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Real and complex indices of vector fields on complete intersection curves with isolated singularity

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Abstract

If $(V, 0)$ is an isolated complete intersection singularity and X a holomorphic vector field tangent to V, one can define an index of X, the so-called GSV index, which generalizes the Poincaré–Hopf index. We prove that the GSV index coincides with the dimension of a certain explicitly constructed vector space, if X is deformable in a certain sense and V is a curve. We also give a sufficient algebraic criterion for X to be deformable in this way. If one considers the real analytic case one can also define an index of X which is called the real GSV index. Under the condition that X has the deformation property, we prove a signature formula for the index generalizing the Eisenbud–Levine Theorem.

1. Introduction

1.1 Classical results

Assume that the continuous map germ $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ defines an isolated zero. Then the map $g/\|g\|\colon S_\delta^{n-1} \to S^{n-1}$ of spheres around the origin has a degree, the so-called Poincaré–Hopf index $\operatorname{ind}_{\mathbb{R}^n,0}(g)$ of g. If g is analytic one has algebraic interpretations of this index, that we first want to describe. If $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is holomorphic, let Q_g be the algebra obtained by factoring $\mathscr{O}_{\mathbb{C}^n,0}$ by the ideal generated by the components of g. One has the following theorem.

Theorem 1.1 [AGV85, GH78].

$$\operatorname{ind}_{\mathbb{C}^n,0}(g) = \dim_{\mathbb{C}} Q_g.$$

Here we have made the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ of course. Now let $\mathscr{E}_{\mathbb{R}^n,0}$ be the ring of real analytic function germs on $(\mathbb{R}^n, 0)$ and further $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be finite and real analytic, in the sense that Q_g is finite dimensional as an $\mathbb{R}$-vector space and where Q_g is the algebra obtained by factoring $\mathscr{E}_{\mathbb{R}^n,0}$ with the ideal generated by the components of g in this case. If one denotes by J_g the determinant of the Jacobian of g, one has the following famous theorem:

Theorem 1.2 (Eisenbud–Levine Theorem). *Let $l\colon Q_g \to \mathbb{R}$ be a linear form with $l(J_g) > 0$. Then*

$$\operatorname{ind}_{\mathbb{R}^n,0}(g) = \operatorname{signature}\langle\, , \rangle_l.$$

Here $\langle\, , \rangle_l$ is the induced bilinear form defined by $\langle h_1, h_2\rangle_l := l(h_1 \cdot h_2)$.

1.2 Generalization to complete intersections

Now let $(V, 0) := (\{f_1 = \cdots = f_q = 0\}, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity of a complete intersection (ICIS) and $X := \sum_{i=1}^n X_i\, \partial/\partial z_i$ be the germ of a holomorphic vector field on $(\mathbb{C}^n, 0)$ tangent to V, say $Xf = Cf$ with an isolated zero on V. In this situation one can also define an

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index $\mathrm{ind}_{V,0}(X)$, called the (complex) GSV index (see [ASV98, BG94, GSV91]); and it is the Poincaré–Hopf index when V is smooth. The definition of the index is as follows:

Choose a sufficiently small sphere S_δ around the origin in $\mathbb{C}^n$ which intersects V transversally and consider the link $K = V \cap S_\delta$ of V. The vectors $X, \nabla f_1, \dots, \nabla f_q$ are linearly independent for all points of K and we have a well defined map

$$(X, \nabla f_1, \dots, \nabla f_q) \colon K \to W_{q+1}(\mathbb{C}^n),$$

where $W_{q+1}(\mathbb{C}^n)$ denotes the manifold of $(q+1)$-frames in $\mathbb{C}^n$ and we consider the complex gradients of course. We have

$$H_{2n-2q-1}(K) \cong \mathbb{Z}, \quad H_{2n-2q-1}(W_{q+1}(\mathbb{C}^n)) \cong \mathbb{Z},$$

and therefore the map has a degree. We let K be oriented as the boundary of the complex manifold $V \setminus \{0\}$ here. The index $\mathrm{ind}_{V,0}(X)$ of X is defined to be the degree of this map. (If V is a curve, K can have more components; we will then sum over the degrees of the components.)

We now want to formulate our main theorems. We need a definition first.

Definition 1.3. We have that X is called a *good vector field* (with respect to V), if there is a holomorphic deformation X_t of X, so that for all $t \in \mathbb{C}^q$ sufficiently close to zero X_t is tangent to the t-fibre V_t of f. Then X_t is called a *good deformation* of X.

We will prove a sufficient criterion for a vector field to be good, which states that X is good whenever all coefficients of the matrix C are contained in the ideal generated by the maximal minors of the Jacobian of f in $\mathscr{O}_{\mathbb{C}^n,0}$. It follows from the definition of the index that it equals the sum of the indices of a good deformation on a smooth fibre.

After a linear generic change of coordinates one can assume that $(f_1, \dots, f_q, X_1, \dots X_{n-q})$ is a regular $\mathscr{O}_{\mathbb{C}^n,0}$-sequence (see [LSS95]), and we always assume the coordinates to be chosen in this way in this paper. Let $\mathscr{B}_0 := \mathscr{O}_{\mathbb{C}^n,0}/(f_1, \dots, f_q, X_1, \dots, X_{n-q})$. Due to the chosen coordinates $\mathscr{B}_0$ is finite dimensional as a complex vector space. We also set $\mathscr{C}_0 := \mathscr{B}_0/\mathrm{ann}_{\mathscr{B}_0}(DF)$, where

$$DF := \det\left(\frac{\partial(f_1, \dots, f_q)}{\partial(z_{n-q+1}, \dots, z_n)}\right).$$

We prove (in § 4.3) an index formula for vector fields in the case $q = n - 1$:

Theorem 1.4. *Let X be a good vector field and V a curve. Then*

$$\mathrm{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{C}_0.$$

Now let

$$(V^{\mathbb{R}}, 0) := (\{f_1^{\mathbb{R}} = \dots = f_q^{\mathbb{R}} = 0\}, 0) \subset (\mathbb{R}^n, 0),$$

defined by real analytic function germs. If f denotes the complexification of $f^{\mathbb{R}}$ we assume that f defines an ICIS of dimension $n-q$. Furthermore let the real analytic vector field $X^{\mathbb{R}}$ be tangent to $(V_{\mathbb{R}}, 0)$ with an algebraic isolated zero on $(V^{\mathbb{R}}, 0)$. One defines the real GSV index of $X^{\mathbb{R}}$ similarly to the complex index (see [ASV98]), and denotes this index by $\mathrm{ind}_{V^{\mathbb{R}},0}(X^{\mathbb{R}})$ if $n-q$ is odd and by $\mathrm{ind}^2_{V^{\mathbb{R}},0}(X^{\mathbb{R}})$ if $n-q$ is even. For topological reasons one can only define a $(\mathrm{mod}\, 2)$-index if $n-q$ is even. The definition of $X^{\mathbb{R}}$ to be good is as in the complex case using real analytic deformations.

For the case $q = n-1$ we prove (in § 5.1) the following formula generalizing the Eisenbud–Levine Theorem:

Theorem 1.5. *Let $V^{\mathbb{R}}$ be a curve, $X^{\mathbb{R}}$ a good vector field and $l \colon \mathscr{C}_0^{\mathbb{R}} \to \mathbb{R}$ a linear form with $l(c_1) > 0$. Then*

$$\mathrm{ind}_{V^{\mathbb{R}},0}(X^{\mathbb{R}}) = \mathrm{signature}\langle\, ,\rangle_l.$$

Here $\mathscr{C}_0^{\mathbb{R}}$ is defined as in the complex case using $\mathscr{E}_{\mathbb{R}^n,0}$ instead of $\mathscr{O}_{\mathbb{C}^n,0}$, c_1 is the coefficient of t in the formal power series expansion of $\det(\Bbbk + tDX^{\mathbb{R}})/\det(\Bbbk + tC)$, where $DX^{\mathbb{R}}$ is the Jacobian of $X^{\mathbb{R}}$, and $\langle\,,\rangle_l$ is the induced bilinear form defined as in the classical case. Finally, C is defined by the tangency condition $X^{\mathbb{R}}f^{\mathbb{R}} = Cf^{\mathbb{R}}$.

2. Residues of holomorphic vector fields

To prove our main theorems we need a few results on residues of holomorphic vector fields that we want to collect in this section.

Let $g\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ be a holomorphic map germ with isolated zero. Then the residue $\operatorname{res}^g_{\mathbb{C}^n,0}(h)$ of any $h \in \mathscr{O}_{\mathbb{C}^n,0}$ with respect to g is defined as

$$\operatorname{res}^g_{\mathbb{C}^n,0}(h) := \frac{1}{(2\pi i)^n}\int_\Gamma \frac{h\,dz_1\wedge\cdots\wedge dz_n}{g_1\cdot\ldots\cdot g_n},$$

where Γ is the real n-cycle $\Gamma := \{|g_i| = \epsilon_i,\ i = 1,\ldots,n\}$ for $\epsilon_i \in \mathbb{R}_{>0}$ small enough with orientation given by $d(\arg g_1)\wedge\cdots\wedge d(\arg g_n) \geqslant 0$. Sometimes we denote this residue also by

$$\operatorname{res}_{\mathbb{C}^n,0}\begin{bmatrix} h \\ g_1\cdots g_n\end{bmatrix}.$$

If we denote by J_g the Jacobian determinant of g, we have the following classical result:

Theorem 2.1 [AGV85, GH78].

i) *The residue* $\operatorname{res}^g_{\mathbb{C}^n,0}\colon Q_g \to \mathbb{C}$ *defines a linear form.*

ii) *The induced bilinear form* $\langle\,,\rangle_{\operatorname{res}^g_{\mathbb{C}^n,0}}$ *is non-degenerate.*

iii) *The index* $\operatorname{ind}_{\mathbb{C}^n,0}(g) = \dim_{\mathbb{C}} Q_g = \operatorname{res}^g_{\mathbb{C}^n,0}(J_g)$.

If we consider linear forms $l\colon Q \to \mathbb{F}$ on commutative $\mathbb{F}$-algebras for an arbitrary field in this paper, the induced bilinear form $\langle\,,\rangle_l$ on Q is always the bilinear form defined by $\langle h_1,h_2\rangle_l := l(h_1\cdot h_2)$. The second statement in the theorem is usually called 'local (Grothendieck) duality' and it states that Q_g is a Gorenstein algebra. This means that the annihilator of the maximal ideal, the socle, of Q_g is one-dimensional and it is well known that it is generated by the class of J_g. One immediately concludes that for any linear form $l\colon Q_g \to \mathbb{C}$ with $l(J_g) \neq 0$ the induced pairing $\langle\,,\rangle_l$ is non-degenerate.

As in the second part of the Introduction, let $(V,0) := (\{f_1 = \cdots = f_q = 0\},0) \subset (\mathbb{C}^n,0)$ be an isolated singularity of a complete intersection (ICIS) and $X := \sum_{i=1}^n X_i\,\partial/\partial z_i$ be the germ of a holomorphic vector field on $(\mathbb{C}^n,0)$ tangent to V, say $Xf = Cf$ with an isolated zero on V. Further let

$$\Sigma := \{f_1 = \cdots = f_q = 0, |X_1| = \epsilon_1,\ldots,|X_{n-q}| = \epsilon_{n-q}\}$$

be a small real $(n-q)$-cycle with orientation determined by

$$d(\arg X_1)\wedge\cdots\wedge d(\arg X_{n-q}) \geqslant 0.$$

Then we define the relative residue of any $h \in \mathscr{O}_{\mathbb{C}^n,0}$ with respect to X to be

$$\operatorname{res}^X_{V,0}(h) := \frac{1}{(2\pi i)^{n-q}}\int_\Sigma \frac{h\,dz_1\wedge\cdots\wedge dz_{n-q}}{X_1\cdot\ldots\cdot X_{n-q}}.$$

The absolute residue of h with respect to X is defined as

$$\operatorname{res}^X_{\mathbb{C}^n,0}(h) := \operatorname{res}_{\mathbb{C}^n,0}\begin{bmatrix} h \\ X_1\cdots X_{n-q}f_1\cdots f_q\end{bmatrix}.$$

Now let c_{n-q} be the coefficient of t^{n-q} in the formal power series expansion of

$$\det(\mathbb{1} + tDX)/\det(\mathbb{1} + tC),$$

where DX is the Jacobian of X. We have the following theorem proven in [LSS95].

Theorem 2.2 [LSS95].

$$\operatorname{ind}_{V,0}(X) = \operatorname{res}^X_{V,0}(c_{n-q}).$$

The author has proven in [Kle02] that one always has $\operatorname{res}^X_{V,0}(h) = \operatorname{res}^X_{\mathbb{C}^n,0}(hDF)$, where we have set

$$DF := \det\left(\frac{\partial(f_1, \ldots, f_q)}{\partial(z_{n-q+1}, \ldots, z_n)}\right).$$

This means that we also have $\operatorname{ind}_{V,0}(X) = \operatorname{res}^X_{\mathbb{C}^n,0}(c_{n-q}\, DF)$, which is one of the main tools in the proof of our main theorems.

3. Vector fields tangent to smooth varieties

To prove our main theorems we first prove them for the smooth situation, which is done in this section, and use good deformations to generalize to the singular case. We also look at the socle of $\mathscr{C}_0$. We use the notation as introduced in the first and second sections.

3.1 The complex situation

Let $1 \leqslant i_1 < \cdots < i_q \leqslant n$ be fixed and $1 \leqslant j_1 < \cdots < j_{n-q} \leqslant n$ be the complement of $i_1, \ldots, i_q$ in $\{1, \ldots, n\}$. Furthermore we set

$$DI := \det\left(\frac{\partial(f_1, \ldots, f_q)}{\partial(z_{i_1}, \ldots, z_{i_q})}\right)$$

and let σ_I be the permutation defined by

$$\sigma_I := \begin{pmatrix} 1 & \ldots & n-q & n-q+1 & \ldots & n \\ j_1 & \ldots & j_{n-q} & i_1 & \ldots & i_q \end{pmatrix},$$

where I is the multiindex $I := (i_1, \ldots, i_q)$. We also set

$$\mathscr{B}^I_0 := \frac{\mathscr{O}_{\mathbb{C}^n,0}}{(f_1, \ldots, f_q, X_{j_1}, \ldots, X_{j_{n-q}})}$$

and if $DI(0) \neq 0$

$$\gamma_I := \frac{\operatorname{sign}\sigma_I}{DI} \det\left(\frac{\partial(X_{j_1}, \ldots, X_{j_{n-q}}, f_1, \ldots, f_q)}{\partial(z_1, \ldots, z_n)}\right).$$

Lemma 3.1. *Let $DI(0) \neq 0$. Then*

$$\operatorname{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{B}^I_0.$$

Proof. By the implicit mapping theorem it is not hard to show that $X|_V$ corresponds to the vector field

$$Y := X_{j_1} \circ \psi \frac{\partial}{\partial y_1} + \cdots + X_{j_{n-q}} \circ \psi \frac{\partial}{\partial y_{n-q}}$$

on $(\mathbb{C}^{n-q}, 0)$, where $\psi\colon (\mathbb{C}^{n-q}, 0) \to (V, 0)$ is a biholomorphic map as in the implicit mapping theorem. From

$$\frac{\mathscr{O}_{\mathbb{C}^{n-q},0}}{(X_{j_1} \circ \psi, \ldots, X_{j_{n-q}} \circ \psi)} \cong \mathscr{B}^I_0$$

the claim follows. □

Lemma 3.2. *Let $DI(0) \neq 0$. Then we have for any $h \in \mathscr{O}_{\mathbb{C}^n,0}$*

$$\operatorname{res}^X_{\mathbb{C}^n,0}(h\, DF) = \operatorname{sign}\sigma_I \operatorname{res}_{\mathbb{C}^n,0}\left[\begin{matrix} h\, DI \\ X_{j_1}\dots X_{j_{n-q}} f_1 \dots f_q \end{matrix}\right].$$

Proof. By the transformation formula for residues [GH78], we have to show that there is a matrix A with

$$(X_1, \dots, X_{n-q}, f_1, \dots, f_q)^{\mathrm{t}} A (X_{j_1}, \dots, X_{j_{n-q}}, f_1, \dots, f_q)^{\mathrm{t}}$$

and $\det A = \operatorname{sign}\sigma_I DF/DI$. From $Xf = Cf$ we get

$$\begin{pmatrix} X_{i_1} \\ \vdots \\ X_{i_q} \end{pmatrix} = Cf - \left(\frac{\partial(f_1, \dots, f_q)}{\partial(z_{i_1}, \dots, z_{i_q})}\right)^{-1} \frac{\partial(f_1, \dots, f_q)}{\partial(z_{j_1}, \dots, z_{j_{n-q}})} \begin{pmatrix} X_{j_1} \\ \vdots \\ X_{j_{n-q}} \end{pmatrix}. \tag{1}$$

Now let $i_1, \dots, i_k \in \{1, \dots, n-q\}$ and $i_{k+1}, \dots, i_q \in \{n-q+1, \dots, n\}$. Then it follows that $j_1, \dots, j_{n-q-k} \in \{1, \dots, n-q\}$ and $j_{n-q-k+1}, \dots, j_{n-q} \in \{n-q+1, \dots, n\}$. For $k = 0$ the claim follows immediately. With (1) we obtain a matrix B with

$$(X_{j_1}, \dots, X_{j_{n-q+k}}, X_{i_1}, \dots, X_{i_k}, f_1, \dots, f_q)^{\mathrm{t}} = B(X_{j_1}, \dots, X_{j_{n-q}}, f_1, \dots, f_q)^{\mathrm{t}}.$$

If $\sigma' \in S_{n-q}$ is the permutation with

$$\sigma' = \begin{pmatrix} 1 & \dots & n-q+k & n-q+k+1 & \dots & n-q \\ j_1 & \dots & j_{n-q-k} & i_1 & \dots & i_k \end{pmatrix},$$

we have $\det A = \operatorname{sign}\sigma' \det B$. Then $\det B$ is the determinant of the upper right $(k \times k)$ block of the matrix

$$-\left(\frac{\partial(f_1, \dots, f_q)}{\partial(z_{i_1}, \dots, z_{i_q})}\right)^{-1} \frac{\partial(f_1, \dots, f_q)}{\partial(z_{j_1}, \dots, z_{j_{n-q}})}.$$

If $d_{l,m}$ is the determinant of the matrix obtained by replacing the lth column of

$$\frac{\partial(f_1, \dots, f_q)}{\partial(z_{i_1}, \dots, z_{i_q})}$$

by the mth column of

$$\frac{\partial(f_1, \dots, f_q)}{\partial(z_{j_1}, \dots, z_{j_{n-q}})},$$

and if

$$D := (d_{l,m})_{l=1,\dots,k}^{m=n-q-k+1,\dots,n-q},$$

we have to show that

$$\det D = (DI)^{k-1} \det\left(\frac{\partial(f_1, \dots, f_q)}{\partial(z_{j_{n-q-k+1}}, \dots, z_{j_{n-q}}, z_{i_{k+1}}, \dots, z_{i_q})}\right).$$

This can be done by induction over k where $k = 1$ is obvious. The conclusion is straightforward and we do not want to write it down here. ☐

Lemma 3.3. *If V is smooth, then*

$$\operatorname{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{C}_0.$$

Proof. Let $DI(0) \neq 0$. By Lemma 3.1 we have

$$\operatorname{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{B}^I_0.$$

If we map a class $[h] \in \mathscr{C}_0$ to the class $[h]'$ of h in $\mathscr{B}_0^I$ we obtain an isomorphism of $\mathbb{C}$-algebras, since by Lemma 3.2 and local duality the following holds:

$$\begin{aligned} [h] = 0 \quad \text{in } \mathscr{C}_0 &\iff hDF \in \mathscr{O}_{\mathbb{C}^n,0}(f_1,\dots,f_q,X_1,\dots,X_{n-q}) \\ &\iff \operatorname{res}^X_{\mathbb{C}^n,0}(gh\,DF) = 0 \quad \text{for all } g \in \mathscr{O}_{\mathbb{C}^n,0} \\ &\iff \operatorname{res}_{\mathbb{C}^n,0}\begin{bmatrix} gh\,DI \\ X_{j_1}\dots X_{j_{n-q}} f_1 \dots f_q \end{bmatrix} = 0 \quad \text{for all } g \in \mathscr{O}_{\mathbb{C}^n,0} \\ &\iff h\,DI \in \mathscr{O}_{\mathbb{C}^n,0}(X_{j_1},\dots X_{j_{n-q}}, f_1,\dots,f_q) \\ &\iff [h]' = 0 \quad \text{in } \mathscr{B}_0^I \end{aligned}$$

□

Lemma 3.4. *Let $DI(0) \neq 0$. Then in $\mathscr{C}_0$ the equation $c_{n-q} = \gamma_I$ holds.*

Proof. By the transformation formula for residues and Lemma 3.1 we have

$$\operatorname{res}^X_{\mathbb{C}^n,0}(DF\gamma_I) = \operatorname{ind}_{V,0}(X)$$

and further for $h \in \mathfrak{m}_{\mathscr{O}_{\mathbb{C}^n,0}}$

$$\operatorname{res}^X_{\mathbb{C}^n,0}(DF\gamma_I h) = \operatorname{res}_{\mathbb{C}^n,0}\begin{bmatrix} h \det\left(\dfrac{\partial(X_{j_1},\dots,X_{j_{n-q}},f_1,\dots,f_q)}{\partial(z_1,\dots,z_n)}\right) \\ X_{j_1}\dots X_{j_{n-q}} f_1 \dots f_q \end{bmatrix} = 0$$

and therefore $DF\gamma_I$ generates the one-dimensional socle of $\mathscr{B}_0$. By the remarks of the Introduction we have

$$\operatorname{res}^X_{\mathbb{C}^n,0}(DF c_{n-q}) = \operatorname{ind}_{V,0}(X),$$

and therefore $DF c_{n-q} \neq 0$ in $\mathscr{B}_0$. Since V is smooth there is a small deformation X_t of X tangent to V, so that X_t has only simple zeros p_i for sufficiently small t on V in a small neighbourhood of the origin. We can also assume that for these zeros $DI(p_i) \neq 0$ holds. For $h \in \mathfrak{m}_{\mathscr{O}_{\mathbb{C}^n,0}}$ we have

$$\begin{aligned} \operatorname{res}^X_{\mathbb{C}^n,0}(DF c_{n-q} h) &= \operatorname{res}_{\mathbb{C}^n,0}\begin{bmatrix} \operatorname{sign}\sigma_I DI c_{n-q} h \\ X_{j_1}\cdots X_{j_{n-q}} f_1 \cdots f_q \end{bmatrix} \\ &= \lim_{t\to 0}\sum_i h(p_i)\operatorname{ind}_{V,p_i}(X_t) \\ &= 0. \end{aligned}$$

This follows from the continuous principle for residues, from the fact that the algebras

$$\frac{\mathscr{O}_{\mathbb{C}^n,p_i}}{(X_{t,j_1},\dots,X_{t,j_{n-q}},f_1\dots,f_q)}$$

are one-dimensional and by the transformation formula that $\operatorname{sign}\sigma_I DI c_{n-q}(t)$ is a unit in these algebras. Therefore $c_{n-q}DF$ generates the one-dimensional socle of $\mathscr{B}_0$ too and since we have

$$\operatorname{res}^X_{\mathbb{C}^n,0}(DF c_{n-q}) = \operatorname{res}^X_{\mathbb{C}^n,0}(DF\gamma_I)$$

it follows that $DF(c_{n-q} - \gamma_I) = 0$ in $\mathscr{B}_0$ and therefore $c_{n-q} = \gamma_I$ in $\mathscr{C}_0$. □

3.2 The real analytic situation

Lemma 3.5. *Let $(V^{\mathbb{R}},0)$ be smooth and $l\colon \mathscr{C}_0^{\mathbb{R}} \to \mathbb{R}$ a linear form with $l(c_{n-q}) > 0$. Then we have $\operatorname{ind}_{V^{\mathbb{R}},0}(X^{\mathbb{R}}) = \operatorname{signature}\langle\, ,\rangle_l$.*

Proof. Let $DI(0) \neq 0$. With the implicit mapping theorem it is not hard to show that the vector field $X^{\mathbb{R}}|_{V^{\mathbb{R}}}$ corresponds to

$$Y := X^{\mathbb{R}}_{j_1}\circ\psi\frac{\partial}{\partial y_1} + \cdots + X^{\mathbb{R}}_{j_{n-q}}\circ\psi\frac{\partial}{\partial y_{n-q}},$$

where $\psi\colon (\mathbb{R}^{n-q},0) \to (V^{\mathbb{R}},0)$ is a diffeomorphism with $\psi_{j_k}(y) = y_{j_k}$ for $k = 1,\dots,n-q$. By the chain rule one has

$$\frac{\partial(\psi_{i_1},\dots,\psi_{i_q})}{\partial(y_1,\dots,y_{n-q})} - \left(\frac{\partial(f_1^{\mathbb{R}},\dots,f_q^{\mathbb{R}})}{\partial(x_{i_1},\dots,x_{i_q})}\circ\psi\right)^{-1} \frac{\partial(f_1^{\mathbb{R}},\dots,f_q^{\mathbb{R}})}{\partial(x_{j_1},\dots,x_{j_{n-q}})}\circ\psi.$$

Applying the chain rule again we get

$$\frac{\partial(Y_1,\dots,Y_{n-q})}{\partial(y_1,\dots,y_{n-q})}\,\frac{\partial(X_{j_1}^{\mathbb{R}},\dots,X_{j_{n-q}}^{\mathbb{R}})}{\partial(x_{j_1},\dots,x_{j_{n-q}})}\circ\psi$$
$$- \frac{\partial(X_{j_1}^{\mathbb{R}},\dots,X_{j_{n-q}}^{\mathbb{R}})}{\partial(x_{i_1},\dots,x_{i_q})}\circ\psi \left(\frac{\partial(f_1^{\mathbb{R}},\dots,f_q^{\mathbb{R}})}{\partial(x_{i_1},\dots,x_{i_q})}\circ\psi\right)^{-1} \frac{\partial(f_1^{\mathbb{R}},\dots,f_q^{\mathbb{R}})}{\partial(x_{j_1},\dots,x_{j_{n-q}})}\circ\psi.$$

A well known lemma from linear algebra states

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det A\cdot\det(D - CA^{-1}B),$$

where A and D are square and A is invertible. The application of this lemma shows that the determinant of the Jacobian of Y is given by $\gamma_I\circ\psi$. By the Eisenbud–Levine Theorem it follows that for any linear form

$$\varphi\colon \frac{\mathscr{E}_{\mathbb{R}^{n-q},0}}{(Y_1,\dots,Y_{n-q})} \to \mathbb{R}$$

with $\varphi(\gamma_I\circ\psi) > 0$ the statement $\operatorname{ind}_{V^{\mathbb{R}},0}(X) = \operatorname{signature}\langle\,,\rangle_\varphi$ holds. The isomorphism of algebras given by ψ shows that we have for any linear form $\Phi\colon \mathscr{B}_0^{I\mathbb{R}} \to \mathbb{R}$ with $\Phi(\gamma_I) > 0$ the formula $\operatorname{ind}_{V^{\mathbb{R}},0}(X) = \operatorname{signature}\langle\,,\rangle_\Phi$ and this is also true in $\mathscr{C}_0^{\mathbb{R}}$, because the isomorphism of algebras in Lemma 3.3 also gives an isomorphism of the corresponding real algebras. On the other hand one has in $\mathscr{C}_0$ the equation $c_{n-q} = \gamma_I$ and this equation also holds in $\mathscr{C}_0^{\mathbb{R}}$. Therefore the statement follows. □

4. An algebraic formula for the complex index

4.1 Good vector fields

First we want to prove a sufficient criterion for good vector fields.

Proposition 4.1 (Sufficient criterion for good vector fields). *Let all coefficients of the matrix C be contained in the ideal generated by the maximal minors of the Jacobian of f in $\mathscr{O}_{\mathbb{C}^n,0}$. Then X is a good vector field.*

Proof. We prove a bit more: there is a deformation X_t of X such that $X_t(f-t) = C(f-t)$ holds. For $(i_1,\dots,i_q) \in \{1,\dots,n\}^q$ and $(j_1,\dots,j_{q-1}) \in \{1,\dots,n\}^{q-1}$ define

$$f_{i_1,\dots,i_q} := \det\left(\frac{\partial(f_1,\dots,f_q)}{\partial(z_{i_1},\dots,z_{i_q})}\right) \quad\text{and}\quad f^k_{j_1,\dots,j_{q-1}} := \det\left(\frac{\partial(f_1,\dots,\hat{f}_k,\dots,f_q)}{\partial(z_{j_1},\dots,z_{j_{q-1}})}\right).$$

If $C = (c_{l,m})$ let

$$c_{l,m} = \sum_{(i_1,\dots,i_q)\in\{1,\dots,n\}^q} c^{l,m}_{i_1,\dots,i_q} f_{i_1,\dots,i_q}.$$

For $k = 1,\dots,q-1$, we set

$$\delta^{i,j}_{j_1,\dots,j_{q-1}} := \delta^{j_k,j}_{j_1,\dots,j_{k-1},i,j_{k+1},\dots,j_{q-1}} : \delta^j_{i,j_1,\dots,j_{q-1}}$$

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and further

$$\delta^{l}_{i_1,\dots,i_q} := -\sum_{m=1}^{q} \sum_{(i_1,\dots,i_q)\in\{1,\dots,n\}^q} c^{l,m}_{i_1,\dots,i_q} t_m.$$

We define the deformation by

$$X_{t,i} := X_i + \sum_{j=1}^{q} \sum_{(j_1,\dots,j_{q-1})\in\{1,\dots,n\}^{q-1}} \delta^{i,j}_{j_1,\dots,j_{q-1}} (-1)^{j+1} f^{j}_{j_1,\dots,j_{q-1}}.$$

Then we have

$$\sum_{i=1}^{n} \frac{\partial f_l}{\partial z_i} X_{t,i} \sum_{m=1}^{n} c_{l,m} f_m + \sum_{(i_1,\dots,i_q)\in\{1,\dots,n\}^q} \delta^{l}_{i_1,\dots,i_q} f_{i_1,\dots i_q} = \sum_{m=1}^{n} c_{l,m}(f_m - t_m).$$ □

In the case of a hypersurface this means that, if c is contained in the ideal generated by the partials of f, the vector field is good. If

$$c = \alpha_1 \frac{\partial f}{\partial z_1} + \cdots + \alpha_n \frac{\partial f}{\partial z_n},$$

the deformation is simply defined as $X_{t,i} := X_i - t\alpha_i$. We have

$$X_t(f-t) = cf - \sum_{i=1}^{n} t\alpha_i \frac{\partial f}{\partial z_i} = c(f-t).$$

4.2 The socle of $\mathscr{C}_0$

Lemma 4.2. *The residue* $\operatorname{res}^X_{V,0}(\cdot)$ *defines a linear form on* $\mathscr{C}_0$ *such that*

$$\operatorname{res}^X_{V,0}(hg) = 0 \text{ for all } h \in \mathscr{O}_{\mathbb{C}^n,0} \Rightarrow g = 0 \text{ in } \mathscr{C}_0$$

holds.

Proof. We have $\operatorname{res}^X_{V,0}(h) = \operatorname{res}^X_{\mathbb{C}^n,0}(h\, DF)$ and this means that the residue $\operatorname{res}^X_{V,0}(\cdot)$ vanishes on $\operatorname{ann}_{\mathscr{B}_0}(DF)$. Furthermore $\operatorname{res}^X_{V,0}(hg) = 0$ for all $h \in \mathscr{O}_{\mathbb{C}^n,0}$ implies $\operatorname{res}^X_{\mathbb{C}^n,0}(hg\, DF) = 0$ for all $h \in \mathscr{O}_{\mathbb{C}^n,0}$. Now local duality gives

$$g\, DF \in \mathscr{O}_{\mathbb{C}^n,0}(X_1,\dots,X_{n-q},f_1,\dots,f_q)$$

and therefore $g \in \operatorname{ann}_{\mathscr{B}_0}(DF)$. □

Proposition 4.3. *The socle of* $\mathscr{C}_0$ *is generated by the class of* c_{n-q}.

Proof. Lemma 4.2 states that $\operatorname{res}^X_{V,0}(\cdot)$ induces a non-degenerate bilinear form on $\mathscr{C}_0$, which means that $\mathscr{C}_0$ has a one-dimensional socle if $\mathscr{C}_0$ is not trivial. On the other hand for any $h \in \mathfrak{m}_{\mathscr{O}_{\mathbb{C}^n,0}}$ we have

$$\operatorname{res}^X_{V,0}(hc_{n-q}) = \lim_{t\to 0} \sum_i h(p_i) \operatorname{ind}_{V_t,p_i}(X_t) = 0,$$

which means that c_{n-q} generates the socle. □

4.3 Proof of Theorem 1.4

For a good vector field, $\operatorname{ind}_{V,0}(X)$ is the sum of Poincaré–Hopf indices of X_t on a smooth fibre V_t of f, where one sums over all zeros of X_t which tend to zero. This follows directly from the definition of the index.

If $q = n-1$ we denote by m_i the minor of the Jacobian matrix of f obtained by cancelling the ith column. We have $DF = m_1$ of course. Recall that we always assume the coordinates to be chosen so that $(f_1,\dots,f_{n-1},X_1)$ is a regular $\mathscr{O}_{\mathbb{C}^n,0}$-sequence. For simplicity we also may assume that $m_i(0)=0$ for $i=1,\dots,n$.

Lemma 4.4. *Let $q=n-1$. Then the following hold:*

i) *$(f_1,\dots,f_{n-1},DF)$ is a regular $\mathscr{O}_{\mathbb{C}^n,0}$-sequence.*

ii) *The dimension of $\mathscr{C}_0$ does not depend on the choice of coordinates, provided that the $\mathscr{O}_{\mathbb{C}^n,0}$-sequence $(f_1,\dots,f_{n-1},X_1)$ is regular in each system of coordinates.*

iii) *After linear changes of coordinates in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$ one can assume that $(f_1,\dots,f_{n-2},DF,m_2)$ is a regular $\mathscr{O}_{\mathbb{C}^n,0}$-sequence.*

Proof. i) All computations are done in the ring $\mathscr{O}_{V,0}=\mathscr{O}_{\mathbb{C}^n,0}/(f_1,\dots,f_{n-1})$. We have to show that DF is not a zero divisor in this ring. Applying Cramer's rule to the equation $Xf=Cf$ we obtain the equations

$$(-1)^i m_i X_1 = -DF\,X_i \tag{2}$$

for $i=1,\dots,n$. Now let $g\,DF=0$ in $\mathscr{O}_{V,0}$. Multiplication with X_i, Equation (2) and using that X_1 is not a zero divisor in $\mathscr{O}_{V,0}$ give $gm_i=0$ for all $i=1,\dots,n$. The ring obtained by dividing $\mathscr{O}_{V,0}$ by all m_i is artinian and therefore there must be complex numbers $\alpha_1,\dots,\alpha_n$ so that $h:=\alpha_1 m_1+\cdots+\alpha_n m_n$ is not a zero divisor in $\mathscr{O}_{V,0}$. On the other hand we have $gh=0$ in $\mathscr{O}_{V,0}$ and therefore $g=0$ in $\mathscr{O}_{V,0}$.

ii) Let $\phi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$, $\phi(y)=z$, be biholomorphic and $\psi:=\phi^{-1}$. We denote by Y the vector field computed in y-coordinates and by DF^y the minor computed in y-coordinates. Standard computations give

$$\begin{pmatrix} Y_1\circ\psi\\ \vdots\\ Y_n\circ\psi\end{pmatrix}=\frac{\partial(\psi_1,\dots,\psi_n)}{\partial(z_1,\dots,z_n)}\begin{pmatrix} X_1\\ \vdots\\ X_n\end{pmatrix}$$

and

$$DF^y\circ\psi=\sum_{j=1}^{n}(-1)^{j+1}(\det D\phi)\circ\psi\,\frac{\partial\psi_1}{\partial z_j}m_j.$$

Set $\mathscr{B}'_0:=\mathscr{O}_{\mathbb{C}^n,0}/(f_1,\dots,f_{n-1},Y_1\circ\psi)$ and $\mathscr{C}'_0:=\mathscr{B}'_0/\operatorname{ann}_{\mathscr{B}'_0}(DF^y\circ\psi)$. We construct an epimorphism $\varphi\colon\mathscr{C}_0\to\mathscr{C}'_0$. For any $g\in\mathscr{O}_{\mathbb{C}^n,0}$ we also denote by g the classes of g in these algebras and define $\varphi(g):=g$. Again all computations are done in the ring $\mathscr{O}_{V,0}$. We want to show that φ is well defined. Let $g\,DF=\alpha X_1$. Multiplying with m_i, using Equation (2) and the fact that DF is not a zero divisor in $\mathscr{O}_{V,0}$, we obtain $gm_i=(-1)^{i+1}\alpha X_i$ for $i=1,\dots,n$. Then for each i we get

$$\begin{aligned} gDF^y\circ\psi&=(\det D\phi)\circ\psi\;\sum_{i=1}^{n}(-1)^{i+1}\frac{\partial\psi_1}{\partial z_i}gm_i\\ &=(\det D\phi)\circ\psi\;\sum_{i=1}^{n}\alpha X_i\frac{\partial\psi_1}{\partial z_i}\\ &=\alpha(\det D\phi)\circ\psi\;Y_1\circ\psi.\end{aligned}$$

This shows that φ is well defined. The surjectivity is obvious and the other direction analogous. Therefore $\dim_{\mathbb{C}}\mathscr{C}_0$ does not depend on the choice of coordinates in $\mathbb{C}^n$. If one considers changes of coordinates in the image space $\mathbb{C}^{n-1}$, the invariance of $\dim_{\mathbb{C}}\mathscr{C}_0$ is obvious.

iii) After a general linear change of coordinates in $\mathbb{C}^{n-1}$, see [Loo84], one can assume that $(f_1,\dots,f_{n-2})$ defines an ICIS and the 1-form df_{n-1} has an isolated zero on this ICIS. Now Lemma 3.4

in [Kle02] shows that, after a linear change of coordinates in $\mathbb{C}^n$, $(f_1, \dots, f_{n-2}, DF, m_2)$ is a regular $\mathscr{O}_{\mathbb{C}^n,0}$-sequence. □

Proof of Theorem 1.4. Let $q = n-1$ and consider good deformations X_t of X. For small neighbourhoods U (respectively T) of the origins in $\mathbb{C}^n$ (respectively $\mathbb{C}^{n-1}$) consider $Z \subset U \times T$ defined by

$$Z := \{f_{t,1} = \cdots = f_{t,n-1} = X_{t,1} = 0\},$$

where we have set $f_{t,i} := f_i - t_i$. Let $\pi\colon Z \to T$ be the finite projection. We define

$$\mathscr{B}_{t,p} := \frac{\mathscr{O}_{\mathbb{C}^n,p}}{(f_{t,1}, \dots, f_{t,n-1}, X_{t,1})}, \quad \mathscr{B}_t := \bigoplus_{p \in \pi^{-1}(t)} \mathscr{B}_{t,p}, \quad \mathscr{B} := \bigcup_{t \in T} \mathscr{B}_t.$$

$\mathscr{B}$ has the natural structure of a holomorphic vector bundle over T which is induced by the locally free sheaf $\pi_* \mathscr{O}_Z$. Similarly we define

$$\mathscr{C}_{t,p} := \frac{\mathscr{B}_{t,p}}{\operatorname{ann}_{\mathscr{B}_{t,p}}(DF)}, \quad \mathscr{C}_t := \bigoplus_{p \in \pi^{-1}(t)} \mathscr{C}_{t,p}, \quad \mathscr{C} := \bigcup_{t \in T} \mathscr{C}_t.$$

We want to show that $\mathscr{C}$ has the natural structure of a holomorphic vector bundle of rank $\operatorname{ind}_{V,0}(X)$ over T.

By Lemma 4.4, part ii, we may assume that the coordinates are chosen as in Lemma 4.4, part iii. Via π we view $\mathscr{O}_{Z,0}/(DF)$ as a finitely generated $\mathscr{O}_{T,0}$-module and claim that

$$\operatorname{depth}_{\mathscr{O}_{T,0}} \frac{\mathscr{O}_{Z,0}}{(DF)} = n-1.$$

For $k = 1, \dots, n-1$ let

$$t_k g = 0 \quad \text{in } \frac{\mathscr{O}_{Z,0}}{(DF, t_1, \dots, t_{k-1})}.$$

This means that there are representatives with

$$t_k g = \alpha X_{t,1} \quad \text{in } \frac{\mathscr{O}_{\mathbb{C}^{2n-1},0}}{(f_{t,1}, \dots, f_{t,n-1}, t_1, \dots, t_{k-1}, DF)}.$$

Applying Cramer's rule to the tangency equation $Xf = Cf$ we obtain

$$t_k g m_2 = 0 \quad \text{in } \frac{\mathscr{O}_{\mathbb{C}^{2n-1},0}}{(f_{t,1}, \dots, f_{t,n-1}, t_1, \dots, t_{k-1}, DF)}.$$

By Lemma 4.4, part i and the choice of coordinates, t_k and m_2 are not zero divisors in the last algebra since $(f_{t,1}, \dots, f_{t,n-1}, t_1, \dots, t_{n-2}, DF, m_2)$ is a regular $\mathscr{O}_{\mathbb{C}^{2n-1},0}$-sequence. This shows the claim.

Now the Syzygy Theorem and the Auslander–Buchsbaum formula show that $\mathscr{O}_{Z,0}/(DF)$ is a free $\mathscr{O}_{T,0}$-module. We have an exact sequence of $\mathscr{O}_{T,0}$-modules

$$0 \to \frac{\mathscr{O}_{Z,0}}{\operatorname{ann}_{\mathscr{O}_{Z,0}}(DF)} \to \mathscr{O}_{Z,0} \to \frac{\mathscr{O}_{Z,0}}{(DF)} \to 0.$$

The Depth Lemma, see [JP00, 6.5.18], shows that

$$\operatorname{depth}_{\mathscr{O}_{T,0}} \frac{\mathscr{O}_{Z,0}}{\operatorname{ann}_{\mathscr{O}_{Z,0}}(DF)} = n-1,$$

which by the Syzygy Theorem and the Auslander–Buchsbaum formula again means that the $\mathscr{O}_{T,0}$-module $\mathscr{O}_{Z,0}/\operatorname{ann}_{\mathscr{O}_{Z,0}}(DF)$ is free. We have seen that the coherent $\mathscr{O}_T$-module

$$\mathscr{F} := \pi_* \mathscr{O}_Z / \operatorname{ann}_{\mathscr{O}_Z}(DF)$$

is free for T chosen small enough. By [Dou68] this is equivalent to the statement that the function $\nu\colon T \to \mathbb{N}$ defined by

$$\nu(t) := \dim_{\mathbb{C}} \mathscr{F}_t \otimes_{\mathscr{O}_{T,t}} \mathbb{C}$$

is constant. Now from the exact sequence

$$0 \to \mathscr{F}_t \to (\pi_* \mathscr{O}_Z)_t \to (\pi_* \mathscr{O}_Z/(DF))_t \to 0$$

we obtain by tensoring with $\mathbb{C}$ the last part of the long exact sequence of torsion

$$0 \to \mathscr{F}_t \otimes_{\mathscr{O}_{T,t}} \mathbb{C} \to \mathscr{B}_t \to \mathscr{B}_t/(DF) \to 0,$$

where we have used that $(\pi_* \mathscr{O}_Z/(DF))_t$ is a free $\mathscr{O}_{T,t}$-module, which is equivalent, see [Dou68], to the statement that

$$\operatorname{Tor}_1^{\mathscr{O}_{T,t}}((\pi_* \mathscr{O}_Z/(DF))_t, \mathbb{C}) = 0.$$

Since we also have an exact sequence

$$0 \to \mathscr{C}_t \to \mathscr{B}_t \to \mathscr{B}_t/(DF) \to 0,$$

this means that $\nu(t) = \dim_{\mathbb{C}} \mathscr{C}_t$ for all $t \in T$. By Lemma 3.3, for regular t, $\nu(t)$ equals the sum of Poincaré–Hopf indices of X_t on the t-fibre of f for $t \neq 0$ which is equal to $\operatorname{ind}_{V,0}(X)$ and therefore $\dim_{\mathbb{C}} \mathscr{C}_0 = \operatorname{ind}_{V,0}(X)$. The map $\cdot DF\colon \mathscr{B} \to \mathscr{B}$ between vector bundles has constant rank and provides $\mathscr{C}$ with the natural structure of a holomorphic vector bundle. □

4.4 Examples and remarks

We want to give a few examples of good vector fields on curves. We can always take the exterior product of the rows of the Jacobian matrix of f. In this way one obtains a vector field with isolated zero on the singularity, where the matrix C is trivial and the index equals zero.

An example of a family of non-trivial good vector fields on a plane curve is the following:

$$D_k: \ f = x^2 y + y^{k-1}, \quad k \geqslant 4$$

and

$$X = (k-2)x^{m+1} \frac{\partial}{\partial x} + 2x^m y \frac{\partial}{\partial y}, \quad m \geqslant 3,$$

with $c = 2(k-1)x^m$. We have an exact sequence

$$0 \to \operatorname{ann}_{\mathscr{B}_0}(DF) \to \mathscr{B}_0 \xrightarrow{\cdot DF} \mathscr{B}_0 \to \frac{\mathscr{B}_0}{(DF)} \to 0$$

and therefore

$$\dim_{\mathbb{C}} \operatorname{ann}_{\mathscr{B}_0}(DF) = \dim_{\mathbb{C}} \frac{\mathscr{B}_0}{(DF)}.$$

It is easy to compute that $\dim_{\mathbb{C}} \mathscr{B}_0/(DF) = 2(k-1)$ and $\dim_{\mathbb{C}} \mathscr{B}_0 = (k-1)(m+1)$ holds and therefore $\operatorname{ind}_{V,0}(X) = (k-1)(m-1)$.

An example of a family of good vector fields on a space curve is

$$f_1 := x^2 + y^2 + z^2, \quad f_2 := xy,$$

with

$$X := z^l (x-y) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right), \quad l \geqslant 1.$$

Here we have

$$C = \begin{pmatrix} 2z^l(x-y) & 0 \\ 0 & 2z^l(x-y) \end{pmatrix}.$$

In this case (f_1, f_2, X_3) is a regular sequence and we have to permute the coordinates. This means that the index is given by the dimension of the factor space obtained from factoring the algebra $\mathscr{O}_{\mathbb{C}^3,0}/(f_1, f_2, X_3)$ by the annihilator of

$$\det \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}.$$

In this case we get a non-trivial index, which one may compute with computer algebra programs such as Singular [GPS01]. Note that $\dim_{\mathbb{C}} \mathscr{C}_0$ is always equal to $\dim_{\mathbb{C}} \mathscr{B}_0 - \dim_{\mathbb{C}} \mathscr{B}_0/(DF)$. The last two dimensions are easy to compute with computer algebra.

We now want to explain why Theorem 1.4 is false in the general case of a complete intersection. Consider the case $n = 3$ and $q = 1$ and set $f_i := \partial f/\partial z_i$ for $i = 1, 2, 3$. We use the notation as in the proof of Theorem 1.4, i.e. $\mathscr{O}_{Z,0} := \mathscr{O}_{4,0}/(f - t, X_{t,1}, X_{t,2})$ and (f, X_1, X_2) is a regular sequence in $\mathscr{O}_{3,0}$, and consider the coherent $\mathscr{O}_T$-module $\pi_* \mathscr{O}_Z/(f_3)$ where T is a small neighbourhood of 0 in $\mathbb{C}$. After shrinking T we can assume that this sheaf is locally free over all $t \neq 0$. Obviously t is not a zero divisor in $\mathscr{F}_0$ and therefore we have a law of conservation of numbers as in the proof of Theorem 1.4 and we get

$$\mathrm{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{F}_0 \otimes_{\mathscr{O}_{T,0}} \mathbb{C}.$$

We further have the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathscr{O}_{T,0}}(\pi_* \mathscr{O}_{Z,0}/(f_3), \mathbb{C}) \to \mathscr{F}_0 \otimes_{\mathscr{O}_{T,0}} \mathbb{C} \to \mathscr{B}_0 \to \mathscr{B}_0/(f_3) \to 0,$$

and this means that

$$\mathrm{ind}_{V,0}(X) = \dim_{\mathbb{C}} \mathscr{C}_0 + \dim_{\mathbb{C}} \mathrm{Tor}_1^{\mathscr{O}_{T,0}}(\pi_* \mathscr{O}_{Z,0}/(f_3), \mathbb{C}).$$

We now show that $\dim_{\mathbb{C}} \mathrm{Tor}_1^{\mathscr{O}_{T,0}}(\pi_* \mathscr{O}_{Z,0}/(f_3), \mathbb{C}) > 0$ if the hypersurface is not smooth, which is equivalent to the statement that t is a zero divisor in $\mathscr{O}_{Z,0}/(f_3)$. Set $\mathscr{O}_{4,0} := \mathbb{C}\{z_1, z_2, z_3, t\}$. The functions $(f - t, f_1, f_2, f_3)$ define an isolated zero in $(\mathbb{C}^4, 0)$ and therefore these define a regular $\mathscr{O}_{4,0}$-sequence. We have

$$f_1 X_{t,1} + f_2 X_{t,2} = 0$$

in the ring $\mathscr{O}_{4,0}/(f - t, f_3)$. It follows that there are $\gamma_1, \gamma_2 \in \mathscr{O}_{4,0}$ such that $X_{t,1} = \gamma_1 f_2$ and $X_{t,2} = \gamma_2 f_1$ in $\mathscr{O}_{4,0}/(f - t, f_3)$ holds. Inserting this in the tangency equation shows that there is a $\gamma \in \mathscr{O}_{4,0}$ such that $X_{t,1} = \gamma f_2$ and $X_{t,2} = -\gamma f_1$ in $\mathscr{O}_{4,0}/(f - t, f_3)$ holds. On the other hand $\mathscr{O}_{Z,0}/(f_3, t)$ is artinian and therefore $(f - t, \gamma, f_3, t)$ is a weak regular $\mathscr{O}_{4,0}$-sequence. This means that we have an exact sequence

$$0 \to \frac{\mathscr{O}_{4,0}}{(f - t, f_1, f_2, f_3)} \xrightarrow{\cdot\gamma} \frac{\mathscr{O}_{4,0}}{(f - t, \gamma f_1, \gamma f_2, f_3)} \to \frac{\mathscr{O}_{4,0}}{(f - t, \gamma, f_3)} \to 0$$

with

$$\frac{\mathscr{O}_{4,0}}{(f - t, \gamma f_1, \gamma f_2, f_3)} \cong \frac{\mathscr{O}_{Z,0}}{(f_3)}.$$

The sequence shows that $\mathrm{depth}_{\mathscr{O}_{T,0}} \mathscr{O}_{Z,0}/(f_3) = 0$. If $\gamma(0) \neq 0$ this is obvious. If $\gamma(0) = 0$ we can apply the Depth Lemma using

$$\mathrm{depth}_{\mathscr{O}_{T,0}} \frac{\mathscr{O}_{4,0}}{(f - t, f_1, f_2, f_3)} = 0$$

and

$$\mathrm{depth}_{\mathscr{O}_{T,0}} \frac{\mathscr{O}_{4,0}}{(f - t, \gamma, f_3)} = 1.$$

5. A signature formula for the real index

Now let

$$(V^{\mathbb{R}},0) := (\{f_1^{\mathbb{R}} = \cdots = f_q^{\mathbb{R}} = 0\},0) \subset (\mathbb{R}^n,0)$$

be a geometric complete intersection of dimension $n-q$, denote by V and f the complexifications and assume that f defines an ICIS. Furthermore let the real analytic vector field $X^{\mathbb{R}}$ be tangent to $(V_{\mathbb{R}},0)$ with an algebraic isolated zero on $(V^{\mathbb{R}},0)$. As before T is a small neighbourhood of the origin in $\mathbb{C}^q$ and let $T^{\mathbb{R}}$ be the corresponding subset in $\mathbb{R}^q$. We also assume that $X^{\mathbb{R}}$ is good in the real analytic sense. We keep the notation of the previous section for all complexifications and for real t we denote the real algebra corresponding to $\mathscr{C}_{t,p}$ by $\mathscr{C}_{t,p}^{\mathbb{R}}$, if $X^{\mathbb{R}}|_{V_t^{\mathbb{R}}}(p) = 0$ and $p \in \mathbb{R}^n$ holds.

5.1 Proof of Theorem 1.5

First we prove a law of conservation of numbers for the signature. Theorem 1.5 then follows as a corollary. We also remark that the sufficient criterion for good vector fields also holds in the real analytic case. The proof is word for word as in the complex case.

Proposition 5.1. *Let $q = n-1$, $X^{\mathbb{R}}$ be a good vector field and $l\colon \mathscr{C}_0^{\mathbb{R}} \to \mathbb{R}$ a linear form with $l(c_1) > 0$, and for any regular value $t \in T^{\mathbb{R}}$ of $f^{\mathbb{R}}$ and any p with $X_t^{\mathbb{R}}|_{V_t^{\mathbb{R}}}(p) = 0$ let $l_{t,p}\colon \mathscr{C}_{t,p}^{\mathbb{R}} \to \mathbb{R}$ be a linear form with $l_{t,p}(c_{t,p,1}) > 0$. Then*

$$\operatorname{signature}\langle\,,\rangle_l = \sum_{\{X_t^{\mathbb{R}}|_{V_t^{\mathbb{R}}}(p)=0\}} \operatorname{signature}\langle\,,\rangle_{l_{t,p}},$$

where the sum goes over the zeros tending to zero.

Proof. We consider the vector bundle $\mathscr{C}$ over T defined in the proof of Theorem 1.4 and denote by τ the map given by complex conjugation. For any $t \in T^{\mathbb{R}}$ we consider the set of invariant multigerms $h \in \mathscr{C}_t$. These are the multigerms with $\tau \circ h = h \circ \tau$. We denote this set by $\mathscr{C}_t^{\mathbb{R}}$. We have

$$\mathscr{C}_t^{\mathbb{R}} = (\oplus_k D_k) \oplus (\oplus_l E_l)\,, \tag{3}$$

where each component D_k corresponds to an algebra $\mathscr{C}_{t,p_k}^{\mathbb{R}}$ for a real zero p_k of $X|_V$ and where each component

$$E_l = (\mathscr{C}_{t,q_l} \oplus \mathscr{C}_{t,\overline{q_l}})^{\mathbb{R}}$$

corresponds to a pair of complex conjugated zeros and $(\mathscr{C}_{t,q_l} \oplus \mathscr{C}_{t,\overline{q_l}})^{\mathbb{R}}$ is the subset of invariant elements of $(\mathscr{C}_{t,q_l} \oplus \mathscr{C}_{t,\overline{q_l}})$. It consists of elements of the form

$$h = \sum a_I z^I + \sum \overline{a_I} z^I.$$

Here q_l (respectively $\overline{q_l}$) are not real of course. If μ is the real dimension of $\mathscr{C}_t^{\mathbb{R}}$ then μ is given by $\dim_{\mathbb{C}} \mathscr{C}_0$. The set $\mathscr{C}^{\mathbb{R}} := \bigcup_{t\in T^{\mathbb{R}}} \mathscr{C}_t^{\mathbb{R}}$ has, for T chosen small enough, the natural structure of a real analytic vector bundle of rank μ over $T^{\mathbb{R}}$. We can continue l real analytically to a family l_t and obtain a real analytic family of non-degenerate bilinear forms $\langle\,,\rangle_{l_t}$. Equation (3) gives an orthogonal decomposition. By dividing the algebra E_l by its maximal ideal one obtains $\mathbb{C}$ and therefore E_l contributes nothing to the signature; see [EL77]. Therefore the signature of $\langle\,,\rangle_{l_t}$ is the sum of signatures of $\langle\,,\rangle_{l_{t,p}}$ that are defined as the restrictions to the components D_k. On the other hand we have $l_{t,p}(c_{t,p,1}) > 0$ and therefore the claim follows by continuity of signatures and by the Eisenbud–Levine Theorem if we choose a fixed regular value $t \in T^{\mathbb{R}}$ of f. ☐

Proof of Theorem 1.5. For a good vector field the index counts the sum of indices of a good deformation of the vector field on a regular fibre in a neighbourhood of the origin by the properties of the real index given in Theorem 2.10 in [ASV98]. Now the claim follows from Lemma 3.5 and Proposition 5.1. □

We want to consider an example. Let $f^{\mathbb{R}}(x,y) := x^2 - y^2$ and $X^{\mathbb{R}} := x^2\,\partial/\partial x + xy\,\partial/\partial y$. A good deformation is given by

$$X_t^{\mathbb{R}} := (x^2 - t)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

with $c_t = c = 2x$. Set $F_t := V_t^{\mathbb{R}} \cap \overline{B_\delta}$ where B_δ is a small ball around the origin in $\mathbb{R}$. Then F_t consists of two branches of a hyperbola and we have $\chi(F_t) = 2$. If l is a linear form as in Theorem 1.5 we obtain signature$\langle\, ,\rangle_l = 0$. Let $t = 1$, $B_\delta := \{x^2 + y^2 = 3\}$ and $F := F_1$. Then $X^{\mathbb{R}}$ deforms to

$$\tilde{X}^{\mathbb{R}} := (x^2 - 1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

The boundary points of F are $P_1 = (\sqrt{2}, 1)$, $P_2 = (\sqrt{2}, -1)$, $P_3 = (-\sqrt{2}, -1)$ and $P_4 = (-\sqrt{2}, 1)$. At the points P_1 and P_2 the vector field $\tilde{X}^{\mathbb{R}}$ points outwards, but inwards at the points P_3 and P_4. From the symmetry of the problem (only the directions of $\tilde{X}^{\mathbb{R}}$ are not symmetric) we find that the sum of the indices of $\tilde{X}^{\mathbb{R}}$ vanishes on F and this is what Theorem 1.5 says. This can also be computed explicitly: the zeros of $\tilde{X}^{\mathbb{R}}$ on F are $(-1,0)$ and $(1,0)$. We can parametrize both branches via $\varphi_\pm(s) := (\pm\sqrt{1+s^2}, s)$ and write $\tilde{X}^{\mathbb{R}}$ in the coordinate given by s. One immediately sees that the index in $(-1,0)$ has the value -1, and the value 1 in $(1,0)$.

If we want to count the Euler characteristic of F_t we have to choose a good vector field whose deformation points outwards at all boundary points. This means that we have to choose a good vector field which points outwards at all boundary points of the intersection of the singular fibre with a small closed ball around the origin.

5.2 Relations to results of Gómez-Mont and Mardesić

Gómez-Mont and Mardesić have proven similar signature formulas [GM99, GM97]. These formulas hold for vector fields on isolated hypersurface singularities which have an isolated zero not only on the variety but also in the ambient space. We want to compare these formulas with our formula for vector fields on plane curves. Let X be a real analytic vector field in $(\mathbb{R}^n, 0)$ with an isolated zero and $(V,0) : (\{f = 0\}, 0) \subset (\mathbb{R}^n, 0)$ an odd-dimensional hypersurface with algebraic isolated singularity. Further let X be tangent to V, i.e. $Xf = cf$. We omit the upper $\mathbb{R}$ to indicate that we are working in the real analytic category. Define

$$\mathbb{A} := \frac{\mathscr{E}_{\mathbb{R}^n,0}}{(f_1, \dots f_n)} \quad \text{and} \quad \mathbb{B} := \frac{\mathscr{E}_{\mathbb{R}^n,0}}{(X_1, \dots, X_n)}.$$

Here the f_i are the partials of f. Let H_f be the Hessian determinant of f. Then $\det DX$ and H_f generate the socles of $\mathbb{B}$ (respectively $\mathbb{A}$). Now we have well determined classes

$$H_f^{\text{rel}} := \frac{H_f}{c} \in \frac{\mathbb{A}}{\text{ann}_{\mathbb{A}}(c)}, \quad \det DX^{\text{rel}} := \frac{\det DX}{c} \in \frac{\mathbb{B}}{\text{ann}_{\mathbb{B}}(c)}$$

defined in the obvious way, which generate the socles of these algebras. Let

$$l_1 \colon \frac{\mathbb{A}}{\text{ann}_{\mathbb{A}}(c)} \to \mathbb{R}, \quad l_2 \colon \frac{\mathbb{B}}{\text{ann}_{\mathbb{B}}(c)} \to \mathbb{R}$$

be linear forms with $l_1(H_f^{\text{rel}}) > 0$ and $l_2(\det DX^{\text{rel}}) > 0$. We have the following result.

Theorem 5.2 (Gómez-Mont and Mardesić).

$$\text{ind}_{V,0}(X) = \text{signature}\langle\, ,\rangle_{l_2} - \text{signature}\langle\, ,\rangle_{l_1}.$$

To compare this result with our theorem we additionally assume $n = 2$, the $\mathscr{E}_{2,0} := \mathbb{R}\{x, y\}$-sequence (f, X_1) to be regular and the vector field to be good. We first give an explicit construction of all vector fields fulfilling both conditions. Denote the good deformation by X_t. We also set $\mathscr{E}_{3,0} := \mathbb{R}\{x, y, t\}$ and $\mathscr{E}_{V,0} := \mathscr{E}_{3,0}/(f - t)$. The tangency condition gives

$$X_{t,1} f_1 + X_{t,2} f_2 = 0 \text{ in } \mathscr{E}_{V,0}.$$

Since (f_1, f_2) is a regular $\mathscr{E}_{V,0}$-sequence, it follows immediately that there are $\tilde{\gamma}, \tilde{\delta_1}, \tilde{\delta_2} \in \mathscr{E}_{3,0}$ such that

$$X_t = (\tilde{\gamma} f_2 + \tilde{\delta_1}(f - t))\frac{\partial}{\partial x} + (-\tilde{\gamma} f_1 + \tilde{\delta_2}(f - t))\frac{\partial}{\partial y}.$$

Setting $t = 0$ we obtain that there are $\delta_1, \delta_2, \gamma \in \mathscr{E}_{2,0}$ such that

$$X = (\gamma f_2 + \delta_1 f)\frac{\partial}{\partial x} + (-\gamma f_1 + \delta_2 f)\frac{\partial}{\partial y}.$$

We have $c = \delta_1 f_1 + \delta_2 f_2$ and this means signature$\langle\, ,\rangle_{l_1} = 0$. We now claim that $\operatorname{ann}_{\mathbb{B}}(c) = \mathbb{B}(\gamma, f)$. Using

$$\begin{aligned} cf &= f_1 X_1 + f_2 X_2, \\ c\gamma &= \delta_2 X_1 - \delta_1 X_2, \end{aligned}$$

we obtain $\mathbb{B}(\gamma, f) \subset \operatorname{ann}_{\mathbb{B}}(c)$. Now let $cg = \alpha_1 X_1 + \alpha_2 X_2$. Multiplication with f gives

$$(f_1 g - \alpha_1 f)X_1 + (f_2 g - \alpha_2 f)X_2 = 0.$$

Since (X_1, X_2) is a regular $\mathscr{E}_{2,0}$-sequence there must be an $h \in \mathscr{E}_{2,0}$ with $f_2 g - \alpha_2 f = hX_1$. Since (f, X_1) is a regular $\mathscr{E}_{2,0}$-sequence this also holds for f, f_2. Therefore we have $g = h\gamma$ in $\mathscr{E}_{2,0}/(f)$, which shows the claim. That $\mathscr{B}_0^{\mathbb{R}}(\gamma) = \operatorname{ann}_{\mathscr{B}_0^{\mathbb{R}}}(f_2)$ is obvious. We have obtained

$$\mathscr{C}_0^{\mathbb{R}} = \frac{\mathbb{B}}{\operatorname{ann}_{\mathbb{B}}(c)} = \frac{\mathscr{E}_{2,0}}{(\gamma, f)}.$$

To prove that Theorems 1.5 and 5.2 produce the same values one has to check that there is a positive real number r such that $rcc_1 = \det DX$ in $\mathbb{B}$. We have

$$c \cdot c_1 = c \text{ trace } DX - c^2 = c \det \frac{\partial(\gamma, f)}{\partial(x, y)}$$

in $\mathbb{B}$. We verify the existence of such an r in the following example.

Set $f := x^2 + y^{k+1}$, $\gamma := y$, $\delta_1 := -(k + 1)$, $\delta_2 := y^l$. We have

$$X_1 = -(k + 1)x^2 \quad \text{and} \quad X_2 = -2xy + x^2 y^l + y^{l+k+1}.$$

One computes

$$\det DX = -2(k + 1)(l + k + 1)xy^{l+k}$$

and

$$c \det \frac{\partial(\gamma, f)}{\partial(x, y)} = -2(k + 1)xy^{l+k}$$

in $\mathbb{B}$. Here we have $r = l + k + 1$.

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