T-duality without isometry via extended gauge symmetries of 2D sigma models

Athanasios Chatzistavrakidis,[a,b] **Andreas Deser**[a] **and Larisa Jonke**[c]

[a]*Institut für Theoretische Physik, Leibniz Universität Hannover,*
Appelstraße 2, 30167 Hannover, Germany

[b]*Van Swinderen Institute for Particle Physics and Gravity, University of Groningen,*
Nijenborgh 4, 9747 AG Groningen, The Netherlands

[c]*Division of Theoretical Physics, Rudjer Bošković Institute,*
Bijenička 54, 10000 Zagreb, Croatia

E-mail: a.chatzistavrakidis@rug.nl,
andreas.deser@itp.uni-hannover.de, larisa@irb.hr

Abstract: Target space duality is one of the most profound properties of string theory. However it customarily requires that the background fields satisfy certain invariance conditions in order to perform it consistently; for instance the vector fields along the directions that T-duality is performed have to generate isometries. In the present paper we examine in detail the possibility to perform T-duality along non-isometric directions. In particular, based on a recent work of Kotov and Strobl, we study gauged 2D sigma models where gauge invariance for an extended set of gauge transformations imposes weaker constraints than in the standard case, notably the corresponding vector fields are not Killing. This formulation enables us to follow a procedure analogous to the derivation of the Buscher rules and obtain two dual models, by integrating out once the Lagrange multipliers and once the gauge fields. We show that this construction indeed works in non-trivial cases by examining an explicit class of examples based on step 2 nilmanifolds.

Contents

1   Introduction

Dualities play a prominent role in many corners of modern theoretical physics (see ref. [1] for a very interesting recent discussion). In string theory dualities are instrumental in understanding the structure of the theory and study its fundamental properties. Notably, T-duality [2] is a symmetry of string theory that relates compactified backgrounds with inverse radii. The string background fields transform under T-duality according to a set of rules determined by Buscher in the seminal papers [3, 4]. This symmetry was subsequently proven a true symmetry between conformal field theories in ref. [5].

The approach followed by Buscher requires the existence of global isometries in the 2D sigma model, which are subsequently gauged. This approach was also followed in more involved cases, such as when there is a non-Abelian set of vector fields [6–13]. In all cases there is a set of invariance conditions and constraints to be obeyed. Most importantly the vector fields generate isometries, which seems to be necessary in order to write down a sigma model that is gauge invariant under standard gauge transformations.

Recently a new twist appeared in the construction of gauged sigma models. Kotov and Strobl (KS) [14] proved the existence of gauged symmetries in sigma models which do not correspond to global ones. This formulation is based on gauge symmetries associated to Lie algebroids,[1] a field pioneered by Strobl in the context of Yang-Mills theories [15] and

---

[1]For the purposes of this paper it will be sufficient to think of Lie algebroids simply as a generalization of Lie algebras with X-dependent structure functions instead of structure constants for a bracket that satisfies the Jacobi identity.

studied further in refs. [16–20]. The essence of this formulation relies on an extension of the standard infinitesimal gauge transformations for the gauge fields A^a to include a part proportional to DX^i, where X^i are the world sheet scalars and DX^i is the gauge covariant derivative on the world sheet obtained by minimal coupling. Then it is possible to construct an action which is invariant under these extended gauge transformations upon a milder condition than isometry. We will explain the basics of this formulation in section 2.

The above remarkable result immediately indicates that one can revisit Buscher's procedure in a more general context where isometries are not present. Earlier attempts to understand dualities on general backgrounds include Poisson-Lie T-duality [21–24], generalized T-duality for cases with no globally defined Killing vectors [8], and an approach based on "covariant coordinates" [25]. In the string theory context this is an important problem, given that one often encounters backgrounds that do not have isometries but one would like to know their T-dual backgrounds. In this paper we take this challenge. In particular we employ the formulation of KS and study gauged sigma models without isometry. These include a set of gauge fields as well as Lagrange multipliers in the same spirit as in Buscher's procedure. The conditions and constraints to be obeyed are determined and shown to be milder than the isometric case. This allows us to obtain two dual models, one by integrating out the Lagrange multipliers thus obtaining the original ungauged model, and one by integrating out the gauge fields. The latter yields a dual model which we describe in precise terms.

Given that several constraints appear in the formulation, it is natural to worry whether any non-trivial cases exist at all, namely whether the formulation is empty of non-trivial examples and isometry is always restored. We prove by an explicit toy example that this is not the case. This example is based on a well-known manifold used in studies of string duality, the 3D Heisenberg nilmanifold. This is a parallelizable manifold with a global section of its tangent bundle. The vector fields that form a basis for any such section are known and while one of them is Killing, the other two are not. Nevertheless, in the formulation established in the present paper we are able to T-dualize along all three directions. We perform this procedure in detail and discuss the dual model. Furthermore, we show that this is not an isolated example; the full class of step 2 nilmanifolds can be treated the same way, as we show in section 5. The examples we examine in this paper are not proper string backgrounds as they are not conformal; however they are often discussed in literature since they appear as T-dual of tori with H flux and have simple yet non-trivial geometric description.

2 Action and gauge symmetry

2.1 Preliminaries

Let us consider the standard σ-model action for the bosonic sector of closed string theory at leading order in α',

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} \mathrm{d}X^i \wedge \star \mathrm{d}X^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \,, \tag{2.1}$$

where $\Sigma_2 = \partial\Sigma_3$ is the 2D world sheet and $X = (X^i) : \Sigma_2 \to \mathrm{M}$ is the map from the world sheet to the target space M. Here and in the following we ignore the dilaton coupling, which enters the action at linear order in α' and leave a discussion on this issue for the future, since it requires different techniques than the ones we introduce here.

The standard approach to T-duality begins with the assumption of *global* target space symmetries generated by vector fields $v_a = v_a^i \partial_i$. This means that the action is required to be invariant under the global transformations

$$\delta_\epsilon X^i = v_a^i(X)\epsilon^a \,, \tag{2.2}$$

where ϵ^a are rigid transformation parameters. It is well-known that the invariance of the action (2.1) is not automatic but imposes the constraints

$$\mathcal{L}_{v_a} g = 0 \,, \tag{2.3}$$
$$\iota_{v_a} H = \mathrm{d}\theta_a \,, \tag{2.4}$$

for some arbitrary 1-forms $\theta_a = \theta_{ai}\mathrm{d}X^i$. This is true regardless whether the vector fields generate an Abelian or a non-Abelian algebra. We will let them here satisfy a non-Abelian one with structure constants C_{ab}^c,

$$[v_a, v_b] = C_{ab}^c v_c \,. \tag{2.5}$$

The next step is to gauge the above global symmetry. This is performed via the usual minimal coupling to gauge fields (1-forms) A^a, where the de Rham differentials on the world sheet are substituted by

$$DX^i = \mathrm{d}X^i - v_a^i(X)A^a \,, \tag{2.6}$$

and the local (gauge) transformations are given as

$$\begin{aligned} \delta_\epsilon X^i &= v_a^i(X)\epsilon^a(X) \,, \\ \delta_\epsilon A^a &= \mathrm{d}\epsilon^a(X) + C_{bc}^a A^b \epsilon^c(X) \,, \end{aligned} \tag{2.7}$$

with $\epsilon^a = \epsilon^a(X)$ the gauge parameters. The corresponding gauged action [6] includes additional fields but we will not discuss its precise form yet because we are going to present a more general result below. However let us mention that gauge invariance imposes additional constraints on top of (2.3) and (2.4). All these conditions and constraints will appear as a certain limit of the more general formulation that we present immediately below.

2.2 Gauging without isometry

As described in ref. [14], it is possible to write down gauged 2D σ-models even when there is no isometry to begin with. This is rather unconventional from a standard gauge theory viewpoint, where normally we gauge a symmetry that is already there as a rigid one. Here we refer to local symmetries that do not possess a global counterpart.

To be precise, let us consider the gauged action

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \tag{2.8}$$
$$- \int_{\Sigma_2} (\theta_a + \mathrm{d}\eta_a) \wedge A^a + \int_{\Sigma_2} \frac{1}{2} (\iota_{\upsilon_{[a}} \theta_{b]} + C_{ab}^c \eta_c) A^a \wedge A^b - \int_{\Sigma_2} \omega_{bi}^a \eta_a A^b \wedge DX^i \ .$$

The explanation of the ingredients is as follows. First of all, we defined the 1-forms

$$DX^i = \mathrm{d}X^i - \upsilon_a^i A^a,$$
$$\theta_a = \theta_{ai} \mathrm{d}X^i, \tag{2.9}$$

where A^a are again the gauge fields, and a set of auxiliary scalar fields η_a. The vector fields υ_a satisfy

$$[\upsilon_a, \upsilon_b] = C_{ab}^c(X) \upsilon_c \,,$$

where now the structure functions C_{ab}^c are not necessarily constants, namely we allow them to depend on X^i. This provides a straightforward generalization to sections in arbitrary Lie algebroids.[2] Finally, ω_{bi}^a are the components of a connection 1-form $\omega_b^a = \omega_{bi}^a \mathrm{d}X^i$ that twines the spacetime indices with the gauge ones. Its role will be clarified immediately below. Note that for vanishing ω_{bi}^a the action (2.8) is precisely the one considered in refs. [6, 13], where T-duality *with* isometry was studied.[3] The geometric interpretation of ω as a connection 1-form was first introduced in [20] (see also [26]); one can then introduce the corresponding exterior covariant derivative

$$D^\omega = d + \omega \wedge \tag{2.10}$$

and curvature

$$R_b^a = D^\omega \omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c \,. \tag{2.11}$$

The transformation properties of ω_{bi}^a are the same as for the spin connection. In particular, being an 1-form, it transforms covariantly in the index i.

Let us now specify the gauge transformations for the fields X^i, A^a and η_a. These have the form

$$\delta_\epsilon X^i = \upsilon_a^i \epsilon^a \,,$$
$$\delta_\epsilon A^a = \mathrm{d}\epsilon^a + C_{bc}^a A^b \epsilon^c + \omega_{bi}^a \epsilon^b DX^i \,,$$
$$\delta_\epsilon \eta_a = -\iota_{\upsilon_{(a}} \theta_{b)} \epsilon^b - C_{ab}^c \epsilon^b \eta_c + \upsilon_a^i \omega_{bi}^d \eta_d \epsilon^b \,, \tag{2.12}$$

for X-dependent parameters $\epsilon^a(X)$. It is directly observed that the gauge transformation for the gauge field A^a is extended in comparison to the standard one by an ω-dependent

---

[2]We point out that this is a *possible* generalization which is implemented here, but not a *necessary* one. The formulation we present is already a generalization of the standard one even in the Lie algebra case.

[3]In ref. [6] the fields η_a do not play a crucial role. This was revisited in ref. [13] where these fields are present and transform non-trivially under gauge transformations. This will be the case in our formulation too.

term proportional to DX^i. This is a key ingredient of the present formulation. Note that in 2D one can add a term proportional to $\star DX^i$; this is currently under investigation [27].

The action (2.8) is invariant under the above ω-extended gauge transformations provided that the following conditions hold

$$\mathcal{L}_{v_a} g = \omega_a^b \vee \iota_{v_b} g \,, \tag{2.13}$$

$$\iota_{v_a} H = \mathrm{d}\theta_a + \theta_b \wedge \omega_a^b - \eta_b R_a^b \,, \tag{2.14}$$

where $\vee$ denotes the symmetric product.[4] Similarly to the standard case there is a set of additional constraints, which now become

$$\mathcal{L}_{v_{[a}}\theta_{b]} = C_{ab}^d \theta_d - \iota_{v_d}\theta_{[a}\omega_{b]}^d - \iota_{v_{[a}}\omega_{b]}^d \theta_d - D_{ab}^c \eta_c \,, \tag{2.15}$$

$$\frac{1}{3}\iota_{v_a}\iota_{v_b}\iota_{v_c} H = \iota_{v_{[a}}C_{bc]}^d \theta_d - 2\iota_{v_{[a}}\omega_b^d \iota_{v_{c]}}\theta_d - 2\tilde{D}_{abc}^e \eta_e \,, \tag{2.16}$$

where we defined the shorthand notation

$$D_{ab}^e = \mathrm{d}C_{ab}^e + C_{ab}^c \omega_c^e + 2C_{d[a}^e \omega_{b]}^d + 2\iota_{v_d}\omega_{[b}^e \omega_{a]}^d + 2\mathcal{L}_{v_{[b}}\omega_{a]}^e + \iota_{v_{[a}}R_{b]}^e \,,$$

$$\tilde{D}_{abc}^e = \iota_{v_{[a}}\iota_{v_b}R_{c]}^e \,. \tag{2.17}$$

This result is obtained using the identity

$$C_{[ab}^d C_{c]d}^e + v_{[c}^k \partial_{\underline{k}} C_{ab]}^e = 0 \,, \tag{2.18}$$

which is the Jacobi identity in the Lie algebroid case where the structure functions are not constant. Note that sending ω_{bi}^a to zero and the functions C_{bc}^a to constants restores the isometric case and all the conditions fully agree[5] with the results of ref. [13]. One apparent difference between our formulation and previously studied ones is the explicit dependence of the constraints (2.15) and (2.16) on the scalar fields η_a. These scalar fields are essentially the analogues of the Lagrange multipliers introduced in Buscher's procedure, which become the coordinates of the T-dual model upon integration of the gauge fields.

At this stage it is useful to discuss the field strength of the gauge fields A^a. Recall that the 2-form that multiplies the Lagrange multipliers in Buscher's procedure is precisely the field strength of the corresponding gauge fields. In the present formulation this turns out to be

$$\mathcal{F}^a := \mathrm{d}A^a + \frac{1}{2}C_{bc}^a A^b \wedge A^c - \omega_{bi}^a A^b \wedge DX^i \,, \tag{2.19}$$

which is the same as the one considered in ref. [20]. A straightforward calculation confirms the result of [20] on the gauge transformation of this field strength:

$$\delta_\epsilon \mathcal{F}^a = (C_{bc}^a - \omega_{ci}^a v_b^i)\epsilon^c \mathcal{F}^b + R_{bij}^a \epsilon^b DX^i \wedge DX^j + D_{bci}^a \epsilon^c DX^i \wedge A^b \,, \tag{2.20}$$

---

[4]This means that eq. (2.13) reads in components as $(\mathcal{L}_{v_a} g)_{ij} = \omega_{a(i}^b v_{\underline{b}}^k g_{j)k}$ and it is obviously covariant, since ω_{ai}^b is an 1-form.

[5]Note that our conventions and notation are slightly different.

Although in the present paper we do not consider dynamics for the gauge fields, the above transformation rule is very suggestive. A covariant transformation rule for the field strength $\mathcal{F}^a$ requires

$$R^a_b = 0 \quad \text{and} \quad D^a_{bc} = 0 \, , \tag{2.21}$$

namely the flatness of the connection ω. In that case it is immediately observed that the η dependence in the constraints (2.15) and (2.16) drops out. This will be the case in the explicit examples that will be presented in later sections, where we will also make some essential comments about this flatness condition.

3 T-duality

In the previous section we considered the gauged action for a σ-model and discussed under which conditions it is gauge invariant. Now we would like to follow the spirit of Buscher's approach to T-duality and obtain the two T-dual models that stem from this action. In order to do so, we have to integrate out two different sets of fields. The original model should be obtained upon integration of the Lagrange multipliers η_a and gauge fixing, while the dual model is obtained by integrating out the gauge fields A^a.

3.1 Recovering the ungauged model

In order to recover the ungauged original model (2.1) we follow the steps described in detail below. First we lift the full action modulo the kinetic term to three dimensions:[6]

$$
\begin{aligned}
S = &\int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k - \\
&- \int_{\Sigma_3} \mathrm{d}(\theta_a + \mathrm{d}\eta_a - \omega^b_a \eta_b) \wedge A^a + \int_{\Sigma_3} (\theta_a + \mathrm{d}\eta_a - \omega^b_a \eta_b) \wedge \mathrm{d}A^a + \\
&+ \int_{\Sigma_3} \frac{1}{2} \mathrm{d}(\iota_{\upsilon_{[a}} \theta_{b]} + C^c_{ab} \eta_c - 2\omega^c_{[b\underline{i}} \upsilon^i_{a]} \eta_c) \wedge A^a \wedge A^b - \\
&- \int_{\Sigma_3} (\iota_{\upsilon_{[a}} \theta_{b]} + C^c_{ab} \eta_c - 2\omega^c_{[b\underline{i}} \upsilon^i_{a]} \eta_c) A^a \wedge \mathrm{d}A^b .
\end{aligned} \tag{3.1}
$$

Next, we covariantize the de Rham differentials of the H term and obtain:

$$
\begin{aligned}
S = &\int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} DX^i \wedge DX^j \wedge DX^k + \\
&+ \int_{\Sigma_3} \left(\frac{1}{2} H_{ijk} \upsilon^i_a A^a \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k - \frac{1}{2} H_{ijk} \upsilon^i_a \upsilon^j_b A^a \wedge A^b \wedge \mathrm{d}X^k + \right. \\
&\left. \qquad\qquad + \frac{1}{6} H_{ijk} \upsilon^i_a \upsilon^j_b \upsilon^k_c A^a \wedge A^b \wedge A^c \right) - \\
&- \int_{\Sigma_3} \mathrm{d}(\theta_a + \mathrm{d}\eta_a - \omega^b_a \eta_b) \wedge A^a + \int_{\Sigma_3} (\theta_a + \mathrm{d}\eta_a - \omega^b_a \eta_b) \wedge \mathrm{d}A^a + \\
&+ \int_{\Sigma_3} \frac{1}{2} \mathrm{d}(\iota_{\upsilon_{[a}} \theta_{b]} + C^c_{ab} \eta_c - 2\omega^c_{[b\underline{i}} \upsilon^i_{a]} \eta_c) \wedge A^a \wedge A^b - \\
&- \int_{\Sigma_3} (\iota_{\upsilon_{[a}} \theta_{b]} + C^c_{ab} \eta_c - 2\omega^c_{[b\underline{i}} \upsilon^i_{a]} \eta_c) A^a \wedge \mathrm{d}A^b .
\end{aligned} \tag{3.2}
$$

---

[6]Using differentiation and Stoke's theorem, and ignoring possible global issues.

Using the constraints imposed by gauge invariance and after a long and tedious calculation the action can be written in the following form:

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} DX^i \wedge DX^j \wedge DX^k +$$
$$+ \int_{\Sigma_3} (\theta_a - \iota_{v_{[b}} \theta_{a]} A^b) \wedge (\mathrm{d}A^a + \frac{1}{2} C^a_{bc} A^b \wedge A^c - \omega^a_{bi} A^b \wedge DX^i) + \qquad (3.3)$$
$$+ \int_{\Sigma_3} (\mathrm{d}\eta_a - C^c_{ba} \eta_c A^b - \omega^b_a \eta_b + 2\omega^c_{[a\underline{i}} v^i_{b]} \eta_c A^b) \wedge \left(\mathrm{d}A^a + \frac{1}{2} C^a_{bc} A^b \wedge A^c - \omega^a_{bi} A^b \wedge DX^i \right).$$

Now we integrate the Lagrange multiplier η_a from the gauged action. The equation of motion for η_a is:

$$\mathcal{F}^a = \mathrm{d}A^a + \frac{1}{2} C^a_{bc} A^b \wedge A^c - \omega^a_{bi} A^b \wedge DX^i = 0. \qquad (3.4)$$

This is the field strength we discussed in the previous section, which is simply the standard F^a of the non-Abelian gauge fields A^a when $\omega^a_{bi} = 0$. Inserting (3.4) in the form (3.3) of the action gives

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} DX^i \wedge DX^j \wedge DX^k . \qquad (3.5)$$

This result is identified with the original model in the same spirit as in the (Abelian or non-Abelian) isometric case [5, 9]. In particular, since $\mathcal{F}^a = 0$ the gauge fields must be pure gauges. Since at this stage we are working on-shell, these gauges may be fixed. The simplest gauge choice, which is the same as the one that was considered in refs. [5, 9], is $A^a = 0$. Then one immediately recovers the original model. Different gauge choices are of course allowed too, and then the original model in different coordinate systems is recovered.[7]

Finally let us note that introducing the shorthand notation[8]

$$K^c_{ab} = 2\iota_{v_{[b}} \omega^c_{a]} - C^c_{ab}, \qquad (3.6)$$

the action (3.3) can be rewritten in the form

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j + \int_{\Sigma_3} \frac{1}{6} H_{ijk} DX^i \wedge DX^j \wedge DX^k +$$
$$+ \int_{\Sigma_3} \mathcal{F}^a \wedge \theta_{ai} DX^i + \int_{\Sigma_3} \mathcal{F}^a \wedge (D^\omega \eta_a + K^c_{ab} \eta_c A^b). \qquad (3.7)$$

Comparing with the Abelian, isometric case (i.e. setting ω and C^a_{bc} to zero) and disregarding η_a we recover the action given earlier in the literature, e.g. [8]. Thus (3.7) is a natural generalization thereof, obtained by replacing $\mathrm{d}A^a$ by the appropriate field strength $\mathcal{F}^a$ and introducing the auxiliaries η_a in a covariant way. Moreover, since eq. (3.3) gives our starting action (2.8), this serves as an additional geometric motivation for the introduction of the ω- and η-dependent terms.

---

[7]Note that since the metric depends on X^i this procedure has to be carefully performed. We will provide a detailed account on that in a class of examples later on.

[8]Neither the structure constants C^a_{bc} nor the connection ω^a_b transform as tensors, but the combination K^a_{bc} does.

3.2 Obtaining the dual model

Let us now turn our attention to the dual model. This is obtained by integrating out the gauge fields A^a from the action. Varying the action (2.8) with respect to A^a results in the equations of motion

$$-g_{ij}v_a^i \star \mathrm{d}X^j + g_{ij}v_a^i v_b^j \star A^b + (\iota_{v_{[a}}\theta_{b]} - K_{ab}^c\eta_c)A^b + \theta_a + D^\omega\eta_a = 0 \,. \tag{3.8}$$

Similarly to [8, 13], it is useful to define the following tensors:

$$G_{ab} = v_a^i g_{ij} v_b^j \,, \tag{3.9}$$

$$D_{ab} = \iota_{v_{[a}}\theta_{b]} - K_{ab}^c\eta_c \,, \tag{3.10}$$

and

$$\xi_a = \theta_a + D^\omega\eta_a \,, \tag{3.11}$$

$$v_a^* = g_{ij}v_a^i\mathrm{d}X^j \,, \tag{3.12}$$

which now contain the components of ω. The equation of motion takes the simpler form

$$\star v_a^* - \xi_a = G_{ab} \star A^b + D_{ab}A^b \,. \tag{3.13}$$

Inserting this into the action (2.8) yields an expression linear in the gauge fields:

$$S = \int_{\Sigma_2} \frac{1}{2}g_{ij}\mathrm{d}X^i \wedge \star\mathrm{d}X^j - \int_{\Sigma_2} \frac{1}{2}A^a \wedge (\star v_a^* - \xi_a) + \int_{\Sigma_3} \frac{1}{6}H_{ijk}\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \,. \tag{3.14}$$

The next step requires solving equation (3.13). This can be done as follows. We make the general Ansatz

$$A^a = M^{ab}v_b^\star + N^{ab}\xi_b + P^{ab} \star v_b^\star + Q^{ab} \star \xi_b \,, \tag{3.15}$$

with coefficients to be determined. Inserting this Ansatz in the equation of motion and using $\star^2 = 1$ we obtain the matrix equations

$$
\begin{aligned}
GM + DP &= 1\,, \\
GN + DQ &= 0\,, \\
GP + DM &= 0\,, \\
GQ + DN &= -1\,,
\end{aligned}
\tag{3.16}
$$

with G and D given in (3.9) and (3.10). The solution of this system gives

$$Q = -(G - DG^{-1}D)^{-1} \,, \tag{3.17}$$

and the rest of the unknowns are determined in terms of Q as

$$
\begin{aligned}
M &= -Q\,, \\
N &= -G^{-1}DQ\,, \\
P &= G^{-1}DQ \,.
\end{aligned}
\tag{3.18}
$$

Then A^a is determined and may be inserted in the action. The resulting action of the dual model is

$$S = \int_{\Sigma_2} \left(\frac{1}{2} (G - DG^{-1}D)^{ab} e_a \wedge \star e_b - \frac{1}{2} \left(G^{-1}D(G - DG^{-1}D)^{-1} \right)^{ab} e_a \wedge e_b \right), \quad (3.19)$$

where

$$e_a = \mathrm{d}\eta_a + \theta_a - (\omega_{ai}^b \eta_b + (G^{-1}D)_a^b \upsilon_b^k g_{ki}) \mathrm{d}X^i . \quad (3.20)$$

We observe that the dual action comprises a coframe that mixes the original coframe with the dual one. Thus the generic result is that the coordinates of the original and the dual model appear mixed and cannot always be disentangled. We will have more to say about this in the following section, where we study a non-trivial example.

4 Example — 3d nilmanifold

In the previous section we presented the formulation that leads to two dual models in the absence of isometry. Evidently this depends crucially on the connection 1-form coefficients ω_{bi}^a. In particular, when these coefficients vanish isometry is restored. Therefore in order to be able to argue that this formulation is not an empty and useless theoretical method it is necessary to show that it works in non-trivial cases. This is not obvious, given that a lot of constraints were imposed and thus one might worry that they do not allow for non-vanishing ω_{bi}^a. In the present section we work out an explicit example which serves as an existence proof and supports the non-triviality of our considerations. In the next section we discuss a larger class of examples.

4.1 The background and the action

Let us consider the geometry of the 3D Heisenberg manifold and set H to zero. This means that the ungauged action is simply

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij}(X) \mathrm{d}X^i \wedge \star \mathrm{d}X^j, \quad (4.1)$$

where the metric is

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2 - x^1 \mathrm{d}x^3)^2 + (\mathrm{d}x^3)^2, \quad (4.2)$$

in a particular coordinate system where the global 1-forms of the coframe and the corresponding dual vector fields are:

$$e^a = \{ \mathrm{d}x^1, \mathrm{d}x^2 - x^1 \mathrm{d}x^3, \mathrm{d}x^3 \},$$
$$\upsilon_a = \{ \partial_1, \partial_2, \partial_3 + x^1 \partial_2 \}, \quad (4.3)$$

and they satisfy

$$\mathrm{d}e^2 = -e^1 \wedge e^3 \quad \text{and} \quad [\upsilon_1, \upsilon_3] = \upsilon_2 . \quad (4.4)$$

Note that this is a case where the structure functions are constant. We are going to use the vectors v_a to perform the T-duality and to this end we calculate the Lie derivative of the metric along them:

$$\mathcal{L}_{v_1} g = -dx^2 \otimes dx^3 - dx^3 \otimes dx^2 + 2x^1 dx^3 \otimes dx^3 ,$$
$$\mathcal{L}_{v_2} g = 0 ,$$
$$\mathcal{L}_{v_3} g = dx^1 \otimes dx^2 + dx^2 \otimes dx^1 - x^1 dx^1 \otimes dx^3 - x^1 dx^3 \otimes dx^1 . \tag{4.5}$$

We note that only v_2 is a Killing vector. Recall that performing a standard T-duality transformation along this Killing direction one gets the well-known case of a 3-torus with H flux. Moreover, it should be mentioned that the vector field ∂_3, which is not one of the v_a we considered, is also Killing and T-duality along this direction yields the case with non-geometric Q_1^{23} flux. Here we take a different route.

Following the general approach of section 2, we gauge the action (4.1) along all three vectors v_a. In the present example we consider $\theta_a = 0$ for simplicity. Then the gauged action reads as

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij} DX^i \wedge \star DX^j - \int_{\Sigma_2} d\eta_a \wedge A^a + \int_{\Sigma_2} \frac{1}{2} C_{ab}^c \eta_c A^a \wedge A^b - \int_{\Sigma_2} \omega_{bi}^a \eta_a A^b \wedge DX^i , \tag{4.6}$$

where the metric is given above, $C_{13}^2 = 1$ is the only non-vanishing component of C_{bc}^a, and

$$DX^1 = dX^1 - A^1 , \quad DX^2 = dX^2 - A^2 - X^1 A^3 , \quad DX^3 = dX^3 - A^3 . \tag{4.7}$$

The next step is to determine the ω_{bi}^a such that all the conditions and constraints are satisfied. Performing this task, the constraints impose the only non-vanishing components of ω_{bi}^a to be

$$\omega_{31}^2 = -\omega_{13}^2 = 2 . \tag{4.8}$$

Then the system of equations is consistent and the action is gauge invariant.

It is interesting to note that in this example the connection 1-form ω is a very particular one. Using the basis e^a of 1-forms it is a direct task to determine the connection Ω compatible with this basis by the tetrad postulate:

$$D^{\Omega} e^a = de^a + \Omega_b^a \wedge e^b = 0 . \tag{4.9}$$

The only non-trivial relation $de^2 = -e^1 \wedge e^3$ gives $\Omega_{31}^2 = -\Omega_{13}^2 = \frac{1}{2}$. Thus, the connection ω in the gauged sigma model is a constant multiple of the connection compatible with the orthonormal basis: $\omega = 4\Omega$. Moreover, the curvature 2-form of ω_b^a vanishes, which is in accord with the gauge covariance of the field strength $\mathcal{F}^a$, as discussed in section 2. It is reasonable to worry that then one can always set ω_b^a to zero by a suitable gauge transformation. However, once the vector fields v_a that implement the gauge symmetry are chosen, this is not possible any more. Of course, the geometry of the model might also possess sets of Killing vector fields, as in the present example, for which ω_b^a vanishes; it is however a legitimate choice *not* to perform T-duality along them.

4.2 Back to the original model

In order to obtain the original model we should integrate out the Lagrange multiplier η_a. The corresponding equations of motion are

$$dA^1 = 0\,,$$
$$dA^2 + A^1 \wedge A^3 = 2A^3 \wedge DX^1 - 2A^1 \wedge DX^3\,,$$
$$dA^3 = 0\,. \tag{4.10}$$

Plugging them in the gauged action we obtain

$$S = \int_{\Sigma_2} \frac{1}{2} g_{ij}(X) DX^i \wedge \star DX^j\,. \tag{4.11}$$

At this stage we consider the gauge fixing procedure that was described in section 3. As mentioned there, one choice that fixes the gauge of A^a is to set them to zero on-shell. Then the original ungauged action is recovered. More generally, since A^1 and A^3 are closed we can choose a gauge where

$$A^1 = \kappa_1 dX^1 \quad \text{and} \quad A^3 = \kappa_3 dX^3\,, \tag{4.12}$$

for some real constants κ_1, κ_3. Then

$$DX^1 = (1 - \kappa_1)dX^1 := dY^1\,,$$
$$DX^3 = (1 - \kappa_3)dX^3 := dY^3\,, \tag{4.13}$$

where we defined the new coordinates Y^1, Y^3 as indicated by the last equations. It remains to gauge fix A^2. In order to do this consistently first we note that the action (4.11) is now written as

$$\frac{1}{2} \int_{\Sigma_2} \left(dY^1 \wedge \star dY^1 + DX^2 \wedge \star DX^2 + \left(1 + \frac{(Y^1)^2}{(1-\kappa_1)^2} \right) dY^3 \wedge \star dY^3 - \frac{2Y^1}{1-\kappa_1} DX^2 \wedge \star dY^3 \right)\,. \tag{4.14}$$

This allows us to determine a gauge choice for A^2 such that the original model is recovered:

$$A^2 = -(\kappa_1 + \kappa_3 - \kappa_1 \kappa_3) X^1 dX^3\,. \tag{4.15}$$

Indeed then the action takes its final form in the new coordinate system (with $Y^2 := X^2$):

$$\int_{\Sigma_2} \frac{1}{2} g_{ij}(Y) dY^i \wedge \star dY^j\,, \tag{4.16}$$

as desired. The remaining consistency check is that the chosen A^2 satisfies its equation of motion. Studying this equation we find that it is obeyed by any set of (κ_1, κ_3) that satisfy the relation

$$\kappa_1 + \kappa_3 = 2\kappa_1 \kappa_3\,. \tag{4.17}$$

Clearly, the simplest consistent choice is $(\kappa_1, \kappa_3) = (0, 0)$ as we noted before, but any other choice is equally consistent, giving the original model in different coordinate systems.

4.3 The dual model

Finally let us obtain the dual model, which is the most interesting instance of our analysis. First we have to integrate out the gauge fields. Thus we vary the action (4.6) with respect to A^a and obtain the equations of motion

$$
\begin{aligned}
A^1 &= \mathrm{d}X^1 - \star(\mathrm{d}\eta_1 - 3\eta_2 A^3 + 2\eta_2 \mathrm{d}X^3)\,, \\
A^2 &= \mathrm{d}X^2 - X^1 \mathrm{d}X^3 - \star\mathrm{d}\eta_2\,, \\
A^3 &= \mathrm{d}X^3 - \star(\mathrm{d}\eta_3 + 3\eta_2 A^1 - 2\eta_2 \mathrm{d}X^1)\,.
\end{aligned}
\tag{4.18}
$$

These equations are coupled but we can easily decouple them and find

$$
\begin{aligned}
A^1 &= \mathrm{d}X^1 - \frac{3\eta_2}{1+9\eta_2^2}(\mathrm{d}\eta_3 + \eta_2 \mathrm{d}X^1) - \frac{1}{1+9\eta_2^2}\star(\mathrm{d}\eta_1 - \eta_2 \mathrm{d}X^3)\,, \\
A^2 &= \mathrm{d}X^2 - X^1 \mathrm{d}X^3 - \star\mathrm{d}\eta_2\,, \\
A^3 &= \mathrm{d}X^3 + \frac{3\eta_2}{1+9\eta_2^2}(\mathrm{d}\eta_1 - \eta_2 \mathrm{d}X^3) - \frac{1}{1+9\eta_2^2}\star(\mathrm{d}\eta_3 + \eta_2 \mathrm{d}X^1)\,.
\end{aligned}
\tag{4.19}
$$

It is obvious that convenience of notation suggests to define

$$
\begin{aligned}
e_1 &= \mathrm{d}\eta_1 - \eta_2 \mathrm{d}X^3\,, \\
e_3 &= \mathrm{d}\eta_3 + \eta_2 \mathrm{d}X^1\,.
\end{aligned}
\tag{4.20}
$$

Note that $\mathrm{d}e_1 \neq 0$ and $\mathrm{d}e_3 \neq 0$. Inserting the equations of motion in the action, a tedious calculation leads to the simple result

$$
S = \int_{\Sigma_2} \frac{1}{2}\left(\mathrm{d}\eta_2 \wedge \star\mathrm{d}\eta_2 + \frac{1}{1+9\eta_2^2}(e_1 \wedge \star e_1 + e_3 \wedge \star e_3) + \frac{6\eta_2}{1+9\eta_2^2}e_1 \wedge e_3\right)\,.
\tag{4.21}
$$

Let us comment on this result. The action of the dual model looks like a Q-flux background. Indeed the background fields in the basis $\{e_1, e_2 = \mathrm{d}\eta_2, e_3\}$ are

$$
g = e_2 \otimes e_2 + \frac{1}{1+9\eta_2^2}(e_1 \otimes e_1 + e_3 \otimes e_3)\,,
\tag{4.22}
$$

$$
B = \frac{6\eta_2}{1+9\eta_2^2}e_1 \wedge e_3\,,
\tag{4.23}
$$

which are globally ill-defined if the direction η_2 is assumed compact. At first sight this seems to be a short cut for the following chain of standard T-dualities

$$
C^2_{13} \quad \overset{T_2}{\to} \quad H_{123} \quad \overset{T_1}{\to} \quad C^1_{23} \quad \overset{T_3}{\to} \quad Q^{13}_2\,.
\tag{4.24}
$$

However, this is not quite true. This can be argued via the following reasoning. When a T-duality is performed the new coordinate along the dualized direction is the corresponding Lagrange multiplier. When we write the action of the dual model in terms of these Lagrange

multipliers η_a the result is

$$S = \int_{\Sigma_2} \frac{1}{2} \left(\mathrm{d}\eta_2 \wedge \star\mathrm{d}\eta_2 + \frac{1}{1+9\eta_2^2}(\mathrm{d}\eta_1 \wedge \star\mathrm{d}\eta_1 + \mathrm{d}\eta_3 \wedge \star\mathrm{d}\eta_3) + \frac{6\eta_2}{1+9\eta_2^2}\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_3 \right) +$$
$$+ \int_{\Sigma_2} \frac{1}{2}\frac{1}{1+9\eta_2^2} \left(\eta_2^2(\mathrm{d}X^1 \wedge \star\mathrm{d}X^1 + \mathrm{d}X^3 \wedge \star\mathrm{d}X^3) + 2\eta_2(\mathrm{d}X^1 \wedge \star\mathrm{d}\eta_3 - \mathrm{d}X^3 \wedge \star\mathrm{d}\eta_1) \right) +$$
$$+ \int_{\Sigma_2} \frac{3\eta_2^2}{1+9\eta_2^2} \left(\eta_2\mathrm{d}X^1 \wedge \mathrm{d}X^3 - \mathrm{d}X^1 \wedge \mathrm{d}\eta_1 - \mathrm{d}X^3 \wedge \mathrm{d}\eta_3 \right) \ . \tag{4.25}$$

In order to compare with the literature, note that the first line of this action is precisely what one obtains when the last step of the duality chain (4.24) is performed, namely the standard non-geometric Q-flux background. If that was the case, namely had it been true that $e_a = \mathrm{d}\eta_a$, one could rename the coordinates $\{\eta_a\} \to \{X^i\}$ as is customary and the result would have been a T-fold [28]. On the other hand, had it been true that $e_1 = \mathrm{d}\eta_1 - \eta_2\mathrm{d}\eta_3$ and $e_3 = \mathrm{d}\eta_3$ the result would have been a twisted T-fold [29], namely the non-geometric T-dual of a nilmanifold carrying additional H flux. However, the additional two lines in eq. (4.25), obtained via the non-isometric procedure, mix the coordinates of the original model with the coordinates of the dual model. This means that only η_2 could be renamed, but not the rest. The fact that the model can be written in the form (4.21) however means that it is associated to a rank 3 subbundle of the cotangent bundle of the manifold $M \times \tilde{M}$, where M is the original (nil)manifold and $\tilde{M}$ the T-fold associated to the dual coordinates η_a. This statement requires a more precise geometric interpretation, which will be provided elsewhere.

5 A class of examples

The toy example we studied in the previous section provides an existence proof for non-trivial cases where the present formulation applies. Furthermore it indicates that there exists a considerably large class of additional examples based on nilmanifolds. Here we formulate non-isometric T-duality for an arbitrary step 2 nilmanifold in any dimension.

In all cases we are working with pure geometries, namely we set $H = 0$. We write the ungauged action in the form

$$S = \int_{\Sigma_2} \frac{1}{2}\delta_{ab}e^a \wedge \star e^b \,, \tag{5.1}$$

where e^a is a global coframe. In a coordinate basis where $e^a = e_i^a\mathrm{d}x^i$, where e_i^a are the (inverse) vielbeins, the metric takes the form

$$g = \delta_{ab}e_i^a e_j^b\mathrm{d}x^i \otimes \mathrm{d}x^j \,, \tag{5.2}$$

and the action becomes the same as in the previous sections. The set of vector fields that we use for T-duality is the one given by the dual frame, i.e.

$$\langle v_a, e^b \rangle = \delta_a^b \ . \tag{5.3}$$

A useful relation is

$$\mathcal{L}_{v_a}e^b = -C_{ac}^b e^c \,, \tag{5.4}$$

where C_{ab}^c are the structure constants of the algebra of vector fields,

$$[v_a, v_b] = C_{bc}^a v_c \,, \tag{5.5}$$

which also appear in the Maurer-Cartan equations

$$\mathrm{d}e^a = -\frac{1}{2} C_{bc}^a e^b \wedge e^c \,. \tag{5.6}$$

Then it is simple to compute the Lie derivative of the metric along these vector fields. This yields the result

$$\mathcal{L}_{v_a} g = -2 \sum_c C_{ab}^c e^b \vee e^c \,. \tag{5.7}$$

On the other hand, we have to solve the condition

$$\mathcal{L}_{v_a} g = \omega_a^b \vee \iota_{v_b} g \,. \tag{5.8}$$

This means that

$$\sum_c \omega_{ab}^c e^b \vee e^c = -2 \sum_c C_{ab}^c e^b \vee e^c \,. \tag{5.9}$$

This equation is solved by

$$\omega_{ab}^c = -2 C_{ab}^c \,, \tag{5.10}$$

which is consistent with the results of our previous example. Moreover, assuming again $\theta_a = 0$, all the constraints imposed by gauge invariance are satisfied. Then the gauged action is

$$\begin{aligned}
S &= \int_2 \left(\frac{1}{2} \delta_{ab} E^a \wedge \star E^b - \mathrm{d}\eta_a \wedge A^a + \frac{1}{2} C_{ab}^c \eta_c A^a \wedge A^b - \omega_{ab}^c \eta_c A^a \wedge E^b \right) \\
&= \int_2 \left(\frac{1}{2} \delta_{ab} E^a \wedge \star E^b - \mathrm{d}\eta_a \wedge A^a + \frac{1}{2} C_{ab}^c \eta_c A^a \wedge (A^b + 4E^b) \right) \,, \tag{5.11}
\end{aligned}$$

where $E^a = e^a - A^a$.

As before, the original model is recovered by integrating out the Lagrange multipliers η_a. This leads to the equations of motion

$$\mathrm{d}A^a + \frac{1}{2} C_{bc}^a A^b \wedge (A^c + 4E^c) = 0 \,, \tag{5.12}$$

which are then inserted to the gauged action and yield

$$S = \int_{\Sigma_2} \frac{1}{2} \delta_{ab} E^a \wedge \star E^b \,. \tag{5.13}$$

Now we have to follow a gauge fixing procedure. We make the general Ansatz

$$A^a = \kappa_b^a e^b + \lambda_d^c C_{bc}^a X^b e^d \,, \tag{5.14}$$

for sets of real constants κ and λ. In order to proceed, we have to use some properties of step 2 nilmanifolds. To this end we consider the splitting of the indices $a = (a_0, \bar{a})$ such that $C_{bc}^{a_0} = 0$ and $C_{bc}^{\bar{a}} \neq 0$. This is always possible because by definition there is always

a subset of vanishing structure constants for nilmanifolds, due to nilpotency. Using the fundamental step 2 relation

$$C^a_{bc} C^c_{de} = 0 \,, \tag{5.15}$$

which is true even without summation in the index c, it is evident that $C^a_{\bar{b}c} = 0$ for all indices with a bar. Under this splitting, the Ansatz for the gauge field becomes

$$A^{a_0} = \kappa^{a_0}_b e^b \,,$$
$$A^{\bar{a}} = \kappa^{\bar{a}}_b e^b + \lambda^{c_0}_d C^{\bar{a}}_{b_0 c_0} X^{b_0} e^d \,. \tag{5.16}$$

First let us choose $\kappa^{\bar{a}}_b = 0, \kappa^{a_0}_{\bar{b}} = 0$ and $\lambda^{a_0}_{\bar{b}} = 0$. Then we compute

$$\mathrm{d}E^{a_0} = 0 \tag{5.17}$$

identically, and

$$\mathrm{d}E^{\bar{a}} = -\frac{1}{2} C^{\bar{a}}_{b_0 c_0} E^{b_0} \wedge E^{c_0} \,, \tag{5.18}$$

provided that

$$C^{\bar{a}}_{c_0 [b_0} \lambda^{c_0}_{d_0]} = C^{\bar{a}}_{p_0 c_0} \kappa^{c_0}_{[d_0} (1 - \frac{1}{2}\kappa)^{p_0}_{b_0]} \,. \tag{5.19}$$

Then the Ansatz for the gauge fields leads to

$$\mathrm{d}E^a = -\frac{1}{2} C^a_{bc} E^b \wedge E^c \,. \tag{5.20}$$

This means that the action (5.13) is precisely the action of the original model. As before, it remains to guarantee that the gauge field satisfies the equations of motion. This is an identity for A^{a_0}, while for $A^{\bar{a}}$ it yields the condition

$$C^{\bar{a}}_{p_0 c_0} \kappa^{c_0}_{[d_0} (1 - \kappa)^{p_0}_{b_0]} = 0 \,. \tag{5.21}$$

Of course the simplest choice is to set all constants to zero, but this is clearly not the only option. Let us note that for step 2 nilmanifolds of dimension $d \leq 7$, which are anyway the interesting cases, the number of unknowns in the above system is larger than the number of equations that constrain them. Above 7 dimensions we cannot make a general claim since there is no classification of nilmanifolds [30]; however it is true that when the index $\bar{a}$ takes only one or two values, then the result holds in any dimension. On the other hand, when the index $\bar{a}$ takes $q \geq 3$ values a simple counting of unknowns and equations shows that the result is true up to d dimensions with $\lfloor \frac{q}{q-2}(q-1-\sqrt{q-1}) \rfloor + 1 \leq d \leq \lfloor \frac{q}{q-2}(q-1+\sqrt{q-1}) \rfloor$, where the symbol $\lfloor \cdot \rfloor$ denotes the integer part.

Now let us integrate out the gauge fields. The corresponding equations of motion are

$$(\delta_{ab} - 3C^c_{ab} \eta_c \star) A^b = (\delta_{ab} - 2C^c_{ab} \eta_c \star) e^b - \star \mathrm{d}\eta_a \,. \tag{5.22}$$

In order to insert this equation in the action we have to determine A^a. Following the same procedure as in the general case, the result is

$$A^a = e^a + 3\delta^{aq} C^r_{qp} \eta_r S^{pb} (-\mathrm{d}\eta_b + C^d_{bc} \eta_d e^c) + S^{ab} \star (-\mathrm{d}\eta_b + C^d_{bc} \eta_d e^c) \,, \tag{5.23}$$

where we defined

$$S^{ab} = (\delta_{ab} + 9C_{bp}^d C_{aq}^c \delta^{pq} \eta_d \eta_c)^{-1} \; . \tag{5.24}$$

Inserting this expression in the action we obtain

$$S = \int_{\Sigma_2} \left(\frac{1}{2} S^{ab} (\mathrm{d}\eta_a - C_{ac}^d \eta_d e^c) \wedge \star (\mathrm{d}\eta_b - C_{bc}^d \eta_d e^c) - \right. $$
$$\left. - \frac{3}{2} \delta^{bm} C_{mp}^d \eta_d S^{pa} (\mathrm{d}\eta_a - C_{aq}^c \eta_c e^q) \wedge (\mathrm{d}\eta_b - C_{bs}^r \eta_r e^s) \right) \; . \tag{5.25}$$

This is a generalization for any step 2 nilmanifold of the dual model we found in the example of the previous section.

6   Conclusions

The range of validity of the Buscher rules for the T-duality of string background fields is limited to the case where isometries are present and additional invariance conditions are imposed. In this paper we used a recent idea about gauge symmetries of 2D sigma models without corresponding global symmetry [14] to study T-duality in a more general setting.[9] In particular we were able to identify the conditions and constraints that guarantee that a bosonic sigma model with a metric and B-field is gauge invariant under an extended set of gauge transformations even when one does not have isometries at hand. All these conditions are milder than their counterparts in the isometric case. The next step was to follow the standard procedure of Buscher in this non-standard setting. Integrating out the Lagrange multipliers from the gauged action and gauge fixing lead back to the original ungauged model. On the other hand, integrating out the gauge fields from the action yields a dual model which was precisely identified.

Since several constraints are involved in the formulation, it is natural to worry whether there is any room for non-trivial applications. As a proof of existence, we studied a particular geometry which is often considered in string theory as a useful toy model. This geometry corresponds to the 3D Heisenberg nilmanifold and carries no H flux. In that case we determined a solution of all conditions and constraints that allows to T-dualize along vector fields that are not Killing. This led to a dual model that mixes the cotangent spaces of the original and dual spaces, yielding a generalization of a T-fold or a twisted T-fold, as discussed for example in refs. [28, 29]. Additionally we showed that this is not an isolated case, but in fact all step 2 nilmanifolds in dimensions up to and including 7 provide a class of working examples.

Although the above results are encouraging, there are certain limitations in their scope as presented in this paper, and it is useful to mention some of them. First of all, the dilaton was ignored. The transformation of the dilaton involves an 1-loop computation since the corresponding coupling appears at first order in α', which should be examined. Moreover, we did not discuss at all the potential equivalence of the two dual models as conformal field

---

[9]Additional symmetries appearing in 2D sigma models and their role in the context of T-duality are under investigation [31].

theories, which is true in the standard case and requires a careful consideration for global issues of the procedure [5]. Furthermore a better understanding of the underlying geometric structures is due. In the cases we examined we found a mixing of the original coordinates and the would-be dual coordinates, which is indicative of doubled formulations, such as the doubled sigma models considered by Hull [32] or the ones recently studied in [33]. Last but not least, it would be very interesting to apply this formalism in the case of the triple T-dual of a torus with H flux, or equivalently to the T-dual of the Q flux background where no isometry is available (recent attempts to understand this problem include [34–36]), and even more so in cases of true string backgrounds. We will report on these and other issues in future publications.

Acknowledgments

References

[1] J. Polchinski, *Dualities of fields and strings*, arXiv:1412.5704 [INSPIRE].

[2] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[3] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59 [INSPIRE].

[4] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466 [INSPIRE].

[5] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053] [INSPIRE].

[6] C.M. Hull and B.J. Spence, *The gauged nonlinear sigma model with Wess-Zumino term*, *Phys. Lett.* **B 232** (1989) 204 [INSPIRE].

[7] C.M. Hull and B.J. Spence, *The geometry of the gauged sigma model with Wess-Zumino term*, *Nucl. Phys.* **B 353** (1991) 379 [INSPIRE].

[8] C.M. Hull, *Global aspects of T-duality, gauged sigma models and T-folds*, *JHEP* **10** (2007) 057 [hep-th/0604178] [INSPIRE].

[9] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, *Nucl. Phys.* **B 403** (1993) 377 [hep-th/9210021] [INSPIRE].

[10] E. Alvarez, L. Álvarez-Gaumé, J.L.F. Barbon and Y. Lozano, *Some global aspects of duality in string theory*, *Nucl. Phys.* **B 415** (1994) 71 [hep-th/9309039] [INSPIRE].

[11] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On non-Abelian duality*, *Nucl. Phys.* **B 424** (1994) 155 [hep-th/9403155] [INSPIRE].

[12] E. Plauschinn, *T-duality revisited*, *JHEP* **01** (2014) 131 [arXiv:1310.4194] [INSPIRE].

[13] E. Plauschinn, *On T-duality transformations for the three-sphere*, *Nucl. Phys.* **B 893** (2015) 257 [arXiv:1408.1715] [INSPIRE].

[14] A. Kotov and T. Strobl, *Gauging without initial symmetry*, *J. Geom. Phys.* **99** (2016) 184 [arXiv:1403.8119] [INSPIRE].

[15] T. Strobl, *Algebroid Yang-Mills theories*, *Phys. Rev. Lett.* **93** (2004) 211601 [hep-th/0406215] [INSPIRE].

[16] M. Bojowald, A. Kotov and T. Strobl, *Lie algebroid morphisms, Poisson sigma models and off-shell closed gauge symmetries*, *J. Geom. Phys.* **54** (2005) 400 [math/0406445] [INSPIRE].

[17] A. Kotov and T. Strobl, *Generalizing geometry — Algebroids and sigma models*, arXiv:1004.0632 [INSPIRE].

[18] V. Salnikov and T. Strobl, *Dirac sigma models from gauging*, *JHEP* **11** (2013) 110 [arXiv:1311.7116] [INSPIRE].

[19] A. Kotov, V. Salnikov and T. Strobl, *2D gauge theories and generalized geometry*, *JHEP* **08** (2014) 021 [arXiv:1407.5439] [INSPIRE].

[20] C. Mayer and T. Strobl, *Lie algebroid Yang-Mills with matter fields*, *J. Geom. Phys.* **59** (2009) 1613 [arXiv:0908.3161] [INSPIRE].

[21] C. Klimčík and P. Ševera, *Dual nonAbelian duality and the Drinfeld double*, *Phys. Lett.* **B 351** (1995) 455 [hep-th/9502122] [INSPIRE].

[22] C. Klimčík, *Poisson-Lie T duality*, *Nucl. Phys. Proc. Suppl.* **46** (1996) 116 [hep-th/9509095] [INSPIRE].

[23] C. Klimčík and P. Ševera, *Poisson-Lie T duality and loop groups of Drinfeld doubles*, *Phys. Lett.* **B 372** (1996) 65 [hep-th/9512040] [INSPIRE].

[24] K. Sfetsos, *Canonical equivalence of nonisometric sigma models and Poisson-Lie T duality*, *Nucl. Phys.* **B 517** (1998) 549 [hep-th/9710163] [INSPIRE].

[25] L. Davidović and B. Sazdović, *T-dualization in a curved background in absence of a global symmetry*, *JHEP* **11** (2015) 119 [arXiv:1505.07301] [INSPIRE].

[26] A. Kotov and T. Strobl, *Curving Yang-Mills-Higgs gauge theories*, *Phys. Rev.* **D 92** (2015) 085032 [INSPIRE].

[27] A. Chatzistavrakidis, A. Deser, L. Jonke and T. Strobl, *Gauging of foliations and universal gauge theory for bosonic strings*, in preparation.

[28] C.M. Hull and R.A. Reid-Edwards, *Non-geometric backgrounds, doubled geometry and generalised T-duality*, *JHEP* **09** (2009) 014 [arXiv:0902.4032] [INSPIRE].

[29] R.A. Reid-Edwards, *Flux compactifications, twisted tori and doubled geometry*, *JHEP* **06** (2009) 085 [arXiv:0904.0380] [INSPIRE].

[30] M. Goze and Y. Khakimdjanov, *Nilpotent Lie algebras*, Mathematics and its Applications volume 361, Kluwer Academic Publishers Group, Dordrecht, The Netherlands (1996).

[31] P. Ševera and T. Strobl, *T-duality without symmetries and DFT without coordinates*, work in progress.

[32] C.M. Hull, *A geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065 [hep-th/0406102] [INSPIRE].

[33] A. Chatzistavrakidis, L. Jonke and O. Lechtenfeld, *Sigma models for genuinely non-geometric backgrounds*, *JHEP* **11** (2015) 182 [arXiv:1505.05457] [INSPIRE].

[34] R. Blumenhagen, A. Deser, D. Lüst, E. Plauschinn and F. Rennecke, *Non-geometric fluxes, asymmetric strings and nonassociative geometry*, *J. Phys.* **A 44** (2011) 385401 [arXiv:1106.0316] [INSPIRE].

[35] D. Andriot, M. Larfors, D. Lüst and P. Patalong, *(Non-)commutative closed string on T-dual toroidal backgrounds*, *JHEP* **06** (2013) 021 [arXiv:1211.6437] [INSPIRE].

[36] I. Bakas and D. Lüst, *T-duality, quotients and currents for non-geometric closed strings*, *Fortsch. Phys.* **63** (2015) 543 [arXiv:1505.04004] [INSPIRE].