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Fully populated VCM or the hidden parameter

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Abstract: Least-squares estimates are trustworthy with minimal variance if the correct stochastic model is used. Due to computational burden, diagonal models that neglect correlations are preferred to describe the elevation dependency of the variance of GPS observations. In this contribution, an improved stochastic model based on a parametric function to take correlations between GPS phase observations into account is presented. Built on an adapted and flexible Mátern function accounting for spatiotemporal variabilities, its parameters can be fixed thanks to Maximum Likelihood Estimation or chosen apriori to model turbulent tropospheric refractivity fluctuations. In this contribution, we will show in which cases and under which conditions corresponding fully populated variance covariance matrices (VCM) replace the estimation of a tropospheric parameter. For this equivalence "augmented functional versus augmented stochastic model" to hold, the VCM should be made sufficiently large which corresponds to computing small batches of observations. A case study with observations from a medium baseline of 80 km divided into batches of 600 s shows improvement of up to 100 mm for the 3Drms when fully populated VCM are used compared with an elevation dependent diagonal model. It confirms the strong potential of such matrices to improve the least-squares solution, particularly when ambiguities are let float.

Keywords: correlations, equivalence stochastic functional model, GNSS phase measurement, hidden tropospheric parameter, least-squares, Mátern covariance function, stochastic model

1 Introduction

Because of an overdetermined system of equations with more observations than unknowns, GNSS measurements are often processed with least-squares estimation methods. The functional model which describes the relationship between the observations and the parameters to be estimated is well-known (Hofmann-Wellenhof et al. 2001); the same cannot be said for the stochastic model. However, the correct modelling of non-deterministic effects can be considered as a prerequisite in order to reach a minimum variance of the estimates. Heteroscedasticity of GPS residuals (Bischoff et al. 2005) is widely assumed and the elevation dependency of the variance of GNSS observations is described thanks to cosine, exponential or CNO/SNR based functions, see exemplarily Vermeer (1997), Wang et al. (1998) or Luo et al. (2014). Even if many factors act on correlating the observations, such as the atmosphere (Schön and Brunner 2008), or the receiver itself (Bona 2000, Amiri-Simkooei and Tiberius 2007), correlations remain mostly disregarded. Besides computational demanding iterative procedures on the residuals (Koch 1999, Teunissen and Amiri-Simkooei 2008), empirical models for correlations between GNSS measurements have been concretely used (El-Rabbany 1994, Howind et al. 1999). However, due to a lack of an accurate and plausible description, correlations are often neglected. Additionally, diagonal variance covariance matrices (VCM) are less difficult to handle than fully populated VCM accounting for correlations. Since the least-squares solution remains unbiased even with approximated stochastic models as long as the residuals are zero-mean, no main differences are expected at the estimates level in ideal cases. This was confirmed for example by Radovanovic (2001). However, when correlations are neglected, the least-squares estimator is less efficient and significance tests biased (Williams et al. 2003). The consequences are for example an overoptimistic precision, a worthier ambiguity resolution or outlier detection (Kermarrec and Schön 2017, Amiri-Simkooei et al. 2016, Li et al. 2017 for Beidou). The development of a better and realistic stochastic model is a way to face this issue (Tralli and Lichten 1990).

Based on a Mátern covariance function and physical considerations, Kermarrec and Schön (2017) proposed a new approach to describe elevation dependent correlations in an understandable manner. This function has two main parameters: the smoothness and a correlation parameter and thus allows a greater flexibility with respect

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to simpler non elevation dependent functions, such as the first order Gauss Markov model (AR(1)) proposed by El-Rabbany (1994). To model atmospheric effects, the parameters can be fixed to given values following Kermarrec and Schön (2014).

In this contribution, we mathematically derive how integrating fully populated VCM built with this function in the least-squares adjustment can impact the least-squares solution. Indeed, in some particular cases, not only the test statistics become more trustworthy and less biased under such an improved stochastic model but also the estimates themselves can be impacted. Thus, the structure of the paper is as follows: in the first part, we will describe shortly the proposed correlation model. The second part explains the concept of the "hidden elevation dependent parameter" to present when such a function can replace a non-estimable tropospheric parameter. In the third part and thanks to an example, we will more concretely highlight the impact on the solution of non-diagonal VCM build with the proposal. The appendix deals with the problem of precision and ambiguity resolution.

2 Stochastic model: a proposal for correlations

2.1 Mathematical background

The point positioning problem is usually solved by first linearizing the observation equations w.r.t. the unknown parameters. Based on approximate parameter values, the so-called linearized functional model is obtained that describes the mathematical relationship between the estimates and the observations. After rearranging, the Observed Minus Computed (OMC) term can be computed which is the difference between actual observations and modelled observations. The corresponding equation is:

$$\mathbf{y} = \mathbf{A}\Delta\mathbf{x} + \varepsilon \tag{1}$$

In this contribution, we assume a relative positioning scenario with GNSS phase observations. We call $\mathbf{y}$ the $n \times 1$ vector of Observed-Minus-Computed (OMC) double differences, ε the $n{\times}1$ error vector. We assume that the error term has zero mean and a normal distribution, $E\left(\varepsilon\varepsilon^{\mathbf{T}}\right) = \sigma^2\mathbf{W}_0$ where $\mathbf{W}_0$ is the positive definite and fully populated cofactor matrix of the double differences and E the mathematical expectation. σ^2 is the apriori variance factor. Dealing with phase measurements is inherently ambiguous, the ambiguities are estimated in a first step as float, i.e. part of the functional model. The design matrix for GNSS positioning can be thus partitioned as $\mathbf{A} = [\ \mathbf{A_C} \quad \mathbf{A_A}\ ]$. The $(n, 3)$ matrix $\mathbf{A_C}$ and the (n, n_{amb}) matrix $\mathbf{A_A}$ describe the design matrices of the coordinates and ambiguities, respectively, where n and n_{amb} are the number of double differences and the number of double differenced ambiguities, respectively. If a tropospheric parameter has to be estimated, the design matrix is extended accordingly, as described for example in Kermarrec and Schön (2016). Similarly to the design matrix, the correction vector for the unknown parameters $\Delta\mathbf{x} = [\ \Delta\mathbf{x_C} \quad \mathbf{x_A}\ ]$ is divided into a correction on the estimated coordinates and the float ambiguity.

The Generalized Least Squares Estimator (GLSE) reads $\Delta\hat{\mathbf{x}}_0 = \left(\mathbf{A^T}\mathbf{W}_0{}^{-1}\mathbf{A}\right)^{-1}\mathbf{A^T}\mathbf{W}_0{}^{-1}\mathbf{y}$ (Koch 1999). In practise, $\mathbf{W}_0$ is unknown and replaced by an assumption or apriori variance covariance matrix (VCM) which we call $\hat{\mathbf{W}}$. As a consequence, the feasible generalized least-squares solution (FGLSE) is given by:

$$\Delta\hat{\mathbf{x}} = \left(\mathbf{A^T}\hat{\mathbf{W}}^{-1}\mathbf{A}\right)^{-1}\mathbf{A^T}\hat{\mathbf{W}}^{-1}\mathbf{y} \tag{2}$$

The apriori cofactor matrix of the estimated parameter $\Delta\hat{\mathbf{x}}$ is $\mathbf{Q}_{\hat{\mathbf{x}}}=\left(\mathbf{A^T}\hat{\mathbf{W}}^{-1}\mathbf{A}\right)^{-1}$, partitioned as follows into an ambiguity and coordinates part:

$$\mathbf{Q}_{\hat{\mathbf{x}}} = \begin{bmatrix} \mathbf{Q_C} & \mathbf{Q_{CA}} \\ \mathbf{Q_{CA}} & \mathbf{Q_A} \end{bmatrix} \tag{3}$$

Calling $\mathbf{v}$ the vector of residuals and $n-u$ the degree of freedom, the aposteriori variance factor for the FGLSE is given by

$$\hat{\sigma}^2_{\mathbf{W}} = \frac{\left(\mathbf{y}-\mathbf{A}\Delta\hat{\mathbf{x}}\right)^{\mathbf{T}}\hat{\mathbf{W}}^{-1}\left(\mathbf{y}-\mathbf{A}\Delta\hat{\mathbf{x}}\right)}{n-u} = \frac{\mathbf{v^T}\hat{\mathbf{W}}^{-1}\mathbf{v}}{n-u} \tag{4}$$

The least-squares estimator is unbiased, consistent and efficient if the least-squares assumptions are not violated, particularly if the residuals are 0-mean and the correct stochastic model is used (Williams et al. 2003). In case of GNSS positioning, heteroscedasticity should be taken into account in the modelling as well as correlations between measurements, when needed. It is thus of central importance for a trustworthy positioning to avoid misspecifications of the stochastic model and describe the temporal relationship between observations.

Fixing the ambiguities to integer

For a high accuracy of the solution, the float ambiguity vector should be fixed to integer. Various strategies can be used from a simple rounding to more complicated methods such as the FARA (Erickson 1992) or the Lambda method (Teunissen 1995). To prevent from a wrong fixing to integer, the fixed ambiguity vector has to be validated.

This can be done for example thanks to discriminant tests such as the ratio test (Verhagen and Teunissen 2013). Eventually, a Fixed Failure-rate Ratio Test (Wang and Feng 2013) or look-up tables (Teunissen and Verhagen 2009) can be used. When not otherwise mentioned, we made use of the Lambda method to fix the ambiguity and use a simple ratio test with a threshold of 0.5 (Wei and Schwarz 1995). As will be shown, the results of this contribution are not impacted by the fixing or validation method. For the sake of completeness however, a short analysis of the impact of correlations on the ratio test is proposed in the appendix.

2.1.1 A proposal to model temporal correlations

An adapted version of the model developed by Kermarrec and Schön (2014) is chosen to describe temporal elevation dependent correlations of GNSS phase measurements. The reader is referred to Kermarrec and Schön (2017) for more details on the choice of this function as well as a comparison with existing strategies such as the model from El-Rabbany (1994) or empirical ARMA processes (Luo et al. 2012).

The covariance C between two observations of satellites with PRN i and j at time t and $t+\tau$ reads:

$$C_{it}^{jt+\tau} = \frac{\rho_{weight}\delta}{\sin\left(El_i\left(t\right)\right)\sin\left(El_j\left(t+\tau\right)\right)}\left(\alpha\tau\right)^{\nu}K_{\nu}\left(\alpha\tau\right) \quad (5)$$

El_i and El_j are the elevations of the satellite with PRN i and j respectively, ρ_{weight} is a weighting factor modelling the covariance between different satellites. In this contribution, we choose to fix $\rho_{weight} = 1$ for correlations between observations from one satellite with itself (i.e. i=j) and disregard correlations between different satellites, i.e. $\rho_{weight} = 0$ if $i \neq j$. Although our model allows to account for cross-correlations based on physical considerations (Kermarrec and Schön 2014), it is unnecessary to account for them for the following derivation about the augmented stochastic model. δ is a scaling parameter so that the variance equals 1 for satellites at 90° elevation. α is called the correlation parameter $\left[s^{-1}\right]$ and ν the smoothness. The modified Bessel function of order ν (Abramowitz and Stegun 1972) is denoted by K_ν. Through this contribution, we will refer to the set $[\alpha, \nu]$ as the "Mátern parameters set". This covariance function is derived from a rational spectral density function (Kermarrec and Schön 2014) and thus the corresponding VCM $\hat{\mathbf{W}}_{\mathbf{UD},corr}$ of undifferenced phase observations are positive definite (Mátern 1960).

The spectral density of the Mátern covariance function is given by:

$$S\left(\omega\right) = \frac{2^{\nu-1}\gamma\Gamma\left(\nu+d/2\right)\alpha^{2\nu}}{\pi^{d/2}\left(\omega^2+\alpha^2\right)^{\nu+d/2}} \quad (6)$$

where $\omega^2 = \omega_1^2 + \omega_2^2 + \ldots + \omega_d^2$ is the angular frequency, Γ the Gamma function (Abramowitz and Segun 1972). The dimension of the field d is 1 in case of time series of observations. From Eq. (6), it can be seen that the behaviour of $S(\omega)$ by letting $\omega \to 0$ is both influenced by the smoothness ν and the correlation parameters α, whereas ν plays a more important role in filtering high frequencies (i.e. as $\omega \to \infty$).

Since the Mátern covariance function in Eq. (5) is weighted by an elevation dependent factor, the covariance is different for each satellite or satellite pairs. The Mátern parameters can be computed by Maximum Likelihood Estimation (Kermarrec and Schön 2017) and are thus depending on the observations (L1, L2, eventually P1 or P2) or alternatively fixed. The value $\nu = 1/2$ corresponds for instance to a first order Gauss Markov process, i.e. an exponential function as proposed by El-Rabbany (1994). The values of $[\alpha, \nu] = [0.008, 5/6]$ following Kermarrec and Schön (2014) and Wheelon (2001) were shown to model tropospheric correlations due to the turbulent variations of the refractivity index.

Through this contribution, we will make use of the set $[\alpha, \nu] = [0.01, 1.05]$ to model elevation dependent correlations due to atmospheric effects. The reasons of this particular choice are shortly highlighted:

- The mean-square differentiability of the field is ensured (Stein 1999) which is for physical reasons an interesting property of the covariance function. Indeed, seeing a GPS unit as a combination of resistors, capacitors and inductors, the differentiability of the current intensity on time and so the measured quantity has to be given. The voltage of the inductor is for instance proportional to the time derivative of the current which may thus be finite (Kermarrec et al. 2017).
- By taking a slightly higher smoothness than 5/6 (i.e. the "tropospheric" value), the correlation parameter α has to be reduced making use of the non-orthogonality property of the Mátern covariance function (Gelfand et al. 2011). This result was confirmed by Kermarrec and Schön (2017).

Both for the sake of numerical stability when inverting fully populated matrices and for modeling additional white noise, the undifferenced VCM $\hat{\mathbf{W}}_{\mathbf{UD},fully}$ are built as a linear combination of $\hat{\mathbf{W}}_{\mathbf{UD},corr}$ and the identity matrix $\mathbf{I}$

modelling white noise as follows:

$$\hat{\mathbf{W}}_{\mathbf{UD},fully} = (1-\beta)\hat{\mathbf{W}}_{\mathbf{UD},corr} - \beta\mathbf{I} \tag{7}$$

β is a positive noise factor between 0 and 1 which can be estimated from the OMC or fixed apriori. This proposal corresponds to an elementary model as proposed by Schwieger (2007). Undifferenced matrices $\hat{\mathbf{W}}_{\mathbf{UD},fully}$ can be built for each satellite with Eq. (7) for a chosen number of epochs η_{epoch} where the satellite is visible, i.e. the batch length. The whole matrix is referred to as the FULLY VCM. The corresponding diagonal VCM $\hat{\mathbf{W}}_{\mathbf{UD},elev}$ where only heteroscedasticity is taken into account is called the ELEV model. Its elements are corresponding to the diagonal of $\hat{\mathbf{W}}_{\mathbf{UD},corr}$, i.e. the commonly used cosine model. Subsequently, the cofactor matrix for a relative positioning scenario with double differences reads $\hat{\mathbf{W}} = \mathbf{M}^{\mathbf{T}}\hat{\mathbf{W}}_{\mathbf{UD}}\mathbf{M}$, where $\mathbf{M}$ is the matrix operator of double differencing.

3 The hidden parameter

Dealing with OMC, we assume that the ionosphere and the troposphere are firstly modelled with enough accuracy in the pre-processing step (Hoffmann and Wellenhof 1999). In some cases, e.g. for medium-long baselines from approximately 20 km length, tropospheric effects do not cancel out by double differencing. Thus a differential tropospheric parameter is estimated as part of the functional model. Due to its small variations between epochs, one value is computed per satellite for session from 1 hour of observations, i.e. one batch of observations. As the temporal resolution is restricted as a consequence of the lack of separability between parameters, particularly with the Up component, usually no additional parameter is estimated for shorter sessions. Unfortunately, the effect of the troposphere still impacts the coordinates for sessions shorter than one hour.

In this section, we will show how a fully populated VCM computed with Eq. (7) can replace the estimation of a tropospheric parameter for sessions shorter than one hour in an elegant way. The mathematical derivation proposed by Blewitt (1998) is presented and extended to the particular case of GPS.

3.1 Augmented functional model versus augmented stochastic model

Augmented functional model

In order to improve the solution of Eq. (1), an additional parameter $\Delta\mathbf{z}$ can be taken into account. For the GPS case, we can consider $\Delta\mathbf{z}$ to be a differential tropospheric parameter. In that case the augmented model reads:

$$\mathbf{y} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{z} + \boldsymbol{\epsilon} \tag{8}$$

$\mathbf{B}$ is the design matrix with dimension $\eta_{sat} \times \eta_{epoch}$ corresponding to $\Delta\mathbf{z}$ where η_{sat} is the number of visible satellites. If Eq. (8) is written in terms of partitioned matrices, it can be shown by applying the lemma on matrix inversions for symmetric matrices that the solution $\Delta\hat{\mathbf{x}}$ is given by $\Delta\hat{\mathbf{x}} = (\mathbf{A}^{\mathbf{T}}\hat{\mathbf{W}}^{-1}\mathbf{P}\mathbf{A})^{-1}\hat{\mathbf{W}}^{-1}\mathbf{P}\mathbf{y}$ with $\mathbf{P} = \mathbf{I} - \mathbf{B}(\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1}$ being a projection operator.

We can thus define a reduced weight matrix as

$$\hat{\mathbf{W}}_{\mathbf{red}}^{-1} = \hat{\mathbf{W}}^{-1}\mathbf{P} = \hat{\mathbf{W}}^{-1} - \hat{\mathbf{W}}^{-1}\mathbf{B}(\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1} \tag{9}$$

If the estimates are expressed as $\Delta\hat{\mathbf{x}} = (\mathbf{A}^{\mathbf{T}}\hat{\mathbf{W}}_{\mathbf{red}}^{-1}\mathbf{A})^{-1}\hat{\mathbf{W}}_{\mathbf{red}}^{-1}\mathbf{y}$, a parallel with Eq. (2) can be drawn. With the knowledge of $\hat{\mathbf{W}}_{\mathbf{red}}$, it is thus possible to compute $\Delta\hat{\mathbf{x}}$ without having to compute $\Delta\hat{\mathbf{z}}$. This is exactly what we aim to achieve in the GPS case for short sessions, due to the lack of separability between the tropospheric and Up parameters. Unfortunately, the reduced weight matrix $\hat{\mathbf{W}}_{\mathbf{red}}^{-1}$ is singular. As a consequence, assessing the stochastic model which would lead to such a VCM and allows for the direct computation of $\hat{\mathbf{W}}_{\mathbf{red}}$ is impossible.

Augmented stochastic model

This difficulty can be overcome by seeing the augmented parameter $\Delta\mathbf{z}$ as a source of noise, i.e. a "process noise", similarly to what is done in Kalman filtering. Concretely, we define $\boldsymbol{\epsilon}_{red}$ as an augmented noise, i.e. $\boldsymbol{\epsilon}_{red} = \mathbf{B}\Delta\mathbf{z} + \boldsymbol{\epsilon}$. As a consequence, the augmented stochastic model reads

$$\begin{aligned}\hat{\mathbf{W}}_{\mathbf{red}}^{\star} &= E(\boldsymbol{\epsilon}_{red}\boldsymbol{\epsilon}_{red}^{T}) \\ &= E((\mathbf{B}\Delta\mathbf{z} + \boldsymbol{\epsilon})(\mathbf{B}\Delta\mathbf{z} + \boldsymbol{\epsilon})^{\mathbf{T}}) \\ &= E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}) + \mathbf{B}\mathbf{E}(\Delta\mathbf{z}\Delta\mathbf{z}^{\mathbf{T}})\mathbf{B}^{\mathbf{T}} \\ &= \hat{\mathbf{W}} + \mathbf{B}\hat{\mathbf{W}}_{\mathbf{z}}\mathbf{B}^{\mathbf{T}}\end{aligned} \tag{10}$$

where $\hat{\mathbf{W}}_{\mathbf{z}}$ is the apriori covariance matrix of the additional parameter. To make a parallel with Eq. (9), $\hat{\mathbf{W}}_{\mathbf{red}}^{\star}$ can be inverted so that

$$\begin{aligned}\hat{\mathbf{W}}_{\mathbf{red}}^{\star-1} &= (\hat{\mathbf{W}} + \mathbf{B}\hat{\mathbf{W}}_{\mathbf{z}}\mathbf{B}^{\mathbf{T}})^{-1} \\ &= (\hat{\mathbf{W}}^{-1} - \hat{\mathbf{W}}^{-1}\mathbf{B}(\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1}\mathbf{B} + \hat{\mathbf{W}}_{\mathbf{z}}^{-1})^{-1}\mathbf{B}^{\mathbf{T}}\hat{\mathbf{W}}^{-1})\end{aligned} \tag{11}$$

It can be seen from Eq. (11) that the equivalence between $\hat{\mathbf{W}}_{\mathbf{red}}^{\star-1}$ and $\hat{\mathbf{W}}_{\mathbf{red}}^{-1}$ holds only if $\hat{\mathbf{W}}_{\mathbf{z}}^{-1}$ is made small or alternatively if $\hat{\mathbf{W}}_{\mathbf{z}}$ dominates in Eq. (10). Moreover, $\hat{\mathbf{W}}_{\mathbf{z}}$ should not contain apriori information on the variance of the process noise (Blewitt 1998) which is already taken into account in **B**.

The "hidden" tropospheric parameter

In this section, we aim to present didactically how matrices built with Eq. (7) are corresponding to an augmented stochastic model, i.e. a "hidden" estimation of a tropospheric parameter. This highlights how taking correlations into account for short batches can replace the estimation of this additional "non-estimable" parameter.

To this end, we first note that the matrix **B** is filled with the squared root of the elements of $\hat{\mathbf{W}}_{\mathbf{UD,elev}}$ (Kermarrec and Schön 2016). Furthermore, in Eq. (10), we make the assumption that $E\left(\varepsilon\varepsilon^T\right) = \mathbf{I}$ and assume homoscedasticity of the errors defined in Eq. (1). We build $E(\Delta\mathbf{z}\Delta\mathbf{z}^\mathbf{T})$ based on a simplified version of Eq. (5), i.e. $C_t^{t+\tau} = \delta(\alpha\tau)^\nu K_\nu(\alpha\tau)$ and choose the Mátern parameters corresponding to a tropospheric modelling with $[\alpha, \nu] = [0.01, 1.05]$, thus identical for all satellites (Kermarrec and Schön 2014). We intentionally disregard the elevation dependency. Therefore $\hat{\mathbf{W}}_\mathbf{z}$ is fulfilled under the aforementioned condition to account for correlations introduced by the data's dependence on the process noise with "no prior information on the variance of the process noise" (Blewitt 1999).

Returning shortly to section 2, we notice that the elevation dependent factor in Eq. (5) is based on a cosine function whose square root is also used to fill **B**. Therefore we can write $\hat{\mathbf{W}}_{\mathbf{UD},fully} = \mathbf{B}\hat{\mathbf{W}}_\mathbf{z}\mathbf{B}^T$ and express the VCM of the augmented noise as

$$\hat{\mathbf{W}}^\star_{\mathbf{red}} = \mathbf{I} + \hat{\mathbf{W}}_{\mathbf{UD,fully}} \tag{12}$$

Condition for the equivalence

We have seen that for the equivalence to hold, $\hat{\mathbf{W}}_\mathbf{z}$ should be built to account for correlations, so that the process noise dominates in Eq. (11). This can be seen starting for example from the equivalent diagonal model presented the appendix, where correlations appear to act similarly to a large weighting factor of the corresponding diagonal matrix, in this case the identity matrix. If correlations are neglected, $\hat{\mathbf{W}}_\mathbf{z} = \mathbf{I}$. Thus the equivalence is much weaker, besides the fact that it does not correspond anymore to a covariance matrix for the tropospheric parameter. Similarly, if the correlation length is much smaller than the batch length, the corresponding fully populated VCM are sparse and nearly correspond to a diagonal VCM, i.e. the 0-value of the covariance is rapidly reached with respect to the batch length.

Using the proposed Mátern parameter set $[\alpha, \nu] = [0.01, 1.05]$, the corresponding correlation length is approximately 600 s. As a consequence, we propose to define a batch-size limit for the equivalence to hold fixed to 3600 s (1 hour of observations). This is also the often assumed condition whether to estimate a tropospheric parameter, independently of the data rate. Eventually, it is possible to decrease α or increase ν to fill the matrix more strongly. Besides the fact that it deviates strongly from a tropospheric correlation model, it has the disadvantage of impacting also the aposteriori variance factor (Kermarrec and Schön 2017) and can thus only be used under the control that no overestimation occurs which will correspond to an underestimation of the precision (Appendix).

From the reduced matrix to a VCM for GPS phase measurements

We note that in Eq. (12) $\hat{\mathbf{W}}^\star_{\mathbf{red}}$ is not corresponding to a cofactor matrix for GPS, i.e. a value of 1 for the variance for a satellite at 90° is not given anymore. Hence, although the estimates will not be influenced by the scaling (Kutterer 1999), the results of statistical tests such as the overall model test cannot be compared anymore with the usually used ELEV model. Thus we use instead a scaled matrix so that the reduced matrix reads $\hat{\mathbf{W}}^\star_{\mathbf{red}} = \beta_{\mathbf{red}}\mathbf{I} + (\mathbf{1}-\beta_{\mathbf{red}})\hat{\mathbf{W}}_{\mathbf{UD,fully}}$, β_{red} being a noise parameter between 0 and 1. By doing so, we slightly weaken the equivalence by decreasing the impact of $\hat{\mathbf{W}}_{\mathbf{UD,fully}}$ (Kermarrec and Schön 2017). This is unproblematic using the proposed Mátern parameter set and mentioned batch length limit. Eventually the weakening could be compensated by decreasing α from 0.005, using the non-orthogonality of $[\alpha, \nu]$ (Stein 1999).

The circle is now complete as the same expression is obtained as in Eq. (7). As a consequence, when correlations are taken into account with the proposed model of Eq. (5), we account for a tropospheric parameter without estimating it explicitly, a "hidden" parameter.

Note that we could have taken $E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T) = \hat{\mathbf{W}}_{\mathbf{UD,elev}}$ in Eq. (10), which would have corresponded to an elevation dependent noise following Radovanovic (2001). This choice is left to the reader. The authors have a preference for an identity noise matrix to make a parallel with the Tikhonov regularization.

3.2 An additional interpretation of fully populated VCM

In the previous section, we have explained how using fully populated VCM can replace the estimation of a tropospheric parameter, the equivalence being valid as long as the VCM is made sufficiently large, i.e. for short batches.

It is worth additionally mentioning that in case of short batches in GPS positioning, the ideal assumption for the least-squares estimator to be unbiased are often not reached (Rao and Toutenburg 1999, Koch 1999). For example, non-normal errors of the residuals may significate that F-distributions cannot be assumed for the aposteriori variance factor but either student distribution (Williams et al. 2013). Moreover, the condition that the residuals are zero-mean may not be fulfilled, particularly for long baselines when observations have drifts due to unmodelled remaining effects. Fortunately, when fully populated matrices build with Eq. (5) are taken into account in the least-squares adjustment, a filtering of such unwanted effects is obtained. This can be seen thanks to Eq. (6), e.g. the smoothness and the correlation parameter impact the frequency content of the observations. As a consequence, using FGLSE with the FULLY model instead of the purely diagonal ELEV model, a decrease of the error of the least-squares solution is obtained corresponding to a lower loss of efficiency. This leads to a more trustworthy position with an associated non overoptimistic precision and better test statistics such as overall model, outlier detection tests or ambiguity validation tests. (see appendix for more details).

3.3 Ambiguity fixed

Through the development of the equivalence, we have considered a global model and assumed that the ambiguity is estimated as float together with the position and not fixed in advance (Eq. (1)). If the integer ambiguities are known in advance, the equivalence still holds. As it is not made used of the less biased float ambiguity under a more correct stochastic model particularly for short batches, the solution (i.e. coordinates) obtained with different VCM will be less different.

4 A case study

The concept of the hidden parameter is not straightforward to validate. Indeed as its name indicates, it corresponds to cases where no parameter can be estimated. In order to overcome this issue, a methodology is proposed based on decreasing the batch length and comparing the solution found under fully populated VCM with respect to a diagonal VCM in cases where the true position is known.

4.1 Observations

Data from the European Permanent Network EPN (Bruyninx et al. 2012) from two stations KRAW and ZYWI are chosen as example for a medium baseline (80km) positioning scenario. OMC observations are computed with 30s rate observations and a cut-off of 3°. The ionospheric and tropospheric delays are partially estimated in a preprocessing step with the Klobuchar and Hopfield models, respectively. A relative positioning scenario is considered and the North East Up (NEU) coordinates are estimated at GPS day DOY220, year 2015. The starting time is GPS-SOD 6000s and was taken arbitrarily. It was shown not to impact the conclusions, i.e. the geometry playing a minor role in the results of our comparison (Kermarrec and Schön 2017, Appendix 2). The reference values are the long term station coordinates from the EPN solution.

4.2 Methodology

We compute the least-squares results given by the FULLY VCM described in section 3 and the diagonal ELEV matrices. We place ourselves in a case where it is assumed that no tropospheric parameter can be estimated so that batches have a length of maximum 100 epochs at 30 s. In case of longer batches, an additional tropospheric parameter should be taken into consideration as the equivalence does not hold anymore, i.e. the FULLY model does not replace the tropospheric parameter. Five batch lengths were selected to show the influence of the fully populated VCM on the float solution when no tropospheric parameter is estimated:

1. 20 batches with 100 epochs (100-epochs-case, 60000 s)
2. 25 batches with 80 epochs (80-epochs-case, 60000 s)
3. 33 batches with 60 epochs (60-epochs-case, 59400 s)
4. 50 batches with 40 epochs (40-epochs-case, 60000 s)
5. 100 batches with 20 epochs (20-epochs-case, 60000 s)

As previously mentioned, a batch approach is retained, i.e. one solution is computed for each batch. The aim of this methodology is to show how decreasing the batch length, i.e. strengthening the equivalence augmented stochastic versus functional model, will impact the positioning.

To this end, a global estimator of the least-squares solution is retained. The reference being in our case the 0 vector since the position was known exactly, the 3Drms is computed for each batch and averaged over all batches for both stochastic models of consideration. The 3Drms difference between the ELEV and FULLY is then formed, i.e. $\sum_{i=1}^{m}(3Drms_{FULLY}(i))-\sum_{i=1}^{m}(3Drms_{ELEV}(i))$ where m is the number of batches corresponding to case 1-5. As the estimation of a tropospheric parameter mainly influences the height component, we similarly compute the rms difference for the height component only. For short batches, the

F-distribution of the ratio $\frac{\hat{\sigma}^2_{\hat{W}}}{\sigma^2_0}$ may not be given anymore (Williams et al. 2003, Kermarrec et al. 2017). Thus we only compute the mean of the aposteriori variance factor over all batches and compare it with the assumed apriori value to assess roughly the trustworthiness of the solution. We took $\sigma_0 = 4$ mm, i.e. a relevant and plausible value for double differences observations.

We choose to let the ambiguities float in order to have a "global" functional model and make use of the better estimated float ambiguities when improving the stochastic model. Moreover, a comparison of the results with different stochastic models is easier to follow as the fixing to the correct ambiguities strongly improve the final solution. Fixing the ambiguities in advance in a preprocessing step leads to less strong differences between ELEV and FULLY model following the results of Kermarrec and Schön (2017). Nevertheless, using fully populated VCM, more batches can be fixed with respect to the ELEV model as described in the appendix. As a consequence, the conclusions of the case study will not be impacted by this choice.

For the sake of completeness and although unrealistic, we add the results given when an additional tropospheric parameter is estimated with the ELEV model for the 40-epochs case.

4.3 Results

The results of the case study are presented in Table 1.

Impact of decreasing the batch length The impact of the stochastic model on the positioning decreases for longer batches. For the 100-epochs-case for example, a 3Drms difference of 0.1 mm is obtained which grows to 106 mm for the 20-epochs-case, highlighting the strong impact of the FULLY populated VCM. If the difference increases, $\sum_{i=1}^{m}(3Drms_{FULLY}(i))$ decreases and becomes closer to the 0 value. As mentioned in section 3, this result gives weight to the equivalence augmented stochastic – functional model as soon as the FULLY VCM are "full". Additionally and for case 1 for example, the value of $E\left(\hat{\sigma}_{WFULLY}\right) = 4.1$ mm is close to the chosen $\sigma_0 = 4$ mm so that the solution can be considered as trustworthy. This is not the case for the ELEV VCM where $E\left(\hat{\sigma}_{WELEV}\right) = 6.5$ mm highlights a model misspecification due to the biased aposteriori variance factor. The same conclusions hold true for the other cases, although the differences between ELEV and FULLY decreased as expected. Improving the stochastic model by means of correlations is thus of main importance to obtain both less biased test statistics and a better positioning.

Using the equivalence and without weakening the data strength, the Up component is strongly improved by up to 37 mm for the 20-epochs-case. This highlights the main importance of using fully populated matrices for short batches. This difference decreases to 10 mm for the 60-epochs-case and is nearly 0 for the 100-epochs case, i.e. for longer batches the use of fully populated model do not replace the estimation of an additional parameter.

Estimating a tropospheric parameter for short batches

In case an additional tropospheric parameter is nevertheless estimated - for case 4 for example -, as done for longer batches, we note that $E(\hat{\sigma}_{WELEV}) = 7.1$ mm. Thus a model misspecification is guessed using the ELEV VCM which is confirmed by the difference between the 3Drms FULLY-ELEV which is up to 90 mm higher than without estimating a parameter. As a consequence, the FULLY model is without a doubt a better alternative than the ELEV model.

Fixing the ambiguities to integer

If the ambiguities are fixed in advance in a preprocessing step, the differences between the models decrease. For the case 1 for example, a 3Drms difference of only 7 mm is obtained. Thus the effect of the FULLY model still impacts the solution but at a lower level. If the ambiguities are estimated as float and fixed for each batch using the ratio test with a threshold of 0.5 (Wei and Schwarz 1995), the fixing to integer can be improved by 5-10% following the results of the simulation presented in the appendix. As a consequence, improving the stochastic model will have a "snowball effect" on the 3Drms, the results of test statistics (ambiguity, outlier detection test, overall model test) being less biased as shown in the appendix for the ambiguity validation test (see also Li et al. 2016). Thus we definitively advice using such models, independently of the strategy used and particularly for short batches when the troposphere is expected to influence the results.

5 Conclusion

In this contribution, we made use of a weighted Mátern covariance function to describe the elevation dependent correlations of GNSS phase observations. For correlations due to turbulent tropospheric variations of the index of refractivity, the Mátern parameters (smoothness and correlation length) can be fixed apriori based on physical considerations. This function was mathematically shown to correspond to taking an additional tropospheric parame-

Table 1: 3Drms differences (FULLY-ELEV) model and rms differences for the Up component. Five different batch lengths are computed. The ambiguities are let float.

No trop.	Case 1	Case 2	Case 3	Case 4	Case 5
3Drms difference FULLY-ELEV [mm]	0.1	16.9	23.0	32.9	106.0
rms difference Up component FULLY-ELEV [mm]	−0.4	5	23.1	14.4	37.4
trop.	Case 4				
3Drms difference FULLY-ELEV [mm]	113.1				
rms difference Up component FULLY-ELEV [mm]	69.6				

ter into account without having to estimate it separately. This equivalence augmented stochastic model-functional model can be used as soon as the separability between the tropospheric parameter and the Up component is not ensured in the least-squares adjustment. It is thus particularly interesting for estimating the Up component with a higher trustworthiness in case of short batches of observations.

In a case study using double differenced observations from a 80 km baseline, this equivalence leads to an improvement of up to 10 cm for observations divided in batches of 20 epochs at 30 s with respect to an elevation dependent diagonal VCM when using the float ambiguities. Taking correlations into account leads thus in a noticeable way to an improvement of the positioning solution for short batches, particularly when the ambiguities cannot be fixed to integer with enough reliability and let float. The impact decreases for longer batches and if the ambiguities are fixed. However, less biased test statistics and a less overoptimistic precision is still obtained with respect to the purely diagonal model. The equivalence holds as soon as the covariance is made sufficiently large. This condition was translated for the GPS case and shown to be plausible for batches up to 3600 s length.

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A Appendix 1

The equivalent diagonal model

In this appendix, some insights on how correlations act on the apriori cofactor of the estimates (called the precision) and the ratio test are given. For didactic purposes, we use an AR(1) model for GPS phase correlations which corresponds to a smoothness of 1/2 in our proposal. In that particular case, the inverse of the corresponding VCM can be exactly expressed thanks to the known or estimated autocorrelation coefficient ρ_{AR} (Rao and Toutenburg 1999). In Kermarrec and Schön (2016), it is explained how correlations can be taken into account thanks to a reduced diagonal VCM.

The inverse of the equivalent VCM for the VCM from an AR(1) process reads:

$$\mathbf{W}^{-1}_{\mathbf{AR(1)\_EQUI}} = \frac{1}{1-\rho_{AR}^2}\begin{bmatrix} 1-\rho_{AR} & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1-\rho_{AR})^2 & 0 & \ddots & 0 & 0 \\ 0 & 0 & (1-\rho_{AR})^2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & (1-\rho_{AR})^2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1-\rho_{AR} \end{bmatrix}$$

To derive the inverse of the FULLY VCM, we assume low variations of the satellite elevation. Thus the elevation dependent factor of the covariance matrix derived thanks to the proposed model can be factorized. As a consequence, the elements of the corresponding equivalent diagonal VCM sorted per epochs for one satellite are:

- First and last diagonal values: $\frac{1}{\sin(El_i(t))^2}(1+\rho_{AR})$
- All other diagonal values: $\frac{1}{\sin(El_i(t))^2}\left(\frac{1+\rho_{AR}}{1-\rho_{AR}}\right))$

As highlighted in Luati and Proietti (2011), the equivalent VCM thus has two diagonal values -at the beginning and the end of a batch of observations- that are lower than the middle diagonal values, all elements being simply proportional to the ELEV model, i.e. corresponding to a higher weighting as $\left(\frac{1+\rho_{AR}}{1-\rho_{AR}}\right) > 1$.

Precision of the least-squares results using FULLY

This particularity of the equivalent VCM (or its inverse) has the consequence that when correlations are taken into account, the impact of the extreme values on the results is getting negligible for long batches. Thus the matrix $\mathbf{Q}_{\hat{\mathbf{x}}\mathbf{FULLY}} = (\mathbf{A}^{\mathbf{T}}\hat{\mathbf{W}}_{\mathbf{FULLY}}^{-1}\mathbf{A})^{-1}$ and $\mathbf{Q}_{\hat{\mathbf{x}}\mathbf{ELEV}} = (\mathbf{A}^{\mathbf{T}}\hat{\mathbf{W}}_{\mathbf{ELEV}}^{-1}\mathbf{A})^{-1}$ are only linked by a scaling factor depending on the correlation length. This result is derived for an AR(1) model and may be slightly different for higher smoothness where more diagonal entries than only the 2 first values are different than the middles values. Luatti and Proietti (2011) show for example that for an AR(3) model 3 first values were different. Thus, even for short batches, a scaling factor can link with a good approximation $\mathbf{Q}_{\hat{\mathbf{x}}\mathbf{FULLY}}$ and $\mathbf{Q}_{\hat{\mathbf{x}}\mathbf{ELEV}}$ when our proposed model is used. As a consequence, the error ellipsoids will have slightly the same orientation in space and the precision with a FULLY model will be more realistic, i.e. no overestimation as for diagonal VCM will occur. The least-squares solution is therefore more trustworthy.

Impact on ambiguity resolution of FULLY

The second consequence of this result can be shown at the ambiguity fixing level. Indeed, when using the Fixed Failure Rate Ratio Test (FFRT) with a FULLY model to estimate an accurate threshold (Wang and Feng 2013), it is expected that the same value as with an ELEV VCM will be found.

Independently of the chosen threshold, the impact of misspecifying the stochastic model up to neglecting correlations on the ratio test defined as $R = \frac{||\hat{\mathbf{x}}_{\mathbf{A},\mathbf{fix}}^{1}-\hat{\mathbf{x}}_{\mathbf{A},\mathbf{float}}||_{\mathbf{Q}_{\mathbf{A}}}}{||\hat{\mathbf{x}}_{\mathbf{A},\mathbf{fix}}^{2}-\hat{\mathbf{x}}_{\mathbf{A},\mathbf{float}}||_{\mathbf{Q}_{\mathbf{A}}}} = \frac{d_1}{d_2} \leq \mu_R$ (Euler and Schaffrin 1991) can be assessed. We call $\hat{\mathbf{x}}_{\mathbf{A},\mathbf{fix}}^{\mathbf{1}}$, $\hat{\mathbf{x}}_{\mathbf{A},\mathbf{fix}}^{\mathbf{2}}$ the two vectors of integer candidates that are corresponding to the two smallest values of the distance between the float and two fixed ambiguity vectors in the metric of the covariance matrix.

To assess the impact of the FULLY model on the ambiguity fixing, we make use of Monte Carlo simulations where time series corresponding to a true VCM with $[\alpha, \nu] = [0.01, 1]$ are computed. In order to assess the sensitivity of the model, the parameters $[\alpha, \nu]$ are varied around the true set where it can be shown from Eq. (6) that increasing corresponds to neglecting correlations. A constellation of 8 satellites observed during 3000s was taken in consideration and a relative positioning strategy used. To the 10000 simulated time series corresponding to the correlation structure of reference, the same but arbitrary ambiguity vector was added. The following results are independent of the constellations or the batch length (Kermarrec and Schön 2017 Appendix 2).

From Fig. 1, it can be clearly seen that neglecting correlations corresponds to a small increase of the ratio test by 0.1 and thus to a slight decrease of the probability to fix the ambiguities for a given similar threshold. This fact may be amplified in real cases when the least-squares assumption are slightly violated. This effect is emphasized when the correlation parameter is smaller than the reference, highlighting the importance of non-underestimating. In the ideal case of simulations, the ambiguities were fixed correctly with the Lambda method whether correlations are correctly taken in consideration or neglected. This may not be the case for real cases, particularly for small batches and thus correlations when present should not be disregarded as developed previously. As a consequence, it is expected that taking correlations into account leads to less biased ambiguity validation tests and thus allows an increase of the ambiguity success rate with respect to using a diagonal VCM for an assumed fix threshold.

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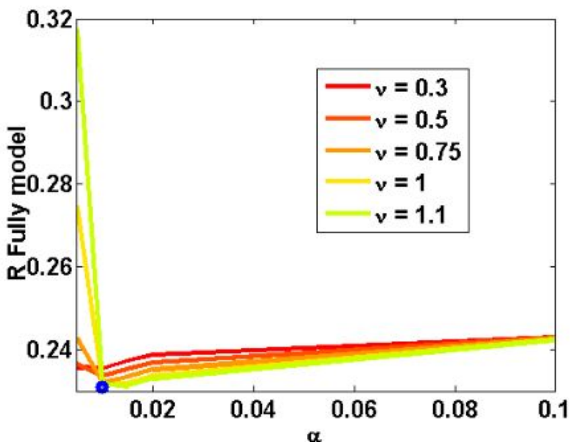

Figure 1: Results of the Monte Carlo simulations study with 10000 iterations per Mátern parameters set. Simulated observations are correlated with $[\alpha, \nu]_0 = [0.01, 1]$ (blue point), the Mátern parameters of the estimated VCM are varied. The mean value of the ratio R is presented.

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