equations from phCS theory on $\mathcal{F}^{5|8}_M$. Now we have all ingredients for discussing phCS theory on deformed CR supertwistor space $\mathcal{F}^{5|8}_M$. The deformed mini-supertwistor space $\mathcal{P}^{2|4}_M$ fits into a double fibration

$$\begin{tikzcd}
& \mathcal{F}^{5|8}_M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{P}^{2|4}_M & & \mathbb{R}^{3|8}
\end{tikzcd} \tag{III.134}$$

similarly to the undeformed case $M = 0$. Recall that we had a holomorphic integral form on $\mathcal{P}^{2|4}$ locally defined by

$$\Omega|_{\mathcal{V}_\pm} \;=\; \pm \mathrm{d}w^1_\pm \wedge \mathrm{d}w^2_\pm \mathrm{d}\eta^\pm_1 \cdots \mathrm{d}\eta^\pm_4 . \tag{III.135}$$

One can extend Ω to a nonvanishing holomorphic volume form

$$\Omega^M|_{\mathcal{V}_\pm} \;=\; \pm \mathrm{d}w^1_\pm \wedge \mathrm{d}w^2_\pm \mathrm{d}\tilde{\eta}^\pm_1 \cdots \mathrm{d}\tilde{\eta}^\pm_4 \tag{III.136}$$

on $\mathcal{P}^{2|4}_M$ if and only if $\mathrm{tr}M = 0$ [74]. This is the reason why we imposed this condition from the very beginning. Similarly to the discussion of phCS theory on $\mathcal{F}^{5|8}$ in Sec. III.3, we consider a subsupermanifold $\mathcal{X}_M$ of $\mathcal{F}^{5|8}_M$ which is defined by the constraints $\bar{\tilde{\eta}}^\pm_i = 0$. Clearly, the latter equations are equivalent to $\bar{\eta}^\pm_i = 0$ and therefore $\mathcal{X}_M$ is diffeomorphic to $\mathcal{X}$. Note that the pull-back of the holomorphic integral form (III.136) to $\mathcal{F}^{5|8}_M$ coincides with $\pi_1^*\Omega$,

$$\tilde{\Omega}^M|_{\hat{\mathcal{V}}_\pm} \;:=\; \pi_1^*\Omega^M|_{\hat{\mathcal{V}}_\pm} \;=\; \pm\Theta^1_\pm \wedge \Theta^2_\pm \mathrm{d}\tilde{\eta}^\pm_1 \cdots \mathrm{d}\tilde{\eta}^\pm_4 \;=\; \pm\Theta^1_\pm \wedge \Theta^2_\pm \mathrm{d}\eta^\pm_1 \cdots \mathrm{d}\eta^\pm_4 \tag{III.137}$$

which is due to (III.111) and the tracelessness of M.

From here on, we proceed as in Sec. III.3 and consider a trivial rank r complex vector bundle over the CR supertwistor space $\mathcal{F}^{5|8}_M$ with a connection $\mathcal{A}_{\mathscr{T}_M}$ along the integrable distribution $\mathscr{T}_M$ defined in (III.126) and (III.128). By assuming that $\bar{V}^i_\pm \lrcorner \mathcal{A}_{\mathscr{T}_M} = 0$ and $\bar{V}^i_\pm(\bar{\mathcal{W}}^\pm_a \lrcorner \mathcal{A}_{\mathscr{T}_M}) = 0$, we may define the action functional

$$S \;=\; \int_{\mathcal{X}_M} \tilde{\Omega}^M \wedge \mathrm{tr}\left\{\mathcal{A}_{\mathscr{T}_M} \wedge \mathrm{d}_{\mathscr{T}_M}\mathcal{A}_{\mathscr{T}_M} + \tfrac{2}{3}\mathcal{A}_{\mathscr{T}_M} \wedge \mathcal{A}_{\mathscr{T}_M} \wedge \mathcal{A}_{\mathscr{T}_M}\right\} \tag{III.138}$$

of deformed phCS theory. The equations of motion keep the form (III.57) up to relabeling $\mathscr{T}$ by $\mathscr{T}_M$ and in components $\mathcal{A}^\pm_a := \bar{\mathcal{W}}^\pm_a \lrcorner \mathcal{A}_{\mathscr{T}_M}$, we have

$$\begin{aligned}
\bar{W}^\pm_1 \mathcal{A}^\pm_2 - \bar{W}^\pm_2 \mathcal{A}^\pm_1 + [\mathcal{A}^\pm_1, \mathcal{A}^\pm_2] \;&=\; 0,\\
\bar{W}^\pm_2 \mathcal{A}^\pm_3 - \bar{W}^\pm_3 \mathcal{A}^\pm_2 + [\mathcal{A}^\pm_2, \mathcal{A}^\pm_3] \mp 2\gamma^2_\pm \mathcal{A}^\pm_1 \;&=\; M_j{}^i \eta^\pm_i \frac{\partial}{\partial \eta^\pm_j}\mathcal{A}^\pm_2,\\
\bar{W}^\pm_1 \mathcal{A}^\pm_3 - \bar{W}^\pm_3 \mathcal{A}^\pm_1 + [\mathcal{A}^\pm_1, \mathcal{A}^\pm_3] \;&=\; M_j{}^i \eta^\pm_i \frac{\partial}{\partial \eta^\pm_j}\mathcal{A}^\pm_1,
\end{aligned} \tag{III.139}$$

where the vector fields (III.129) have already been substituted. The dependence of the components $\mathcal{A}^\pm_a$ on $\lambda_\pm, \bar{\lambda}_\pm$ and $\eta^\pm_i$ is of the same form as the one given in (III.61) but with coefficient functions obeying M-deformed equations.

Substituting the expansions of the form (III.61) for $\mathcal{A}_a^\pm$ and our vector fields $\bar{W}_a^\pm$ into (III.139), we obtain mass-deformed supersymmetric Bogomolny equations:

$$
\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} &= -\tfrac{\mathrm{i}}{2}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\Phi},\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\chi}{}^i_{\dot\gamma} - \tfrac{1}{2}M_j{}^i\overset{\circ}{\chi}{}^j_{\dot\alpha} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{\Phi},\overset{\circ}{\chi}{}^i_{\dot\alpha}],\\
\overset{\circ}{\triangle}\overset{\circ}{W}{}^{ij} + M_k{}^{[i}M_l{}^{[j]}\overset{\circ}{W}{}^{k]l} &= -\tfrac{1}{4}[\overset{\circ}{\Phi},[\overset{\circ}{W}{}^{ij},\overset{\circ}{\Phi}]] - \mathrm{i}M_k{}^{[i}[\overset{\circ}{\Phi},\overset{\circ}{W}{}^{j]k}] - \epsilon^{\dot\alpha\dot\beta}\{\overset{\circ}{\chi}{}^i_{\dot\alpha},\overset{\circ}{\chi}{}^j_{\dot\beta}\},\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{\chi}_{i\dot\gamma} - \tfrac{1}{2}M_i{}^j\overset{\circ}{\chi}_{j\dot\alpha} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{\chi}_{i\dot\alpha},\overset{\circ}{\Phi}] + 2[\overset{\circ}{W}_{ij},\overset{\circ}{\chi}{}^j_{\dot\alpha}],\\
\epsilon^{\dot\beta\dot\gamma}\overset{\circ}{\nabla}_{\dot\alpha\dot\beta}\overset{\circ}{G}_{\dot\gamma\dot\delta} &= -\tfrac{\mathrm{i}}{2}[\overset{\circ}{G}_{\dot\alpha\dot\delta},\overset{\circ}{\Phi}] + \{\overset{\circ}{\chi}{}^i_{\dot\alpha},\overset{\circ}{\chi}_{i\dot\delta}\} - \tfrac{1}{2}[\overset{\circ}{W}_{ij},\overset{\circ}{\nabla}_{\dot\alpha\dot\delta}\overset{\circ}{W}{}^{ij}] +\\
&\quad + \tfrac{\mathrm{i}}{4}\epsilon_{\dot\alpha\dot\delta}[\overset{\circ}{W}_{ij},[\overset{\circ}{\Phi},\overset{\circ}{W}{}^{ij}]] + \tfrac{1}{2}\epsilon_{\dot\alpha\dot\delta}M_m{}^k[\overset{\circ}{W}_{kl},\overset{\circ}{W}{}^{lm}].
\end{aligned}
\tag{III.140}
$$

Eqs. (III.139) show that, as in the undeformed case (III.58), some of the fields appearing in the expansions of $\mathcal{A}_a^\pm$ are not independent degrees of freedom but composite fields. In fact, we find

$$
\begin{aligned}
\overset{\circ}{W}{}^{ij}_{\dot\alpha\dot\beta} &= -\left(\partial_{(\dot\alpha\dot\beta)}\overset{\circ}{W}{}^{ij} + [\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta},\overset{\circ}{W}{}^{ij}] - \epsilon_{\dot\alpha\dot\beta}M_k{}^{[i}\overset{\circ}{W}{}^{j]k}\right),\\
\overset{\circ}{\chi}{}^{ijk}_{\dot\alpha(\dot\beta\dot\gamma)} &= -\tfrac{1}{2}\left(\partial_{(\dot\alpha(\dot\beta)}\overset{\circ}{\chi}{}^{ijk}_{\dot\gamma)} + [\overset{\circ}{\mathcal{B}}_{\dot\alpha(\dot\beta},\overset{\circ}{\chi}{}^{ijk}_{\dot\gamma)}] + \tfrac{3}{2}\epsilon_{\dot\alpha(\dot\beta}M_l{}^{[i}\overset{\circ}{\chi}{}^{jk]l}_{\dot\gamma)}\right),\\
\overset{\circ}{G}{}^{ijkl}_{\dot\alpha(\dot\beta\dot\gamma\dot\delta)} &= -\tfrac{1}{3}\left(\partial_{(\dot\alpha(\dot\beta)}\overset{\circ}{G}{}^{ijkl}_{\dot\gamma\dot\delta)} + [\overset{\circ}{\mathcal{B}}_{\dot\alpha(\dot\beta},\overset{\circ}{G}{}^{ijkl}_{\dot\gamma\dot\delta)}] - 2\epsilon_{\dot\alpha(\dot\beta}M_m{}^{[i}\overset{\circ}{G}{}^{jkl]m}_{\dot\gamma\dot\delta)}\right).
\end{aligned}
\tag{III.141}
$$

Finally, upon substituting our superfield expansions for $\mathcal{A}_a^\pm$ into the action (III.138) and integrating over the odd coordinates and over the Riemann sphere, we end up with

$$
S = S_0 - \tfrac{1}{2}\int \mathrm{d}^3x\ \mathrm{tr}\left\{\overset{\circ}{\chi}_{i\dot\alpha}M_j{}^i\overset{\circ}{\chi}{}^{j\dot\alpha} - \overset{\circ}{W}_{ij}M_k{}^iM_l{}^{[j}\overset{\circ}{W}{}^{k]l} + \mathrm{i}\overset{\circ}{\Phi}M_k{}^i[\overset{\circ}{W}_{ij},\overset{\circ}{W}{}^{jk}]\right\}, \tag{III.142}
$$

where S_0 is the action functional for the massless supersymmetric Bogomolny equations as given by (III.68).

§III.18 Summary. We have described a one-to-one correspondence between gauge equivalence classes of local solutions to the supersymmetric Bogomolny equations with massive fermions and scalar fields and equivalence classes of $\mathscr{T}_M$-flat bundles over the CR supertwistor space $\mathcal{F}_M^{5|8}$ which are holomorphically trivial on each $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}_M^{5|8}$. We have also described a one-to-one correspondence between the equivalence classes of $\mathscr{T}_M$-flat complex vector bundles over $\mathcal{F}_M^{5|8}$ and of holomorphic vector bundles over the deformed mini-supertwistor space $\mathcal{P}_M^{2|4}$. The assumption that these bundles become holomorphically trivial on projective lines translates in the Dolbeault description into a one-to-one correspondence between gauge equivalence classes of solutions to the field equations of i) hBF theory on the deformed mini-supertwistor space $\mathcal{P}_M^{2|4}$, ii) phCS theory on the CR supertwistor space $\mathcal{F}_M^{5|8}$ and iii) massive supersymmetric Bogomolny model on Euclidean three-dimensional space $\mathbb{R}^3$.

In fact, this section's discussion can be understood as a corollary of Thm. III.2.: complex structure deformations on the mini-supertwistor space induce, for appropriately chosen CR structures, CR structure deformations on the CR supertwistor space and, of course, vice versa. Our above discussion is the translation of Thm. III.2. to this deformed setting.

III.5 Solution generating techniques

In the preceding sections, we have presented in detail the relations between supersymmetric Bogomolny monopole equations on the Euclidean space $\mathbb{R}^3$ and field equations of phCS theory on the CR supertwistor space $\mathcal{F}^{5|8}$ as well as hBF theory on the mini-supertwistor space $\mathcal{P}^{2|4}$. We have shown that the moduli spaces of solutions to the field equations of these three theories are bijective. Furthermore, we introduced mass-deformed versions of these field theories. In this section, we want to show how the twistor correspondences described in the previous sections can be used for constructing explicit solutions to the supersymmetric Bogomolny equations. In fact, any solution to the standard Bogomolny equations given as a pair $(\mathring{\mathcal{A}}_{\dot\alpha\dot\beta},\mathring{\Phi})$ of a gauge potential and a Higgs field can be extended to a solution including the remaining fields of the supersymmetrically extended Bogomolny equations in a nontrivial fashion. The subsequent discussion is devoted to this issue. However, we are not considering this task in full generality but merely give some flavor of how the algorithms work. For simplicity, we also restrict ourselves to the case when only the fields $\mathring{\mathcal{A}}_{\dot\alpha\dot\beta}$, $\mathring{\Phi}$ and $\mathring{G}_{\dot\alpha\dot\beta}$ are non-zero. In this case, the supersymmetric Bogomolny equations (III.66) simplify to

$$\begin{aligned}
-\tfrac{1}{2}\epsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}\mathring{\mathcal{A}}_{\dot\beta\dot\delta}-\partial_{(\dot\beta\dot\delta)}\mathring{\mathcal{A}}_{\dot\alpha\dot\gamma}+[\mathring{\mathcal{A}}_{\dot\alpha\dot\gamma},\mathring{\mathcal{A}}_{\dot\beta\dot\delta}]\right) &= -\tfrac{\mathrm{i}}{2}\left(\partial_{(\dot\alpha\dot\beta)}\mathring{\Phi}+[\mathring{\mathcal{A}}_{\dot\alpha\dot\beta},\mathring{\Phi}]\right),\\
\epsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}\mathring{G}_{\dot\beta\dot\delta}+[\mathring{\mathcal{A}}_{\dot\alpha\dot\gamma},\mathring{G}_{\dot\beta\dot\delta}]\right) &= -\tfrac{\mathrm{i}}{2}[\mathring{G}_{\dot\alpha\dot\beta},\mathring{\Phi}].
\end{aligned}\tag{III.143}$$

First, we discuss Abelian solutions to these equations, which correspond to Dirac monopole-antimonopole systems. After this, we present two algorithms which generate non-Abelian solutions.

§III.19 Abelian solutions. In the Abelian case, the system (III.143) simplifies further to

$$\begin{aligned}\epsilon^{\dot\gamma\dot\delta}\left(\partial_{(\dot\alpha\dot\gamma)}\mathring{\mathcal{A}}_{\dot\beta\dot\delta}-\partial_{(\dot\beta\dot\delta)}\mathring{\mathcal{A}}_{\dot\alpha\dot\gamma}\right) &= \mathrm{i}\partial_{(\dot\alpha\dot\beta)}\mathring{\Phi},\\ \epsilon^{\dot\gamma\dot\delta}\partial_{(\dot\alpha\dot\gamma)}\mathring{G}_{\dot\beta\dot\delta} &= 0.\end{aligned} \tag{III.144}$$

It is convenient to rewrite these equations in terms of the real coordinates x^r on $\mathbb{R}^3$ with $r=1,2,3$ as

$$\begin{aligned}\tfrac{1}{2}\epsilon_{rst}(\partial_s\mathring{\mathcal{A}}_t-\partial_t\mathring{\mathcal{A}}_s) &= \partial_r\mathring{\Phi},\\ \partial_r\mathring{G}_r &= 0,\\ \epsilon_{rst}\partial_s\mathring{G}_t &= 0.\end{aligned} \tag{III.145}$$

From the second equation of (III.145), it follows that

$$\mathring{G}_r = \tfrac{1}{2}\epsilon_{rst}(\partial_s\mathring{\mathcal{A}}'_t-\partial_t\mathring{\mathcal{A}}'_s), \tag{III.146}$$

and from third one, we obtain

$$\mathring{G}_r = -\partial_r\mathring{\Phi}', \tag{III.147}$$

where the sign in (III.147) was chosen to match the fact that in four dimensions, $\mathring{G}_r$ corresponds to an anti-self-dual two-form with components $\mathring{G}_{\mu\nu}=\bar\eta^r_{\mu\nu}\mathring{G}_r$ and helicity -1, where $\bar\eta^r_{\mu\nu}$ are the 't Hooft tensors. Here, $\mathring{\mathcal{A}}'_r$ and $\mathring{\Phi}'$ are a vector and a scalar, respectively. Therefore, Eqs. (III.145) can be rewritten as

$$\begin{aligned}\tfrac{1}{2}\epsilon_{rst}(\partial_s\mathring{\mathcal{A}}_t-\partial_t\mathring{\mathcal{A}}_s) &= \partial_r\mathring{\Phi},\\ \tfrac{1}{2}\epsilon_{rst}(\partial_s\mathring{\mathcal{A}}'_t-\partial_t\mathring{\mathcal{A}}'_s) &= -\partial_r\mathring{\Phi}'.\end{aligned} \tag{III.148}$$

It is well known that the first equation describes Dirac monopoles while the second one Dirac antimonopoles (see, e.g., Atiyah et al. [22] and references therein). Thus, the action (III.68) with only the fields $\mathring{f}_{\dot\alpha\dot\beta}$, $\mathring{\Phi}$ and $\mathring{G}_{\dot\alpha\dot\beta}$ being non-zero can be considered as a proper action for the description of monopole-antimonopole systems.

Let us consider a configuration of m_1 Dirac monopoles and m_2 antimonopoles located at points $a_i=(a_i^1,a_i^2,a_i^3)$ with $i=1,\dots,m_1$ and $i=m_1+1,\dots,m_1+m_2$, respectively. Moreover, we assume for simplicity that $a_i^{1,2}\neq a_j^{1,2}$ for $i\neq j$. Such a configuration is then described by the fields

$$\begin{aligned}
\overset{\circ}{\mathcal{A}}{}^N &= \sum_{j=1}^{m_1} \overset{\circ}{\mathcal{A}}{}^{N,j}\ , \qquad \overset{\circ}{\mathcal{A}}{}^S = \sum_{j=1}^{m_1} \overset{\circ}{\mathcal{A}}{}^{S,j}, \qquad \overset{\circ}{\Phi}{}^N = \overset{\circ}{\Phi}{}^S = \sum_{j=1}^{m_1} \frac{\mathrm{i}}{2r_j},\\
\overset{\circ}{\mathcal{A}}{}'_N &= \sum_{j=m_1+1}^{m_1+m_2} \bar{\overset{\circ}{\mathcal{A}}}{}^{N,j}, \quad \overset{\circ}{\mathcal{A}}{}'^S = \sum_{j=m_1+1}^{m_1+m_2} \bar{\overset{\circ}{\mathcal{A}}}{}^{S,j}, \quad \overset{\circ}{\Phi}{}'^N = \overset{\circ}{\Phi}{}^S = \sum_{j=m_1+1}^{m_1+m_2} \frac{\mathrm{i}}{2r_j},
\end{aligned} \tag{III.149}$$

where $\overset{\circ}{\mathcal{A}}{}^{N,j} = \overset{\circ}{\mathcal{A}}{}^{N,j}_m \mathrm{d}x^m$ and $\overset{\circ}{\mathcal{A}}{}^{S,j} = \overset{\circ}{\mathcal{A}}{}^{S,j}_m \mathrm{d}x^m$ with

$$\begin{aligned}
\overset{\circ}{\mathcal{A}}{}^{N,j}_1 &= \frac{\mathrm{i}x_j^2}{2r_j(r_j+x_j^3)}, \quad \overset{\circ}{\mathcal{A}}{}^{N,j}_2 = \frac{-\mathrm{i}x_j^1}{2r_j(r_j+x_j^3)}, \quad \overset{\circ}{\mathcal{A}}{}^{N,j}_3 = 0,\\
\overset{\circ}{\mathcal{A}}{}^{S,j}_1 &= -\frac{\mathrm{i}x_j^2}{2r_j(r_j-x_j^3)}, \quad \overset{\circ}{\mathcal{A}}{}^{S,j}_2 = \frac{\mathrm{i}x_j^1}{2r_j(r_j-x_j^3)}, \quad \overset{\circ}{\mathcal{A}}{}^{S,j}_3 = 0,
\end{aligned} \tag{III.150}$$

$$x_j^s = x^s - a_j^s\ , \quad r_j^2 = \delta_{rs} x_j^r x_j^s\ . \tag{III.151}$$

Here, N and S denote the following two regions in $\mathbb{R}^3$:

$$\begin{aligned}
\mathbb{R}^3_{N,m_1+m_2} &:= \mathbb{R}^3 \setminus \bigcup_{i=1}^{m_1+m_2} \left\{x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \le a_i^3\right\},\\
\mathbb{R}^3_{S,m_1+m_2} &:= \mathbb{R}^3 \setminus \bigcup_{i=1}^{m_1+m_2} \left\{x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \ge a_i^3\right\}
\end{aligned} \tag{III.152}$$

and bar stands for complex conjugation. Note that

$$\mathbb{R}^3_{N,m_1+m_2} \cup \mathbb{R}^3_{S,m_1+m_2} = \mathbb{R}^3 \backslash \{a_1, \ldots, a_{m_1+m_2}\} \tag{III.153}$$

and the configuration (III.149), (III.150) has delta-function sources at the points a_i with $i = 1, \ldots, m_1 + m_2$.

§III.20 Non-Abelian solutions via a contour integral. For the gauge group $SU(2)$, one can consider the Wu-Yang point monopole [268] and its generalizations to configurations describing m_1 monopoles and m_2 antimonopoles [207]. This solution, which is singular at points a_i, $i = 1, \ldots, m_1 + m_2$, is a solution to Eqs. (III.143) for $\mathfrak{su}(2)$-valued fields. However, it is just an Abelian configuration in disguise, as it is equivalent to the multi-monopole configuration (III.149), (III.150) [207].

One can construct true non-Abelian solutions to (III.143) as follows. Let us first consider a configuration $\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta} = 0 = \overset{\circ}{\Phi}$ and $\overset{\circ}{G}_{\dot\alpha\dot\beta} \neq 0$. Then from (III.143) one obtains the equation

$$\epsilon^{\dot\beta\dot\gamma} \partial_{(\dot\alpha\dot\beta)} \overset{\circ}{G}_{\dot\gamma\dot\delta} = 0. \tag{III.154}$$

All solutions to this equation can be described in the twistor approach [192] via a contour integral

$$\overset{\circ}{G}_{\dot\alpha\dot\beta} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \lambda^+_{\dot\alpha}\lambda^+_{\dot\beta}\overset{\circ}{\Upsilon}_{+-}(w^1_+, w^2_+), \tag{III.155}$$

where $\overset{\circ}{\Upsilon}_{+-}(w^1_+, w^2_+)$ is a Lie-algebra valued meromorphic function of $w^1_+ = \lambda^+_{\dot\alpha}\lambda^+_{\dot\beta} y^{\dot\alpha\dot\beta}$ and $w^2_+ = \lambda_+$ holomorphic in the vicinity of the curve $\mathscr{C} \cong S^1 \subset \mathbb{C}P^1$. From (III.155) it follows that nontrivial contributions to $\overset{\circ}{G}_{\dot\alpha\dot\beta}$ are only given by those $\overset{\circ}{\Upsilon}_{+-}$ which define elements of the cohomology group $H^1(\mathcal{P}^2, \mathfrak{gl}(r, \mathcal{O}(-4))$. It is easy to see that (III.155) satisfies (III.154) due to the identity

$$\lambda^{\dot\beta}_+ \partial_{(\dot\alpha\dot\beta)} \overset{\circ}{\Upsilon}_{+-} \;=\; \frac{\partial \overset{\circ}{\Upsilon}_{+-}}{\partial w^1_+} \lambda^{\dot\beta}_+ \lambda^+_{\dot\beta}\lambda^+_{\dot\alpha} \;=\; 0,$$

which appears after pulling the derivatives $\partial_{(\dot\alpha\dot\beta)}$ under the integral.

Consider now a fixed solution $(\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}, \overset{\circ}{\Phi})$ of the Bogomolny equations given by the first line of (III.143). One may take, e.g., the $SU(2)$ BPS monopole [209, 59]. In the twistor approach, we can find functions $\hat\psi_\pm$ solving the linear system

$$\lambda^{\dot\beta}_\pm (\partial_{(\dot\alpha\dot\beta)} + \overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta} - \tfrac{\mathrm{i}}{2}\epsilon_{\dot\alpha\dot\beta}\overset{\circ}{\Phi})\hat\psi_\pm \;=\; 0 \qquad \text{and} \qquad \partial_{\bar\lambda_\pm}\hat\psi_\pm \;=\; 0, \tag{III.156}$$

which is equivalent to the linear system of phCS theory. These $\hat\psi_\pm$ are known explicitly for many cases, e.g., for our chosen example of the $SU(2)$ BPS monopole, they have been given by Ward in [251, 252]. Using $\hat\psi_\pm$, we can introduce dressed fields $\overset{\circ}{G}_{\dot\alpha\dot\beta}$ by the formula

$$\overset{\circ}{G}_{\dot\alpha\dot\beta} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \lambda^+_{\dot\alpha}\lambda^+_{\dot\beta}\hat\psi_+ \overset{\circ}{\Upsilon}_{+-}\hat\psi_-^{-1}, \tag{III.157}$$

where $\overset{\circ}{\Upsilon}$ is chosen as above. One can straightforwardly check that with this choice,

$$\epsilon^{\dot\beta\dot\gamma}\left(\partial_{(\dot\alpha\dot\beta)}\overset{\circ}{G}_{\dot\gamma\dot\delta} + [\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta} - \tfrac{\mathrm{i}}{2}\epsilon_{\dot\alpha\dot\beta}\overset{\circ}{\Phi}, \overset{\circ}{G}_{\dot\gamma\dot\delta}]\right) \;=\; 0, \tag{III.158}$$

and therefore the configuration $(\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}, \overset{\circ}{\Phi}, \overset{\circ}{G}_{\dot\alpha\dot\beta})$ satisfies (III.143). The explicit form of a $\overset{\circ}{G}_{\dot\alpha\dot\beta}$ for a given $\overset{\circ}{\mathcal{A}}_{\dot\alpha\dot\beta}$ and $\overset{\circ}{\Phi}$ is obtained by performing the contour integral (III.157) along $\mathscr{C}$ after a proper choice of the Lie-algebra valued function $\overset{\circ}{\Upsilon}$. Recall that the configuration $(\overset{\circ}{\mathcal{A}}_s, \overset{\circ}{\Phi})$ will be real, i.e., the fields will take values in the Lie algebra $\mathfrak{su}(r)$, if the matrix-valued functions $\hat\psi_\pm$ in (III.156) satisfy the reality condition as the one induced by (II.23) and $\det(\hat\psi_+^{-1}\hat\psi_-) = 1$. Imposing a proper reality condition on the function $\overset{\circ}{\Upsilon}_{+-}$ will ensure the skew-Hermiticity of $\overset{\circ}{G}_s$.

§III.21 Solutions via nilpotent dressing transformations in the Čech approach. In this section, we will present a novel algorithm for constructing solutions to Eqs. (III.143) based on the twistor description of hidden symmetry algebras in the self-dual SYM theory in four dimensions [266] – cf. our discussion presented in Chap. V. Recall that we have described a one-to-one correspondence between equivalence classes of transition functions of $\mathscr{T}$-flat vector bundles over the CR supertwistor space $\mathcal{F}^{5|8}$ obeying certain triviality conditions and gauge equivalence classes of solutions to the supersymmetric Bogomolny equations on $\mathbb{R}^3$. We can, however, associate with any open subset $\hat{\mathcal{V}}_+\cap\hat{\mathcal{V}}_-\subset\mathcal{F}^{5|8}$ an infinite number of classes of transition functions, which in turn yield an infinite number of gauge equivalence classes of solutions to the supersymmetric Bogomolny equations. Therefore, one naturally meets with a possibility of constructing new solutions from a given one, that is, with dressing transformations. In the remainder of this chapter, we discuss a particular example of such a construction but first we briefly introduce some necessary background material. At this stage, we present the latter in a more applied formulation and without precise mathematical terminology. For a more thorough exposition, we refer to Chap. V, where also a Čech cohomological interpretation of the underlying structure is given.

We consider the linear system (III.79), which can be rewritten as

$$(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm \;=\; 0,\quad \partial_{\bar\lambda_\pm}\hat\psi_\pm \;=\; 0 \quad\text{and}\quad (\bar V^i_\pm+\hat{\mathcal{A}}^i_\pm)\hat\psi_\pm \;=\; 0, \tag{III.159}$$

where we have defined

$$\bar V^\pm_{\dot\alpha} \;:=\; \lambda^{\dot\beta}_\pm\partial_{(\dot\alpha\dot\beta)} \qquad\text{and}\qquad \hat{\mathcal{A}}^\pm_{\dot\alpha} \;:=\; \bar V^\pm_{\dot\alpha}\lrcorner\hat{\mathcal{A}}_{\mathscr{T}}, \tag{III.160}$$

as before.[4] From arguments similar to those used subsequent to (III.79), we have $\hat{\mathcal{A}}^\pm_{\dot\alpha}=\lambda^{\dot\beta}_\pm\mathcal{B}_{\dot\alpha\dot\beta}$ and $\hat{\mathcal{A}}^i_\pm=\lambda^{\dot\alpha}_\pm\mathcal{A}^i_{\dot\alpha}$ with λ-independent superfields $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$. The compatibility conditions for the linear system (III.159) are Eqs. (III.81). From this linear system one also derives that $\tilde f_{+-}=\hat\psi_+^{-1}\hat\psi_-$ is $\mathscr{T}$-flat, i.e.,

$$\bar V^\pm_{\dot\alpha}\tilde f_{+-} \;=\; 0,\qquad \partial_{\bar\lambda_\pm}\tilde f_{+-} \;=\; 0 \qquad\text{and}\qquad \bar V^i_\pm\tilde f_{+-} \;=\; 0. \tag{III.161}$$

[4]Note that the first and third equations of (III.79) are equivalent to

$$\lambda^{\dot\alpha}_\pm(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm \;=\; 0 \qquad\text{and}\qquad \hat\lambda^{\dot\alpha}_\pm(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm \;=\; 0,$$

respectively, which together imply $(\bar V^\pm_{\dot\alpha}+\hat{\mathcal{A}}^\pm_{\dot\alpha})\hat\psi_\pm=0$.

The key idea is to study infinitesimal deformations of the transition function $\tilde{f}_{+-}$ of the $\mathscr{T}$-flat vector bundle preserving (III.161) and the triviality properties discussed above. More explicitly, given such a function $\tilde{f}_{+-} = \hat{\psi}_+^{-1}\hat{\psi}_-$ (with $\partial_{\bar{\lambda}_\pm}\hat{\psi}_\pm = 0$), we consider

$$\tilde{f}_{+-} + \delta\tilde{f}_{+-} \;=\; (\hat{\psi}_+ + \delta\hat{\psi}_+)^{-1}(\hat{\psi}_- + \delta\hat{\psi}_-), \tag{III.162}$$

where δ represents some generic infinitesimal deformation. Note that any infinitesimal $\mathscr{T}$-flat deformation (that is, preserving (III.161)) is allowed since for small perturbations, the trivializability property of the bundle $\tilde{\mathcal{E}}$ on the holomorphic curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}^{5|8}$ is preserved (cf. Chap. V). Upon introducing the Lie-algebra valued function

$$\phi_{+-} \;:=\; \hat{\psi}_+(\delta\tilde{f}_{+-})\hat{\psi}_-^{-1} \tag{III.163}$$

and linearizing (III.162), we have to find a splitting

$$\phi_{+-} \;=\; \phi_+ - \phi_-, \tag{III.164}$$

where the Lie-algebra valued functions $\phi_\pm$ can be extended to holomorphic functions in $\lambda_\pm$, which yields

$$\delta\hat{\psi}_\pm \;=\; -\phi_\pm\hat{\psi}_\pm. \tag{III.165}$$

To find these $\phi_\pm$ from ϕ_{+-} means to solve the infinitesimal Riemann-Hilbert problem. Clearly, such solutions are not unique, as we have the freedom

$$\phi_{+-} \;=\; \phi_+ - \phi_- \;=\; (\phi_+ + \omega) - (\phi_- + \omega) \;=:\; \tilde{\phi}_+ - \tilde{\phi}_-, \tag{III.166}$$

with $\tilde{\phi}_\pm := \phi_\pm + \omega$, where the function ω is independent of $\lambda_\pm$. This freedom can be used to preserve the transversal gauge condition (II.28). The preservation of this gauge is in fact needed in order to compare the deformed superfields and the ones one has started with as one wishes to derive the induced transformations of the component fields.

Linearizing (III.159), we get

$$\delta\hat{\mathcal{A}}^\pm_{\dot\alpha} \;=\; \bar{\nabla}^\pm_{\dot\alpha}\phi_\pm \qquad \text{and} \qquad \delta\hat{\mathcal{A}}^i_\pm \;=\; \bar{\nabla}^i_\pm\phi_\pm, \tag{III.167}$$

where we have introduced the operators $\bar{\nabla}^\pm_{\dot\alpha} := \bar{V}^\pm_{\dot\alpha} + \hat{\mathcal{A}}^\pm_{\dot\alpha}$ and $\bar{\nabla}^i_\pm := \bar{V}^i_\pm + \hat{\mathcal{A}}^i_\pm$. From (III.159), (III.161) and (III.162) it follows that

$$\bar{\nabla}^\pm_{\dot\alpha}\phi_{+-} \;=\; 0 \;=\; \bar{\nabla}^i_\pm\phi_{+-}, \tag{III.168}$$

and we eventually arrive at the formulas

$$\delta\mathcal{B}_{\dot\alpha\dot\beta} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \frac{\bar\nabla^+_{\dot\alpha}\phi_+}{\lambda_+\lambda_+^{\dot\beta}} \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot\alpha} \;=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \frac{\bar\nabla^i_+\phi_+}{\lambda_+\lambda_+^{\dot\alpha}}, \tag{III.169}$$

where the contour is $\mathscr{C} = \{\lambda_+ \in \mathbb{C}P^1\,|\,|\lambda_+| = 1\}$. Thus, the consideration of infinitesimal deformations of the transition function of some $\mathscr{T}$-flat vector bundle over the CR supertwistor space $\mathcal{F}^{5|8}$ obeying certain triviality conditions gives by virtue of the integral formulas (III.169) infinitesimal deformations of the components $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$, which satisfy – by construction – the linearized constraint equations (III.82) and thus the supersymmetric Bogomolny equations (III.66). Once again, we have a one-to-one correspondence between equivalence classes of local solutions, with equivalence induced on the gauge theory side by infinitesimal gauge transformations and on the twistor side by transformations of the form $\phi_\pm = \psi_\pm\chi_\pm\psi_\pm^{-1}$, where the $\chi_\pm$s are functions globally defined on $\hat{\mathcal{V}}_\pm \subset \mathcal{F}^{5|8}$ and annihilated by all vector fields from the distribution $\mathscr{T}$. Putting it differently, we have just presented the "infinitesimal" version of Thm. III.2..

Let us now exemplify our discussion by describing how to construct explicit solutions to (III.143). Consider a $\mathscr{T}$-flat vector bundle $\tilde{\mathcal{E}} \to \mathcal{F}^{5|8}$ of rank r which is holomorphically trivial when restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{F}^{5|8}$. Assume further that a transition function $\tilde f_{+-}$ of $\tilde{\mathcal{E}}$ is chosen such that all the fields $\mathring{\chi}^i_{\dot\alpha}$, $\mathring{W}^{ij}$, $\mathring{\chi}_{i\dot\alpha}$ and $\mathring{G}_{\dot\alpha\dot\beta}$ vanish identically, i.e., we start with the field equation

$$\mathring{f}_{\dot\alpha\dot\beta} \;=\; -\tfrac{\mathrm{i}}{2}\mathring{\nabla}_{\dot\alpha\dot\beta}\mathring{\Phi}. \tag{III.170}$$

Without loss of generality, we may assume that the transition function of $\tilde{\mathcal{E}}$ can be split as $\tilde f_{+-} = \hat\psi_+^{-1}\hat\psi_-$, where the $\hat\psi_\pm$s do not depend on the fermionic coordinates $\eta^\pm_i$.

Suppose now that

$$\delta\tilde f_{+-} \;:=\; -\tfrac{1}{4!}\epsilon^{j_1\cdots j_4}\eta^+_{j_1}\cdots\eta^+_{j_4}[X,\tilde f_{+-}], \tag{III.171}$$

where $X \in \mathfrak{gl}(r,\mathbb{C})$. Considering vector bundles subject to the reality conditions induced by (II.23), one restricts the perturbations to those preserving these conditions. In our example, they read explicitly as

$$\delta\tilde f_{+-} \;=\; -\tfrac{1}{4!}\epsilon^{j_1\cdots j_4}\left(\eta^+_{j_1}\cdots\eta^+_{j_4} + \eta^-_{j_1}\cdots\eta^-_{j_4}\right)[X,\tilde f_{+-}], \tag{III.172}$$

with $X \in \mathfrak{su}(r)$. For illustrating reasons and to simplify equations, we shall be continuing with (III.171). Then a short calculation reveals that any splitting (III.164) is of the form

$$\phi_{+-} \;=\; \phi_+ - \phi_- \;=\; -\tfrac{1}{4!}\epsilon^{j_1\cdots j_4}\eta^+_{j_1}\cdots\eta^+_{j_4}(\mathring{\phi}_+ - \mathring{\phi}_-), \tag{III.173}$$

with $\overset{\circ}{\phi}_\pm := -[X,\hat\psi_\pm]\hat\psi_\pm^{-1}$. Introducing the shorthand notation

$$\eta^{\dot\gamma_1\cdots\dot\gamma_4} \;:=\; -\tfrac{1}{4!}\epsilon^{j_1\cdots j_4}\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_4}_{j_4}, \tag{III.174}$$

we find

$$\phi_{+-} \;=\; \eta^{\dot2\dot2\dot2\dot2}\overset{\circ}{\phi}{}^4_{+-} + 4\eta^{\dot2\dot2\dot2\dot1}\overset{\circ}{\phi}{}^3_{+-} + 6\eta^{\dot2\dot2\dot1\dot1}\overset{\circ}{\phi}{}^2_{+-} + 4\eta^{\dot2\dot1\dot1\dot1}\overset{\circ}{\phi}{}^1_{+-} + \eta^{\dot1\dot1\dot1\dot1}\overset{\circ}{\phi}{}^0_{+-}, \tag{III.175}$$

where we have used the fact that $\eta^{\dot\gamma_1\cdots\dot\gamma_4}$ is totally symmetric. In addition, we defined

$$\overset{\circ}{\phi}{}^m_{+-} \;:=\; \lambda_+^m\overset{\circ}{\phi}_+ - \lambda_+^m\overset{\circ}{\phi}_- \;:=\; \overset{\circ}{\phi}{}^m_+ - \overset{\circ}{\phi}{}^m_-. \tag{III.176}$$

The functions $\overset{\circ}{\phi}{}^m_\pm$ can be Laurent-expanded as ($m \geq 0$)

$$\overset{\circ}{\phi}{}^m_\pm \;=\; \sum_{n=0}^\infty \lambda_+^{\pm n}\overset{\circ}{\phi}{}^{m(n)}_\pm \tag{III.177}$$

with

$$\overset{\circ}{\phi}{}^{m(n)}_+ \;=\; \begin{cases}\delta_{m,0}\overset{\circ}{\phi}{}^{0(0)}_+ & n=0\\ \overset{\circ}{\phi}{}^{0(n-m)}_+ - \overset{\circ}{\phi}{}^{0(m-n)}_- & n>0\end{cases} \qquad\text{and}\qquad \overset{\circ}{\phi}{}^{m(n)}_- \;=\; \overset{\circ}{\phi}{}^{0(m+n)}_-. \tag{III.178}$$

Combining the expansion

$$\phi_\pm \;=\; \sum_{n=0}^\infty \lambda_+^{\pm n}\phi_\pm^{(n)} \tag{III.179}$$

with (III.173)–(III.178), we therefore find

$$\phi_-^{(n)} \;=\; \eta^{\dot2\dot2\dot2\dot2}\overset{\circ}{\phi}{}^{0(4+n)}_- + 4\eta^{\dot2\dot2\dot2\dot1}\overset{\circ}{\phi}{}^{0(3+n)}_- + 6\eta^{\dot2\dot2\dot1\dot1}\overset{\circ}{\phi}{}^{0(2+n)}_- + 4\eta^{\dot2\dot1\dot1\dot1}\overset{\circ}{\phi}{}^{0(1+n)}_- + \eta^{\dot1\dot1\dot1\dot1}\overset{\circ}{\phi}{}^{0(n)}_-, \tag{III.180}$$

and a similar expression for $\phi_+^{(n)}$.

At this point, we have to choose an ω to preserve the transversal gauge. Explicitly, a possible ω is given by

$$\omega \;=\; -\eta^{\dot2\dot2\dot2\dot1}\overset{\circ}{\phi}{}^{0(3)}_- - 3\eta^{\dot2\dot2\dot1\dot1}\overset{\circ}{\phi}{}^{0(2)}_- - 3\eta^{\dot2\dot1\dot1\dot1}\overset{\circ}{\phi}{}^{0(1)}_- - \eta^{\dot1\dot1\dot1\dot1}\overset{\circ}{\phi}{}^{0(0)}_-. \tag{III.181}$$

Let us justify this result. In (III.166), we noticed a freedom in splitting the Lie-algebra valued function ϕ_{+-} and we claimed that it can be used to guarantee the transversal gauge condition (II.28) which, of course, translates to the requirement $\eta_i^{\dot\alpha}\delta\mathcal{A}^i_{\dot\alpha} = 0$. In fact, from the second equation of (III.167), we obtain

$$\begin{aligned} \delta\mathcal{A}^i_{\dot 1} &= \nabla^i_{\dot 1}\phi^{(0)}_+ - \nabla^i_{\dot 2}\phi^{(1)}_+ &= \nabla^i_{\dot 1}\phi^{(0)}_-, \\ \delta\mathcal{A}^i_{\dot 2} &= \nabla^i_{\dot 2}\phi^{(0)}_+ &= -\nabla^i_{\dot 1}\phi^{(1)}_- + \nabla^i_{\dot 2}\phi^{(0)}_-, \end{aligned} \tag{III.182}$$

where we have inserted (III.179). The contraction of these equations with $\eta_i^{\dot\alpha}$ yields the constraints

$$\mathscr{D}\phi^{(0)}_- + \mathscr{D}\omega = \eta_i^{\dot 2}\nabla^i_{\dot 1}\phi^{(1)}_- \qquad \text{and} \qquad \mathscr{D}\phi^{(0)}_+ + \mathscr{D}\omega = \eta_i^{\dot 1}\nabla^i_{\dot 2}\phi^{(1)}_+. \tag{III.183}$$

Here, we used the fact that $\tilde\phi^{(0)}_\pm = \phi^{(0)}_\pm + \omega$ and $\tilde\phi^{(1)}_\pm = \phi^{(1)}_\pm$, respectively, and recalled the definition $\mathscr{D} = \eta_i^{\dot\alpha}\nabla^i_{\dot\alpha}$. Thus, a splitting (III.166) with an ω satisfying (III.183) yields a deformation of the gauge potential which respects the transversal gauge condition. In our present example, Eqs. (III.183) simplify to

$$\mathscr{D}\phi^{(0)}_- + \mathscr{D}\omega = \eta_i^{\dot 2}\partial^i_{\dot 1}\phi^{(1)}_- \qquad \text{and} \qquad \mathscr{D}\phi^{(0)}_+ + \mathscr{D}\omega = \eta_i^{\dot 1}\partial^i_{\dot 2}\phi^{(1)}_+. \tag{III.184}$$

Since our particular deformation (III.171) is of fourth order in the odd coordinates, we may assume that $\omega = \eta^{\dot\gamma_1\cdots\dot\gamma_4}\omega_{\dot\gamma_1\cdots\dot\gamma_4}$. Then, after some algebraic manipulations, the expansions of $\phi^{(n)}_\pm$ given by (III.180) together with (III.184) and the ansatz for ω lead to (III.181).

The perturbations $\delta\mathcal{B}_{\dot\alpha\dot\beta}$ and $\delta\mathcal{A}^i_{\dot\alpha}$ are obtained from Eqs. (III.169) according to

$$\begin{aligned} \delta\mathcal{B}_{\dot\alpha\dot 1} &= \overset{\circ}{D}_{\dot\alpha\dot 1}(\phi^{(0)}_- + \omega) &= \overset{\circ}{D}_{\dot\alpha\dot 1}(\phi^{(0)}_+ + \omega) - \overset{\circ}{D}_{\dot\alpha\dot 2}\phi^{(1)}_+, \\ \delta\mathcal{B}_{\dot\alpha\dot 2} &= -\overset{\circ}{D}_{\dot\alpha\dot 1}\phi^{(1)}_- + \overset{\circ}{D}_{\dot\alpha\dot 2}(\phi^{(0)}_- + \omega) &= \overset{\circ}{D}_{\dot\alpha\dot 2}(\phi^{(0)}_+ + \omega) \end{aligned} \tag{III.185}$$

and

$$\begin{aligned} \delta\mathcal{A}^i_{\dot 1} &= \partial^i_{\dot 1}(\phi^{(0)}_- + \omega) &= \partial^i_{\dot 1}(\phi^{(0)}_+ + \omega) - \partial^i_{\dot 2}\phi^{(1)}_+ \ , \\ \delta\mathcal{A}^i_{\dot 2} &= -\partial^i_{\dot 1}\phi^{(1)}_- + \partial^i_{\dot 2}(\phi^{(0)}_- + \omega) &= \partial^i_{\dot 2}(\phi^{(0)}_+ + \omega). \end{aligned} \tag{III.186}$$

Remember that $\overset{\circ}{D}_{\dot\alpha\dot\beta} = \partial_{(\dot\alpha\dot\beta)} + \overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta}$. Consider now the expansions

$$\begin{aligned} \delta\mathcal{B}_{\dot\alpha\dot\beta} &= \delta\overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta} + \sum_{k\geq 1}\frac{1}{k!}\,\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_k}_{j_k}\,\delta[\overset{\circ}{\dot\alpha\dot\beta}]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}, \\ \delta\mathcal{A}^i_{\dot\alpha} &= \sum_{k\geq 1}\frac{k}{(k+1)!}\,\eta^{\dot\gamma_1}_{j_1}\cdots\eta^{\dot\gamma_k}_{j_k}\,\delta[\overset{\circ}{{}^{\,i}_{\dot\alpha}}]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}, \end{aligned} \tag{III.187}$$

where the brackets $[\ \ ]^{j_1\cdots j_k}_{\dot\gamma_1\cdots\dot\gamma_k}$ are composite expressions of some superfields, cf. also our discussion given in §II.6. Since our particular deformation of the transition function implies that $\phi_\pm + \omega = \mathcal{O}(\eta^4)$, the resulting deformations of $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$ are of the form $\mathcal{B}_{\dot\alpha\dot\beta} = \mathcal{O}(\eta^4)$ and $\mathcal{A}^i_{\dot\alpha} = \mathcal{O}(\eta^3)$, respectively. In transversal gauge, the explicit superfield expansions of $\mathcal{B}_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha}$ show that $\delta\mathring{\mathcal{B}}_{\dot\alpha\dot\beta} = \delta\mathring{\chi}^{\,i}_{\dot\alpha} = \delta\mathring{W}^{ij} = \delta\mathring{\chi}_{i\dot\alpha} = 0$. Together with the recursion relations (II.30) (of course, applied to the present setting obtained via dimensional reduction), they moreover imply that the variation of all higher order terms than of fourth (respectively, of third) order of $\mathcal{A}_{\alpha\dot\alpha}$ (respectively, of $\mathcal{A}^i_{\dot\alpha}$) in the η-expansions vanish. Hence, from (III.187) we find

$$\begin{aligned} \delta\mathcal{B}_{\dot\alpha\dot\beta} &= \tfrac{1}{2\cdot 4!}\epsilon^{j_1j_2j_3j_4}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\eta^{\dot\gamma_4}_{j_4}\epsilon_{\dot\beta\dot\gamma_1}\mathring{D}_{\dot\alpha\dot\gamma_2}\delta\mathring{G}_{\dot\gamma_3\dot\gamma_4}, \\ \delta\mathcal{A}^i_{\dot\alpha} &= \tfrac{3}{4}\epsilon^{ij_1j_2j_3}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\epsilon_{\dot\alpha\dot\gamma_1}\delta\mathring{G}_{\dot\gamma_2\dot\gamma_3}. \end{aligned} \tag{III.188}$$

Comparing these equations with (III.185) and the η-expansions of $\phi^{(0)}_\pm$, $\phi^{(1)}_\pm$ and ω given earlier, we arrive at

$$\delta\mathring{G}_{\dot1\dot1} = 2\mathring{\phi}^{0(1)}_-, \qquad \delta\mathring{G}_{\dot1\dot2} = 2\mathring{\phi}^{0(2)}_- \qquad \text{and} \qquad \delta\mathring{G}_{\dot2\dot2} = 2\mathring{\phi}^{0(3)}_- \tag{III.189}$$

together with the field equations

$$\mathring{f}_{\dot\alpha\dot\beta} = -\tfrac{\mathrm{i}}{2}\mathring{\nabla}_{\dot\alpha\dot\beta}\mathring{\Phi} \qquad \text{and} \qquad \epsilon^{\dot\beta\dot\gamma}\mathring{\nabla}_{\dot\alpha\dot\beta}\delta\mathring{G}_{\dot\gamma\dot\delta} = -\tfrac{\mathrm{i}}{2}[\delta\mathring{G}_{\dot\alpha\dot\delta},\mathring{\Phi}]. \tag{III.190}$$

Since Eqs. (III.143) are linear in $\mathring{G}_{\dot\alpha\dot\beta}$, we hence have generated a solution to (III.143) starting from a solution to the first equation of (III.143), that is, we may identify $\delta\mathring{G}_{\dot\alpha\dot\beta}$ with $\mathring{G}_{\dot\alpha\dot\beta}$. Thus, knowing the explicit splitting $\tilde{f}_{+-} = \hat{\psi}_+^{-1}\hat{\psi}_-$, we can define functions $\mathring{\phi}_\pm = -[X,\hat{\psi}_\pm]\hat{\psi}_\pm^{-1}$ which then in turn yield $\mathring{G}_{\dot\alpha\dot\beta}$.

Chapter IV

Super Yang-Mills theory

Main subject of this chapter is the twistorial description of the maximally supersymmetric Yang-Mills theory in four space-time dimensions, that is, of $\mathcal{N}=4$ (respectively, $\mathcal{N}=3$) SYM theory [103, 63]. Notice that $\mathcal{N}=4$ and $\mathcal{N}=3$ SYM theories represent the same physical theory. The only difference between both formulations lies in what part of the R-symmetry group is made manifest: for $\mathcal{N}=4$ it is $SU(4)$ while for $\mathcal{N}=3$ it is just the subgroup $U(1)\times SU(3)$. The twistor description of $\mathcal{N}=3$ SYM theory dates back to Witten's work [259] done in the late 70ies of the last century. The idea is roughly to discuss equivalence classes of certain holomorphic vector bundles over superambitwistor space $\mathbb{L}^{5|6}$ which in turn are in one-to-one correspondence with gauge equivalence classes of solutions to the $\mathcal{N}=3$ SYM equations. The superambitwistor description involving $\mathbb{L}^{5|8}$ does not yield the $\mathcal{N}=4$ SYM equations directly. That is why we want to focus on the $\mathcal{N}=3$ formulation of $\mathcal{N}=4$ SYM theory. Moreover, in this respect it is worth mentioning that Manin [169] generalized the theorems of Witten [259] and Isenberg et al. [126] by showing that there is a one-to-one correspondence between equivalence classes of certain holomorphic vector bundles over superambitwistor space $\mathbb{L}^{5|2\mathcal{N}}$ – for $\mathcal{N}\leq 3$ – which admit an extension to a $(3-\mathcal{N})$-th formal neighborhood of $\mathbb{L}^{5|2\mathcal{N}}$ in $\mathbb{P}^{3|\mathcal{N}}\times\mathbb{P}^{3|\mathcal{N}}_*$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended SYM equations in four dimensions. We first discuss $\mathcal{N}=3$ SYM theory and then briefly comment on SYM theories with less supersymmetry.

IV.1 $\mathcal{N}=3$ super Yang-Mills theory

Let us begin our discussion by recalling that in §I.13 we have defined superambitwistor space $\mathbb{L}^{5|6}$ in terms of flag supermanifolds. In particular, $\mathbb{L}^{5|6}$ is participating in the following double fibration:

$$\begin{tikzcd} & \mathbb{F}^{6|12} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{L}^{5|6} & & \mathbb{M}^{4|12} \end{tikzcd} \tag{IV.1}$$

Moreover, we have shown that superambitwistor space can be viewed as a hypersurface in $\mathbb{P}^{3|3} \times \mathbb{P}^{3|3}_*$ determined by the zero locus

$$z^\alpha \rho_\alpha - w^{\dot\alpha}\pi_{\dot\alpha} + 2\theta^i \eta_i \ = \ 0, \tag{IV.2}$$

where $[z^\alpha, \pi_{\dot\alpha}, \eta_i, \rho_a, w^{\dot\alpha}, \theta^i]$ are homogeneous coordinates on $\mathbb{P}^{3|3} \times \mathbb{P}^{3|3}_*$. In order to be able to continuing as in the preceding two chapters, we first need to give local coordinates at each stage of (IV.1).

§IV.1 Local coordinates. In §I.18, we have already introduced local coordinates on $\mathbb{M}^{4|12}$. In particular, we had $(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \theta^{i\alpha})$ on $\mathcal{M}^{4|12} \cong \mathbb{C}^{4|12}$. As in Chap. I, we are now making use of the two projections $\pi_{1,2}$ to introduce the affine parts of $\mathbb{F}^{6|12}$ and $\mathbb{L}^{5|6}$ according to

$$\mathcal{F}^{6|12} \ := \ \pi_2^{-1}(\mathbb{C}^{4|12}) \qquad \text{and} \qquad \mathcal{L}^{5|6} \ := \ \pi_1(\pi_2^{-1}(\mathbb{C}^{4|12})). \tag{IV.3}$$

Analogously to (I.8), one may show that

$$\mathcal{F}^{6|12} \ \cong \ \mathbb{C}^{4|12} \times Y, \qquad \text{with} \qquad Y \ := \ \mathbb{C}P^1 \times \mathbb{C}P^1_*. \tag{IV.4}$$

This makes it obvious that $\mathcal{F}^{6|12}$ can be covered by four coordinate patches which we denote by $\hat{\mathfrak{W}} = \{\hat{\mathcal{W}}_a\}$, with $a, b, \ldots = 1, \ldots, 4$. Letting $\lambda_\pm \in U_\pm$ and $\mu_\pm \in V_\pm$ be local coordinates on

$$\mathbb{C}P^1 \times \mathbb{C}P^1_* \ = \ \underbrace{(U_+ \times V_+)}_{=:\,W_1} \cup \underbrace{(U_+ \times V_-)}_{=:\,W_2} \cup \underbrace{(U_- \times V_+)}_{=:\,W_3} \cup \underbrace{(U_- \times V_-)}_{=:\,W_4}, \tag{IV.5}$$

where $U_\pm$ (respectively, $V_\pm$) are the canonical patches covering $\mathbb{C}P^1$ (respectively, $\mathbb{C}P^1_*$), we thus may take

$$\begin{aligned} &(x^{\alpha\dot\alpha}, \lambda_+, \mu_+, \eta_i^{\dot\alpha}, \theta^{i\alpha}) \qquad \text{on} \qquad \hat{\mathcal{W}}_1,\\ &(x^{\alpha\dot\alpha}, \lambda_+, \mu_-, \eta_i^{\dot\alpha}, \theta^{i\alpha}) \qquad \text{on} \qquad \hat{\mathcal{W}}_2,\\ &(x^{\alpha\dot\alpha}, \lambda_-, \mu_+, \eta_i^{\dot\alpha}, \theta^{i\alpha}) \qquad \text{on} \qquad \hat{\mathcal{W}}_3,\\ &(x^{\alpha\dot\alpha}, \lambda_-, \mu_-, \eta_i^{\dot\alpha}, \theta^{i\alpha}) \qquad \text{on} \qquad \hat{\mathcal{W}}_4. \end{aligned} \tag{IV.6}$$

In the sequel, we shall collectively denote them by $(x^{\alpha\dot\alpha}, \lambda_{(a)}, \mu_{(a)}, \eta_i^{\dot\alpha}, \theta^{i\alpha})$ on the patch $\hat{\mathcal{W}}_a = \mathbb{C}^{4|12} \times W_a \subset \mathcal{F}^{6|12}$.

Next we need coordinates on $\mathcal{L}^{5|6}$. By our above construction, $\mathcal{L}^{5|6}$ is an open subset of $\mathbb{L}^{5|6}$ and as such it can be viewed as a degree two hypersurface in $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ – in fact, $\mathcal{L}^{5|6} = \mathbb{L}^{5|6} \cap (\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3})$. Here, $\mathcal{P}^{3|3}$ represents the supertwistor space as given in (I.37) and $\mathcal{P}_*^{3|3}$ is the dual supertwistor space

$$\mathcal{O}_{\mathbb{C}P^1_*}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}_{\mathbb{C}P^1_*}(1) \otimes \mathbb{C}^3 \;\to\; \mathbb{C}P^1_*. \tag{IV.7}$$

As before, we denote the cover of $\mathcal{P}^{3|3}$ by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ and moreover, that of $\mathcal{P}_*^{3|3}$ by $\mathfrak{V} = \{\mathcal{V}_+, \mathcal{V}_-\}$. Then the product $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ is covered by four patches according to

$$\mathfrak{U} \times \mathfrak{V} \;=\; \{\mathcal{U}_+ \times \mathcal{V}_+, \mathcal{U}_+ \times \mathcal{V}_-, \mathcal{U}_- \times \mathcal{V}_+, \mathcal{U}_- \times \mathcal{V}_-\}, \tag{IV.8}$$

and we may set

$$\mathcal{L}^{5|6} \;=\; \bigcup_{a=1}^{4} \mathcal{W}_a, \quad \text{with} \quad \mathfrak{W} \;=\; (\mathfrak{U} \times \mathfrak{V}) \cap \mathbb{L}^{5|6} \quad \text{and} \quad \mathfrak{W} \;=\; \{\mathcal{W}_a\}. \tag{IV.9}$$

Let now $(z^\alpha_\pm, z^3_\pm, \eta_i^\pm)$ (respectively, $(w^{\dot\alpha}_\pm, w^3_\pm, \theta^i_\pm)$) be local coordinates on $\mathcal{P}^{3|3}$ (respectively, on $\mathcal{P}_*^{3|3}$), that is, we take

$$\begin{aligned} z^\alpha_+ &:= \frac{z^\alpha}{\pi_{\dot 1}}, & z^3_+ &:= \frac{\pi_{\dot 2}}{\pi_{\dot 1}} & \text{and} \quad \eta_i^+ &:= \frac{\eta_i}{\pi_{\dot 1}} & \text{on} \quad & \mathcal{U}_+,\\ z^\alpha_- &:= \frac{z^\alpha}{\pi_{\dot 2}}, & z^3_- &:= \frac{\pi_{\dot 1}}{\pi_{\dot 2}} & \text{and} \quad \eta_i^- &:= \frac{\eta_i}{\pi_{\dot 2}} & \text{on} \quad & \mathcal{U}_-, \end{aligned} \tag{IV.10}$$

and

$$\begin{aligned} w^{\dot\alpha}_+ &:= \frac{w^{\dot\alpha}}{\rho_1}, & w^3_+ &:= \frac{\rho_2}{\rho_1} & \text{and} \quad \theta^i_+ &:= \frac{\theta^i}{\rho_1} & \text{on} \quad & \mathcal{V}_+,\\ w^{\dot\alpha}_- &:= \frac{w^{\dot\alpha}}{\rho_2}, & w^3_- &:= \frac{\rho_1}{\rho_2} & \text{and} \quad \theta^i_- &:= \frac{\theta^i}{\rho_2} & \text{on} \quad & \mathcal{V}_-. \end{aligned} \tag{IV.11}$$

The induced coordinates on $\mathcal{L}^{5|6}$ are then given by

$$\begin{aligned} (z^\alpha_{(1)}, z^3_{(1)}, \eta_i^{(1)}, w^{\dot\alpha}_{(1)}, w^3_{(1)}, \theta^i_{(1)})|_{\mathcal{L}^{5|6}} &= (z^\alpha_+, z^3_+, \eta_i^+, w^{\dot\alpha}_+, w^3_+, \theta^i_+)|_{\mathcal{L}^{5|6}} & \text{on} \quad \mathcal{W}_1,\\ (z^\alpha_{(2)}, z^3_{(2)}, \eta_i^{(2)}, w^{\dot\alpha}_{(2)}, w^3_{(2)}, \theta^i_{(2)})|_{\mathcal{L}^{5|6}} &= (z^\alpha_+, z^3_+, \eta_i^+, w^{\dot\alpha}_-, w^3_-, \theta^i_-)|_{\mathcal{L}^{5|6}} & \text{on} \quad \mathcal{W}_2,\\ (z^\alpha_{(3)}, z^3_{(3)}, \eta_i^{(3)}, w^{\dot\alpha}_{(3)}, w^3_{(3)}, \theta^i_{(3)})|_{\mathcal{L}^{5|6}} &= (z^\alpha_-, z^3_-, \eta_i^-, w^{\dot\alpha}_+, w^3_+, \theta^i_+)|_{\mathcal{L}^{5|6}} & \text{on} \quad \mathcal{W}_3,\\ (z^\alpha_{(4)}, z^3_{(4)}, \eta_i^{(4)}, w^{\dot\alpha}_{(4)}, w^3_{(4)}, \theta^i_{(4)})|_{\mathcal{L}^{5|6}} &= (z^\alpha_-, z^3_-, \eta_i^-, w^{\dot\alpha}_-, w^3_-, \theta^i_-)|_{\mathcal{L}^{5|6}} & \text{on} \quad \mathcal{W}_4, \end{aligned} \tag{IV.12}$$

which are transformed on nonempty intersections $\mathcal{W}_a \cap \mathcal{W}_b$ in an obvious way which is obtained from the transformation laws on $\mathcal{P}^{3|3}$ and $\mathcal{P}_*^{3|3}$, respectively. Of course, these coordinates are not independent as they are subject to (cf. Eq. (IV.2))

$$z^\alpha_{(a)}\rho^{(a)}_\alpha - w^{\dot\alpha}_{(a)}\pi^{(a)}_{\dot\alpha} + 2\theta^i_{(a)}\eta^{(a)}_i \;=\; 0, \tag{IV.13}$$

where $\pi^{(1)}_{\dot\alpha} = \pi^{(2)}_{\dot\alpha} = \pi^+_{\dot\alpha}$, $\pi^{(3)}_{\dot\alpha} = \pi^{(4)}_{\dot\alpha} = \pi^-_{\dot\alpha}$ and $\rho^{(1)}_\alpha = \rho^{(3)}_\alpha = \rho^+_\alpha$, $\rho^{(2)}_\alpha = \rho^{(4)}_\alpha = \rho^-_\alpha$ together with $(\pi^+_{\dot\alpha}) = {}^t(1, z^3_+)$, $(\pi^-_{\dot\alpha}) = {}^t(z^3_-, 1)$ and $(\rho^+_\alpha) = {}^t(1, w^3_+)$, $(\rho^-_\alpha) = {}^t(w^3_-, 1)$. Note that there is no summation over a.

Altogether, we therefore obtain the following double fibration

$$\begin{tikzcd} & \mathcal{F}^{6|12} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{L}^{5|6} & & \mathbb{C}^{4|12} \end{tikzcd} \tag{IV.14}$$

together with the two holomorphic projections

$$\begin{aligned} \pi_1 \,:\, (x^{\alpha\dot\alpha}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_\alpha, \eta^{\dot\alpha}_i, \theta^{i\alpha}) \;&\mapsto\; (z^\alpha_{(a)} = (x^{\alpha\dot\alpha} - \theta^{i\alpha}\eta^{\dot\alpha}_i)\lambda^{(a)}_{\dot\alpha}, z^3_{(a)} = \lambda_{(a)}, \\ & w^{\dot\alpha}_{(a)} = (x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta^{\dot\alpha}_i)\mu^{(a)}_\alpha, w^3_{(a)} = \mu_{(a)}, \eta^{(a)}_i = \eta^{\dot\alpha}_i\lambda^{(a)}_{\dot\alpha}, \theta^i_{(a)} = \theta^{i\alpha}\mu^{(a)}_\alpha), \\ \pi_2 \,:\, (x^{\alpha\dot\alpha}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_\alpha, \eta^{\dot\alpha}_i, \theta^{i\alpha}) \;&\mapsto\; (x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i, \theta^{i\alpha}); \end{aligned} \tag{IV.15}$$

cf. also our discussion presented in §I.3, §I.12 and §I.18. In these equations, we have used the notation $\lambda^{(1)}_{\dot\alpha} = \lambda^{(2)}_{\dot\alpha} = \lambda^+_{\dot\alpha}$, $\lambda^{(3)}_{\dot\alpha} = \lambda^{(4)}_{\dot\alpha} = \lambda^-_{\dot\alpha}$ and $\mu^{(1)}_\alpha = \mu^{(3)}_\alpha = \mu^+_\alpha$, $\mu^{(2)}_\alpha = \mu^{(4)}_\alpha = \mu^-_\alpha$.

In fact, Eqs. (IV.15) illustrate Prop. I.4.: a fixed point $p \in \mathcal{L}^{5|6}$ corresponds to a super null line $\mathbb{C}^{1|6}_p \subset \mathbb{C}^{4|12}$ and furthermore, a fixed point $(x, \eta, \theta) \in \mathbb{C}^{4|12}$ corresponds to a holomorphic embedding of $Y_{x,\eta,\theta} \hookrightarrow \mathcal{L}^{5|6}$. To see that $\mathbb{C}^{1|6}_p$ is null, we solve

$$\begin{aligned} z^\alpha_{(a)} &= (x^{\alpha\dot\alpha} - \theta^{i\alpha}\eta^{\dot\alpha}_i)\lambda^{(a)}_{\dot\alpha}, & w^{\dot\alpha}_{(a)} &= (x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta^{\dot\alpha}_i)\mu^{(a)}_\alpha, \\ \eta^{(a)}_i &= \eta^{\dot\alpha}_i\lambda^{(a)}_{\dot\alpha}, & \theta^i_{(a)} &= \theta^{i\alpha}\mu^{(a)}_\alpha \end{aligned} \tag{IV.16}$$

for a generic $p \in \mathcal{L}^{5|6}$,

$$\begin{aligned} x^{\alpha\dot\alpha} &= \hat x^{\alpha\dot\alpha} + \varepsilon\mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)} + \varepsilon_i\lambda^{\dot\alpha}_{(a)}\hat\theta^{i\alpha} + \varepsilon^i\mu^\alpha_{(a)}\hat\eta^{\dot\alpha}_i, \\ \eta^{\dot\alpha}_i &= \hat\eta^{\dot\alpha}_i + \varepsilon_i\lambda^{\dot\alpha}_{(a)}, \qquad \theta^{i\alpha} = \hat\theta^{i\alpha} + \varepsilon^i\mu^\alpha_{(a)}, \end{aligned} \tag{IV.17}$$

where ε, ε_i and ε^i are arbitrary. Here, $(\hat x^{\alpha\dot\alpha}, \hat\eta^{\dot\alpha}_i, \hat\theta^{i\alpha})$ denotes a particular solution to (IV.16). This shows that $\mathbb{C}^{1|6}_p$ is null. Hence, $\mathcal{L}^{5|6}$ is the space of all super null lines in $\mathbb{C}^{4|12}$.

§IV.2 Remark. Though we are mainly interested in $\mathcal{N} = 3$, let us again stick to generic values of $\mathcal{N}$. In Secs. I.1 and I.3, we have seen that the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ is an open subset in $\mathbb{C}P^{3|\mathcal{N}}$ describing small deformations of the Riemann sphere $\mathbb{C}P^1$ inside $\mathbb{C}P^{3|\mathcal{N}}$. In fact, a similar argument can be given for $\mathcal{L}^{5|2\mathcal{N}}$. Let us consider the following sequence of embeddings:

$$Y \overset{\varphi}{\hookrightarrow} \mathbb{L}^{5|2\mathcal{N}} \hookrightarrow \mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*. \tag{IV.18}$$

Then we have an exact sequence

$$0 \;\to\; N \;\to\; N_Y \;\to\; \varphi^* N_{\mathbb{L}^{5|2\mathcal{N}}} \;\to\; 0, \tag{IV.19}$$

where N_Y is the normal sheaf of Y in $\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*$ and $N_{\mathbb{L}^{5|2\mathcal{N}}}$ that of $\mathbb{L}^{5|2\mathcal{N}}$ in $\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}^{3|\mathcal{N}}_*$. In particular, we have

$$\begin{aligned} N_Y \;&\cong\; \mathrm{pr}_1^* \mathcal{P}^{3|\mathcal{N}} \oplus \mathrm{pr}_2^* \mathcal{P}^{3|\mathcal{N}}_*, \\ \varphi^* N_{\mathbb{L}^{5|2\mathcal{N}}} \;&\cong\; \mathcal{O}_Y(1,1). \end{aligned}$$

Here, $\mathrm{pr}_{1,2}$ are again the two projections from $Y = \mathbb{C}P^1 \times \mathbb{C}P^1_*$ to the first and second factors and $\mathcal{O}_Y(1,1)$ is the sheaf of sections of the divisor line bundle of the diagonal $\mathbb{C}P^1 \subset Y$. Thus, N denotes the normal sheaf of Y in $\mathbb{L}^{5|2\mathcal{N}}$. In fact, $\mathcal{L}^{5|2\mathcal{N}}$ is a rank $3|2\mathcal{N}$ holomorphic vector bundle over Y and moreover, $N \cong \mathcal{L}^{5|2\mathcal{N}}$. Hence,

$$0 \;\to\; \mathcal{L}^{5|2\mathcal{N}} \;\to\; \mathrm{pr}_1^* \mathcal{P}^{3|\mathcal{N}} \oplus \mathrm{pr}_2^* \mathcal{P}^{3|\mathcal{N}}_* \;\to\; \mathcal{O}_Y(1,1) \;\to\; 0. \tag{IV.20}$$

The sequence (IV.20) induces a long exact cohomology sequence

$$\begin{aligned} 0 \;\to\; H^0(Y, \mathcal{L}^{5|2\mathcal{N}}) \;\to\; H^0(Y, N_Y) \;\xrightarrow{\kappa}\; H^0(Y, \mathcal{O}_Y(1,1)) \;\to \\ \to\; H^1(Y, \mathcal{L}^{5|2\mathcal{N}}) \;\to\; H^1(Y, N_Y) \;\to\; H^1(Y, \mathcal{O}_Y(1,1)) \;\to\; \cdots . \end{aligned} \tag{IV.21}$$

As before, we let $\pi_{\dot\alpha}$ and ρ_α be homogeneous coordinates on Y. An element of $H^0(Y, N_Y)$ is described by a $4|2\mathcal{N}$-tuple $(z^\alpha, w^{\dot\alpha}, \eta_i, \theta^i)$, where z^α and $w^{\dot\alpha}$ are even homogeneous bidegree $(1,0)$ respectively, bidegree $(0,1)$ polynomials in $(\pi_{\dot\alpha}, \rho_\alpha)$ while η_i and θ^i are odd polynomials of bidegree $(1,0)$ and $(0,1)$, respectively. The map

$$\kappa \,:\, H^0(Y, N_Y) \;\to\; H^0(Y, \mathcal{O}_Y(1,1))$$

is then given by

$$\kappa \,:\, (z^\alpha, w^{\dot\alpha}, \eta_i, \theta^i) \;\mapsto\; z^\alpha \rho_\alpha - w^{\dot\alpha} \pi_{\dot\alpha} + 2\theta^i \eta_i \tag{IV.22}$$

Clearly, this mapping is surjective. Hence, the sequence (IV.21) splits to give a short exact sequence

$$0 \;\to\; H^0(Y, \mathcal{L}^{5|2\mathcal{N}}) \;\to\; H^0(Y, N_Y) \;\to\; H^0(Y, \mathcal{O}_Y(1,1)) \;\to\; 0. \tag{IV.23}$$

Using Künneth's formula[1], one readily verifies that

$$H^0(Y, \mathcal{O}_Y(m,n)) \cong \mathbb{C}^{(m+1)(n+1)}$$
$$H^1(Y, \mathcal{O}_Y(m,n)) \cong 0 \tag{IV.24}$$

for $m, n \geq 0$. By virtue of (IV.23), we therefore find

$$H^0(Y, \mathcal{L}^{5|2\mathcal{N}}) \cong \mathbb{C}^{4|4\mathcal{N}}. \tag{IV.25}$$

Similarly, one may show that $H^1(Y, \mathcal{L}^{5|2\mathcal{N}}) \cong 0$ (cf. Eqs (IV.24)).

§IV.3 Penrose-Ward transform. After having presented the setup, we shall now come to the description of $\mathcal{N} = 3$ SYM theory. For this, we consider the double fibration (IV.14) together with a rank r holomorphic vector bundle $\mathcal{E} \to \mathcal{L}^{5|6}$ which is characterized by the transition functions $f = \{f_{ab}\}$ and its pull-back $\pi_1^*\mathcal{E}$ to the supermanifold $\mathcal{F}^{6|12}$. We denote again the pulled-back transition functions by the same letter f. By definition of a pull-back, the transition functions f are constant along $\pi_1 : \mathcal{F}^{6|12} \to \mathcal{L}^{5|6}$. The relative tangent sheaf $\mathscr{T} := (\Omega^1(\mathcal{F}^{5|6})/\pi_1^*\Omega^1(\mathcal{L}^{5|6}))^*$ is of rank $1|6$ and freely generated by

$$D_{(a)} = \mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)}\partial_{\alpha\dot\alpha}, \qquad D^i_{(a)} = \lambda^{\dot\alpha}_{(a)} D^i_{\dot\alpha} \qquad \text{and} \qquad D^{(a)}_i = \mu^\alpha_{(a)} D_{i\alpha}, \tag{IV.26}$$

where $\partial_{\alpha\dot\alpha} := \partial/\partial x^{\alpha\dot\alpha}$ and

$$D^i_{\dot\alpha} = \partial^i_{\dot\alpha} + \theta^{i\alpha}\partial_{\alpha\dot\alpha} \qquad \text{and} \qquad D_{i\alpha} = \partial_{i\alpha} + \eta^{\dot\alpha}_i\partial_{\alpha\dot\alpha}, \tag{IV.27}$$

with $\partial^i_{\dot\alpha} := \partial/\partial\eta^{\dot\alpha}_i$ and $\partial_{i\alpha} := \partial/\partial\theta^{i\alpha}$. Hence, the transition functions of $\pi_1^*\mathcal{E}$ are annihilated by the vector fields (IV.26). Clearly, they are also annihilated by $\bar\partial_{\mathcal{F}}$.

Next we want to assume that the vector bundle $\mathcal{E} \to \mathcal{L}^{5|6}$ is smoothly trivial and in addition $\mathbb{C}^{4|12}$-trivial, i.e., holomorphically trivial when restricted to any submanifold[2] $Y_{x,\eta,\theta} \hookrightarrow \mathcal{L}^{5|6}$. Together, these conditions imply that there exist some $\psi = \{\psi_a\} \in C^0(\mathfrak{W}, \mathfrak{S})$, which define trivializations of $\pi_1^*\mathcal{E}$, such that $f = \{f_{ab}\}$ can be decomposed as

$$f_{ab} = \psi_a^{-1}\psi_b \tag{IV.28}$$

[1] Recall the following fact. Let $\mathcal{E}_i$ be a holomorphic vector bundle over X_i for $i = 1, 2$. Furthermore, denote the two projections from $X = X_1 \times X_2$ to X_i by pr_i and consider the bundle $\mathcal{E} = \mathrm{pr}_1^*\mathcal{E}_1 \oplus \mathrm{pr}_2^*\mathcal{E}_2$. Künneth's formula then says:

$$H^k(X, \mathcal{E}) \cong \bigoplus_{p+q=k} H^p(X_1, \mathcal{E}_1) \otimes H^q(X_2, \mathcal{E}_2).$$

[2] See also Prop. I.4.

and

$$\bar{\partial}_{\mathcal{F}}\psi_a \;=\; 0. \tag{IV.29}$$

In particular, ψ_a depends holomorphically on λ_a and μ_a. Applying the vector fields (IV.26) to (IV.28), we realize that by virtue of an extension of Liouville's theorem to $\mathbb{C}P^1\times\mathbb{C}P^1_*$, the expressions

$$\psi_a D_{(a)}\psi_a^{-1} \;=\; \psi_b D_{(a)}\psi_b^{-1}, \quad \psi_a D^i_{(a)}\psi_a^{-1} \;=\; \psi_b D^i_{(a)}\psi_b^{-1}, \quad \psi_a D_i^{(a)}\psi_a^{-1} \;=\; \psi_b D_i^{(a)}\psi_b^{-1}$$

must be at most linear in $\lambda_{(a)}$ and $\mu_{(a)}$. Therefore, we may introduce a relative connection one-form $\mathcal{A}_{\mathscr{T}}\in\Gamma(\mathcal{F}^{6|12},\Omega^1_{\mathscr{T}}(\mathcal{F}^{6|12})\otimes\operatorname{End}\pi_1^*\mathcal{E})$ such that

$$\begin{aligned}
D\lrcorner\mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{W}}_a} &:= \mathcal{A}_{(a)} \;=\; \psi_a D_{(a)}\psi_a^{-1} \;=\; \mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)}\mathcal{A}_{\alpha\dot\alpha},\\
D^i\lrcorner\mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{W}}_a} &=: \mathcal{A}^i_{(a)} \;=\; \psi_a D^i_{(a)}\psi_a^{-1} \;=\; \lambda^{\dot\alpha}_{(a)}\mathcal{A}^i_{\dot\alpha},\\
D_i\lrcorner\mathcal{A}_{\mathscr{T}}|_{\hat{\mathcal{W}}_a} &:= \mathcal{A}_i^{(a)} \;=\; \psi_a D_i^{(a)}\psi_a^{-1} = \mu^\alpha_{(a)}\mathcal{A}_{i\alpha},
\end{aligned} \tag{IV.30}$$

and hence

$$\begin{aligned}
\mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)}(\partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha})\psi_a &= 0,\\
\lambda^{\dot\alpha}_{(a)}(D^i_{\dot\alpha}+\mathcal{A}^i_{\dot\alpha})\psi_a &= 0,\\
\mu^\alpha_{(a)}(D_{i\alpha}+\mathcal{A}_{i\alpha})\psi_a &= 0,\\
\bar{\partial}_{\mathcal{F}}\psi_a &= 0.
\end{aligned} \tag{IV.31}$$

The compatibility conditions for the linear system (IV.31) read as

$$\begin{aligned}
\{\nabla^i_{(\dot\alpha},\nabla^j_{\dot\beta)}\} &= 0,\\
\{\nabla_{i(\alpha},\nabla_{j\beta)}\} &= 0,\\
\{\nabla_{i\alpha},\nabla^j_{\dot\beta}\}-2\delta_i^j\nabla_{\alpha\dot\beta} &= 0,
\end{aligned} \tag{IV.32}$$

where we have introduced

$$\nabla^i_{\dot\alpha} \;:=\; D^i_{\dot\alpha}+\mathcal{A}^i_{\dot\alpha}, \quad \nabla_{i\alpha} \;:=\; D_{i\alpha}+\mathcal{A}_{i\alpha} \quad\text{and}\quad \nabla_{\alpha\dot\alpha} \;:=\; \partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha}. \tag{IV.33}$$

Eqs. (IV.32) are the constraint equations of $\mathcal{N}=3$ SYM theory.

Next let us discuss how to obtain the functions ψ_a in (IV.31) from a given gauge potential. Formally, a solution is given by

$$\psi_a \;=\; P\exp\left(-\int_{\mathscr{C}}\mathcal{A}\right). \tag{IV.34}$$

Here, "P" denotes the path-ordering symbol and

$$\mathcal{A} \;=\; E^{\alpha\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} + E_i^{\dot\alpha}\mathcal{A}_{\dot\alpha}^i + E^{i\alpha}\mathcal{A}_{i\alpha}, \tag{IV.35}$$

where the basis $(E^{\alpha\dot\alpha}, E_i^{\dot\alpha}, E^{i\alpha})$ is dual to $(\partial_{\alpha\dot\alpha}, D_{\dot\alpha}^i, D_{i\alpha})$. It is explicitly given by

$$E^{\alpha\dot\alpha} \;=\; \mathrm{d}x^{\alpha\dot\alpha} + \eta_i^{\dot\alpha}\mathrm{d}\theta^{i\alpha} + \theta^{i\alpha}\mathrm{d}\eta_i^{\dot\alpha}, \qquad E_i^{\dot\alpha} \;=\; \mathrm{d}\eta_i^{\dot\alpha} \qquad \text{and} \qquad E^{i\alpha} \;=\; \mathrm{d}\theta^{i\alpha}. \tag{IV.36}$$

The contour $\mathscr{C}$ is a any real curve within a superlight-ray $\mathbb{C}^{1|6}$ from a point $(\hat{x}, \hat{\eta}, \hat{\theta})$ to a point (x, η, θ), with

$$\begin{aligned} x^{\alpha\dot\alpha}(s) \;&=\; \hat{x}^{\alpha\dot\alpha} + s(\varepsilon\mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)} + \varepsilon_i\lambda^{\dot\alpha}_{(a)}\theta^{i\alpha} + \varepsilon^i\mu^\alpha_{(a)}\eta_i^{\dot\alpha}), \\ \eta_i^{\dot\alpha}(s) \;&=\; \hat{\eta}_i^{\dot\alpha} + s\varepsilon_i\lambda^{\dot\alpha}_{(a)}, \qquad \theta^{i\alpha}(s) \;=\; \hat{\theta}^{i\alpha} + s\varepsilon^i\mu^\alpha_{(a)}, \end{aligned} \tag{IV.37}$$

for $s \in [0,1]$; the choice of the contour plays no role, since the curvature is zero when restricted to the superlight-ray. Furthermore, $(\varepsilon, \varepsilon_i, \varepsilon^i)$ are some free parameters.

Before discussing the superfield expansions of the above gauge potentials, let us say a few words about reality conditions. So far, we have entirely worked in a complex setting. However, we eventually want to discuss real $\mathcal{N} = 3$ SYM theory on Minkowski space. In order to achieve this, we need to extend our in §I.20 introduced antiholomorphic involution τ_M to $\pi_1^*\mathcal{E} \to \mathcal{F}^{6|12}$, that is, we have to require

$$f_{12}^\dagger \;=\; f_{31}, \qquad f_{14}^\dagger \;=\; f_{41} \qquad \text{and} \qquad f_{23}^\dagger \;=\; f_{23} \tag{IV.38}$$

on appropriate intersections. In these equations, we have used the shorthand notation $f_{ab}^\dagger := [f_{ab}(\tau_M(\cdots))]^\dagger$. Upon imposing (IV.38), we find

$$\psi_1^\dagger \;=\; \psi_1^{-1}, \qquad \psi_2^\dagger \;=\; \psi_3^{-1} \qquad \text{and} \qquad \psi_4^\dagger \;=\; \psi_4^{-1} \tag{IV.39}$$

and hence,

$$\begin{aligned} \tau_M(\mathcal{A}_{\alpha\dot\beta}) \;&=\; \mathcal{A}^\dagger_{\beta\dot\alpha} \;=\; \mathcal{A}_{\alpha\dot\beta}, \\ \tau_M(\mathcal{A}^i_{\dot\alpha}) \;&=\; -\mathcal{A}^\dagger_{i\alpha} \;=\; \mathcal{A}^i_{\dot\alpha}, \\ \tau_M(\mathcal{A}_{i\alpha}) \;&=\; -\mathcal{A}^{i\dagger}_{\dot\alpha} \;=\; \mathcal{A}_{i\alpha}, \end{aligned} \tag{IV.40}$$

as desired. In this case, the gauge group $GL(r,\mathbb{C})$ is reduced to the unitary group $U(r)$ and as we have seen in §II.5, the additional requirement $\det(f_{ab}) = 1$ yields $SU(r)$.

§IV.4 Remark. If $f=\{f_{ab}\}$ is independent of say $\mu_{(a)}$ then one may assume without loss of generality that $f_{12}=f_{34}=\mathbb{1}_r$ and $f_{13}=f_{24}=f_{+-}(x_R^{\alpha\dot\alpha}\lambda^+_{\dot\alpha},\lambda_+,\eta_i^{\dot\alpha}\lambda^+_{\dot\alpha})$ [235]. All other transition functions are obtained from those by virtue of the cocycle conditions. In this case, the linear system (IV.31) reduces to that of self-dual SYM theory given in (II.16). Hence, by assuming that all transition function are independent of $\mu_{(a)}$, one ends up with self-dual SYM theory. Similarly, by imposing independence of $\lambda_{(a)}$, one eventually obtains the anti-self-dual SYM equations.

§IV.5 Field expansions, field equations and action functional. Let us now show that the constraint equations (IV.32) imply the equations of motion of $\mathcal{N}=3$ SYM theory (and vice versa). The analysis is, however, basically the same as the one given in §II.6, for instance. Therefore, we shall be rather brief and merely quote results. More details about the derivation can be found in Harnad et al. [109, 110, 111].

Eqs. (IV.32) can be formally solved according to

$$\begin{aligned}\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\} &= 2\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}W_k,\\ \{\nabla_{i\alpha},\nabla_{j\beta}\} &= 2\epsilon_{\alpha\beta}\epsilon_{ijk}W^k,\\ \{\nabla_{i\alpha},\nabla^j_{\dot\alpha}\} &= 2\delta_i^j\nabla_{\alpha\dot\alpha}.\end{aligned} \tag{IV.41}$$

It follows then from Bianchi identities that the odd spinor superfields are given by

$$\begin{aligned}\psi_\alpha &= \tfrac{1}{3}\nabla_{i\alpha}W^i,\\ \psi_{\dot\alpha} &= \tfrac{1}{3}\nabla^i_{\dot\alpha}W_i,\\ \chi_{i\dot\alpha} &= -\tfrac{1}{2}\epsilon^{\alpha\beta}[\nabla_{\alpha\dot\alpha},\nabla_{i\beta}],\\ \chi^i_\alpha &= -\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}[\nabla_{\alpha\dot\alpha},\nabla^i_{\dot\beta}].\end{aligned} \tag{IV.42}$$

As before, one imposes the transversal gauge condition

$$\eta_i^{\dot\alpha}\mathcal{A}^i_{\dot\alpha}+\theta^{i\alpha}\mathcal{A}_{i\alpha} \;=\; 0 \tag{IV.43}$$

to remove superfluous gauge degrees of freedom associated with the odd coordinates. This time, the recursion operator takes the form

$$\mathscr{D} \;=\; \eta_i^{\dot\alpha}\nabla^i_{\dot\alpha}+\theta^{i\alpha}\nabla_{i\alpha} \;=\; \eta_i^{\dot\alpha}\partial^i_{\dot\alpha}+\theta^{i\alpha}\partial_{i\alpha}. \tag{IV.44}$$

With the help of Bianchi identities one may prove the following recursion relations:

$$
\begin{aligned}
\mathscr{D}\mathcal{A}_{\alpha\dot\alpha} &= -\epsilon_{\alpha\beta}\theta^{i\beta}\chi_{i\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\eta_i^{\dot\beta}\chi_\alpha^i,\\
(1+\mathscr{D})\mathcal{A}^i_{\dot\alpha} &= 2\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\eta_j^{\dot\beta}W_k + 2\theta^{i\alpha}\mathcal{A}_{\alpha\dot\alpha},\\
(1+\mathscr{D})\mathcal{A}_{i\alpha} &= 2\epsilon_{\alpha\beta}\epsilon_{ijk}\theta^{j\beta}W^k + 2\eta_i^{\dot\alpha}\mathcal{A}_{\alpha\dot\alpha},\\
\mathscr{D}W_i &= \eta_i^{\dot\alpha}\psi_{\dot\alpha} + \epsilon_{ijk}\theta^{j\alpha}\chi^k_\alpha,\\
\mathscr{D}W^i &= \theta^{i\alpha}\psi_\alpha + \epsilon^{ijk}\eta_j^{\dot\alpha}\chi_{k\dot\alpha},\\
\mathscr{D}\psi_\alpha &= \epsilon_{\alpha\beta}\epsilon_{ijk}\theta^{i\beta}[W^j,W^k] + 2\eta_i^{\dot\alpha}\nabla_{\alpha\dot\alpha}W^i,\\
\mathscr{D}\psi_{\dot\alpha} &= \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\eta_i^{\dot\beta}[W_j,W_k] + 2\theta^{i\alpha}\nabla_{\alpha\dot\alpha}W_i,\\
\mathscr{D}\chi^i_\alpha &= -2\epsilon^{ijk}\eta_j^{\dot\alpha}\nabla_{\alpha\dot\alpha}W_k + 2\theta^{i\beta}f_{\alpha\beta} + \epsilon_{\alpha\beta}\theta^{j\beta}(\delta^i_j[W^k,W_k] - 2[W_j,W^i]),\\
\mathscr{D}\chi_{i\dot\alpha} &= -2\epsilon_{ijk}\theta^{j\alpha}\nabla_{\alpha\dot\alpha}W^k + 2\eta_i^{\dot\beta}f_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\beta}(\delta^j_i[W_k,W^k] - 2[W^j,W_i]),
\end{aligned}
\tag{IV.45}
$$

where again $f_{\alpha\beta}$ (respectively, $f_{\dot\alpha\dot\beta}$) represents the self-dual (respectively, anti-self-dual) part of the field strength. Note that these equations resemble the supersymmetry transformations, but nevertheless they should not be confused with them. Now one can start to iterate these equations to obtain all the superfields order by order in the odd coordinates. Using formulas similar to Eqs. (II.33) and (II.34), one eventually finds:

$$
\begin{aligned}
\mathcal{A}^i_{\dot\alpha} &= \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\eta_j^{\dot\alpha}\overset{\circ}{W}_k + \theta^{i\alpha}\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \tfrac23\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\eta_j^{\dot\beta}\eta_k^{\dot\gamma}\overset{\circ}{\psi}_{\dot\gamma} -\\
&\qquad - \tfrac43\epsilon_{\dot\alpha\dot\beta}\theta^{i\alpha}\eta_j^{\dot\beta}\overset{\circ}{\chi}{}^j_\alpha + \tfrac23\epsilon_{\dot\alpha\dot\beta}\theta^{j\alpha}\eta_j^{\dot\beta}\overset{\circ}{\chi}{}^i_\alpha - \tfrac23\epsilon_{\alpha\beta}\theta^{i\alpha}\theta^{j\beta}\overset{\circ}{\chi}_{j\dot\alpha} + \cdots,\\
\mathcal{A}_{i\alpha} &= \epsilon_{\alpha\beta}\epsilon_{ijk}\theta^{j\alpha}\overset{\circ}{W}{}^k + \eta_i^{\dot\alpha}\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha} + \tfrac23\epsilon_{\alpha\beta}\epsilon_{ijk}\theta^{j\beta}\theta^{k\gamma}\overset{\circ}{\psi}_\gamma +\\
&\qquad + \tfrac43\epsilon_{\alpha\beta}\theta^{j\beta}\eta_i^{\dot\alpha}\overset{\circ}{\chi}_{j\dot\alpha} - \tfrac23\epsilon_{\alpha\beta}\theta^{j\beta}\eta_j^{\dot\alpha}\overset{\circ}{\chi}_{i\dot\beta} - \tfrac23\epsilon_{\dot\alpha\dot\beta}\eta_i^{\dot\alpha}\eta_j^{\dot\beta}\overset{\circ}{\chi}{}^j_\alpha + \cdots.
\end{aligned}
\tag{IV.46}
$$

Upon substituting these expansions into the constraint equations (IV.41), we obtain the equations of motion of $\mathcal{N}=3$ SYM theory

$$
\begin{aligned}
\epsilon^{\alpha\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\psi}_\beta+[\mathring{\chi}_{i\dot\alpha},\mathring{W}^i] &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\psi}_{\dot\beta}+[\mathring{\chi}^i_\alpha,\mathring{W}_i] &= 0,\\
\epsilon^{\alpha\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}^j_\beta+[\mathring{\chi}_{i\dot\alpha},\mathring{W}_k]\epsilon^{ijk}-[\mathring{\psi}_{\dot\alpha},\mathring{W}^j] &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}_{j\dot\beta}+[\mathring{\chi}^i_\alpha,\mathring{W}^k]\epsilon_{ijk}-[\mathring{\psi}_\alpha,\mathring{W}_j] &= 0,\\
\mathring{\Box}\mathring{W}_j+[[\mathring{W}^i,\mathring{W}_j],\mathring{W}_i]-\tfrac12[[\mathring{W}^i,\mathring{W}_i],\mathring{W}_j]+\tfrac12\epsilon^{\dot\alpha\dot\beta}\{\mathring{\chi}_{j\dot\alpha},\mathring{\psi}_{\dot\beta}\}-\tfrac14\epsilon_{ijk}\epsilon^{\alpha\beta}\{\mathring{\chi}^i_\alpha,\mathring{\chi}^k_\beta\} &= 0,\\
\mathring{\Box}\mathring{W}^j+[[\mathring{W}_i,\mathring{W}^j],\mathring{W}^i]-\tfrac12[[\mathring{W}_i,\mathring{W}^i],\mathring{W}^j]+\tfrac12\epsilon^{\alpha\beta}\{\mathring{\chi}^j_\alpha,\mathring{\psi}_\beta\}-\tfrac14\epsilon^{ijk}\epsilon^{\dot\alpha\dot\beta}\{\mathring{\chi}_{i\dot\alpha},\mathring{\chi}_{k\dot\beta}\} &= 0,\\
\epsilon^{\alpha\beta}\mathring{\nabla}_{\alpha\dot\gamma}\mathring{f}_{\beta\gamma}+\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\gamma\dot\alpha}\mathring{f}_{\dot\beta\dot\gamma}+\{\mathring{\chi}^k_\gamma,\mathring{\chi}_{k\dot\gamma}\}+\{\mathring{\psi}_\gamma,\mathring{\psi}_{\dot\gamma}\}+[\mathring{W}^i,\mathring{\nabla}_{\gamma\dot\gamma}\mathring{W}_i]+[\mathring{W}_i,\mathring{\nabla}_{\gamma\dot\gamma}\mathring{W}^i] &= 0,
\end{aligned}
\tag{IV.47}
$$

where we have defined

$$\mathring{\Box}\ :=\ \tfrac12\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\nabla}_{\beta\dot\beta}. \tag{IV.48}$$

It can now be shown by induction and with the help of the recursion operator $\mathscr{D}$ that Eqs. (IV.47) are in one-to-one correspondence with the constraint equations (IV.41). For details, see [109, 110, 111].

Finally, it remains to give an appropriate action functional producing the equations of motion (IV.47). A straightforward calculation reveals that they can be obtained by varying

$$
\begin{aligned}
S\ =\ \int \mathrm{d}^4x\,\mathrm{tr}\Big\{&\mathring{f}_{\alpha\beta}\mathring{f}^{\alpha\beta}+\mathring{f}_{\dot\alpha\dot\beta}\mathring{f}^{\dot\alpha\dot\beta}+\mathring{\chi}^{i\alpha}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}^{\dot\alpha}_i+2\mathring{W}^i\mathring{\Box}\mathring{W}_i+\mathring{W}_i\{\mathring{\chi}^i_\alpha,\mathring{\psi}^\alpha\}+\\
&+\mathring{W}^i\{\mathring{\chi}_{i\dot\alpha},\mathring{\psi}^{\dot\alpha}\}+\tfrac12\epsilon_{ijk}\{\mathring{\chi}^i_\alpha,\mathring{\chi}^{j\alpha}\}\mathring{W}^k+\tfrac12\epsilon^{ijk}\{\mathring{\chi}_{i\dot\alpha},\mathring{\chi}_{j\dot\beta}\}\mathring{W}_k-\\
&-[\mathring{W}_i,\mathring{W}^j][\mathring{W}_j,\mathring{W}^i]+\tfrac12[\mathring{W}_i,\mathring{W}^i][\mathring{W}_j,\mathring{W}^j]\Big\}.
\end{aligned}
\tag{IV.49}
$$

§IV.6 HCS theory on $\mathcal{L}^{5|6}$. Using the Dolbeault approach to holomorphic vector bundles, we gave the twistor interpretation of Siegel's action functional of $\mathcal{N}=4$ self-dual SYM theory in §II.7. It turned out to be the action functional of hCS theory on supertwistor space $\mathcal{P}^{3|4}$. The existence of an appropriate action principle on $\mathcal{P}^{3|4}$ was due to the formal Calabi-Yau property of the latter. As we have seen, also superambitwistor space $\mathcal{L}^{5|6}$ is a formal Calabi-Yau supermanifold. Thus, there is a globally well-defined nowhere vanishing holomorphic volume form. On the patch $\mathcal{W}_a\subset\mathcal{L}^{5|6}$, it is given by

$$\Omega|_{\mathcal{W}_a}\ =\ \frac{(-)^{\lfloor\frac a2\rfloor}}{2\pi\mathrm{i}}\oint_{\mathscr{C}}\frac{\mathrm{d}^3z_{(a)}\wedge\mathrm{d}^3w_{(a)}\mathrm{d}^3\eta_{(a)}\mathrm{d}^3\theta_{(a)}}{z^\alpha_{(a)}\rho^{(a)}_\alpha-w^{\dot\alpha}_{(a)}\pi^{(a)}_{\dot\alpha}+2\theta^i_{(a)}\eta^{(a)}_i}, \tag{IV.50}$$

where $\mathscr{C}$ is any contour encircling $\mathcal{L}^{5|6} \hookrightarrow \mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$ and $\mathrm{d}^3 z_{(a)} := \mathrm{d}z^1_{(a)} \wedge \mathrm{d}z^2_{(a)} \wedge \mathrm{d}z^3_{(a)}$, etc. The problem which prevents us from writing down an action functional like (II.59) is the "wrong" dimensionality of superambitwistor space: the holomorphic volume form (IV.26) is a form of type $(5|6,0)$ while the Chern-Simons form is of type $(0,3|0)$, as one is again interested in a subsupermanifold $\mathcal{Y} \subset \mathcal{L}^{5|6}$ determined by $\bar{\eta}_i^{(a)} = 0 = \bar{\theta}^i_{(a)}$. Thus, in total one obtains a form of type $(5|6,3|0)$ which, of course, cannot be integrated over $\mathcal{Y}$.

In order to circumvent this problem, Mason et al. [177] (cf. also Ref. [176]) proposed to instead consider a real codimension $2|0$ CR supermanifold[3] in superambitwistor space. Their construction is based on an Euclidean signature as for Minkowski signature the CR supermanifold under consideration will no longer be smooth but have singularities. In this setting it is, however, possible to use the holomorphic volume form (IV.50) to give an action functional for phCS theory (see also our discussion given in Sec. III.2) reproducing the action functional of $\mathcal{N}=3$ SYM theory. Though their construction works for an Euclidean setting, it remains an open question to find an appropriate twistor interpretation of (IV.49) in the case of Minkowski signature.

§IV.7 Summary. Let us now summarize:

Theorem IV.1. *There is a one-to-one correspondence between gauge equivalence classes of local solutions to the $\mathcal{N}=3$ SYM equations on four-dimensional Minkowski space and equivalence classes of holomorphic vector bundles $\mathcal{E}$ over superambitwistor space $\mathcal{L}^{5|6}$ which are smoothly trivial and holomorphically trivial on any submanifold $(\mathbb{C}P^1 \times \mathbb{C}P^1_*)_{x,\eta,\theta} \hookrightarrow \mathcal{L}^{5|6}$.*

If we let $H^1_{\nabla^{0,1}}(\mathcal{L}^{5|6}, \tilde{\mathcal{E}})$ be the moduli space of hCS theory on $\mathcal{L}^{5|6}$ for vector bundles $\tilde{\mathcal{E}}$ smoothly equivalent to $\mathcal{E}$, we may equivalently write

$$H^1_{\nabla^{0,1}}(\mathcal{L}^{5|6}, \tilde{\mathcal{E}}) \;\cong\; \mathcal{M}^{\mathcal{N}=3}_{\mathrm{SYM}}, \tag{IV.51}$$

where $\mathcal{M}^{\mathcal{N}=3}_{\mathrm{SYM}}$ denotes the moduli space of $\mathcal{N}=3$ SYM theory. The latter is obtained from the solution space by quotiening with respect to the group of gauge transformations.

§IV.8 Remark. In Chap. III, we have described the dimensional reduction of the supertwistor space and obtained the mini-supertwistor space. In this setting, we established a correspondence between hBF theory on mini-supertwistor space and a supersymmetric Bogomolny model in three dimensions. As for this model, which was obtained by a

[3]For a definition of a CR supermanifold, see Sec. III.1.

dimensional reduction of $\mathcal{N} = 4$ self-dual SYM theory, one can establish a twistor correspondence for the full $\mathcal{N} = 6$ (respectively, $\mathcal{N} = 8$) SYM theory in three dimensions by using a dimensional reduction of $\mathcal{L}^{5|6}$. In fact, one can establish:

$$\begin{tikzcd}
 & \mathcal{F}^{6|12} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{L}^{5|6} \arrow[dd] & \mathcal{F}^{5|12} \arrow[dl] \arrow[dr] & \mathbb{C}^{4|12} \arrow[dd] \\
 & & \\
\mathcal{L}^{4|6} & & \mathbb{C}^{3|12}
\end{tikzcd} \tag{IV.52}$$

The details about this correspondence, the construction of the field equations, etc. can be found in [220, 221].

IV.2 Thickenings and $\mathcal{N} < 3$ super Yang-Mills theory

For the sake of completeness, we shall now briefly talk about SYM theories with less supersymmetry, that is, for $\mathcal{N} < 3$. In order to discuss them, we first need the notion of formal neighborhoods. This will then allow us to provide a twistor formulation of these theories. However, we stress in advance that we will not prove any of the subsequent statements but rather quote results.

§IV.9 Formal neighborhoods. Let X be a complex supermanifold and Y a sub(super)manifold. Furthermore, let $\mathcal{I}$ be the ideal subsheaf of all holomorphic functions in $\mathcal{O}_X$ which vanish on Y. Then we have a short exact sequence of sheaves

$$0 \;\to\; \mathcal{I} \;\to\; \mathcal{O}_X \;\to\; \mathcal{O}_X/\mathcal{I} \;\to\; 0 \tag{IV.53}$$

on X. The sheaf $\mathcal{O}_X/\mathcal{I}$ can then be identified with $\mathcal{O}_Y$. Generally speaking, there is an isomorphism

$$\mathcal{O}_X|_Y \;\cong\; \mathcal{O}_X/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots \tag{IV.54}$$

given by the Taylor expansion of a germ of f at any point of Y. Note that $\mathcal{I}/\mathcal{I}^2$ can be identified with the conormal sheaf of Y in X.

Next we define

$$\mathcal{O}_Y^{(k)} \;:=\; \mathcal{O}_X/\mathcal{I}^{k+1}. \tag{IV.55}$$

Then $(Y, \mathcal{O}_Y^{(k)})$ is called the k-th formal neighborhood of Y in X. Note that $\mathcal{O}_Y^{(k)}$ can be expanded according to

$$\mathcal{O}_Y^{(k)} \;\cong\; \mathcal{O}_X/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots \oplus \mathcal{I}^k/\mathcal{I}^{k+1}. \tag{IV.56}$$

Putting it differently, the k-th formal neighborhood contains formal Taylor expansions in the normal sheaf direction up to order k.

§IV.10 Manin's theorem for $\mathcal{N} \leq 3$ SYM theory. Let us consider the double fibration

$$\begin{tikzcd} & \mathbb{F}^{6|4\mathcal{N}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{L}^{5|2\mathcal{N}} & & \mathbb{M}^{4|4\mathcal{N}} \end{tikzcd} \tag{IV.57}$$

as introduced in §I.13. Then we may state the following theorem [169]:

Theorem IV.2. *Let U be an open subset of $\mathbb{M}^{4|4\mathcal{N}}$ such that any null line intersects U in a simply connected set. Then there is a one-to-one correspondence between equivalence classes of U-trivial holomorphic vector bundles which admit an extension to a $(3-\mathcal{N})$-th formal neighborhood of $\mathbb{L}^{5|2\mathcal{N}}$ in $\mathbb{P}^{3|\mathcal{N}} \times \mathbb{P}_*^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended SYM equations on U.*

Details about the proof can be found in [169]. Furthermore, related aspects such as superfield expansions, etc., are discussed in Refs. [93, 111].

Chapter V

Hidden symmetries in self-dual super Yang-Mills theory

The purpose of this chapter is the discussion of hidden infinite-dimensional symmetry algebras in $\mathcal{N}$-extended self-dual SYM theory. The main tools we shall be using are built upon the twistor correspondence, which we established in Chap. II.

Since Pohlmeyer's work [196], it has been known that self-dual YM theory possess infinitely many hidden nonlocal symmetries and hence infinitely many conserved nonlocal charges. As was shown in [70, 71, 72, 241, 82, 76], these symmetries are affine extensions of internal symmetries with an underlying Kac-Moody structure. For a review on that matter, we refer also to [83]. Furthermore, in [200] affine extensions of conformal symmetries have been found. As a result, certain Kac-Moody-Virasoro-type algebras associated with space-time symmetries were obtained. A systematic investigation of symmetries based on twistor theory and on Čech and Dolbeault cohomology methods was performed in [202] (see also Refs. [203, 127] and the book [174]). Therein, all symmetries of the self-dual YM equations were derived.

One of the goals of this chapter is a generalization of the above symmetries to the self-dual SYM equations. The subsequent discussion is based on [266, 267]. In particular, we will consider perturbations of transition functions of holomorphic vector bundles over supertwistor space. Using the Penrose-Ward transform, we relate these perturbations to symmetries of $\mathcal{N}$-extended self-dual SYM theory. After some general words on hidden symmetry algebras, we exemplify our discussion by constructing Kac-Moody symmetries which come from affine extensions of internal symmetries. Furthermore, we also consider affine extensions of the superconformal algebra resulting in super Kac-Moody-Virasoro-type symmetries. Moreover, by focussing on certain Abelian subalgebras of the affinely extended superconformal algebra, we introduce supermanifolds which we call generalized supertwistor spaces. These spaces then allow us to introduce so-called hierarchies which describe sets of graded Abelian symmetries. This generalizes the results known for self-dual YM theory [172, 237, 173, 4, 174, 129]. We remark that such symmetries of the latter theory are intimately connected with one-loop maximally helicity violating amplitudes [24, 67, 68, 217, 106].

V.1 Holomorphicity and symmetries

As indicated, the key idea for studying symmetries within the supertwistor framework is, for a given Stein covering $\mathfrak{U} = \{\mathcal{U}_a\}$ of supertwistor space $\mathcal{P}^{3|\mathcal{N}}$, to consider infinitesimal deformations of the transition functions $f = \{f_{ab}\} \in H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H})$ of some smoothly trivial rank r holomorphic vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|\mathcal{N}}$ which in addition is $\mathbb{C}^{4|2\mathcal{N}}$- (respectively, $\mathbb{R}^{4|2\mathcal{N}}$-) trivial.[1] Recall that $\mathfrak{H} = GL(r, \mathcal{O}_{\mathcal{P}^{3|\mathcal{N}}})$. Generically, an infinitesimal deformation looks as

$$f \ \mapsto \ f' \ = \ f + \delta f \qquad \longleftrightarrow \qquad f_{ab} \ \mapsto \ f'_{ab} \ = \ f_{ab} + \delta f_{ab}. \tag{V.1}$$

Clearly, such perturbations have to obey certain criteria:

- They must be holomorphic and moreover preserve the cocycle conditions, that is, $f' = \{f'_{ab}\} \in H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H})$.
- They must preserve the holomorphic triviality on any $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$.
- In case one is interested in real YM fields, they must in addition respect the reality conditions induced by the antiholomorphic involution τ_E introduced in §I.19.

The first point is not really a restriction. Remember that in Chaps. I and II we have seen that there exists a two-set Stein covering of $\mathcal{P}^{3|\mathcal{N}}$. Therefore, no cocycle conditions appear when working with this covering.[2] The second point deserves some discussion. Besides $H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H})$, consider the Abelian group (by addition) $H^1(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{H})$, where $\mathrm{Lie}\,\mathfrak{H} := \mathfrak{gl}(r, \mathcal{O}_{\mathcal{P}^{3|\mathcal{N}}})$. This group parametrizes infinitesimal deformations of the trivial

[1] In §III.21, we have already used the idea of infinitesimal deformations for the construction of solutions to the equations of motion of the supersymmetrized Bogomolny model. Here, we are going to formalize the things used in that paragraph.

[2] In priniciple, there are transformations which can change the cover. Then one needs to consider the common refinement. For details, see [202]. In the sequel, we shall only be interested in deformations which preserve the chosen cover.

bundle $\mathcal{E}_0$.[3] Furthermore, $\dim_{\mathbb{C}} H^1(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{H}) = \infty$, that is, in an arbitrarily small neighborhood of the trivial bundle $\mathcal{E}_0$ there exists an infinite number of holomorphically nontrivial bundles $\mathcal{E}$. If we let $H^1(\mathbb{C}P^1_{x_R,\eta}, \mathrm{Lie}\,\mathfrak{H}_{\mathbb{C}P^1_{x_R,\eta}})$ be the restriction of the cohomology group $H^1(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{H})$ to $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ for some fixed $(x_R, \eta) \in \mathbb{C}^{4|2\mathcal{N}}$, then $H^1(\mathbb{C}P^1_{x_R,\eta}, \mathrm{Lie}\,\mathfrak{H}_{\mathbb{C}P^1_{x_R,\eta}})$ parametrizes infinitesimal deformations of $\mathcal{E}_0$ restricted to $\mathbb{C}P^1_{x_R,\eta}$. However,

$$H^1(\mathbb{C}P^1_{x_R,\eta}, \mathrm{Lie}\,\mathfrak{H}_{\mathbb{C}P^1_{x_R,\eta}}) \;\cong\; 0$$

which follows from the vanishing of $H^1(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1})$. This in turn implies that small enough deformations do not change the trivializability of $\mathcal{E}$ over $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. In fact, this is a version of the Kodaira-Spencer-Nirenberg theorem [137]. Therefore, point two from the above list is likewise no restriction on the allowed perturbations. Altogether, any infinitesimal holomorphic deformation is allowed. The only restriction one has to implement is point three from the above list. If not otherwise stated, we shall always assume that our deformations are compatible with the reality conditions.

§V.1 Kac-Moody symmetries. Let us consider the nonholomorphic fibration $\mathcal{P}^{3|\mathcal{N}} \to \mathbb{R}^{4|2\mathcal{N}}$ given by (II.42). Choose again the canonical cover $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ of the supertwistor space. Given a smoothly and $\mathbb{R}^{4|2\mathcal{N}}$-trivial rank r holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$, we define an action of the one-cochain group $C^1(\mathfrak{U}, \mathfrak{H})$ on $Z^1(\mathfrak{U}, \mathfrak{H})$ by

$$h \;:\; f_{+-} \;\mapsto\; h_{+-} f_{+-} h_{-+}^{-1}, \tag{V.2}$$

where $h = \{h_{+-}, h_{-+}\} \in C^1(\mathfrak{U}, \mathfrak{H})$ and $f = \{f_{+-}\} \in Z^1(\mathfrak{U}, \mathfrak{H})$. Obviously, the group $C^1(\mathfrak{U}, \mathfrak{H})$ acts transitively on $Z^1(\mathfrak{U}, \mathfrak{H})$, since for an arbitrary Čech one-cocycle $f \in Z^1(\mathfrak{U}, \mathfrak{H})$ one can find an $h \in C^1(\mathfrak{U}, \mathfrak{H})$ such that $f_{+-} = h_{+-} h_{-+}^{-1}$ and $f_{-+} = h_{-+} h_{+-}^{-1}$. The stabilizer of the trivial Čech one-cocycle, given by $f = \{\mathbb{1}_r\}$, is the subgroup

$$C^1_\Delta(\mathfrak{U}, \mathfrak{H}) \;:=\; \{h \in C^1(\mathfrak{U}, \mathfrak{H}) \,|\, h_{+-} = h_{-+}\} \tag{V.3}$$

implying that $Z^1(\mathfrak{U}, \mathfrak{H})$ can be identified with the coset

$$Z^1(\mathfrak{U}, \mathfrak{H}) \;\cong\; C^1(\mathfrak{U}, \mathfrak{H}) / C^1_\Delta(\mathfrak{U}, \mathfrak{H}). \tag{V.4}$$

Then the moduli space $H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H})$ is given by a double coset space

$$H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H}) \;\cong\; C^0(\mathfrak{U}, \mathfrak{H}) \backslash C^1(\mathfrak{U}, \mathfrak{H}) / C^1_\Delta(\mathfrak{U}, \mathfrak{H}), \tag{V.5}$$

[3]Note that infinitesimal deformations of an arbitrary holomorphic vector bundle $\mathcal{E} \to X$ are parametrized by $H^1(X, \mathrm{End}\,\mathcal{E})$.

where the zero-cochain group acts according to

$$h\ :\ f_{+-}\ \mapsto\ h_+ f_{+-} h_-^{-1}, \tag{V.6}$$

with $h = \{h_+, h_-\} \in C^0(\mathfrak{U}, \mathfrak{H})$.

Let us now study infinitesimal deformations of the transition function $f = \{f_{+-}\}$. Recalling the definition $\mathfrak{S} = GL(r, \mathcal{S}_{\mathcal{P}^{3|\mathcal{N}}})$, we introduce the subsheaves $\mathfrak{P} \subset \mathfrak{S}$ and $\operatorname{Lie}\mathfrak{P} \subset \operatorname{Lie}\mathfrak{S}$ consisting of those $GL(r, \mathbb{C})$-valued, respectively, $\mathfrak{gl}(r, \mathbb{C})$-valued smooth functions which are holomorphic in $\lambda_\pm$. We have a natural infinitesimal action of the group $C^1(\mathfrak{U}, \mathfrak{H})$ on the space $Z^1(\mathfrak{U}, \mathfrak{H})$ which is induced by the linearization of (V.2). That is, we get

$$\delta f_{+-}\ =\ \delta h_{+-} f_{+-} - f_{+-} \delta h_{-+}, \tag{V.7}$$

where $\delta h = \{\delta h_{+-}, \delta h_{-+}\} \in C^1(\mathfrak{U}, \operatorname{Lie}\mathfrak{H})$ and $f = \{f_{+-}\} \in Z^1(\mathfrak{U}, \mathfrak{H})$. Then we introduce a $\mathfrak{gl}(r, \mathbb{C})$-valued function

$$\phi_{+-}\ :=\ \psi_+ (\delta f_{+-} f_{+-}^{-1}) \psi_+^{-1}\ =\ \psi_+ (\delta f_{+-}) \psi_-^{-1}, \tag{V.8}$$

where $\psi = \{\psi_+, \psi_-\} \in C^0(\mathfrak{U}, \mathfrak{P})$ and $f_{+-} = \psi_+^{-1}\psi_-$, as before. From Eq. (V.7) it is then immediate that

$$\phi_{+-}\ =\ -\phi_{-+}.$$

Moreover, ϕ_{+-} is annihilated by the vector field $\partial_{\bar{\lambda}_\pm}$. Thus, it defines an element $\phi = \{\phi_{+-}\} \in Z^1(\mathfrak{U}, \operatorname{Lie}\mathfrak{P})$. However, any one-cocycle with values in the sheaf $\operatorname{Lie}\mathfrak{P}$ is a one-coboundary since $H^1(\mathcal{P}^{3|\mathcal{N}}, \operatorname{Lie}\mathfrak{P}) \cong 0$. This is an immediate consequence of the vanishing of $H^1(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1})$ and of the acyclicity of the sheaf $\mathcal{S}_{\mathcal{P}^{3|\mathcal{N}}}$. Therefore, we have

$$\phi_{+-}\ =\ \phi_+ - \phi_-, \tag{V.9}$$

where $\phi = \{\phi_+, \phi_-\} \in C^0(\mathfrak{U}, \operatorname{Lie}\mathfrak{P})$. Linearizing $f_{+-} = \psi_+^{-1}\psi_-$, we obtain

$$\delta f_{+-}\ =\ f_{+-}\psi_-^{-1}\delta\psi_- - \psi_+^{-1}\delta\psi_+ f_{+-}$$

and hence by virtue of Eqs. (V.8) and (V.9) we find

$$\delta\psi_\pm\ =\ -\phi_\pm\psi_\pm. \tag{V.10}$$

In summary, given some $\delta h = \{\delta h_{+-}, \delta h_{-+}\} \in C^1(\mathfrak{U}, \operatorname{Lie}\mathfrak{H})$ one derives via (V.7)–(V.9) the perturbations $\delta\psi_\pm$. Moreover, we point out that finding such $\phi_\pm$ means to solve the infinitesimal variant of the Riemann-Hilbert problem. Obviously, the splitting (V.9), (V.10) and hence solutions to the Riemann-Hilbert problem are not unique, as we certainly have the freedom to consider a new $\tilde{\phi}$ shifted by functions $\omega_\pm$, $\tilde{\phi}_\pm = \phi_\pm + \omega_\pm$, such that $\omega_+ = \omega_-$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, that is, $\omega \in Z^0(\mathfrak{U}, \mathrm{Lie}\,\mathfrak{P}) \equiv H^0(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{P})$.

Furthermore, infinitesimal variations of the linear system (II.49) yield

$$\begin{aligned} \delta\mathcal{A}^\pm_\alpha &= \delta\psi_\pm \bar{V}^\pm_\alpha \psi^{-1}_\pm + \psi_\pm \bar{V}^\pm_\alpha \delta\psi^{-1}_\pm, \\ \delta\mathcal{A}^i_\pm &= \delta\psi_\pm \bar{V}^i_\pm \psi^{-1}_\pm + \psi_\pm \bar{V}^i_\pm \delta\psi^{-1}_\pm. \end{aligned} \tag{V.11}$$

Substituting (V.10) into these equations, we arrive at

$$\delta\mathcal{A}^\pm_\alpha \;=\; \bar{\nabla}^\pm_\alpha \phi_\pm \qquad \text{and} \qquad \delta\mathcal{A}^i_\pm \;=\; \bar{\nabla}^i_\pm \phi_\pm. \tag{V.12}$$

Here, we have introduced the definitions

$$\bar{\nabla}^\pm_\alpha \;:=\; \bar{V}^\pm_\alpha + \mathcal{A}^\pm_\alpha \qquad \text{and} \qquad \bar{\nabla}^i_\pm \;:=\; \bar{V}^i_\pm + \mathcal{A}^i_\pm. \tag{V.13}$$

Note that (V.13) acts adjointly in (V.12). Therefore, (V.12) together with (V.9) imply that

$$\bar{\nabla}^\pm_\alpha \phi_{+-} \;=\; 0 \qquad \text{and} \qquad \bar{\nabla}^i_\pm \phi_{+-} \;=\; 0. \tag{V.14}$$

One may easily check that the choice $\phi_\pm = \psi_\pm \chi_\pm \psi^{-1}_\pm$, where $\chi = \{\chi_+, \chi_-\} \in C^0(\mathfrak{U}, \mathrm{Lie}\,\mathfrak{H})$, implies $\delta\mathcal{A}_{\alpha\dot{\alpha}} = 0$ and $\delta\mathcal{A}^i_{\dot{\alpha}} = 0$, respectively. Hence, such $\phi_\pm$ define trivial perturbations. On the other hand, infinitesimal gauge transformations of the form

$$\delta\mathcal{A}_{\alpha\dot{\alpha}} \;=\; \nabla^R_{\alpha\dot{\alpha}}\omega \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot{\alpha}} \;=\; \nabla^i_{\dot{\alpha}}\omega,$$

where ω is some smooth $\mathfrak{su}(r)$-valued function on $\mathbb{R}^{4|2\mathcal{N}}$, imply (for irreducible gauge potentials) that

$$\phi_\pm \;=\; \omega.$$

Hence, such $\phi_\pm$ do not depend on $\lambda_\pm$. In particular, we have $\phi_{+-} = 0$ and hence $\delta f_{+-} = 0$.

Finally, we obtain the formulas

$$\delta\mathcal{A}_{\alpha\dot{\alpha}} \;=\; \frac{1}{2\pi\mathrm{i}} \oint_{\mathscr{C}} \mathrm{d}\lambda_+ \, \frac{\bar{\nabla}^+_\alpha \phi_+}{\lambda_+ \lambda^{\dot{\alpha}}_+} \qquad \text{and} \qquad \delta\mathcal{A}^i_{\dot{\alpha}} \;=\; \frac{1}{2\pi\mathrm{i}} \oint_{\mathscr{C}} \mathrm{d}\lambda_+ \, \frac{\bar{\nabla}^i_+ \phi_+}{\lambda_+ \lambda^{\dot{\alpha}}_+} \tag{V.15}$$

where the contour $\mathscr{C} = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$ encircles $\lambda_+ = 0$. We see that generically the outcome of the transformation $\delta\mathcal{A}$ is a highly nonlocal expression depending on $\mathcal{A}$, i.e., we may write $\delta\mathcal{A} = F[\mathcal{A}, \partial\mathcal{A}, \ldots]$, where F is a functional whose explicit form is determined by $\phi_\pm$.

Summarizing, to any infinitesimal deformation, $f \mapsto f + \delta f$, of the transition function f we have associated a symmetry transformation $\mathcal{A} \mapsto \mathcal{A} + \delta\mathcal{A}$, that is, solutions to the linearized $\mathcal{N}$-extended self-dual SYM equations. In fact, we have obtained a one-to-one correspondence between equivalence classes of deformations of holomorphic vector bundles over supertwistor space which are subject to certain triviality conditions and gauge equivalence classes of solutions to the linearized field equations of $\mathcal{N}$-extended SYM theory. Putting it differently, our above discussion represents the "infinitesimal" version of Thm. II.2.

§V.2 Virasoro-type symmetries. Above we have introduced Kac-Moody symmetries which were generated by the algebra $C^1(\mathfrak{U}, \mathrm{Lie}\,\mathfrak{H})$. In this paragraph we focus on symmetries, which are related to the group of local biholomorphisms of supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. We shall only be interested in this subgroup of the diffeomorphism group of $\mathcal{P}^{3|\mathcal{N}}$ as generic local diffeomorphisms would change the complex structure of the supertwistor space. However, this in turn would induce a change of the conformal structure and a metric on $\mathbb{R}^{4|2\mathcal{N}}$ as was demonstrated in the purely even setting by Penrose [191] and Atiyah et al. [21]. As we want to discuss symmetries of the self-dual SYM equations on $\mathbb{R}^4$, we need to consider those diffeomorphisms which preserve the complex structure, that is, biholomorphisms. Let us denote the group of local biholomorphisms by $\mathcal{H}_{\mathcal{P}^{3|\mathcal{N}}}$. Furthermore, we are again choosing the canonical cover $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ of $\mathcal{P}^{3|\mathcal{N}}$ together with the coordinates $(Z^I_\pm) = (z^\alpha_\pm, z^3_\pm, \eta_i^\pm)$. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, they are related by transition functions $t = \{t^I_{+-}\}$,

$$Z^I_+ \;=\; t^I_{+-}(Z^J_-). \tag{V.16}$$

To $\mathcal{H}_{\mathcal{P}^{3|\mathcal{N}}}$ one associates the algebra $C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ of zero-cochains on $\mathcal{P}^{3|\mathcal{N}}$ with values in the tangent sheaf $T\mathcal{P}^{3|\mathcal{N}}$ of supertwistor space. In order to define an appropriate action of $C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$, let us first consider the algebra $C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ whose elements are collections of vector fields

$$\chi \;=\; \{\chi_{+-}, \chi_{-+}\} \;=\; \left\{\chi^I_{+-}\partial^+_I, \chi^I_{-+}\partial^-_I\right\}, \tag{V.17}$$

where we have abbreviated $\partial^\pm_I := \partial/\partial Z^I_\pm$. In particular, χ_{+-} and χ_{-+} are elements of the algebra $T\mathcal{P}^{3|\mathcal{N}}|_{\mathcal{U}_+\cap\mathcal{U}_-}$ of holomorphic vector fields on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$. Thus,

$C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ can be decomposed according to

$$C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \;\cong\; T\mathcal{P}^{3|\mathcal{N}}|_{\mathcal{U}_+\cap\mathcal{U}_-} \oplus T\mathcal{P}^{3|\mathcal{N}}|_{\mathcal{U}_+\cap\mathcal{U}_-}. \tag{V.18}$$

Kodaira-Spencer deformation theory [137] then tells us that the algebra $C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ acts on the transition functions t^I_{+-} according to

$$\delta t^I_{+-} \;=\; \chi^I_{+-} - \chi^J_{-+}\partial^-_J t^I_{+-} \tag{V.19}$$

which can equivalently be rewritten as

$$\delta t_{+-} \;:=\; \delta t^I_{+-}\partial^+_I \;=\; \chi_{+-} - \chi_{-+}. \tag{V.20}$$

Consider a subalgebra

$$C^1_\Delta(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \;:=\; \{\chi \in C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})\,|\,\chi_{+-} = \chi_{-+}\} \tag{V.21}$$

of the algebra $C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$. Then the space

$$Z^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \;=\; \{\chi \in C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})\,|\,\chi_{+-} = -\chi_{-+}\} \tag{V.22}$$

is given by the quotient

$$Z^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \;\cong\; C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})/C^1_\Delta(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}). \tag{V.23}$$

We stress that the transformations (V.19) change the complex structure of $\mathcal{P}^{3|\mathcal{N}}$ if $\chi_{+-} \neq \chi_+ - \chi_-$, where $\{\chi_+, \chi_-\} \in C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$. Therefore,

$$H^1(\mathcal{P}^{3|\mathcal{N}}, T\mathcal{P}^{3|\mathcal{N}}) \;\cong\; Z^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})/C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \tag{V.24}$$

is the tangent space (at a chosen complex structure) of the moduli space of deformations of the complex structure on $\mathcal{P}^{3|\mathcal{N}}$.

According to (V.20), we may define an action of the algebra $C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ on the transition functions $f = \{f_{+-}\}$ of holomorphic vector bundles over supertwistor space by

$$\delta f_{+-} \;:=\; \chi_{+-}(f_{+-}) - \chi_{-+}(f_{+-}). \tag{V.25}$$

However, as we have just seen, such transformations may change the complex structure on $\mathcal{P}^{3|\mathcal{N}}$. Therefore, we let the algebra $C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ act on $t = \{t^I_{+-}\}$ and $f = \{f_{+-}\}$ via

$$C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \ni \{\chi_+, \chi_-\} \;\mapsto\; \{\chi_+ - \chi_-, \chi_- - \chi_+\} \in C^1(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}}) \tag{V.26}$$

together with Eqs. (V.20) and (V.25), i.e.,

$$\begin{aligned} \delta t_{+-} &= \chi_+ - \chi_-, \\ \delta f_{+-} &= \chi_+(f_{+-}) - \chi_-(f_{+-}). \end{aligned} \tag{V.27}$$

These transformations do preserve the complex structure on supertwistor space.

After this digression, we may now follow the lines presented in §V.1 to arrive at the formulas (V.15) for symmetries. We shall not repeat the argumentation at this point. We refer to symmetries obtained in this way as Virasoro-type symmetries since they are associated with the group of local biholomorphisms.

§V.3 All symmetries. With the help of Čech cohomology and the Penrose-Ward transform, we have shown how Kac-Moody and Virasoro-type symmetries in $\mathcal{N}$-extended self-dual SYM theory arise. In the next paragraph, we will make the symmetry algebras more transparent by proving a theorem relating certain deformation algebras on the twistor side to symmetry algebras on the self-dual SYM side. Furthermore, in Sec. V.2 we then give explicit examples of symmetry transformations and corresponding symmetry algebras. Before dealing with these issues, however, let us summarize our preceding discussion. In fact, above we have given all possible continuous symmetry transformations acting on the solution space of the equations of motion of self-dual SYM theory. If we let $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ be the canonical cover of the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$, $\mathfrak{H} = GL(r, \mathcal{O}_{\mathcal{P}^{3|\mathcal{N}}})$ and $\mathcal{H}_{\mathcal{P}^{3|\mathcal{N}}}$ be the group of local biholomorphisms of $\mathcal{P}^{3|\mathcal{N}}$ associated with the algebra $C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$, we may state the following result:

Theorem V.1. *The full group of continuous (symmetry) transformations acting on the space $Z^1(\mathfrak{U}, \mathfrak{H})$ of smoothly and $\mathbb{R}^{4|2\mathcal{N}}$-trivial holomorphic vector bundles $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ of rank r is a semidirect product*

$$\mathcal{H}_{\mathcal{P}^{3|\mathcal{N}}} \ltimes C^1(\mathfrak{U}, \mathfrak{H}).$$

By virtue of the Penrose-Ward transform, one obtains a corresponding group action on the solution space of $\mathcal{N}$-extended self-dual SYM theory in four dimensions.

A proof of this theorem (for $\mathcal{N} = 0$) can be found in Ref. [202]. The argumentation for $\mathcal{N} > 0$ goes along similar lines.

§V.4 General symmetry algebras. So far, we have worked within a quite abstract scheme. Let us now present a more concrete relationship between certain deformation algebras and hidden symmetry algebras [267]. Suppose we are given some indexed set

$\{\delta_I\}$ of infinitesimal variations $\delta_I f$ of the transition function $f = \{f_{+-}\}$ of our holomorphic vector bundle $\mathcal{E}$ over supertwistor space. Suppose further that the δ_Is satisfy a deformation algebra of the form

$$[\delta_I, \delta_J\} \;=\; f_{IJ}{}^K \delta_K, \tag{V.28}$$

where generically $f_{IJ}{}^K \in C^1(\mathfrak{U}, \mathcal{O}_{\mathcal{P}^{3|\mathcal{N}}})$, with

$$f_{IJ}{}^K \;=\; -(-)^{p_I p_J} f_{JI}{}^K,$$

and $[\cdot, \cdot\}$ denotes the supercommutator.

Theorem V.2. *Given a deformation algebra of the form* (V.28) *with constant* $f_{IJ}{}^K \in \mathbb{C}$, *the corresponding symmetry algebra on the gauge theory side has exactly the same form modulo possible gauge transformations.*

Putting it differently, the linearized Penrose-Ward transform is, besides being an isomorphism between the tangent spaces of the moduli spaces $H^1_{\nabla^{0,1}}(\mathcal{P}^{3|\mathcal{N}}, \mathcal{E})$ and $\mathcal{M}^{\mathcal{N}}_{\mathrm{SDYM}}$ (see also Thm. II.2.), a Lie algebra homomorphism. Let us now prove the theorem.

Proof: Let $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ be a smoothly and $\mathbb{R}^{4|2\mathcal{N}}$-trivial holomorphic vector bundle of rank r. Recall that the transition function $f = \{f_{+-}\} \in H^1(\mathcal{P}^{3|\mathcal{N}}, \mathfrak{H})$ can be split according to $f_{+-} = \psi_+^{-1}\psi_-$, with $\psi = \{\psi_+, \psi_-\} \in C^0(\mathfrak{U}, \mathfrak{P})$. Consider

$$[\delta_1, \delta_2] \;=\; (-)^{p_I p_J} \varepsilon^I \varrho^J [\delta_I, \delta_J\},$$

where ε^I and ϱ^J are the infinitesimal parameters of the transformations δ_1 and δ_2, respectively. By virtue of (V.12), we may write

$$[\delta_1, \delta_2]\mathcal{A}_\pm \;=\; \delta_1(\mathcal{A}_\pm + \delta_2\mathcal{A}_\pm) - \delta_1\mathcal{A}_\pm - \delta_2(\mathcal{A}_\pm + \delta_1\mathcal{A}_\pm) + \delta_2\mathcal{A}_\pm,$$

where $\mathcal{A}_\pm$ stands symbolically for both components $\mathcal{A}^\pm_\alpha = \lambda^{\dot\alpha}_\pm \mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_\pm = \lambda^{\dot\alpha}_\pm \mathcal{A}_{\dot\alpha}$. Then one easily checks that

$$[\delta_1, \delta_2]\mathcal{A}_\pm \;=\; \bar\nabla_\pm \Sigma_{\pm 12}, \qquad \text{with} \qquad \Sigma_{\pm 12} \;:=\; \delta_1\phi^2_\pm - \delta_2\phi^1_\pm + [\phi^1_\pm, \phi^2_\pm],$$

where $\bar\nabla_\pm$ represents any of the covariant derivatives given in (V.13). Note that we use the notation $\phi^1_\pm = \varepsilon^I \phi_{\pm I}$ and similarly for $\phi^2_\pm$. Hence, $\Sigma_{\pm 12} = (-)^{p_I p_J} \varepsilon^I \varrho^J \Sigma_{\pm IJ}$. Next one considers the commutator

$$[\delta_1, \delta_2] f_{+-} \;=\; \delta_1(f_{+-} + \delta_2 f_{+-}) - \delta_1 f_{+-} - \delta_2(f_{+-} + \delta_1 f_{+-}) + \delta_2 f_{+-}.$$

Using definition (V.8) and the resulting splitting (V.9) for the deformations $\delta_{1,2}f_{+-}$, one obtains after some tedious but straightforward algebraic manipulations

$$[\delta_1, \delta_2]f_{+-} \;=\; \psi_+^{-1}(\Sigma_{+12} - \Sigma_{-12})\psi_-,$$

where $\Sigma_{\pm 12}$ has been introduced above. By assumption, it must also be equal to

$$[\delta_1, \delta_2]f_{+-} \;=\; \delta_3 f_{+-}, \qquad \text{with} \qquad \delta_3 \;=\; \varepsilon^I \varrho^J f_{IJ}{}^K \delta_K,$$

i.e.,

$$[\delta_1, \delta_2]f_{+-} \;=\; \psi_+^{-1}(\phi_+^3 - \phi_-^3)\psi_-,$$

where $\phi_\pm^3 = \varepsilon^I \varrho^J f_{IJ}{}^K \phi_{\pm K}$. By comparing this equation with the previously obtained result for $[\delta_1, \delta_2]f_{+-}$, we may conclude that

$$\Sigma_{\pm 12} \;=\; \phi_\pm^3 + \omega^3 \;=\; \varepsilon^I \varrho^J (f_{IJ}{}^K \phi_{\pm K} + \omega_{IJ})$$

since by assumption $f_{IJ}{}^K \in \mathbb{C}$. Here, ω^3 (respectively, ω_{IJ}) belongs to $H^0(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{P})$. Thus, by our above discussion, it represents an infinitesimal gauge transformation. ∎

Hence, any Lie-algebra type deformation algebra on the twistor side yields by virtue of the Penrose-Ward transform a symmetry algebra on the gauge theory side being of the same form. Furthermore, we also realize that if $f_{IJ}{}^K \in C^1(\mathfrak{U}, \mathcal{O}_{\mathcal{P}^{3|\mathcal{N}}})$, the algebra on the gauge theory side will generically no longer close because of the dependence of $f_{IJ}{}^K$ on $(x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha})$ through $(z_\pm^\alpha, z_\pm^3, \eta_i^\pm)$. In fact, in order to be able to compare $\Sigma_{\pm 12}$ with $\phi_\pm^3$ (which will no longer be given by $\varepsilon^I \varrho^J f_{IJ}{}^K \phi_{\pm K}$) from the above proof, one has to Laurent-expand $f_{IJ}{}^K$ in powers of $\lambda_\pm$. The resulting coefficient functions will solely depend on $(x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \in \mathbb{R}^{4|2\mathcal{N}}$. Therefore, the space-time derivatives appearing in the transformations (V.12) will destroy the "would be" algebra on the self-dual SYM side. As a matter of fact, if one allows $f_{IJ}{}^K$ to depend only on $z_\pm^3 = \lambda_\pm$, no such problems will occur. That is, the algebra, though modified, will still close. More details about these issues can be found in the next section when explicitly dealing with affine extensions of superconformal symmetries.

V.2 Explicit examples of hidden symmetry algebras

By now it is clear that self-dual SYM theory possesses infinite-dimensional hidden symmetry algebras. To exemplify our discussion, let us now describe some explicit symmetry algebras. We begin with affine extensions of internal symmetries and afterwards discuss affine extensions related to space-time symmetries. As before, we consider a smoothly and $\mathbb{R}^{4|2\mathcal{N}}$-trivial rank r holomorphic vector bundle $\mathcal{E}$ over supertwistor space, the latter being covered by $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$. The transition function of $\mathcal{E}$ is again given by $f = \{f_{+-}\} = \{\psi_+^{-1}\psi_-\}$, with $\psi = \{\psi_+, \psi_-\} \in C^0(\mathfrak{U}, \mathfrak{P})$.

§V.5 Kac-Moody symmetries. The prime example is Kac-Moody symmetries associated with internal symmetries. Let X_I be any generator of the gauge group $\mathfrak{su}(r)$ with

$$[X_I, X_J] \;=\; f_{IJ}{}^K X_K, \tag{V.29}$$

where the $f_{IJ}{}^K$s are the structure constants of $\mathfrak{su}(r)$. Then we may define the following infinitesimal deformation

$$\delta_I^m f_{+-} \;:=\; \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)[X_I, f_{+-}] \qquad \text{for} \qquad m \in \mathbb{N}_0. \tag{V.30}$$

It is clearly a holomorphic deformation and moreover, one may readily check that it preserves the reality condition (II.23). Notice that by comparing with Eq. (V.7), we see that

$$\delta_I^m h_{+-} \;=\; \delta_I^m h_{-+} \;=\; \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m) X_I$$

define one-cochains $\delta_I^m h \in C^1(\mathfrak{U}, \mathrm{Lie}\,\mathfrak{H})$. For $m = 0$, the transformations of the components of the gauge potential turn out to be (after properly fixing the freedom in solving the infinitesimal Riemann-Hilbert problem; cf. Eq. (V.37))

$$\delta_I^0 \mathcal{A}_{\alpha\dot\alpha} \;=\; [X_I, \mathcal{A}_{\alpha\dot\alpha}] \qquad \text{and} \qquad \delta_I^0 \mathcal{A}_{\dot\alpha}^i \;=\; [X_I, \mathcal{A}_{\dot\alpha}^i]. \tag{V.31}$$

Thus, they represent an internal symmetry transformation. Then a short calculation reveals that

$$[\delta_I^m, \delta_J^n] \;=\; \tfrac{1}{2} f_{IJ}{}^K (\delta_K^{m+n} + (-)^n \delta_K^{|m-n|}), \tag{V.32}$$

which is the analytic half of a centerless Kac-Moody algebra $\widehat{\mathfrak{su}(r)}$. One can bring (V.32) in a more familiar form as follows. Define $\Delta_I^0 := \delta_I^0$ and $\Delta_I^1 := \delta_I^1$. Furthermore, let

$$\Delta_I^m \;:=\; \sum_{k=0}^{m} c_k^m \delta_I^k, \tag{V.33}$$

with some (real) coefficients c_k^m to be determined. Next consider

$$[\Delta_I^m, \Delta_J^1] \;=:\; f_{IJ}{}^K \Delta_K^{m+1}, \tag{V.34}$$

that is, the $(m+1)$-th generator is recursively defined by m-th one. Then

$$[\Delta_I^m, \Delta_J^n] \;=\; f_{IJ}{}^K \Delta_K^{m+n}, \tag{V.35}$$

for all $m, n \geq 0$. In fact, this is a direct consequence of Eq. (V.32) and can be proven by induction with the help of Jacobi identity for the generators Δ_I^m. As this is rather straightforward, we leave the explicit verification to the interested reader.

Next we need to find – by means of the discussion of the previous section – the action of δ_I^m for $m > 0$ on the components $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}_{\dot\alpha}^i$ of the gauge potential. Consider the function ϕ_{+-I}^m as defined in (V.8). We obtain

$$\begin{aligned}
\phi_{+-I}^m &= \phi_{+I}^m - \phi_{-I}^m \\
&= \psi_+ \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)[X_I, f_{+-}]\psi_-^{-1} \;=\; \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)\psi_+[X_I, \psi_+^{-1}\psi_-]\psi_-^{-1} \\
&= -\tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)[X_I, \psi_+]\psi_+^{-1} + \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)[X_I, \psi_-]\psi_-^{-1} \\
&= \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)\phi_{+I}^0 - \tfrac{1}{2}(\lambda_+^m + (-\lambda_-)^m)\phi_{-I}^0,
\end{aligned} \tag{V.36}$$

where in the last step we have introduced the functions $\phi_{\pm I}^0$ which are solutions of the Riemann-Hilbert problem for $m = 0$,

$$\phi_{\pm I}^0 \;=\; -[X_I, \psi_\pm]\psi_\pm^{-1}. \tag{V.37}$$

As the $\phi_{\pm I}^0$s are holomorphic and nonsingular in $\lambda_\pm$ on their respective domains, we expand them in powers of λ_+ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ according to

$$\phi_{\pm I}^0 \;=\; \sum_{n=0}^{\infty} \lambda_+^{\pm n} \phi_{\pm I}^{0(n)}. \tag{V.38}$$

Furthermore, Eqs. (V.15) imply

$$\begin{aligned}
\delta_I^m \mathcal{A}_{\alpha\dot1} &= \nabla_{\alpha\dot1}\phi_{+I}^{m(0)} - \nabla_{\alpha\dot2}\phi_{+I}^{m(1)} \;=\; \nabla_{\alpha\dot1}\phi_{-I}^{m(0)}, \\
\delta_I^m \mathcal{A}_{\alpha\dot2} &= \nabla_{\alpha\dot2}\phi_{+I}^{m(0)} \;=\; -\nabla_{\alpha\dot1}\phi_{-I}^{m(1)} + \nabla_{\alpha\dot2}\phi_{-I}^{m(0)},
\end{aligned} \tag{V.39}$$

and similarly for $\mathcal{A}_{\dot\alpha}^i$.

Upon substituting the expansions (V.38) into (V.36), we find the following splitting of ϕ_{+-I}^m into $\phi_{\pm I}^m$:

$$\phi_{\pm I}^m \;=\; \tfrac{1}{2}(\pm)^m \lambda_+^{\pm m}\phi_{\pm I}^0 + \tfrac{1}{2}(\mp)^m \sum_{n=0}^{\infty} \lambda_+^{\pm n}\phi_{\pm I}^{0(m+n)} - \tfrac{1}{2}(\pm)^m \sum_{n=0}^{m-1} \lambda_+^{\pm(m-n)}\phi_{\mp I}^{0(n)}. \tag{V.40}$$

Recall that solutions to the Riemann-Hilbert problem are not unique. For instance, we could have added to ϕ^m_{+I} any smooth function which does not depend on $\lambda_\pm$. But at the same time we had to add the same function to ϕ^m_{-I}, as well. However, we have learned that such shifts of "zero-modes" will eventually result in gauge transformations. As we are not interested in such trivial symmetries, solution (V.40) turns out to be the appropriate choice. Note that for $m = 0$, Eq. (V.40) is an identity. Expanding the functions $\phi^m_{\pm a}$ in powers of λ_+, we obtain for $m > 0$

$$\phi^{m(n)}_{\pm I} \;=\; \begin{cases} \frac{1}{2}(\mp)^m \phi^{0(m)}_{\pm I} & \text{for} \quad n = 0 \\ \frac{1}{2}\left\{(\pm)^m \phi^{0(n-m)}_{\pm I} + (\mp)^m \phi^{0(m+n)}_{\pm I} - (\pm)^m \phi^{0(m-n)}_{\mp I}\right\} & \text{for} \quad n > 0 \end{cases}. \tag{V.41}$$

Note that $\phi^{0(-n)}_{\pm I} = 0$ for all $n > 0$. Combining these expressions with the transformation laws (V.39), we have thus given the action of δ^m_I on $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$, respectively. Furthermore, by Thm. V.2. we know that the δ^m_Is satisfy

$$[\delta^m_I, \delta^n_J] \;=\; \tfrac{1}{2} f_{IJ}{}^K(\delta^{m+n}_K + (-)^n \delta^{|m-n|}_K) \tag{V.42}$$

modulo gauge transformations. For an explicit derivation, see also Ref. [266]. Following the discussion subsequent to (V.32), we eventually arrive at

$$[\Delta^m_I, \Delta^n_J] \;=\; f_{IJ}{}^K \Delta^{m+n}_K, \tag{V.43}$$

for all $m, n \geq 0$ modulo gauge transformations. Altogether, we have thus obtained an infinite-dimensional affine symmetry algebra in $\mathcal{N}$-extended self-dual SYM theory, which is the analytic half of the affine Lie algebra $\widehat{\mathfrak{su}(r)}$.

§V.6 Virasoro-type symmetries. Let us now discuss affine extensions of superconformal symmetries. However, we first need some preliminaries. The superconformal group for $\mathcal{N} \neq 4$ is locally isomorphic to a real form of the super matrix group $SU(4|\mathcal{N})$. In the case of maximal $\mathcal{N} = 4$ supersymmetries, the supergroup $SU(4|4)$ is not semi-simple and the superconformal group is considered to be a real form of the semi-simple part $PSU(4|4) \subset SU(4|4)$. The generators of the superconformal group are the translation generators $P_{\alpha\dot\alpha}$, $Q_{i\alpha}$ and $Q^i_{\dot\alpha}$, the dilatation generator D, the generators of special conformal transformations $K_{\alpha\dot\alpha}$, $K^{i\alpha}$ and $K^{\dot\alpha}_i$, the rotation generators $J_{\alpha\beta}$ and $J_{\dot\alpha\dot\beta}$, the generators R^j_i of the R-symmetry and the generator of the axial symmetry A. The latter one is absent in the case of maximal $\mathcal{N}=4$ supersymmetries. The superconformal algebra then takes the following form:

$$
\begin{aligned}
[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= \tfrac{1}{2}(\epsilon_{\alpha\beta} J_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} J_{\alpha\beta}) - \tfrac{1}{4}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D, \\
\{Q_{i\alpha}, Q^j_{\dot\alpha}\} &= -2\delta^j_i P_{\alpha\dot\alpha}, \qquad \{K^{i\alpha}, K^{\dot\alpha}_j\} = -2\delta^i_j K^{\alpha\dot\alpha}, \\
\{Q_{i\alpha}, K^{j\beta}\} &= -2\delta^j_i (J_\alpha{}^\beta + \tfrac{1}{4}\delta^\beta_\alpha D) - \tfrac{1}{2}\delta^\beta_\alpha \delta^j_i (1 - \tfrac{4}{\mathcal{N}}) A + \delta^\beta_\alpha R^j_i, \\
[R^i_j, K^{k\alpha}] &= -(\delta^k_j K^{i\alpha} - \tfrac{1}{\mathcal{N}}\delta^i_j K^{k\alpha}), \\
[A, K^{i\alpha}] &= -\tfrac{1}{2}K^{i\alpha}, \qquad [D, K^{i\alpha}] = -\tfrac{1}{2}K^{i\alpha}, \\
[J_{\alpha\beta}, K^{i\gamma}] &= \tfrac{1}{2}\epsilon_{\delta(\alpha}\delta^\gamma_{\beta)} K^{i\delta}, \qquad [P_{\alpha\dot\alpha}, K^{i\beta}] = -\tfrac{1}{2}\delta^\beta_\alpha Q^i_{\dot\alpha} \\
[R^j_i, Q_{k\alpha}] &= \delta^j_k Q_{i\alpha} - \tfrac{1}{\mathcal{N}}\delta^j_i Q_{k\alpha}, \\
[A, Q_{i\alpha}] &= \tfrac{1}{2}Q_{i\alpha}, \qquad [D, Q_{i\alpha}] = \tfrac{1}{2}Q_{i\alpha}, \\
[J_{\alpha\beta}, Q_{i\gamma}] &= -\tfrac{1}{2}\epsilon_{\gamma(\alpha} Q_{i\beta)}, \qquad [Q_{i\beta}, K^{\alpha\dot\alpha}] = -\tfrac{1}{2}\delta^\alpha_\beta K^{\dot\alpha}_i, \\
[R^j_i, R^l_k] &= \delta^j_k R^l_i - \delta^l_i R^j_k, \qquad [J_{\alpha\beta}, J^{\gamma\delta}] = -\delta^{(\gamma}_{(\alpha} J_{\beta)}{}^{\delta)}, \\
[D, P_{\alpha\dot\alpha}] &= P_{\alpha\dot\alpha}, \qquad [D, K^{\alpha\dot\alpha}] = -K^{\alpha\dot\alpha}, \\
[J_{\alpha\beta}, K^{\gamma\dot\gamma}] &= \tfrac{1}{2}\epsilon_{\delta(\alpha}\delta^\gamma_{\beta)} K^{\delta\dot\gamma}, \qquad [J_{\alpha\beta}, P_{\gamma\dot\gamma}] = -\tfrac{1}{2}\epsilon_{\gamma(\alpha} P_{\beta)\dot\gamma},
\end{aligned}
\tag{V.44}
$$

where, as before, parentheses mean normalized symmetrization of the enclosed indices. As is well known, there is a representation of this algebra in terms of vector fields on the anti-chiral superspace $\mathbb{R}^{4|2\mathcal{N}}$ according to

$$
\begin{aligned}
P_{\alpha\dot\alpha} &= \partial^R_{\alpha\dot\alpha}, \qquad Q_{i\alpha} = -2\eta^{\dot\alpha}_i \partial^R_{\alpha\dot\alpha}, \qquad Q^i_{\dot\alpha} = \partial^i_{\dot\alpha}, \\
D &= -x^{\alpha\dot\alpha}\partial^R_{\alpha\dot\alpha} - \tfrac{1}{2}\eta^{\dot\alpha}_i \partial^i_{\dot\alpha}, \\
K^{\alpha\dot\alpha} &= \tfrac{1}{4}x^{\alpha\dot\beta}_R (x^{\beta\dot\alpha}_R \partial^R_{\beta\dot\beta} + \eta^{\dot\alpha}_i \partial^i_{\dot\beta}), \\
K^{i\alpha} &= -\tfrac{1}{2}x^{\alpha\dot\alpha}_R \partial^i_{\dot\alpha}, \qquad K^{\dot\alpha}_i = \eta^{\dot\beta}_i (x^{\alpha\dot\alpha}_R \partial^R_{\alpha\dot\beta} + \eta^{\dot\alpha}_j \partial^j_{\dot\beta}), \\
J_{\alpha\beta} &= \tfrac{1}{2}x^{\gamma\dot\alpha}_R \epsilon_{\gamma(\alpha}\partial^R_{\beta)\dot\alpha}, \qquad J_{\dot\alpha\dot\beta} = \tfrac{1}{2}\left(x^{\alpha\dot\gamma}_R \epsilon_{\dot\gamma(\dot\alpha}\partial^R_{\alpha\dot\beta)} + \eta^{\dot\gamma}_i \epsilon_{\dot\gamma(\dot\alpha}\partial^i_{\dot\beta)}\right), \\
R^j_i &= \eta^{\dot\alpha}_i \partial^j_{\dot\alpha} - \tfrac{1}{\mathcal{N}}\,\delta^j_i \eta^{\dot\alpha}_k \partial^k_{\dot\alpha}, \qquad A = \tfrac{1}{2}\eta^{\dot\alpha}_i \partial^i_{\dot\alpha}.
\end{aligned}
\tag{V.45}
$$

Infinitesimal transformations of the components $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ of the gauge potential under the action of the superconformal group are given by

$$
\delta_{N_I}\mathcal{A}_{\alpha\dot\alpha} = \mathcal{L}_{N_I}\mathcal{A}_{\alpha\dot\alpha} \qquad \text{and} \qquad \delta_{N_I}\mathcal{A}^i_{\dot\alpha} = \mathcal{L}_{N_I}\mathcal{A}^i_{\dot\alpha}, \tag{V.46}
$$

where $N_I = N^{\alpha\dot\alpha}_I \partial^R_{\alpha\dot\alpha} + N^{\dot\alpha}_{I\,i}\partial^i_{\dot\alpha}$ is any generator of the superconformal group, and $\mathcal{L}_{N_I}$ is the Lie derivative[4] along the vector field N_I. Explicitly, Eqs. (V.46) read as

$$\begin{aligned}\mathcal{L}_{N_I}\mathcal{A}_{\alpha\dot\alpha} &= N_I\mathcal{A}_{\alpha\dot\alpha} + \mathcal{A}_{\beta\dot\beta}\partial^R_{\alpha\dot\alpha}N_I^{\beta\dot\beta} + (-)^{p_I+1}\mathcal{A}^i_{\dot\beta}\partial^R_{\alpha\dot\alpha}N^{\dot\beta}_{I\,i},\\ \mathcal{L}_{N_I}\mathcal{A}^i_{\dot\alpha} &= N_I\mathcal{A}^i_{\dot\alpha} + (-)^{p_I}\mathcal{A}_{\beta\dot\beta}\partial^i_{\dot\alpha}N_I^{\beta\dot\beta} + \mathcal{A}^j_{\dot\beta}\partial^i_{\dot\alpha}N^{\dot\beta}_{I\,j}.\end{aligned} \tag{V.47}$$

It is not too difficult to show that for any generator N_I as given in (V.45), the transformations (V.46) together with (V.47) give a symmetry of the $\mathcal{N}$-extended self-dual SYM equations (II.17).

So far, we have given the action of the superconformal group on the components of the gauge potential on Euclidean superspace. The linear system (II.49) is, however, defined on the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$. Therefore, the question is how to lift the action of the superconformal group to supertwistor space but at the same time preserving the linear system (II.49). The answer is at hand. By our discussion given in Sec. V.1, we have to preserve the complex structure on $\mathcal{P}^{3|\mathcal{N}}$. Recall the diffeomorphism $\mathcal{P}^{3|\mathcal{N}} \cong \mathbb{R}^{4|2\mathcal{N}} \times S^2$. Complex structures on the body $\mathbb{R}^{4|0}$ of $\mathbb{R}^{4|2\mathcal{N}}$ are parametrized by a two-sphere $S^2 \cong SO(4)/U(2)$. The latter can be viewed as the complex projective line $\mathbb{C}P^1$ parametrized by the coordinates $\lambda_\pm$. Then a complex structure $\mathcal{J} = (\mathcal{J}_{\alpha\dot\alpha}{}^{\beta\dot\beta})$ on $\mathbb{R}^{4|0}$, compatible with (I.75) and (I.79), is given by

$$\mathcal{J}_{\alpha\dot\alpha}{}^{\beta\dot\beta} = -\mathrm{i}\gamma_\pm\delta^\beta_\alpha(\lambda^\pm_{\dot\alpha}\hat\lambda^{\dot\beta}_\pm + \lambda^{\dot\beta}_\pm\hat\lambda^\pm_{\dot\alpha}), \tag{V.48}$$

where $\gamma_\pm = (1+\lambda_\pm\bar\lambda_\pm)^{-1}$. Notice that in the present case the corresponding Kähler two-form ω is anti-self-dual, i.e., its components are of the form $\omega_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\mathcal{J}_{\dot\alpha\dot\beta}$. If we had chosen the $\mathcal{N}$-extended anti-self-dual SYM equations from the very beginning, the Kähler form would have been self-dual. On the two-sphere S^2 which parametrizes the different complex structures of $\mathbb{R}^{4|0}$, we introduce the standard complex structure $\mathfrak{J}$ which, for instance, on the U_+ patch is given by $\mathfrak{J}_{\lambda_+}{}^{\lambda_+} = \mathrm{i} = -\mathfrak{J}_{\bar\lambda_+}{}^{\bar\lambda_+}$. Thus, the complex structure on the body $\mathbb{R}^{4|0} \times S^2$ of supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ can be taken as $J = (\mathcal{J}, \mathfrak{J})$.

[4] Note that the Lie derivative is defined as in the purely even setting. In particular, $\mathcal{L}_X f := Xf$ and $\mathcal{L}_X Y := [X, Y\}$, where f is some local function and X, Y are local vector fields. Furthermore, one requires that $\mathcal{L}_X$ commutes with the contraction operator. For any differential one-form $\omega = \mathrm{d}Z^I\omega_I$, one may readily verify the following result:

$$Y \lrcorner \mathcal{L}_X\omega = (-)^{p_X p_Y} X(Y \lrcorner \omega) + [Y, X\} \lrcorner \omega.$$

Letting Y be $\partial_I = \partial/\partial Z^I$, one sets $(\mathcal{L}_X\omega)_I := \mathcal{L}_X\omega_I := (-)^{p_X p_I}\partial_I \lrcorner \mathcal{L}_X\omega$. The extra sign has been chosen for convenience.

Having introduced a complex structure on the even part $\mathbb{R}^{4|0}\times S^2$ of $\mathbb{R}^{4|2\mathcal{N}}\times S^2$, we need to extend the above discussion to the full supertwistor space. In order to define a complex structure on $\mathbb{R}^{4|2\mathcal{N}}\times S^2$, recall that only an even amount of supersymmetries is possible, i.e., $\mathcal{N}=0,2$ or 4. Our particular choice of the symplectic Majorana condition induced by (I.78) allows us to introduce a complex structure on the odd part $\mathbb{R}^{0|2\mathcal{N}}$ similar to (V.48), that is,

$$\mathbf{J}^{\dot\alpha}_{i}{}^{j}_{\dot\beta} \;=\; -\mathrm{i}\gamma_\pm\delta^j_i(\lambda^\pm_{\dot\beta}\hat\lambda^{\dot\alpha}_\pm+\lambda^{\dot\alpha}_\pm\hat\lambda^\pm_{\dot\beta}). \tag{V.49}$$

Therefore, $J=(\mathcal{J},\mathbf{J},\mathfrak{J})$ will be the proper choice of a complex structure[5] on supertwistor space.

We can now answer the initial question: the generators N_I given by (V.45) of the superconformal group should be lifted to vector fields $\widetilde{N}_I=\widetilde{N}^{\alpha\dot\alpha}_I\partial_{\alpha\dot\alpha}+\widetilde{N}^{\dot\alpha}_{I\,i}\partial^i_{\dot\alpha}+\widetilde{N}^{\lambda_\pm}_I\partial_{\lambda_\pm}+\widetilde{N}^{\bar\lambda_\pm}_I\partial_{\bar\lambda_\pm}$ on supertwistor space such that the Lie derivative of the complex structure J along the lifted vector fields $\widetilde{N}_I$ vanishes, i.e.,

$$\mathcal{L}_{\widetilde{N}_I}J \;=\; 0. \tag{V.50}$$

Letting $\mathcal{J}_{\dot\alpha}{}^{\dot\beta}=\frac{\gamma_\pm}{2\mathrm{i}}(\lambda^\pm_{\dot\alpha}\hat\lambda^{\dot\beta}_\pm+\lambda^{\dot\beta}_\pm\hat\lambda^\pm_{\dot\alpha})$, we can write (V.50) explicitly as

$$\begin{aligned}
2\,\widetilde{N}_I\mathcal{J}_{\dot\alpha}{}^{\dot\beta}+\mathcal{J}_{\dot\gamma}{}^{\dot\alpha}\partial_{\alpha\dot\beta}\widetilde{N}^{\alpha\dot\gamma}_I-\mathcal{J}_{\dot\beta}{}^{\dot\gamma}\partial_{\alpha\dot\gamma}\widetilde{N}^{\alpha\dot\alpha}_I \;&=\; 0,\\
\mathcal{J}_{\dot\delta}{}^{\dot\gamma}\partial_{\alpha\dot\alpha}\widetilde{N}^{\dot\delta}_{I\,i}-\mathcal{J}_{\dot\alpha}{}^{\dot\beta}\partial_{\alpha\dot\beta}\widetilde{N}^{\dot\gamma}_{I\,i} \;&=\; 0,\\
\mathcal{J}_{\dot\alpha}{}^{\dot\beta}\partial^i_{\dot\gamma}\widetilde{N}^{\alpha\dot\alpha}_I-\mathcal{J}_{\dot\gamma}{}^{\dot\delta}\partial^i_{\dot\delta}\widetilde{N}^{\alpha\dot\beta}_I \;&=\; 0,\\
\delta^i_j\,\widetilde{N}_I\mathcal{J}_{\dot\alpha}{}^{\dot\beta}-(-)^{p_I}\mathcal{J}_{\dot\alpha}{}^{\dot\gamma}\partial^j_{\dot\gamma}\widetilde{N}^{\dot\beta}_{I\,i}+(-)^{p_I}\mathcal{J}_{\dot\gamma}{}^{\dot\beta}\partial^j_{\dot\beta}\widetilde{N}^{\dot\gamma}_{I\,i} \;&=\; 0,
\end{aligned} \tag{V.51}$$

whereas the equations involving $\mathfrak{J}$ imply that the components $\widetilde{N}^{\lambda_\pm}_I$ and $\widetilde{N}^{\bar\lambda_\pm}_I$ are holomorphic in $\lambda_\pm$ and $\bar\lambda_\pm$, respectively. The final expressions for the generators (V.45) lifted to supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ and obeying (V.51) are

$$\begin{aligned}
\widetilde{P}_{\alpha\dot\alpha} \;&=\; P_{\alpha\dot\alpha}, \qquad \widetilde{Q}_{i\alpha} \;=\; Q_{i\alpha}, \qquad \widetilde{Q}^i_{\dot\alpha} \;=\; Q^i_{\dot\alpha},\\
\widetilde{D} \;&=\; D,\\
\widetilde{K}^{\alpha\dot\alpha} \;&=\; K^{\alpha\dot\alpha}+\tfrac14 x^{\alpha\dot\beta}Z_{\dot\beta}{}^{\dot\alpha}, \qquad \widetilde{K}^{i\alpha} \;=\; K^{i\alpha}, \qquad \widetilde{K}^{\dot\alpha}_i \;=\; K^{\dot\alpha}_i+\eta^{\dot\beta}_i Z_{\dot\beta}{}^{\dot\alpha},\\
\widetilde{J}_{\alpha\beta} \;&=\; J_{\alpha\beta}, \qquad \widetilde{J}_{\dot\alpha\dot\beta} \;=\; J_{\dot\alpha\dot\beta}-\tfrac12 Z_{\dot\alpha\dot\beta},\\
\widetilde{R}^j_i \;&=\; R^j_i, \qquad \widetilde{A} \;=\; A,
\end{aligned} \tag{V.52}$$

[5] Clearly, this choice of the complex structure does not exhaust the space of all admissible complex structures. However, in the present case it is enough to restrict ourselves to this class of complex structures. For more details, see [266].

where

$$Z_{\dot{\alpha}\dot{\beta}} \;:=\; \pm\lambda^{\pm}_{\dot{\alpha}}\lambda^{\pm}_{\dot{\beta}}\partial_{\lambda_\pm} \pm \hat{\lambda}^{\pm}_{\dot{\alpha}}\hat{\lambda}^{\pm}_{\dot{\beta}}\partial_{\bar{\lambda}_\pm}. \tag{V.53}$$

Now we can give the infinitesimal transformation of $\psi = \{\psi_+, \psi_-\} \in C^0(\mathfrak{U}, \mathfrak{P})$ which participates in (II.49)

$$\delta_{\widetilde{N}_I}\psi_\pm \;=\; \mathcal{L}_{\widetilde{N}_I}\psi_\pm \;=\; \widetilde{N}_I\psi_\pm, \tag{V.54}$$

where $\widetilde{N}_I$ is any of the generators given in (V.52). It is a straightforward exercise to verify explicitly that the linear system (II.49) is invariant under the transformations (V.46) and (V.54).

Having collected all necessary ingredients, we can now start with constructing affine symmetry algebras related to superconformal symmetries. In order to keep formulas simple, we shall in the remainder of this section work in the complexified gauge algebra, that is, we drop the last point from the list given at the beginning of Sec. V.1. It should be stressed, however, that the subsequent derivation can be modified for a real gauge algebra without any problems. Let $\widetilde{N}_I$ be any vector field of (V.52), with

$$[\widetilde{N}_I, \widetilde{N}_J\} \;=\; f_{IJ}{}^K \widetilde{N}_K, \tag{V.55}$$

where the $f_{IJ}{}^K$s are the structure constants of the superconformal group. Then we define the following perturbation of $f = \{f_{+-}\}$:

$$\delta^m_I f_{+-} \;:=\; \lambda^m_+ \widetilde{N}_I f_{+-} \qquad \text{for} \qquad m \in \mathbb{N}_0. \tag{V.56}$$

Note that the antiholomorphic λ-derivative appearing in $\widetilde{N}_I$ drops out as f_{+-} is holomorphic. Therefore, Eq. (V.56) defines a zero-cochain $\chi = \{\chi_+, \chi_-\} \in C^0(\mathfrak{U}, T\mathcal{P}^{3|\mathcal{N}})$ given by[6]

$$\chi_+ \;=\; \lambda^m_+ \widetilde{N}_I|_{T\mathcal{P}^{3|\mathcal{N}}} \qquad \text{and} \qquad \chi_- \;=\; 0.$$

Furthermore, one may readily check that

$$[\delta^m_I, \delta^n_J\} \;=\; \left(f_{IJ}{}^K + n\, g_I\, \delta^K_J - (-)^{p_I p_J} m\, g_J\, \delta^K_I\right)\delta^{m+n}_K \tag{V.57}$$

upon action on f_{+-}. Here, we have introduced the shorthand notation

$$g_I \;:=\; \lambda^{-1}_+ \widetilde{N}^{\lambda_+}_I. \tag{V.58}$$

[6]Note that for a real gauge algebra, χ_- would not be zero but instead be given by $\chi_- = \overline{\tau_E(\chi_+)}$, where τ_E represents the antiholomorphic involution (I.75) corresponding to Euclidean signature.

Generally speaking, (V.57) can be seen as a centerless Kac-Moody-Virasoro-type algebra.

Next we need to find the action of δ_I^m on the components $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$. The derivation (V.36) gets modified according to

$$\phi^m_{+-I} \;=\; \phi^m_{+I} - \phi^m_{-I} \;=\; \lambda_+^m\phi^0_{+I} - \lambda_+^m\phi^0_{-I}, \tag{V.59}$$

where this time

$$\phi^0_{\pm I} \;=\; -(\widetilde{N}_I\psi_\pm)\psi_\pm^{-1} + (\widetilde{N}_I\psi_+)\psi_+^{-1}|_{\lambda_+=0}. \tag{V.60}$$

Clearly, the second term of this expression is λ-independent, i.e., it belongs to $H^0(\mathcal{P}^{3|\mathcal{N}}, \mathrm{Lie}\,\mathfrak{P})$. In fact, we have simply shifted all the "zero-modes" into ϕ^0_{-I}. Recall that such shifts are always possible and eventually result in gauge transformations. Upon substituting (V.60) into (V.15) and using Eq. (V.38), we find[7]

$$\begin{aligned}\delta_I^0\mathcal{A}_{\alpha\dot1} \;&=\; \nabla_{\alpha\dot1}\phi^{0(0)}_{-I} \;=\; -\nabla_{\alpha\dot2}\phi^{0(1)}_{+I},\\ \delta_I^0\mathcal{A}_{\alpha\dot2} \;&=\; -\nabla_{\alpha\dot1}\phi^{0(1)}_{-I} + \nabla_{\alpha\dot2}\phi^{0(0)}_{-I} \;=\; 0,\end{aligned} \tag{V.61}$$

since $\phi^{0(0)}_{+} = 0$. Similar expressions hold for $\mathcal{A}^i_{\dot\alpha}$. Eq. (V.59) yields, after expanding $\phi^m_{\pm I}$ in powers of $\lambda_\pm$ and comparing with $\phi^0_{\pm I}$, the following coefficient functions:

$$\phi^{m(n)}_{+I} \;=\; \begin{cases}0 & \text{for}\quad n=0\\ \phi^{0(n-m)}_{+I} - \phi^{0(m-n)}_{-I} & \text{for}\quad n>0\end{cases} \qquad \text{and} \qquad \phi^{m(n)}_{-I} \;=\; \phi^{0(m+n)}_{-I}. \tag{V.62}$$

Note that $\phi^{0(-n)}_{\pm I} = 0$ for $n > 0$. These coefficients together with the integral formulas (V.15) give the action of δ_I^m on $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ according to

$$\begin{aligned}\delta_I^m\mathcal{A}_{\alpha\dot1} \;&=\; \nabla_{\alpha\dot1}\phi^{m(0)}_{-I} \;=\; -\nabla_{\alpha\dot2}\phi^{m(1)}_{+I},\\ \delta_I^m\mathcal{A}_{\alpha\dot2} \;&=\; -\nabla_{\alpha\dot1}\phi^{m(1)}_{-I} + \nabla_{\alpha\dot2}\phi^{m(0)}_{-I} \;=\; 0,\end{aligned} \tag{V.63}$$

and similarly for $\mathcal{A}^i_{\dot\alpha}$.

Next we are interested in the underlying algebraic structure. Thm. V.2. is not directly applicable, as we now have to deal with structure functions. Recall that the g_Is appearing in (V.57) are holomorphic functions on $\mathcal{U}_+\cap\mathcal{U}_-$. However, there are only minor modifications to be made. In proving Thm. V.2., we have introduced the functions $\Sigma^{mn}_{\pm IJ}$. Their difference is now given by

$$\Sigma^{mn}_{+IJ} - \Sigma^{mn}_{-IJ} \;=\; C_{IJ}{}^K(\phi^{m+n}_{+K} - \phi^{m+n}_{-K}), \tag{V.64}$$

[7] Note that δ_I^0 is a composition of a superconformal and a particular gauge transformation.

where

$$C_{IJ}{}^K \;:=\; f_{IJ}{}^K + n\, g_I\, \delta_J^K - (-)^{p_I p_J} m\, g_J\, \delta_I^K. \qquad \text{(V.65)}$$

In order to determine $\Sigma^{mn}_{\pm IJ}$, one expands both sides of Eq. (V.64) in powers of λ_+ and compares coefficients. We are only interested in $\Sigma^{mn(0)}_{\pm IJ}$ and $\Sigma^{mn(1)}_{\pm IJ}$, as they determine the action of $[\delta_I^m, \delta_J^n\}$ onto the components of the gauge potential. We find

$$\begin{aligned} \Sigma^{mn(0)}_{\pm IJ} &= C_{IJ}{}^{K(0)} \phi^{m+n(0)}_{\pm K} + C_{IJ}{}^{K(\mp 1)} \phi^{m+n(1)}_{\pm K} - C_{IJ}{}^{K(-1)} \phi^{m+n(1)}_{+K}, \\ \Sigma^{mn(1)}_{\pm IJ} &= C_{IJ}{}^{K(\pm 1)} (\phi^{m+n(0)}_{\pm K} - \phi^{m+n(0)}_{\mp K}) + C_{IJ}{}^{K(0)} \phi^{m+n(1)}_{\pm K} + C_{IJ}{}^{K(1)} \phi^{m+n(2)}_{\mp K}, \end{aligned} \qquad \text{(V.66)}$$

where $C_{IJ}{}^K = \sum_{k=-1}^{1} \lambda_+^k C_{IJ}{}^{K(k)}$ has been inserted. Note that last term in the first line represents again a gauge transformation. It has been adjusted such that $\Sigma^{mn(0)}_{+IJ} = 0$. By virtue of (V.15) and (V.62), we finally arrive at

$$\begin{aligned} [\delta_I^m, \delta_J^n\}\mathcal{A}_{\alpha \dot 1} &= \nabla_{\alpha \dot 1} \sum_{k=-1}^{1} C_{IJ}{}^{K(k)} \phi^{0(m+n+k)}_{-K}, \\ [\delta_I^m, \delta_J^n\}\mathcal{A}_{\alpha \dot 2} &= 0 \end{aligned} \qquad \text{(V.67)}$$

and similarly for $\mathcal{A}^i_{\dot\alpha}$. In deriving this result, the explicit dependence of $C_{IJ}{}^K$ on m and n has been used. Obviously, the algebra closes if and only if the coefficient functions are independent of $(x_R^{\alpha\dot\alpha}, \eta_i^{\dot\alpha})$ – a fact which we have already encountered at the end of Sec. V.1. Therefore, we are left with only a subalgebra of the superconformal algebra. Indeed, we need to exclude the generators $\widetilde{K}^{\alpha\dot\alpha}$ and $\widetilde{K}_i^{\dot\alpha}$ of special conformal transformations. Let us consider the maximal subalgebra of the superconformal algebra which contains neither $\widetilde{K}^{\alpha\dot\alpha}$ nor $\widetilde{K}_i^{\dot\alpha}$. Furthermore, let $h_{IJ}{}^K$ denote the corresponding structure constants. Then we end up with

$$[\delta_I^m, \delta_J^n\} \;=\; h_{IJ}{}^K \delta_K^{m+n} + \sum_{k=-1}^{1} \left(n g_I^{(k)} \delta_J^K - (-)^{p_I p_J} m g_J^{(k)} \delta_I^K \right) \delta_K^{m+n+k}, \qquad \text{(V.68)}$$

where $g_I = \sum_{k=-1}^{1} \lambda_+^k g_I^{(k)}$ is the expansion of g_I defined in (V.58).

V.3 Hierarchies

In Refs. [172, 237, 173, 4, 174, 129] it was shown that a solution to the self-dual YM equations can be embedded into an infinite-parameter family of new solutions by moving it along commuting flows of a so-called self-dual YM hierarchy. The lowest generators of the latter are the generators of space-time translations. Putting it differently, the self-dual YM hierarchy describes infinitely many Abelian symmetries of the self-dual YM equations associated with translational symmetries. Our next topic is the generalization of these ideas to the self-dual SYM equations. For the sake of clarity, we shall again work in a complex setting, that is, we consider the double fibration (II.9). If desired, reality conditions can be imposed.

§V.7 Generalized supertwistor space. Consider the Abelian subalgebra of the superconformal algebra which is spanned by the translation generators $\widetilde{P}_{\alpha\dot{\alpha}}$ and $\widetilde{Q}^i_{\dot{\alpha}}$. On supertwistor space, we may use the coordinates $z^\alpha_\pm = x^{\alpha\dot{\alpha}}_R\lambda^\pm_{\dot{\alpha}}$, $\pi^\pm_{\dot{\alpha}} = \lambda^\pm_{\dot{\alpha}}$ and $\eta^\pm_i = \eta^{\dot{\alpha}}_i\lambda^\pm_{\dot{\alpha}}$. Expressing the generators $\widetilde{P}_{\alpha\dot{\alpha}}$ and $\widetilde{Q}^i_{\dot{\alpha}}$ in terms of these coordinates, we obtain

$$\widetilde{P}_{\alpha\dot{\alpha}} \;=\; \pi^\pm_{\dot{\alpha}}\frac{\partial}{\partial z^\alpha_\pm} \qquad \text{and} \qquad \widetilde{Q}^i_{\dot{\alpha}} \;=\; \pi^\pm_{\dot{\alpha}}\frac{\partial}{\partial \eta^\pm_i}, \tag{V.69}$$

when acting on holomorphic functions of $(z^\alpha_\pm, \pi^\pm_{\dot{\alpha}}, \eta^\pm_i)$ on $\mathcal{P}^{3|\mathcal{N}}$. Then we may define local holomorphic vector fields $\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}$ and $\widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}$ by

$$\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}} \;:=\; \pi^\pm_{\dot{\alpha}_1}\cdots\pi^\pm_{\dot{\alpha}_{m_\alpha}}\frac{\partial}{\partial z^\alpha_\pm} \qquad \text{and} \qquad \widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}} \;:=\; \pi^\pm_{\dot{\alpha}_1}\cdots\pi^\pm_{\dot{\alpha}_{n_i}}\frac{\partial}{\partial \eta^\pm_i}, \tag{V.70}$$

for $m_\alpha, n_i \in \mathbb{N}$. Clearly, the vector fields $\widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}$ and $\widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}$ are totally symmetric under an exchange of their dotted indices. As before, we define a perturbation of the transition function $f = \{f_{+-}\}$ of a smoothly trivial holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ which is trivial along $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ according to[8]

$$f_{+-} \;\mapsto\; \widetilde{P}^\pm_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}} f_{+-} \qquad \text{and} \qquad f_{+-} \;\mapsto\; \widetilde{Q}^i_{\pm\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}} f_{+-}. \tag{V.71}$$

Next one could pull back $f' = f + \delta f$ to the correspondence space and solve the corresponding infinitesimal Riemann-Hilbert problem (which we have already done in the preceding section) to construct the symmetry transformations of the components $\mathcal{A}_{\alpha\dot{\alpha}}$ and $\mathcal{A}^i_{\dot{\alpha}}$ of the gauge potential. We shall, however, proceed differently and instead consider the following dynamical system:

$$\begin{aligned}
\frac{\partial}{\partial t^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}} f_{+-} \;&=\; \delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}} f_{+-} \;=\; \pi^\pm_{\dot{\alpha}_1}\cdots\pi^\pm_{\dot{\alpha}_{m_\alpha}}\frac{\partial}{\partial z^\alpha_\pm} f_{+-},\\
\frac{\partial}{\partial \xi_i^{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}} f_{+-} \;&=\; \delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}} f_{+-} \;=\; \pi^\pm_{\dot{\alpha}_1}\cdots\pi^\pm_{\dot{\alpha}_{n_i}}\frac{\partial}{\partial \eta^\pm_i} f_{+-},
\end{aligned} \tag{V.72}$$

[8]Clearly, such a deformation does not preserve the reality condition $f_{+-}(\cdots) = [f_{+-}(\tau_E(\cdots))]^\dagger$. See also footnote 6.

where $t^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}$ and $\xi_i^{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}$ are parameters. These equations can easily be solved. In fact, the solution to (V.72) reads as

$$f_{+-} \;=\; f_{+-}(z^\alpha_\pm + t^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{m_\alpha}}, \pi^\pm_{\dot{\alpha}} = \lambda^\pm_{\dot{\alpha}}, \eta^\pm_i + \xi_i^{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{n_i}}). \qquad \text{(V.73)}$$

Note that any point of $\mathbb{C}^{4|2\mathcal{N}}$ can be obtained by a shift of the origin and hence we may put, without loss of generality, $x_R^{\alpha\dot{\alpha}}$ and $\eta_i^{\dot{\alpha}}$ to zero. Therefore, (V.73) simplifies to

$$f_{+-} \;=\; f_{+-}(t^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{m_\alpha}}, \lambda^\pm_{\dot{\alpha}}, \xi_i^{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{n_i}}), \qquad \text{(V.74)}$$

where now

$$(t^{\alpha\dot{1}\cdots\dot{1}}, t^{\alpha\dot{2}\dot{1}\cdots\dot{1}}, \xi_i^{\dot{1}\cdots\dot{1}}, \xi_i^{\dot{2}\dot{1}\cdots\dot{1}})$$

are interpreted as coordinates on the anti-chiral superspace $\mathbb{C}^{4|2\mathcal{N}}$ whereas the others are additional moduli sometimes also referred to as "higher times".

For finite sums in (V.74), $m_\alpha, n_i < \infty$, the polynomials

$$z^\alpha_\pm \;=\; t^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{m_\alpha}} \qquad \text{and} \qquad \eta^\pm_i \;=\; \xi_i^{\dot{\alpha}_1\cdots\dot{\alpha}_{n_i}}\lambda^\pm_{\dot{\alpha}_1}\cdots\lambda^\pm_{\dot{\alpha}_{n_i}} \qquad \text{(V.75)}$$

can be regarded as holomorphic sections of the bundle

$$\mathcal{O}_{\mathbb{C}P^1}(m_1)\oplus\mathcal{O}_{\mathbb{C}P^1}(m_2)\oplus\bigoplus_{i=1}^{\mathcal{N}}\Pi\mathcal{O}_{\mathbb{C}P^1}(n_i) \;\to\; \mathbb{C}P^1. \qquad \text{(V.76)}$$

In the following, we shall call this space generalized supertwistor space and denote it by $\mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_\mathcal{N}}$. Note that it can be understood as an open subset in the weighted projective superspace $W\mathbb{C}P^{3|\mathcal{N}}[m_1, m_2, 1, 1|n_1, \ldots, n_\mathcal{N}]$,

$$\begin{aligned} &\mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_\mathcal{N}} \;=\\ &=\; W\mathbb{C}P^{3|\mathcal{N}}[m_1, m_2, 1, 1|n_1, \ldots, n_\mathcal{N}] \setminus W\mathbb{C}P^{1|\mathcal{N}}[m_1, m_2|n_1, \ldots, n_\mathcal{N}]. \end{aligned}$$

See also our discussion presented in Sec. II.3. Thus, $W\mathbb{C}P^{3|\mathcal{N}}[m_1, m_2, 1, 1|n_1, \ldots, n_\mathcal{N}]$ is a compactified version of $\mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_\mathcal{N}}$. Moreover, for the particular combination

$$m_1 + m_2 - (n_1 + \cdots + n_\mathcal{N}) + 2 \;=\; 0,$$

it becomes a formal Calabi-Yau supermanifold. As we shall discuss in detail below, (V.74) can then be interpreted as transition function of a holomorphic vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_\mathcal{N}}$.

§V.8 Penrose-Ward transform. Like supertwistor space, also generalized supertwistor space is a part of a double fibration. For notational convenience, let us denote the generalized supertwistor space by $\mathcal{P}^{3|\mathcal{N}}_{m,n}$. Furthermore, $H^0(\mathbb{C}P^1, \mathcal{P}^{3|\mathcal{N}}_{m,n}) \cong \mathbb{C}^{M|N}$, where $M := m_1 + m_2 + 2$ and $N := n_1 + \cdots + n_{\mathcal{N}} + 2\mathcal{N}$. Then we find

$$\begin{array}{ccc} & \mathcal{F}_R^{M+1|N} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{P}^{3|\mathcal{N}}_{m,n} & & \mathbb{C}^{M|N} \end{array} \tag{V.77}$$

where the correspondence space is again a direct product $\mathcal{F}_R^{M+1|N} \cong \mathbb{C}^{M|N} \times \mathbb{C}P^1$. The natural choice of coordinates on $\mathcal{F}_R^{M+1|N}$ is

$$(x_R^{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{n_i}}). \tag{V.78}$$

Generalized supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{m,n}$ can be covered by two coordinate patches, $\mathfrak{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$, and equipped with local coordinates $(z^\alpha_\pm, z^3_\pm, \eta_i^\pm)$. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, they are related by

$$z^\alpha_+ = \frac{1}{(z^3_-)^{m_\alpha}} z^\alpha_-, \qquad z^3_+ = \frac{1}{z^3_-}, \qquad \text{and} \qquad \eta_i^+ = \frac{1}{(z^3_-)^{n_i}}\eta_i^-. \tag{V.79}$$

Hence, the projections π_1 and π_2 in the fibration (V.77) act as follows:

$$\begin{aligned} \pi_1 : (x_R^{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{n_i}}) &\mapsto (z^\alpha_\pm = x_R^{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha}} \lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{m_\alpha}}, z^3_\pm = \lambda_\pm, \\ &\qquad \eta_i^\pm = \eta_i^{\dot\alpha_1\cdots\dot\alpha_{n_i}} \lambda^\pm_{\dot\alpha_1}\cdots\lambda^\pm_{\dot\alpha_{n_i}}), \\ \pi_2 : (x_R^{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha}}, \lambda^\pm_{\dot\alpha}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{n_i}}) &\mapsto (x_R^{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha}}, \eta_i^{\dot\alpha_1\cdots\dot\alpha_{n_i}}). \end{aligned} \tag{V.80}$$

By virtue of these projections, we obtain the following proposition:

Proposition V.1. *There exist the following geometric correspondences:*

$$\begin{array}{llcl} \text{(i)} & \textit{point } p \textit{ in } \mathcal{P}^{3|\mathcal{N}}_{m,n} & \longleftrightarrow & \mathbb{C}_p^{M-2|N-\mathcal{N}} \textit{ in } \mathbb{C}^{M|N} \\ \text{(ii)} & \mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}_{m,n} & \longleftrightarrow & \textit{point } (x_R, \eta) \textit{ in } \mathbb{C}^{M|N} \end{array}$$

Let $\mathcal{E}$ be a rank r holomorphic vector bundle over $\mathcal{P}^{3|\mathcal{N}}_{m,n}$ and $\pi_1^*\mathcal{E}$ be the pull-back of $\mathcal{E}$ to the correspondence space $\mathcal{F}_R^{M+1|N}$. The covering of the latter is denoted by $\hat{\mathfrak{U}} = \{\hat{\mathcal{U}}_+, \hat{\mathcal{U}}_-\}$. These bundles are defined by transition functions[9] $f = \{f_{+-}\}$ which are annihilated by the vector fields

$$D^\pm_{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}} = \lambda^{\dot\alpha}_\pm \partial^R_{\alpha(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1})} \quad \text{and} \quad D^i_{\pm\dot\alpha_1\cdots\dot\alpha_{n_i-1}} = \lambda^{\dot\alpha}_\pm \partial^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1})} \tag{V.81}$$

[9]Here, we again use the same letter f for both bundles.

as they freely generate the relative tangent sheaf $(\Omega^1(\mathcal{F}_R^{M+1|N})/\pi_1^*\Omega^1(\mathcal{P}^{3|\mathcal{N}}_{m,n}))^*$. The derivatives $\partial^R_{\alpha(\dot\alpha_1\cdots\dot\alpha_{m_\alpha})}$ and $\partial^i_{(\dot\alpha_1\cdots\dot\alpha_{n_i})}$ are defined according to

$$\begin{aligned}\partial^R_{\alpha(\dot\alpha_1\cdots\dot\alpha_{m_\alpha})}\, x_R^{\beta\dot\beta_1\cdots\dot\beta_{m_\beta}} &:= \delta^\beta_\alpha\delta^{\dot\beta_1}_{(\dot\alpha_1}\cdots\delta^{\dot\beta_{m_\beta}}_{\dot\alpha_{m_\alpha})},\\ \partial^i_{(\dot\alpha_1\cdots\dot\alpha_{n_i})}\,\eta_j^{\dot\beta_1\cdots\dot\beta_{n_j}} &:= \delta^i_j\delta^{\dot\beta_1}_{(\dot\alpha_1}\cdots\delta^{\dot\beta_{n_j}}_{\dot\alpha_{n_i})}\end{aligned} \tag{V.82}$$

and can be understood as generalizations of (III.6).

The requirement of smooth triviality of the bundle $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}_{m,n}$ allows us to split the transition function f_{+-} according to

$$f_{+-} \;=\; \psi_+^{-1}\psi_-, \tag{V.83}$$

whereas $\mathbb{C}^{M|N}$-triviality ensures that there exists a $\psi=\{\psi_+,\psi_-\}$ which belongs to $C^0(\hat{\mathfrak{U}},\mathfrak{P})$. Therefore, we may introduce a Lie-algebra valued one-form such that

$$\begin{aligned}\mathcal{A}^\pm_{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}} &:= \lambda^{\dot\alpha}_\pm\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}} \;=\; \psi_\pm D^\pm_{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}}\psi_\pm^{-1},\\ \mathcal{A}^i_{\pm\dot\alpha_1\cdots\dot\alpha_{n_i-1}} &:= \lambda^{\dot\alpha}_\pm\mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}} \;=\; \psi_\pm D^i_{\pm\dot\alpha_1\cdots\dot\alpha_{n_i-1}}\psi_\pm^{-1},\end{aligned} \tag{V.84}$$

and therefore

$$\begin{aligned}(D^\pm_{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}}+\mathcal{A}^\pm_{\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}})\psi_\pm &= 0,\\ (D^i_{\pm\dot\alpha_1\cdots\dot\alpha_{n_i-1}}+\mathcal{A}^i_{\pm\dot\alpha_1\cdots\dot\alpha_{n_i-1}})\psi_\pm &= 0.\end{aligned} \tag{V.85}$$

We note that for $m_\alpha=n_i=1$ this system reduces, of course, to the old one given by (II.16). Moreover, we have the following symmetry properties

$$\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}} \;=\; \mathcal{A}_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1})} \quad\text{and}\quad \mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}} \;=\; \mathcal{A}^i_{\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{n_i-1})}. \tag{V.86}$$

The compatibility conditions for (V.85) read as

$$\begin{aligned}[D^R_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}},D^R_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{m_\beta-1}}]+[D^R_{\alpha\dot\beta\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}},D^R_{\beta\dot\alpha\dot\beta_1\cdots\dot\beta_{m_\beta-1}}] &= 0,\\ [D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}},D^R_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{m_\beta-1}}]+[D^i_{\dot\beta\dot\alpha_1\cdots\dot\alpha_{n_i-1}},D^R_{\beta\dot\alpha\dot\beta_1\cdots\dot\beta_{m_\beta-1}}] &= 0,\\ \{D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}},D^j_{\dot\beta\dot\beta_1\cdots\dot\beta_{n_j-1}}\}+\{D^i_{\dot\beta\dot\alpha_1\cdots\dot\alpha_{n_i-1}},D^j_{\dot\alpha\dot\beta_1\cdots\dot\beta_{n_j-1}}\} &= 0.\end{aligned} \tag{V.87}$$

Here, we have defined the first order differential operators

$$\begin{aligned}D^R_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}} &:= \partial^R_{\alpha(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1})}+\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m_\alpha-1}},\\ D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}} &:= \partial^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1})}+\mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n_i-1}}.\end{aligned} \tag{V.88}$$

We remark that the components

$$\mathcal{A}_{\alpha\dot{\alpha}\dot{1}\cdots\dot{1}} \qquad \text{and} \qquad \mathcal{A}^i_{\dot{\alpha}\dot{1}\cdots\dot{1}}$$

coincide with the components $\mathcal{A}_{\alpha\dot{\alpha}}$ and $\mathcal{A}^i_{\dot{\alpha}}$ of the gauge potential on $\mathbb{C}^{4|2\mathcal{N}}$. In the sequel, we shall refer to (V.87) as the truncated $\mathcal{N}$-extended self-dual SYM hierarchy of level $(m_1, m_2|n_1, \ldots, n_{\mathcal{N}})$. The full hierarchy is then obtained by taking the limit $m_\alpha, n_i \to \infty$.

From Eqs. (V.84) it follows that

$$\begin{aligned} \mathcal{A}_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha-1}} &= \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \frac{\mathcal{A}^+_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{m_\alpha-1}}}{\lambda_+\lambda_+^{\dot{\alpha}}}, \\ \mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{n_i-1}} &= \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \frac{\mathcal{A}^i_{+\dot{\alpha}_1\cdots\dot{\alpha}_{n_i-1}}}{\lambda_+\lambda_+^{\dot{\alpha}}}, \end{aligned} \tag{V.89}$$

where $\mathscr{C} = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$. As in the previous discussion, Eqs. (V.89) make the Penrose-Ward transform explicit.

§V.9 Field expansions and field equations. So far, we have written down the truncated self-dual SYM hierarchies (V.87) quite abstractly as compatibility conditions of a linear system. The next step in our discussion is to construct the equations of motion on superfield level equivalent to (V.87). To do this, we need to identify the field content. At first sight, we expect to find as fundamental field content (in a covariant formulation) the field content of $\mathcal{N}$-extended self-dual SYM theory plus a tower of additional fields depending on the parameters m_α and n_i. However, as we shortly realize, this will not entirely be true. For $n_i > 1$, we instead find that certain combinations of $\mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{n_i-1}}$ play the role of potentials for a lot of the naively expected fields, such that those combinations should be regarded as fundamental fields.

In the remainder of this paragraph, we shall for simplicity consider the case when $m_1 = m_2 =: m$ and $n_1 = \cdots = n_{\mathcal{N}} =: n$. To simplify the subsequent formulas, let us also introduce a shorthand index notation

$$\mathcal{A}_{\alpha\dot{\alpha}\dot{A}} := \mathcal{A}_{\alpha\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}} \qquad \text{and} \qquad \mathcal{A}^I_{\dot{\alpha}} := \mathcal{A}^i_{\dot{\alpha}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}. \tag{V.90}$$

First, we point out that Eqs. (V.87) can concisely be rewritten as

$$\begin{aligned} [D^R_{\alpha\dot{\alpha}\dot{A}}, D^R_{\beta\dot{\beta}\dot{B}}] + [D^R_{\alpha\dot{\beta}\dot{A}}, D^R_{\beta\dot{\alpha}\dot{B}}] = 0, \qquad [D^I_{\dot{\alpha}}, D^R_{\beta\dot{\beta}\dot{B}}] + [D^I_{\dot{\beta}}, D^R_{\beta\dot{\alpha}\dot{B}}] = 0, \\ \{D^I_{\dot{\alpha}}, D^J_{\dot{\beta}}\} + \{D^I_{\dot{\beta}}, D^J_{\dot{\alpha}}\} = 0, \end{aligned} \tag{V.91}$$

which translates to the following superfield definitions

$$\begin{aligned}
[D^R_{\alpha\dot\alpha\dot A}, D^R_{\beta\dot\beta\dot B}] &:= \epsilon_{\dot\alpha\dot\beta} f_{\alpha\dot A\beta\dot B},\\
[D^I_{\dot\alpha}, D^R_{\beta\dot\beta\dot B}] &:= \epsilon_{\dot\alpha\dot\beta}\chi^I_{\beta\dot B},\\
\{D^I_{\dot\alpha}, D^J_{\dot\beta}\} &:= \epsilon_{\dot\alpha\dot\beta} W^{IJ}.
\end{aligned} \tag{V.92}$$

Note that quite generally we have

$$\begin{aligned}
\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B} &= [D^R_{\alpha\dot\alpha\dot A}, D^R_{\beta\dot\beta\dot B}] = \tfrac{1}{2}(\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B} - \mathcal{F}_{\beta\dot\beta\dot B\alpha\dot\alpha\dot A})\\
&= \epsilon_{\dot\alpha\dot\beta} f_{\alpha\dot A\beta\dot B} + \epsilon_{\alpha\beta} f_{\dot\alpha\dot A\dot\beta\dot B} + \mathcal{F}_{\alpha\dot\alpha[\dot A\beta\dot\beta\dot B]},
\end{aligned} \tag{V.93}$$

where

$$\begin{aligned}
f_{\alpha\dot A\beta\dot B} &:= -\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B},\\
f_{\dot\alpha\dot A\dot\beta\dot B} &:= \tfrac{1}{2}\epsilon^{\alpha\beta}\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B},\\
\mathcal{F}_{\alpha\dot\alpha[\dot A\beta\dot\beta\dot B]} &:= \tfrac{1}{2}(\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B} - \mathcal{F}_{\alpha\dot\alpha\dot B\beta\dot\beta\dot A}).
\end{aligned} \tag{V.94}$$

Eq. (V.93) can be simplified further to

$$\mathcal{F}_{\alpha\dot\alpha\dot A\beta\dot\beta\dot B} = \epsilon_{\dot\alpha\dot\beta} f_{\alpha\dot A\beta\dot B} + \epsilon_{\alpha\beta} f_{\dot\alpha(\dot A\dot\beta\dot B)} + \mathcal{F}_{(\alpha\dot\alpha[\dot A\beta)\dot\beta\dot B]}. \tag{V.95}$$

Therefore, the first equation of (V.92) implies that

$$f_{\dot\alpha(\dot A\dot\beta\dot B)} = 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot A\beta)\dot\beta\dot B]} = 0, \tag{V.96}$$

which are the first two of the superfield equations of motion. We point out that for the choice $\dot A = \dot B = (\dot 1 \cdots \dot 1)$ the set (V.96) represents nothing but the self-dual YM equations.

Next we consider the Bianchi identity for the triple $(D^R_{\alpha\dot\alpha\dot A}, D^I_{\dot\beta}, D^R_{\gamma\dot\gamma\dot C})$. We find

$$D^I_{\dot\alpha} f_{\alpha\dot A\beta\dot B} = D^R_{\alpha\dot\alpha\dot A}\chi^I_{\beta\dot B}. \tag{V.97}$$

From this equation we deduce another two field equations,

$$\epsilon^{\alpha\beta} D^R_{\alpha\dot\alpha(\dot A}\chi^I_{\beta\dot B)} = 0 \qquad \text{and} \qquad D^R_{(\alpha\dot\alpha[\dot A}\chi^I_{\beta)\dot B]} = 0. \tag{V.98}$$

The Bianchi identity for $(D^R_{\alpha\dot\alpha\dot A}, D^I_{\dot\beta}, D^J_{\dot\gamma})$ implies

$$D^R_{\alpha\dot\alpha\dot A} W^{IJ} = D^I_{\dot\alpha}\chi^J_{\alpha\dot A}. \tag{V.99}$$

Applying $D^R_{\beta\dot\beta\dot B}$ to (V.99), we obtain upon (anti)symmetrization the following two equations of motion:

$$\begin{aligned}
\tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} D^R_{\alpha\dot\alpha(\dot A} D^R_{\beta\dot\beta\dot B)} W^{IJ} + \epsilon^{\alpha\beta}\{\chi^I_{\alpha(\dot A}, \chi^J_{\beta\dot B)}\} &= 0,\\
\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta} D^R_{(\alpha\dot\alpha[\dot A} D^R_{\beta)\dot\beta\dot B]} W^{IJ} + \{\chi^I_{(\alpha[\dot A}, \chi^J_{\beta)\dot B]}\} &= 0.
\end{aligned} \tag{V.100}$$

Furthermore, the Bianchi identity for the combination $(D^I_{\dot\alpha}, D^J_{\dot\beta}, D^K_{\dot\gamma})$ shows that $D^I_{\dot\alpha}W^{JK}$ determines a superfield which is totally antisymmetric in the indices IJK, i.e.,

$$D^I_{\dot\alpha}W^{JK} \;=:\; \tfrac{1}{2}\chi^{IJK}_{\dot\alpha}. \tag{V.101}$$

Upon acting on both sides by $D^R_{\alpha\dot\alpha\dot A}$ and contracting the dotted indices, we obtain the field equation for $\chi^{IJK}_{\dot\alpha}$

$$\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha\dot A}\chi^{IJK}_{\dot\beta} - 6[W^{[IJ}, \chi^{K]}_{\alpha\dot A}] \;=\; 0. \tag{V.102}$$

The application of $D^I_{\dot\alpha}$ to $\chi^{JKL}_{\dot\beta}$ and symmetrization in $\dot\alpha$ and $\dot\beta$ leads by virtue of (V.101) to a new superfield which is totally antisymmetric in $IJKL$,

$$D^I_{(\dot\alpha}\chi^{JKL}_{\dot\beta)} \;=:\; -G^{IJKL}_{\dot\alpha\dot\beta}. \tag{V.103}$$

Some algebraic manipulations show that

$$D^I_{\dot\alpha}\chi^{JKL}_{\dot\beta} \;=\; D^I_{(\dot\alpha}\chi^{JKL}_{\dot\beta)} + D^I_{[\dot\alpha}\chi^{JKL}_{\dot\beta]} \;=\; -G^{IJKL}_{\dot\alpha\dot\beta} + 3\epsilon_{\dot\alpha\dot\beta}[W^{I[J}, W^{KL]}], \tag{V.104}$$

where equation (V.101) and the definition (V.103) have been used. From this equation, the equation of motion for the superfield $G^{IJKL}_{\dot\alpha\dot\beta}$ can readily be derived. We obtain

$$\epsilon^{\dot\alpha\dot\gamma}D^R_{\alpha\dot\alpha\dot A}G^{IJKL}_{\dot\beta\dot\gamma} - 4\{\chi^{[I}_{\alpha\dot A}, \chi^{JKL]}_{\dot\beta}\} - 6[W^{[JK}, D^R_{\alpha\dot\beta\dot A}W^{LI]}] \;=\; 0. \tag{V.105}$$

As (V.101) implies the existence of the superfield $G^{IJKL}_{\dot\alpha\dot\beta}$, definition (V.103) determines a new superfield $\psi^{IJKLM}_{\dot\alpha\dot\beta\dot\gamma}$ being totally antisymmetric in $IJKLM$ and totally symmetric in $\dot\alpha\dot\beta\dot\gamma$, i.e.,

$$D^I_{(\dot\alpha}G^{JKLM}_{\dot\beta\dot\gamma)} \;=:\; \psi^{IJKLM}_{\dot\alpha\dot\beta\dot\gamma}. \tag{V.106}$$

It is easily shown that

$$D^I_{\dot\alpha}G^{JKLM}_{\dot\beta\dot\gamma} \;=\; D^I_{(\dot\alpha}G^{JKLM}_{\dot\beta\dot\gamma)} - \tfrac{2}{3}\epsilon_{\dot\alpha(\dot\beta}\epsilon^{\dot\delta\dot\epsilon}D^I_{\dot\delta}G^{JKLM}_{\dot\epsilon\dot\gamma)}. \tag{V.107}$$

After some tedious algebra, we obtain from (V.107) the formula

$$D^I_{\dot\alpha}G^{JKLM}_{\dot\beta\dot\gamma} \;=\; \psi^{IJKLM}_{\dot\alpha\dot\beta\dot\gamma} - \tfrac{2}{3}\epsilon_{\dot\alpha(\dot\beta}\left(4[W^{I[J}, \chi^{KLM]}_{\dot\gamma)}] + 3[\chi^{I[JK}_{\dot\gamma)}, W^{LM]}]\right), \tag{V.108}$$

where definition (V.106) has been substituted. This equation in turn implies the equation of motion for $\psi^{IJKLM}_{\dot\alpha\dot\beta\dot\gamma}$,

$$\begin{aligned}\epsilon^{\dot\alpha\dot\delta}D^R_{\alpha\dot\alpha\dot A}\psi^{IJKLM}_{\dot\beta\dot\gamma\dot\delta} &+ 5[\chi^{[I}_{\alpha\dot A}, G^{JKLM]}_{\dot\beta\dot\gamma}] - \\ &- \tfrac{20}{3}[D^R_{\alpha(\dot\beta\dot A}W^{[IJ}, \chi^{KLM]}_{\dot\gamma)}] + \tfrac{10}{3}[W^{[IJ}, D^R_{\alpha(\dot\beta\dot A}\chi^{KLM]}_{\dot\gamma)}] \;=\; 0,\end{aligned} \tag{V.109}$$

which follows after a somewhat lengthy calculation.

Now one can continue this procedure of defining superfields via the action of $D^I_{\dot\alpha}$ and of finding the corresponding equations of motion. Generically, the number of fields one obtains in this way is determined by the parameter n, i.e., the most one can get is

$$\psi^{I_1\cdots I_{\mathcal{N}n}}_{\dot\alpha_1\cdots\dot\alpha_{\mathcal{N}n-2}},$$

which is, as before, totally antisymmetric in $I_1\cdots I_{\mathcal{N}n}$ and totally symmetric in $\dot\alpha_1\cdots\dot\alpha_{\mathcal{N}n-2}$.

Let us collect the superfield equations of motion for the $\mathcal{N}$-extended self-dual truncated SYM hierarchy:

$$\begin{aligned}
f_{\dot\alpha(\dot A\dot\beta\dot B)} &= 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot A\beta)\dot\beta\dot B]} = 0,\\
\epsilon^{\alpha\beta}D^R_{\alpha\dot\alpha(\dot A}\chi^I_{\beta\dot B)} &= 0 \qquad \text{and} \qquad D^R_{(\alpha\dot\alpha[\dot A}\chi^I_{\beta)\dot B]} = 0,\\
\tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha(\dot A}D^R_{\beta\dot\beta\dot B)}W^{IJ}+\epsilon^{\alpha\beta}\{\chi^I_{\alpha(\dot A},\chi^J_{\beta\dot B)}\} &= 0,\\
\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}D^R_{(\alpha\dot\alpha[\dot A}D^R_{\beta)\dot\beta\dot B]}W^{IJ}+\{\chi^I_{(\alpha[\dot A},\chi^J_{\beta)\dot B]}\} &= 0,\\
\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha\dot A}\chi^{IJK}_{\dot\beta}-6[W^{[IJ},\chi^{K]}_{\alpha\dot A}] &= 0,\\
\epsilon^{\dot\alpha\dot\gamma}D^R_{\alpha\dot\alpha\dot A}G^{IJKL}_{\dot\beta\dot\gamma}-4\{\chi^{[I}_{\alpha\dot A},\chi^{JKL]}_{\dot\beta}\}-6[W^{[JK},D^R_{\alpha\dot\beta\dot A}W^{LI]}] &= 0,\\
&\ \ \vdots\\
\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha\dot A}\psi^{I_1\cdots I_{\mathcal{N}n}}_{\dot\beta\dot\alpha_1\cdots\dot\alpha_{\mathcal{N}n-3}}+J^{I_1\cdots I_{\mathcal{N}n}}_{\alpha\dot A\dot\alpha_1\cdots\dot\alpha_{\mathcal{N}n-3}} &= 0,
\end{aligned} \tag{V.110}$$

where the currents $J^{I_1\cdots I_{\mathcal{N}n}}_{\alpha\dot A\dot\alpha_1\cdots\dot\alpha_{\mathcal{N}n-3}}$ are determined in an obvious manner. Clearly, the system (V.110) contains as a subset the $\mathcal{N}$-extended self-dual SYM equations. In particular, for the choice $m=n=1$ it reduces to the latter. Altogether, we have obtained the field content of $\mathcal{N}$-extended self-dual SYM theory plus a number of additional fields together with their superfield equations of motion.

However, as we have already indicated, this is not the end of the story. The system (V.110), though describing the truncated hierarchy, contains a lot of redundant information. For that reason, it should not be regarded as the fundamental system displaying the truncated hierarchy. In fact, the use of the shorthand index notation (V.90) does not entirely reflect all of the possible index symmetry properties of the appearing superfields. In order to incorporate all possibilities, we instead need to write out the explicit form of $I,J,K,\ldots$.

As before, let us impose the transversal gauge condition

$$\eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}\mathcal{A}^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}} \;=\; 0, \tag{V.111}$$

which again reduces super gauge transformations to ordinary ones. Note that in (V.111) only $\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1})}$ contributes, since the fermionic coordinates are totally symmetric under an exchange of their dotted indices. The condition (V.111) then allows to define the recursion operator $\mathscr{D}$ according to

$$\mathscr{D} \;:=\; \eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}} \;=\; \eta_i^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}\partial^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1})}. \tag{V.112}$$

The third equation of (V.92) yields

$$\begin{aligned}(1+\mathscr{D})\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1})} \;&=\; -\epsilon_{\dot\beta(\dot\alpha}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1})\,\dot\beta_1\cdots\dot\beta_{n-1}},\\ \mathscr{D}\mathcal{A}^i_{[\dot\alpha\dot\beta]\dot\alpha_1\cdots\dot\alpha_{n-2}} \;&=\; \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\gamma\dot\beta_1\cdots\dot\beta_{n-1}}W^{ij}_{\dot\gamma\dot\alpha_1\cdots\dot\alpha_{n-2}\,\dot\beta_1\cdots\dot\beta_{n-1}},\end{aligned} \tag{V.113}$$

which states that $\mathcal{A}^i_{(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1})}$ does not have a zeroth order component in the η-expansion while $\mathcal{A}^i_{[\dot\alpha\dot\beta]\dot\alpha_1\cdots\dot\alpha_{n-2}}$ does. Therefore, we obtain as a fundamental superfield

$$\phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}} \;:=\; 2\mathcal{A}^i_{[\dot1\dot2]\dot\alpha_1\cdots\dot\alpha_{n-2}} \qquad \text{for} \qquad n>1. \tag{V.114}$$

Note that as $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}} = \phi^i_{(\dot\alpha_1\cdots\dot\alpha_{n-2})}$ it defines for each i a spin $\frac{n}{2}-1$ superfield of odd parity.

The second equation of (V.92) reads explicitly as

$$[D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}, D^R_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{m-1}}] \;=\; \epsilon_{\dot\alpha\dot\beta}\chi^i_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{m-1}}.$$

The contraction with $\epsilon^{\dot\alpha\dot\alpha_1}$ shows that

$$\chi^i_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{m-1}} \;=\; D^R_{\beta(\dot\alpha_1\dot\beta_1\cdots\dot\beta_{m-1}}\phi^i_{\dot\alpha_2\cdots\dot\alpha_{n-1})}, \tag{V.115}$$

where the symmetrization is only meant between the $\dot\alpha_1\cdots\dot\alpha_{n-1}$. Therefore, the superfield $\chi^i_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\beta\,\dot\beta_1\cdots\dot\beta_{m-1}}$ cannot be regarded as a fundamental field – the superfield (V.114) plays the role of a potential for the former.

Next we discuss the superfield $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}}$. To decide which combinations of it are really fundamental, we need some preliminaries. Consider the index set

$$\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n,$$

which separately is totally symmetric in $\dot\alpha_1\cdots\dot\alpha_n$ and $\dot\beta_1\cdots\dot\beta_n$, respectively. Then we have the useful formula

$$\begin{aligned}\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n &= (\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n) + \sum \text{all possible contractions}\\ &= (\dot\alpha_1\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_n) + \\ &\qquad + A_1\sum_{i,j}\dot\alpha_1\cdots\overbrace{\dot\alpha_i\cdots\dot\alpha_n\dot\beta_1\cdots\dot\beta_j}\cdots\dot\beta_n + \cdots,\end{aligned} \tag{V.116}$$

where the parentheses denote, as before, symmetrization of the enclosed indices and "contraction" means antisymmetrization in the respective index pair. The A_is for $i=1,\ldots,n$ are combinatorial coefficients, whose explicit form is not needed in the sequel. The proof of (V.116) is quite similar to the one of the Wick theorem and we thus leave it to the interested reader.

The third equation of (V.92) is explicitly given by

$$\{D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}, D^j_{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}\} = \epsilon_{\dot\alpha\dot\beta}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}}.$$

After contraction with $\epsilon^{\dot\alpha\dot\alpha_1}$ we obtain

$$W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}} = -D^j_{\dot\alpha_1\dot\beta_1\cdots\dot\beta_{n-1}}\phi^i_{\dot\alpha_2\cdots\dot\alpha_{n-1}}, \tag{V.117}$$

where the definition (V.114) has been inserted. Contracting this equation with $\epsilon^{\dot\alpha_1\dot\beta_1}$, we get

$$\epsilon^{\dot\alpha_1\dot\beta_1}W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}} = -\{\phi^i_{\dot\alpha_2\cdots\dot\alpha_{n-1}}, \phi^j_{\dot\beta_2\cdots\dot\beta_{n-1}}\}. \tag{V.118}$$

Thus, we conclude that $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-2}[\dot\alpha_{n-1}\dot\beta_1]\dot\beta_2\cdots\dot\beta_{n-1}}$ is a composite field and hence not a fundamental one. Using formula (V.116), we may schematically write

$$W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}} = W^{ij}_{(\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1})} + \sum \text{all possible contractions.} \tag{V.119}$$

The contraction terms in (V.119), however, solely consist of composite expressions due to (V.118). Therefore, only the superfield

$$W^{ij}_{(\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1})} = W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1})} \tag{V.120}$$

is fundamental. For each combination $[ij]$ it represents an even superfield with spin $n-1$.

Next we need to consider the superfield defined in (V.101),

$$\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}} = D^i_{\dot\alpha\dot\alpha_1\cdots\dot\alpha_{n-1}}W^{ij}_{\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}}.$$

By extending the formula (V.116) to the index triple

$$\dot\alpha_1 \cdots \dot\alpha_{n-1}\,\dot\beta_1 \cdots \dot\beta_{n-1}\,\dot\gamma_1 \cdots \dot\gamma_{n-1}$$

and by utilizing the symmetry properties of $\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}}$, one can show, by virtue of the above arguments, that only the combination

$$\chi^{ijk}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1})} \;=\; \chi^{[ijk]}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1})} \tag{V.121}$$

of $\chi^{ijk}_{\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}}$ remains as a fundamental superfield. It defines for each $[ijk]$ an odd superfield with spin $\frac{3}{2}n-1$.

Repeating this procedure, we deduce from the definition (V.103) that

$$G^{ijkl}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}\,\dot\delta_1\cdots\dot\delta_{n-1})} \;=\; G^{[ijkl]}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}\,\dot\delta_1\cdots\dot\delta_{n-1})} \tag{V.122}$$

is fundamental and it represents one[10] spin $2n-1$ superfield which is even. All higher order fields, such as (V.106), yield no further fundamental fields due to the antisymmetrization of $ijklm$, etc. In summary, the fundamental field content of the truncated self-dual SYM hierarchies is given by

$$\begin{gathered}\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}},\quad \phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}},\quad W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1})},\\ \chi^{[ijk]}_{(\dot\alpha\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1})},\quad G^{[ijkl]}_{(\dot\alpha\dot\beta\,\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\gamma_1\cdots\dot\gamma_{n-1}\,\dot\delta_1\cdots\dot\delta_{n-1})},\end{gathered} \tag{V.123}$$

where we assume that $n>1$.[11] All other naively expected fields, which for instance appear in (V.110), are composite expressions of the above fields. Note that the field $\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}}$ can be decomposed according to

$$\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}} \;=\; \mathcal{A}_{\alpha(\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1})} + \epsilon_{\dot\alpha(\dot\alpha_1}\Phi_{\alpha\dot\alpha_2\cdots\dot\alpha_{m-1})}. \tag{V.124}$$

Therefore, $\mathcal{A}_{\alpha(\dot\alpha_1\cdots\dot\alpha_m)}$ can be interpreted as a gauge potential while $\Phi_{\alpha\dot\alpha_1\cdots\dot\alpha_{n-2}}$ represents a collection of Higgs fields.

It remains to find the superfield equations of motion for the fields (V.123). This, however, is easily done since we have already derived (V.110). By following the lines which led to (V.110) and by taking into account the definition (V.114), the system (V.110)

[10]Note that $\mathcal{N}\leq 4$.

[11]For $n=1$, the field $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}}$ must be replaced by χ^i_α.

reduces for $n>1$ to

$$
\begin{aligned}
f_{\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{m-1}\dot\beta\dot\beta_1\cdots\dot\beta_{m-1})} \;=\; 0 \qquad \text{and} \qquad \mathcal{F}_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{m-1}\beta)\dot\beta\dot\beta_1\cdots\dot\beta_{m-1}]} &= 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha(\dot\alpha_1\cdots\dot\alpha_{m-1}}D^R_{\beta\dot\beta\dot\beta_1\cdots\dot\beta_{m-1})}\phi^i_{\dot\gamma_1\cdots\dot\gamma_{n-2}} &= 0,\\
D^R_{(\alpha\dot\alpha[\dot\alpha_1\cdots\dot\alpha_{m-1}}D^R_{\beta)(\dot\gamma_1\dot\beta_1\cdots\dot\beta_{m-1}]}\phi^i_{\dot\gamma_2\cdots\dot\gamma_{n-1})} &= 0,\\
\epsilon^{\dot\alpha\dot\beta_1}D^R_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}}W^{[ij]}_{(\dot\beta_1\cdots\dot\beta_{n-2})} - \qquad\qquad & \\
-\{\phi^{[i}_{(\dot\beta_2\cdots\dot\beta_{n-1}},D^R_{\alpha\dot\beta_n\dot\alpha_1\cdots\dot\alpha_{m-1}}\phi^{j]}_{\dot\beta_{n+1}\cdots\dot\gamma_{2n-2})}\} &= 0,\\
\epsilon^{\dot\alpha\dot\beta}D^R_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\beta_{m-1}}\chi^{[ijk]}_{(\dot\beta\dot\beta_1\cdots\dot\beta_{3n-3})} - \qquad\qquad & \\
-\,6[W^{[ij}_{(\dot\beta_1\cdots\dot\beta_{2n-2}},D^R_{\alpha\dot\beta_{2n-1}\dot\alpha_1\cdots\dot\alpha_{m-1}}\phi^{k]}_{\dot\beta_{2n}\cdots\dot\beta_{3n-3})}] &= 0,\\
\epsilon^{\dot\alpha\dot\gamma}D^R_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}}G^{[ijkl]}_{(\dot\beta\dot\gamma\dot\beta_1\cdots\dot\beta_{4n-4})} + \qquad\qquad & \\
+\,4\{D^R_{\alpha(\dot\beta_1\dot\alpha_1\cdots\dot\alpha_{m-1}}\phi^{[i}_{\dot\beta_2\cdots\dot\beta_{n-1}},\chi^{ijk]}_{\dot\beta\dot\beta_n\cdots\dot\beta_{4n-4})}\} + \qquad & \\
+\,6[W^{[ij}_{(\dot\beta_1\cdots\dot\beta_{2n-2}},D^R_{\alpha\dot\beta\dot\alpha_1\cdots\dot\alpha_{m-1}}W^{kl]}_{\dot\beta_{2n-1}\cdots\dot\beta_{4n-4})}] &= 0.
\end{aligned}
\tag{V.125}
$$

These are the superfield equations of motion for the truncated $\mathcal{N}$-extended self-dual SYM hierarchy.

Above we have derived the superfield equations of motion. What remains is to show how the superfields (V.123) are expressed in terms of their zeroth order components

$$
\overset{\circ}{\mathcal{A}}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}},\quad \overset{\circ}{\phi}{}^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}},\quad \overset{\circ}{W}{}^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{2n-2})},\quad \overset{\circ}{\chi}{}^{[ijk]}_{(\dot\alpha_1\cdots\dot\alpha_{3n-2})},\quad \overset{\circ}{G}{}^{[ijkl]}_{(\dot\alpha_1\cdots\dot\alpha_{4n-2})},
\tag{V.126}
$$

and furthermore that the field equations on $\mathbb{C}^4$ (or on $\mathbb{R}^4$ after reality conditions have been imposed), i.e., those equations which are obtained from the set (V.125) by projecting onto the zeroth order components (V.126) of the superfields (V.123), imply the compatibility conditions (V.91). We will, however, be not too explicit in showing this equivalence, since the argumentation goes along similar lines as those given for the $\mathcal{N}$-extended self-dual SYM equations. Here, we just want to give the outline.

In order to write down the superfield expansions, remember that we have imposed the gauge (V.111) which led to the recursion operator $\mathscr{D}$ according to (V.112). Using the formulas (V.92), (V.99), (V.101), (V.103), (V.106) and (V.115), we obtain the following recursion relations:

$$
\begin{aligned}
(1+\mathscr{D})\mathcal{A}^i_{(\dot\alpha_1\cdots\dot\alpha_n)} &= -\epsilon_{\dot\beta(\dot\alpha_1}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}W^{ij}_{\dot\alpha_2\cdots\dot\alpha_n)\,\dot\beta_1\cdots\dot\beta_{n-1}},\\
\mathscr{D}\mathcal{A}_{\alpha\dot\alpha\dot\alpha_1\cdots\dot\alpha_{m-1}} &= -\epsilon_{\dot\alpha\dot\beta}\eta_j^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}D^R_{\alpha\dot\beta_1\dot\alpha_1\cdots\dot\alpha_{m-1}}\phi^i_{\dot\beta_2\cdots\dot\beta_{n-1}},\\
\mathscr{D}\phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}} &= \eta_j^{\dot\gamma\dot\beta_1\cdots\dot\beta_{n-1}}W^{ij}_{\dot\gamma\dot\alpha_1\cdots\dot\alpha_{n-2}\,\dot\beta_1\cdots\dot\beta_{n-1}},\\
\mathscr{D}W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{2n-2})} &= \tfrac{1}{2}\eta_k^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}\chi^{[ij]k}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{2n-2})\,\dot\beta_1\cdots\dot\beta_{n-1}},\\
\mathscr{D}\chi^{[ijk]}_{(\dot\alpha_1\cdots\dot\alpha_{3n-2})} &= -\eta_l^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}G^{[ijk]l}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{3n-2})\,\dot\beta_1\cdots\dot\beta_{n-1}} + \\
&\quad+\ 3\epsilon_{\dot\beta(\dot\alpha_1}\eta_l^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}[W^{l[i}_{\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\alpha_2\cdots\dot\alpha_n},W^{jk]}_{\dot\alpha_{n+1}\cdots\dot\alpha_{3n-2})}],\\
\mathscr{D}G^{[ijkl]}_{(\dot\alpha_1\cdots\dot\alpha_{4n-2})} &= \eta_m^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}\psi^{[ijkl]m}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{4n-2})\,\dot\beta_1\cdots\dot\beta_{n-1}} - \\
&\quad-\ \tfrac{2}{3}\epsilon_{\dot\beta(\dot\alpha_1}\eta_m^{\dot\beta\dot\beta_1\cdots\dot\beta_{n-1}}\Big(4[W^{m[i}_{\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\alpha_2\cdots\dot\alpha_n},\chi^{jkl]}_{\dot\alpha_{n+1}\cdots\dot\alpha_{4n-2})}] + \\
&\qquad+\ 3[\chi^{m[ij}_{\dot\alpha_2\,\dot\beta_1\cdots\dot\beta_{n-1}\,\dot\alpha_3\cdots\dot\alpha_{2n-4}},W^{kl]}_{\dot\alpha_{2n-3}\cdots\dot\alpha_{4n-2})}]\Big)\,.
\end{aligned}
\tag{V.127}
$$

An explanation of these formulas is in order. The right hand sides of Eqs. (V.127) depend not only on the fundamental fields (V.123) but also on composite expressions of those fields: For instance, consider the recursion relation of the field $\phi^i_{\dot\alpha_1\cdots\dot\alpha_{n-2}}$. The right hand side of this equation depends on the superfield $W^{ij}_{\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1}}$. However, as we learned in (V.119), it can be rewritten as the fundamental field $W^{[ij]}_{(\dot\alpha_1\cdots\dot\alpha_{n-1}\,\dot\beta_1\cdots\dot\beta_{n-1})}$ plus contraction terms which are of the form (V.118). Similar arguments hold for the other recursion relations. Therefore, the right hand sides of (V.127) can solely be written in terms of the fundamental fields. However, as these formulas in terms of the fundamental fields look rather messy, we refrain from writing them down. Note that the field $\psi^{[ijkl]m}_{\dot\beta(\dot\alpha_1\cdots\dot\alpha_{4n-2})\,\dot\beta_1\cdots\dot\beta_{n-1}}$ appearing in the last recursion relation consists only of composite expressions of the fields (V.118). Using Eqs. (V.127), one can now straightforwardly determine the superfield expansions by a successive application of the recursion operator $\mathscr{D}$, since if one knows the expansions to n-th order in the fermionic coordinates, the recursions (V.127) yield them to next order. But again, this procedure will lead to both unenlightening and complicated looking expressions, so we do not present them here. Finally, the recursion operator can be used to show the equivalence between the field equations and the constraint equations (V.87). This can be done inductively, i.e., one first assumes that Eqs. (V.125) hold to n-th order in the fermionic coordinates, then one applies $k+\mathscr{D}$ to (V.125), where $k\in\mathbb{N}_0$ is some properly chosen integer, and shows that they also hold to $(n+1)$-th order. For more details, see Ref. [266].

§V.10 Light-cone gauge. Let us now give an alternative interpretation of the hierarchy equations. First, we rewrite the constraint equations (II.17) of $\mathcal{N}$-extended self-dual SYM theory in light-cone gauge. One of the interesting issues of this (non-covariant) gauge is that all the equations for all the fields reduce to equations on a single Lie-algebra valued superfield Ψ.

To be explicit, assume the following expansion

$$\psi_+ \; = \; 1 + \lambda_+ \Psi + \cdots \tag{V.128}$$

on $\hat{\mathcal{U}}_+$. Here, all λ-dependence has been made explicit, i.e., Ψ is defined on $\mathbb{C}^{4|2\mathcal{N}}$. Note that the expansion (V.128) can be obtained from some general ψ_+ by performing the gauge transformation $\psi_+ \mapsto (\psi_+^{(0)})^{-1}\psi_+$. Upon substituting (V.128) into (II.16), we obtain

$$\mathcal{A}_{\alpha\dot{1}} \; = \; \partial_{\alpha\dot{2}}\Psi, \qquad \mathcal{A}_{\alpha\dot{2}} \; = \; 0 \qquad \text{and} \qquad \mathcal{A}^i_{\dot{1}} \; = \; \partial^i_{\dot{2}}\Psi, \qquad \mathcal{A}^i_{\dot{2}} \; = \; 0. \tag{V.129}$$

Plugging (V.129) into the constraint equations (II.17), we find the following set of equations:

$$\begin{aligned} \partial_{1\dot{1}}\partial_{2\dot{2}}\Psi - \partial_{2\dot{1}}\partial_{1\dot{2}}\Psi + [\partial_{1\dot{2}}\Psi, \partial_{2\dot{2}}\Psi] \; &= \; 0, \\ \partial^i_{\dot{1}}\partial_{\alpha\dot{2}}\Psi - \partial_{\alpha\dot{1}}\partial^i_{\dot{2}}\Psi + [\partial^i_{\dot{2}}\Psi, \partial_{\alpha\dot{2}}\Psi] \; &= \; 0, \\ \partial^i_{\dot{1}}\partial^j_{\dot{2}}\Psi + \partial^j_{\dot{1}}\partial^i_{\dot{2}}\Psi + \{\partial^i_{\dot{2}}\Psi, \partial^j_{\dot{2}}\Psi\} \; &= \; 0. \end{aligned} \tag{V.130}$$

Let us now come back to the linear system (V.85) and the constraint equations (V.87) of the truncated self-dual SYM hierarchy. Upon imposing light-cone gauge, that is, upon assuming an expansion of the form

$$\psi_+ \; = \; 1 + \lambda_+ \hat{\Psi} + \cdots , \tag{V.131}$$

where now $\hat{\Psi}$ is defined on $\mathbb{C}^{M|N}$, we find

$$\begin{aligned} \mathcal{A}_{\alpha\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}} \; &= \; \partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}}\hat{\Psi} \qquad \text{and} \qquad \mathcal{A}_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}} \; = \; 0, \\ \mathcal{A}^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}} \; &= \; \partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\hat{\Psi} \qquad \text{and} \qquad \mathcal{A}^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}} \; = \; 0. \end{aligned} \tag{V.132}$$

Therefore, (V.87) turns into the following system:

$$\begin{aligned} &\partial_{\alpha\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}}\partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} - \partial_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}}\hat{\Psi} + \\ &\qquad\qquad + \; [\partial_{\alpha\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{m-1}}\hat{\Psi}, \partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi}] \; = \; 0, \\ &\partial^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} - \partial_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\hat{\Psi} + \\ &\qquad\qquad + \; [\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\hat{\Psi}, \partial_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi}] \; = \; 0, \\ &\partial^i_{\dot{1}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\partial^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\hat{\Psi} + \partial^j_{\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\hat{\Psi} + \\ &\qquad\qquad + \; \{\partial^i_{\dot{2}\dot{\alpha}_1\cdots\dot{\alpha}_{n-1}}\hat{\Psi}, \partial^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\hat{\Psi}\} \; = \; 0. \end{aligned} \tag{V.133}$$

Clearly, if $\hat{\Psi}$ solely depends on space-time coordinates, i.e., $\hat{\Psi}$ is independent of "higher time moduli", all the extra fields disappear and the above system reduces to (V.130).

If we define

$$\delta_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_m}\hat{\Psi} \;:=\; \partial_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_m}\hat{\Psi} \qquad \text{and} \qquad \delta^i_{\dot{\alpha}_1\cdots\dot{\alpha}_n}\hat{\Psi} \;:=\; \partial^i_{\dot{\alpha}_1\cdots\dot{\alpha}_n}\hat{\Psi}, \tag{V.134}$$

the system (V.133) implies

$$\begin{aligned}
\partial_{\alpha\dot{1}\cdots\dot{1}}\delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} - \partial_{\alpha\dot{2}\dot{1}\cdots\dot{1}}\delta_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} \;+ [\partial_{\alpha\dot{2}\dot{1}\cdots\dot{1}}\hat{\Psi}, \delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi}] &= 0,\\
\partial^i_{\dot{1}\cdots\dot{1}}\delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} - \partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\delta_{\beta\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi} \;+ [\partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\hat{\Psi}, \delta_{\beta\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{m-1}}\hat{\Psi}] &= 0,\\
\partial^i_{\dot{1}\cdots\dot{1}}\delta^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\hat{\Psi} - \partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\delta^j_{\dot{1}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\hat{\Psi} \;+ \{\partial^i_{\dot{2}\dot{1}\cdots\dot{1}}\hat{\Psi}, \delta^j_{\dot{2}\dot{\beta}_1\cdots\dot{\beta}_{n-1}}\hat{\Psi}\} &= 0.
\end{aligned} \tag{V.135}$$

If one differentiates these equations with respect to the space-time coordinates, one realizes that the resulting equations coincide with the linearized versions of (V.130). Putting it differently, some equations of the self-dual SYM hierarchy can be interpreted as equations on symmetries for the self-dual SYM equations.

§V.11 Summary. As for $\mathcal{N}$-extended self-dual SYM theory, we may now summarize the above discussion as follows:

Theorem V.3. *There is a one-to-one correspondence between gauge equivalence classes of local solutions to the truncated $\mathcal{N}$-extended self-dual SYM hierarchy of level $(m_1, m_2|n_1, \ldots, n_{\mathcal{N}})$ on four-dimensional space-time and equivalence classes of holomorphic vector bundles $\mathcal{E}$ over generalized supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_{\mathcal{N}}}$ which are smoothly trivial and holomorphically trivial on any projective line $\mathbb{C}P^1_{x_R,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}_{m_1,m_2|n_1,\ldots,n_{\mathcal{N}}}$.*

§V.12 Example. Let us now give an explicit example of a truncated hierarchy which also makes contact with our discussion presented in Sec. II.3. Consider the truncated $\mathcal{N}=2$ self-dual SYM hierarchy of level $(m_1, m_2|n_1, n_2) = (1, 1|2, 2)$. Its field equations are given by

$$\begin{aligned}
\mathring{f}_{\dot{\alpha}\dot{\beta}} &= 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\mathring{\nabla}^R_{\alpha\dot{\alpha}}\mathring{\nabla}^R_{\beta\dot{\beta}}\mathring{\phi}^i &= 0,\\
\epsilon^{\dot{\alpha}\dot{\beta}}\mathring{\nabla}^R_{\alpha\dot{\alpha}}\mathring{W}^{[ij]}_{\dot{\beta}\dot{\gamma}} - \{\mathring{\phi}^{[i}, \mathring{\nabla}^R_{\alpha\dot{\gamma}}\mathring{\phi}^{j]}\} &= 0,
\end{aligned} \tag{V.136}$$

and follow from (V.125). The $\mathcal{N}=2$ self-dual SYM equations, which are the first three equations of (II.36) (with $i, j = 1, 2$), are by construction a "subset" of (V.136). That is, apply to the last equation of (V.136) the operator $\overset{\circ}{\nabla}{}^R_{\beta\dot\delta}$ and contract with $\epsilon^{\alpha\beta}$ to obtain

$$\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\beta\dot\delta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{W}{}^{[ij]}_{\dot\beta\dot\gamma} - \epsilon^{\alpha\beta}\{\overset{\circ}{\nabla}{}_{\beta\dot\delta}{}^R\overset{\circ}{\phi}{}^{[i},\overset{\circ}{\nabla}{}^R_{\alpha\dot\gamma}\overset{\circ}{\phi}{}^{j]}\} \;=\; 0.$$

Eq. (V.115) together with

$$\epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\nabla}{}^R_{\beta\dot\beta} \;=\; \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\gamma}\overset{\circ}{\nabla}{}^R_{\beta\dot\delta}$$

imply

$$\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} \;&=\; 0,\\
\epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^i_{\dot\beta\beta} \;&=\; 0,\\
\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\nabla}{}^R_{\beta\dot\beta}\overset{\circ}{W}{}^{[ij]}_{\dot\gamma\dot\delta} + \epsilon^{\alpha\beta}\{\overset{\circ}{\chi}{}^{[i}_{\dot\gamma\alpha},\overset{\circ}{\chi}{}^{j]}_{\dot\delta\beta}\} \;&=\; 0.
\end{aligned} \tag{V.137}$$

These equations reduce to the $\mathcal{N}=2$ self-dual SYM equations when the dotted indices of $\overset{\circ}{\chi}{}^i_{\dot\alpha\alpha}$ and $\overset{\circ}{W}{}^{[ij]}_{\dot\alpha\dot\beta}$ are chosen to be one. Note that as i,j run only from one to two, the last equation of (V.136) can be rewritten in the form

$$\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} \;&=\; 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{\nabla}{}^R_{\beta\dot\beta}\overset{\circ}{\phi}{}^i \;&=\; 0,\\
\epsilon^{\dot\alpha\dot\beta}\overset{\circ}{\nabla}{}^R_{\alpha\dot\alpha}\overset{\circ}{G}_{\dot\beta\dot\gamma} - \epsilon_{ij}\{\overset{\circ}{\phi}{}^i,\overset{\circ}{\nabla}{}^R_{\alpha\dot\gamma}\overset{\circ}{\phi}{}^j\} \;&=\; 0,
\end{aligned} \tag{V.138}$$

where we have defined an anti-self-dual differential two-form according to $\overset{\circ}{G}_{\dot\alpha\dot\beta} := \epsilon_{ij}\overset{\circ}{W}{}^{[ij]}_{\dot\alpha\dot\beta}$. Note that these are exactly the field equations given in (II.75). Hence, (II.75) can be interpreted to describe the $(1,1|2,2)$ hierarchy of $\mathcal{N}=2$ self-dual SYM theory. Recall also that in this case we have an appropriate action principle leading to (V.138). It is an interesting fact that even though it is not possible to write down an action functional of $\mathcal{N}=2$ self-dual SYM theory ($\mathcal{P}^{3|2}$ is not formally Calabi-Yau), it is possible to find one for a certain hierarchy thereof.

V.4 Nonlocal conservation laws

What remains is to give the nonlocal conservation laws associated with the symmetry transformations discussed above. In what follows, we use the ideas of [4].

§V.13 Conserved nonlocal currents. The compatibility conditions of the linear system (II.16) can equivalently be written as

$$[\bar\nabla^\pm_\alpha, \bar\nabla^\pm_\beta] \;=\; 0, \qquad [\bar\nabla^\pm_\alpha, \bar\nabla^i_\pm] \;=\; 0 \qquad \text{and} \qquad \{\bar\nabla^i_\pm, \bar\nabla^j_\pm\} \;=\; 0. \tag{V.139}$$

Recall further that ϕ_{+-} as given in (V.8) satisfies

$$\bar\nabla^\pm_\alpha \phi_{+-} \;=\; 0 \qquad \text{and} \qquad \bar\nabla^i_\pm \phi_{+-} \;=\; 0, \tag{V.140}$$

where the covariant derivatives act in the adjoint representation. Then we may define

$$J^{+-}_\alpha \;:=\; \operatorname{tr}\left\{\phi_{+-}\delta\mathcal{A}^+_\alpha\right\} \;=\; \operatorname{tr}\left\{\phi_{+-}\nabla^+_\alpha\phi_+\right\}, \tag{V.141}$$

where $\delta\mathcal{A}^+_\alpha = \nabla^+_\alpha\phi_+$ is a symmetry of (V.139) given according to (V.12). Here, "tr" denotes the matrix trace. Notice that this expression is gauge invariant as both, ϕ_{+-} and $\delta\mathcal{A}^+_\alpha$ transform in the adjoint representation. Now it is a straightforward exercise to show that

$$\epsilon^{\alpha\beta}\bar V^+_\alpha J^{+-}_\beta \;=\; 0. \tag{V.142}$$

Indeed, we have

$$\begin{aligned}
\bar V^+_\alpha J^{+-}_\beta &= \operatorname{tr}\left\{-[\mathcal{A}^+_\alpha, \phi_{+-}]\delta\mathcal{A}^+_\beta\right\} + \operatorname{tr}\left\{\phi_{+-}\bar V^+_\alpha\delta\mathcal{A}^+_\beta\right\}\\
&= \operatorname{tr}\left\{-[\mathcal{A}^+_\alpha, \phi_{+-}]\delta\mathcal{A}^+_\beta\right\} + \operatorname{tr}\left\{\phi_{+-}\left(\bar V^+_\beta\delta\mathcal{A}^+_\alpha - [\delta\mathcal{A}^+_\alpha\mathcal{A}^+_\beta] - [\mathcal{A}^+_\alpha\delta\mathcal{A}^+_\beta]\right)\right\}\\
&= \operatorname{tr}\left\{\phi_{+-}\bar V^+_\beta\delta\mathcal{A}^+_\alpha\right\} + \operatorname{tr}\left\{-[\mathcal{A}^+_\beta, \phi_{+-}]\delta\mathcal{A}^+_\alpha\right\}\\
&= \bar V^+_\beta J^{+-}_\alpha.
\end{aligned}$$

This allows us to associate with any symmetry transformation a conserved current according to

$$j_{\alpha\dot\alpha} \;:=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+\, J^{+-}_\alpha \lambda^+_{\dot\alpha}, \tag{V.143}$$

since

$$\partial^{\alpha\dot\alpha} j_{\alpha\dot\alpha} \;=\; 0 \tag{V.144}$$

by virtue of Eq. (V.142). In (V.143), the contour $\mathscr{C} = \{\lambda_+ \in \mathbb{C}P^1 \,|\, |\lambda_+| = 1\}$ encircles $\lambda_+ = 0$. It is important to stress that (V.143) is a superfield. Moreover, Eq. (V.143) can slightly be rewritten to get

$$\begin{aligned}
j_{\alpha\dot\alpha} &= \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+ \operatorname{tr}\left\{\phi_{+-}\bar\nabla^+_\alpha\phi_+\right\}\lambda^+_{\dot\alpha},\\
&= \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+\, \lambda^+_{\dot\alpha}\bar V^+_\alpha\left(\operatorname{tr}\left\{\phi_{+-}\phi_+\right\}\right)
\end{aligned} \tag{V.145}$$

being more in spirit of Penrose's integral formulas. Recall also that due to Eqs. (V.41) and (V.62), certain combinations of the coefficients functions $\phi_{\pm}^{0(m)}$ determine the explicit transformations $\delta\mathcal{A}_{\alpha\dot\alpha}$ and $\delta\mathcal{A}^i_{\dot\alpha}$ and hence contribute to the integrals (V.143) and (V.145), respectively.

Finally, we stress that one may also associate currents with symmetries in the following way. Let Υ_{+-} be any $\mathfrak{gl}(r,\mathbb{C})$-valued function which is annihilated by $\bar{V}^{\pm}_{\alpha}$, $\bar{V}^i_{\pm}$ and $\partial_{\bar\lambda_\pm}$.[12] Then it is not too difficult to see that $\psi_+\Upsilon_{+-}\psi_-^{-1}$ satisfies

$$\bar\nabla^{\pm}_{\alpha}(\psi_+\Upsilon_{+-}\psi_-^{-1}) \;=\; 0 \qquad \text{and} \qquad \bar\nabla^{i}_{\pm}(\psi_+\Upsilon_{+-}\psi_-^{-1}) \;=\; 0, \tag{V.146}$$

the the covariant derivatives act in the adjoint representation. Therefore, we may take

$$\tilde{j}_{\alpha\dot\alpha} \;:=\; \frac{1}{2\pi\mathrm{i}}\oint_{\mathscr{C}} \mathrm{d}\lambda_+\,\mathrm{tr}\left\{\psi_+\Upsilon_{+-}\psi_-^{-1}\delta\mathcal{A}^+_{\alpha}\right\}\lambda^+_{\dot\alpha}, \tag{V.147}$$

which is conserved by the above reasoning.

[12] See also our discussion given in §III.20.

Summary and discussion

In this thesis, we have reported on various aspects of supersymmetric gauge theories within the supertwistor approach. In particular, we first gave a detailed twistor description of $\mathcal{N}$-extended self-dual SYM theory. We in addition discussed some related self-dual models which follow from self-dual SYM theory by suitable reductions. Their twistor formulation requires certain weighted projective superspaces as twistor manifolds. As we have shown, all these manifolds are formally Calabi-Yau thus being naturally equipped with globally well-defined holomorphic volume forms. This property enabled us to also present appropriate action functionals for these models. Besides four-dimensional self-dual models, we also discussed a dimensional reduction to three dimensions. As a result, we obtained certain supersymmetric Bogomolny models. In fact, we generalized Hitchin's twistor construction of non-Abelian monopoles to a (maximally) supersymmetric setting. In addition, appropriate action principles on mini-supertwistor space and Cauchy-Riemann twistor space were given and shown to be equivalent to the action functional reproducing the field equations of the Bogomolny model. In connection to this, we were naturally led to the notion of Cauchy-Riemann supermanifolds and to partially hCS theory. Moreover, we discussed certain complex structure deformations on mini-supertwistor space. As a result, we obtained a supersymmetric Bogomolny model in three space-time dimensions with massive fermionic and scalar fields. Similar to the massless case, we also derived appropriate action principles on the twistor manifolds in question. We furthermore developed novel solution generating techniques by studying infinitesimal deformations of vector bundles on mini-supertwistor and Cauchy-Riemann twistor spaces, respectively. The algorithms were then exemplified in the case where only fields with helicity ± 1 and the Higgs field were nontrivial. As we argued, the corresponding Abelian configurations give rise to the Dirac monopole-antimonopole systems. The fifth chapter of this thesis was devoted to the studies of hidden symmetries of $\mathcal{N}$-extended self-dual SYM theory. We first gave a detailed cohomological interpretation of hidden symmetries of self-dual SYM theory. We saw how general deformation algebras on the twistor side are mapped to corresponding symmetry algebras in self-dual SYM theory. Kac-Moody algebras as affine extensions of internal symmetries were constructed. In addition, we discussed affine extensions of space-time symmetries, that is, we obtained an affinization of the superconformal algebra. The algebra in question turned out to be of Kac-Moody-Virasoro-type. As was argued, the existence of such infinite-dimensional algebras of hidden nonlocal symmetries originates from the fact that the full group of continuous transformations acting on the space of holomorphic vector bundles over supertwistor space is a semi-direct product of the group of local holomorphic automorphisms of the supertwistor space and of the group of one-cochains with respect to a certain covering with values in the sheaf of holomorphic maps of supertwistor space into the gauge group. Besides symmetry algebras, we constructed certain self-dual SYM hierarchies. The basis of this construction was a generalization of twistor space. As we saw, such a hierarchy consists of an infinite system of partial differential equations, where the self-dual SYM equations are embedded in. As was shown, the lowest level flows of the hierarchies in question represent supertranslations. Indeed, the existence of such hierarchies allows us to embed a given solution into an infinite-parameter family of new solutions. Moreover, a detailed derivation of the field equations together with the corresponding superfield expansions for the truncated hierarchies was presented.

However, there are certainly a lot of open issues and questions which deserve further investigations:

- An obvious and challenging task is the generalization of the constructions given in Chap. V to the full SYM theory. In Chap. IV, we saw how solutions to the field equations of $\mathcal{N} = 3$ SYM theory are related to certain holomorphic vector bundles over superambitwistor space $\mathcal{L}^{5|6}$. In principle, the algorithms relating infinitesimal deformations of vector bundles over supertwistor space to symmetries being developed for self-dual SYM theory in Chap. V can also be applied to full $\mathcal{N} = 3$ SYM theory. This is basically because of the vanishing of appropriate cohomology groups. Furthermore, as a matter of fact, Thm. V.2. also applies to the full case which is due to the structural similarity of the corresponding linear system. Hence, once given a suitable deformation algebra (of Lie-algebra type) on the twistor side, one automatically has a hidden symmetry algebra in full $\mathcal{N} = 3$ SYM theory.

- Besides questions associated with symmetry transformations, etc., one in addition needs to write down the related conserved nonlocal currents and charges – not only as superfields but also in components. In Chap. V, we considered particular nonlocal conservation laws associated with any symmetry of the field equations. It is desirable, to generalize such constructions to the full $\mathcal{N}=3$ theory. Presumably, the construction of fermionic conserved currents[13], that is, currents obeying equations of the form $D^{i\dot\alpha}j_{i\dot\alpha}=0=D^\alpha_i j^i_\alpha$ will simplify the discussion, since with the help of those one may derive conservation laws of the form $\partial^{\alpha\dot\alpha}j_{\alpha\dot\alpha}=0$ by way of

$$j_{\alpha\dot\alpha} \;:=\; \int \mathrm{d}^6\theta\mathrm{d}^6\eta\,\theta^i_\alpha j_{i\dot\alpha} + \text{ h.c.}$$

However, to proceed further (also in view of passing to the quantum theory) and to clarify the physical significance of such currents, one clearly needs to find their superfield expansions. In addition, one should then compute the classical Poisson brackets among the corresponding charges. After passing to the quantum regime, a challenging task will be to make contact with the quantum symmetry algebras considered in [84, 85]. Moreover, it would also be interesting to see how such quantum symmetry algebras fit into the context of twistor string theory [263] and how they can be understood within the recently proposed twistor approach to $\mathcal{N}=4$ SYM theory [58].

- Another issue also worthwhile to explore is the construction of hidden symmetry algebras and hierarchies of gravity theories, in particular of conformal supergravity (see, e.g., [98] for a review). The description of conformal supergravity in terms of the supertwistor correspondence has been discussed in [263, 50, 9]. By applying similar techniques as those presented in Chap. V, one will eventually obtain hidden infinite-dimensional symmetry algebras of the conformal supergravity equations, generalizing the results known for self-dual gravity. See Refs. [60, 236, 185, 201, 132, 151, 88, 89], for instance.

- Recall that it was conjectured by Ward [253, 254, 255] that all integrable models in less than four space-time dimensions can be obtained from self-dual YM theory in four dimensions. Examples are the nonlinear Schrödinger equation, the Korteweg-de Vries equation, the sine-Gordon model, etc. In particular, they all follow from the self-dual YM equations upon implementing suitable algebraic ansätze for the

[13] See, e.g., [3] for such constructions in ten-dimensional SYM theory.

gauge potential followed by a dimensional reduction. In a similar spirit, the Ward conjecture can be "supersymmetrically" extended in order to derive the supersymmetric versions of the above-mentioned models. Therefore, it would be of interest to take the $\mathcal{N}$-extended self-dual SYM hierarchy presented in Chap. V and to derive the corresponding super hierarchies of these integrable systems in less than four dimensions.

- Finally, we address additional points for further investigations being related to noncommutative field theories. In [218], we (in collaboration with Christian Sämann) considered $\mathcal{N}=4$ SYM theory on a nonanticommutative superspace, that is, instead of $\mathbb{C}^{m|n}=(\mathbb{C}^m,\mathcal{O}_{\rm red}(\Lambda^\bullet\mathbb{C}^n))$ we took $\mathbb{C}^{m|n}_\hbar=(\mathbb{C}^m,\mathcal{O}_{\rm red}(\mathrm{Cliff}(\mathbb{C}^n)))$, where $\mathrm{Cliff}(\mathbb{C}^n)$ denotes the Clifford algebra of $\mathbb{C}^n$

$$\{\eta_i,\eta_j\} \;=\; \hbar C_{ij} \qquad \text{for} \qquad i,j=1,\ldots,n.$$

Upon introducing an involution corresponding to Euclidean signature, we derived the superfield expansions and the field equations of deformed $\mathcal{N}=4$ SYM theory to first order in the deformation parameter $\hbar$. In showing this, we proposed an extension of the Seiberg-Witten map [223] to superspace. Our derivation was based on the $\mathcal{N}=4$ formulation of the constraint system. Clearly, one may straightforwardly translate our results into the $\mathcal{N}=3$ formulation of $\mathcal{N}=4$ SYM theory. It would then be of interest to establish a nonanticommutative version of the superambitwistor correspondence. In particular, Eqs. (IV.16) induce a nonanticommutative structure on superambitwistor space. In addition, generalizing the results of [177], it should in principle be possible to derive a corresponding action functional of deformed $\mathcal{N}=4$ SYM theory which is still lacking. Moreover, splitting and dressing methods obtained from twistor theory have successfully been applied to the construction of solitons and instantons in noncommutative field theories. See, e.g., Refs. [152, 153, 154, 265, 120, 125, 155, 156, 87, 258]. It would also be interesting to see how the solution generating techniques as considered in, e.g., Chap. III need to be generalized to noncommutative field theories. Partial results on that matter have already been given in [158]. In addition, the methods used in this thesis might also shed light on the question of hidden symmetries in noncommutative/nonanticommutative field theories.

List of symbols

$\mathbb{R}$ ($\mathbb{C}$)	real (complex) numbers
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$, where $\mathbb{Z}$ is the integers
R	supercommutative ring
$M,\ N$	R-modules
$X,\ Y,\ Z,\ \ldots$	(super)manifolds
TX	tangent bundle (respectively, sheaf) of a (super)manifold X
$N_{Y\|X},\ N_Y$	normal bundle (respectively, sheaf) of a (super)manifold Y in a (super)manifold X
$\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \ldots$	sheaves
$\mathcal{E},\ \mathcal{F},\ \mathcal{G},\ \ldots$	vector bundles
$\mathcal{I}$	ideal sheaf
$\mathcal{N}$	sheaf of nilpotents
$\mathcal{U},\ \mathcal{V},\ \mathcal{W},\ \ldots$	open sets
$\mathfrak{U},\ \mathfrak{V},\ \mathfrak{W},\ \ldots$	(open) coverings
$\overline{\mathscr{D}}$	Cauchy-Riemann structure
$\mathscr{T}$	$\mathscr{T}$-structure, distribution
$\mathcal{O}_X$	sheaf of holomorphic functions
$\mathcal{S}_X$	sheaf of smooth functions
$\mathcal{C}_X$	sheaf of $\mathscr{T}$-functions
$\mathcal{O}_{\mathrm{red}} = \mathcal{O}_X/\mathcal{N}$	reduced sheaves
$\mathcal{S}_{\mathrm{red}} = \mathcal{S}_X/\mathcal{N}$	
$\mathcal{C}_{\mathrm{red}} = \mathcal{C}_X/\mathcal{N}$	
$(Y, \mathcal{O}_Y^{(k)} = \mathcal{O}_X/\mathcal{I}^k)$	k-th formal neighborhood of Y in X
$\mathcal{H}_X$	group of local biholomorphisms on a (super)manifold X
Π	parity map

$\mathcal{O}_{\mathbb{C}P^{m\|n}}(-1)$	tautological sheaf on $\mathbb{C}P^{m\|n}$, $\mathcal{O}_{\mathbb{C}P^{m\|n}}(1) = \mathcal{O}_{\mathbb{C}P^{m\|n}}(-1)^*$, $\mathcal{O}_{\mathbb{C}P^{m\|n}}(d) = \mathcal{O}_{\mathbb{C}P^{m\|n}}(1)^{\otimes d}$
$C^q(\mathfrak{U}, \mathcal{A})$	set of q-cochains with values in $\mathcal{A}$
$Z^q(\mathfrak{U}, \mathcal{A})$	set of q-cocycle with values in $\mathcal{A}$
$H^q(\mathfrak{U}, \mathcal{A})$, $H^q(X, \mathcal{A})$	q-th Čech cohomology set with values in $\mathcal{A}$
$\operatorname{End} \mathcal{E}$	endomorphism bundle
$\operatorname{sdet} \mathcal{E}$	superdeterminant line bundle
$c_k(\mathcal{E})$ $(ch_k(\mathcal{E}))$	k-th Chern class (character)
$\Omega^k(X)$	sheaf of differential k-forms
$\Omega^k_0(X)$	sheaf of differential k-forms with compact support
$\Omega^k_{\mathscr{T}}(X)$	sheaf of $\mathscr{T}$-differential k-forms
$\Omega^{p,q}(X)$	sheaf of differential (p, q)-forms
$\Omega^{p,q}_{\mathrm{CR}}(X)$	sheaf of CR differential (p, q)-forms
$\mathrm{Ber}(X)$	sheaf of volume forms
$\mathrm{Ber}_0(X)$	sheaf of volume forms with compact support
$\Sigma^k(X)$	sheaf of integral k-forms
$\Sigma^{p,q}(X)$	sheaf of integral (p, q)-forms
$\Lambda^\bullet$	exterior algebra
$\mathrm{Hom}(M, N)$	group of morphisms between R-modules M and N
$\mathscr{H}om(\mathcal{A}, \mathcal{B})$	sheaf of local morphisms between sheaves $\mathcal{A}$ and $\mathcal{B}$
$\mathrm{Mat}(m\|n, p\|q, R)$	morphisms between free R-modules of rank $m\|n$ and $p\|q$
$GL(m\|n, R)$	Lie group of invertible even automorphisms of $R^{m\|n}$
$\mathfrak{gl}(m\|n, R)$	Lie algebra of $GL(m\|n, R)$
$\mathfrak{H} = GL(m\|n, \mathcal{O}_X)$	sheaf of GL-valued holomorphic functions
$\operatorname{Lie} \mathfrak{H} = \mathfrak{gl}(m\|n, \mathcal{O}_X)$	sheaf of $\mathfrak{gl}$-valued holomorphic functions
$\mathfrak{S} = GL(m\|n, \mathcal{S}_X)$	sheaf of GL-valued smooth functions
$\operatorname{Lie} \mathfrak{S} = \mathfrak{gl}(m\|n, \mathcal{S}_X)$	sheaf of $\mathfrak{gl}$-valued smooth functions
$\mathfrak{C} = GL(m\|n, \mathcal{C}_X)$	sheaf of GL-valued $\mathscr{T}$-functions
$\operatorname{Lie} \mathfrak{C} = \mathfrak{gl}(m\|n, \mathcal{C}_X)$	sheaf of $\mathfrak{gl}$-valued $\mathscr{T}$-functions

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List of related publications

The subsequent list of publications represents the scientific work the author of this thesis was involved in during his Ph.D. studies (chronologically reversed order):

Martin Wolf
Twistors and aspects of integrability of self-dual SYM theory
Proc. of the Intern. Workshop on Supersymmetries and Quantum Symmetries (SQS'05), Eds. E. Ivanov and B. Zupnik, Dubna, 2005, [hep-th/0511230]

With the help of the Penrose-Ward transform, which relates certain holomorphic vector bundles over the supertwistor space to the equations of motion of self-dual SYM theory in four dimensions, we construct hidden infinite-dimensional symmetries of the theory. We also present a new and shorter proof (cf. hep-th/0412163) of the relation between certain deformation algebras and hidden symmetry algebras. This article is based on a talk given by the author at the Workshop on Supersymmetries and Quantum Symmetries 2005 at the BLTP in Dubna, Russia.

Alexander D. Popov, Christian Sämann and Martin Wolf
The topological B model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations
JHEP **0510**, 058 (2005) [hep-th/0505161]

In the recent paper hep-th/0502076, it was argued that the open topological B model whose target space is a complex 2|4-dimensional mini-supertwistor space with D3- and D1-branes added corresponds to a super Yang-Mills theory in three dimensions. Without the D1-branes, this topological B model is equivalent to a dimensionally reduced holomorphic Chern-Simons theory. Identifying the latter with a holomorphic BF-type theory, we describe a twistor correspondence between this theory and a supersymmetric Bogomolny model on $\mathbb{R}^3$. The connecting link in this correspondence is a partially holomorphic Chern-Simons theory on a Cauchy-Riemann supermanifold which is a real one-dimensional fibration over the mini-supertwistor space. Along the way of proving this twistor correspondence, we review the necessary basic geometric notions and construct action functionals for the involved theories. Furthermore, we discuss the geometric aspect of a recently proposed deformation of the mini-supertwistor space, which gives rise to mass terms in the supersymmetric Bogomolny equations. Eventually, we present solution generating techniques based on the developed twistorial description together with some examples and comment briefly on a twistor correspondence for super Yang-Mills theory in three dimensions.

Martin Wolf
On hidden symmetries of a super gauge theory and twistor string theory
JHEP **0502**, 018 (2005) [hep-th/0412163]

We discuss infinite-dimensional hidden symmetry algebras (and hence an infinite number of conserved nonlocal charges) of the $\mathcal{N}$-extented self-dual super Yang-Mills equations for general $\mathcal{N} \leq 4$ by using the supertwistor correspondence. Furthermore, by enhancing the supertwistor space, we construct the $\mathcal{N}$-extended self-dual super Yang-Mills hierarchies, which describe infinite sets of graded Abelian symmetries. We also show that the open topological B model with the enhanced supertwistor space as target manifold will describe the hierarchies. Furthermore, these hierarchies will in turn – by a supersymmetric extension of Ward's conjecture – reduce to the super hierarchies of integrable models in $D < 4$ dimensions.

Alexander D. Popov and Martin Wolf
Topological B model on weighted projective spaces and self-dual models in four dimensions
JHEP **0409**, 007 (2004) [hep-th/0406224]

It was recently shown by Witten on the basis of several examples that the topological B model whose target space is a Calabi-Yau (CY) supermanifold is equivalent to holomorphic Chern-Simons (hCS) theory on the same supermanifold. Moreover, for the supertwistor space $\mathbb{C}P^{3|4}$ as target space, it has been demonstrated that hCS theory on $\mathbb{C}P^{3|4}$ is equivalent to self-dual $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions. We consider as target spaces for the B model the weighted projective spaces $W\mathbb{C}P^{3|2}[1,1,1,1|p,q]$ with two fermionic coordinates of weight p and q, respectively - which are CY supermanifolds for $p+q=4$ – and discuss hCS theory on them. By using twistor techniques, we obtain certain field theories in four dimensions which are equivalent to hCS theory. These theories turn out to be self-dual truncations of $\mathcal{N} = 4$ SYM theory or of its twisted (topological) version.

Christian Sämann and Martin Wolf
Constraint and super Yang-Mills equations on the deformed superspace $\mathbb{R}^{4|16}_\hbar$
JHEP **0403**, 048 (2004) [hep-th/0401147]

It has been known for quite some time that the $\mathcal{N} = 4$ super Yang-Mills equations defined on four-dimensional Euclidean space are equivalent to certain constraint equations on the Euclidean superspace $\mathbb{R}^{4|16}$. In this paper we consider the constraint equations on a deformed superspace $\mathbb{R}^{4|16}_\hbar$ a la Seiberg and derive the deformed super Yang-Mills equations. In showing this, we propose a super Seiberg-Witten map.

Curriculum Vitae

Day of Birth	February 19^{th}, 1979
Place of Birth	Zwickau, Germany
09/2003 – present	University of Hannover, Ph.D. Student
07/2003	Dresden University of Technology and University of Hannover Diplom (M.Sc.) in Physics, Thesis Title: On Topologically Nontrivial Field Configurations in Noncommutative Field Theories Thesis Advisors: Olaf Lechtenfeld, Alexander Popov, Gerhard Soff
06/2002 – 06/2003	University of Hannover, Diploma Student
08/2000 – 05/2002	Duke University, Exchange Student
1999 – 2000	Max Planck Institute for Physics of Complex Systems, Dresden, Research Assistent
08/1999	Dresden University of Technology, Vordiplom (B.Sc.) in Physics
10/1997 – 06/2003	Dresden University of Technology, Diploma Student
06/1997	Johann H. Pestalozzi Gymnasium, Zwickau, Abitur
09/2003 – present	scholarship of the Graduierten Kolleg No. 282 *Quantenfeldtheoretische Methoden in der Teilchenphysik, Gravitation, Statistischen Physik und Quantenoptik* at the University of Hannover
08/2001 – 12/2001	scholarship of the Center for Geometry and Theoretical Physics at the Duke University
08/2000 – 05/2001	scholarship of the DAAD (German Academic Exchange Service) at the Duke University
07/2000 – 06/2003	scholarship of the Studienstiftung des Deutschen Volkes (German National Academic Foundation)