Construction of higher genus CMC surfaces in $\mathbb{R}^3$ via the generalized Whitham flow

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Abstract

In this thesis, we use the generalized Whitham flow to construct symmetric higher genus $g > 1$ CMC surfaces in $\mathbb{R}^3$. It is well-known that the existence of a conformally immersed CMC surface $f : M \to \mathbb{R}^3$ is equivalent to the existence of a family of flat connections satisfying the intrinsic (periodicity of the conformal metric) and extrinsic (periodicity of the immersion) closing conditions. The generalized Whitham flow preserves the intrinsic while varies the extrinsic closing conditions. Thereby, the flow parameter, denoted by ρ, determines the genus of the resulting CMC surface. At the initial value $\rho = 0$ a torus has to be chosen, which in our case is the 3-lobed Wente torus T^2. For tori, the monodromy of the associated family can be parameterized by algebraic data on a hyperelliptic curve, called the spectral curve. The spectral curve of the 3-lobed Wente torus has spectral genus 2 and hyperelliptic reduction allows us to characterize the spectral data in terms of data on elliptic curves. This will help us to derive closing conditions at the initial value $\rho = 0$.

In order to constructed closed symmetric higher genus CMC surfaces in $\mathbb{R}^3$, we will study families of flat connections on higher genus Riemann surfaces as the pullback of Fuchsian systems on the 4-punctured sphere, i.e., logarithmic connections on the holomorphically trivial rank two bundle. By that, the underlying Fuchsian system will be parameterized by flat line bundle connections on a torus. This particularly provides useful coordinates to study closing conditions of higher genus CMC surfaces. Investigating the spectral data shows that we have to open two double points outside the unit circle and increase the genus of the spectral curve to 6. By an implicit function theorem argument, we will show that the closing conditions are satisfied for $\rho \in (-\epsilon, \epsilon)$ and prove the existence of compact and branched higher genus CMC surfaces in $\mathbb{R}^3$.

Kurzzusammenfassung

In der vorliegenden Arbeit wird der 'generalized Whitham flow' genutzt, um symmetrische CMC Flächen in $f : M \to \mathbb{R}^3$ vom Geschlecht $g > 1$ zu konstruieren. Dabei wird von der Eigenschaft Gebrauch gemacht, dass sich CMC Flächen in Raumformen durch Familien von flachen Zusammenhängen, die intrinsische und extrinsische Schließungsbedingungen erfüllen, parametrisieren lassen. Wir werden solche Familien durch den Rücktransport von Fuchschen Systemen auf der 4-fach punktierten Sphäre konstruieren. Parallel dazu werden diese Systeme durch flache Linienbündel Zusammenhängen auf einem Torus, der die 4-fach punktierte Sphäre doppelt überlagert, parametrisiert. Dieses Setup eröffnet eine andere Betrachtungsweise, um die Schließungsbedingungen zu studieren.

Der 'generalized Whitham flow' erhält die intrinsische, während die extrinsische Schließungsbedngung variiert wird. Zum Zeitpunkt $\rho = 0$, wobei ρ der Flussparameter ist, muss ein initial Torus gewählt werden. In unserem Fall ist dies der 3-lobed Wente Torus. Die besondere Eigenschaft von Tori ist, dass die assoziierte Familie ∇^λ in die direkte Summe von Flachen Linienbündel Zusammenhängen splittet. Die Monodromie von ∇^λ, und damit auch die Schließungsbedingungen, können durch algebraische Daten auf einer hyperelliptischen Kurve parametrisiert werden. Diese Kurve wird auch Spektralkurve genannt. Die Spektralkurve des 3-lobed Wente Torus hat Geschlecht 2 und hyperelliptische Reduktion erlaubt es uns die Spektraldaten über elliptische Kurven zu beschreiben. Diese Eigenschaft nutzen wir, um die Schließungsbedingungen des 3-lobed Wente Torus zu charakterisieren und dann über das implizite Funktionen Theorem die Existenz von kompakten und verzweigten Flächen vom höherem Geschlecht $g > 1$ in

$\mathbb{R}^3$ zu beweisen.

Schlagwörter in deutscher und englischer Sprache:

CMC Flächen, Whitham Fluss, Torus, Wente Torus, assoziierte Familie, Monodromy, hyperelliptische Kurve, Spektralkurve, höheres Geschlecht, Symmetrien, Fuchssche Systeme, 4-fach punktierte Sphäre

CMC surfaces, Whitham flow, associated family, monodromy, family of flat connections, higher genus, symmetric surfaces, symmetry, Fuchsian system, 4-punctured sphere, spectral curve

Contents

Chapter 1

Introduction

The study of constant mean curvature (CMC) surfaces is classical. Such surfaces are critical points of the area functional under the constraint of enclosed volume. A particular class of CMC surfaces are minimal surfaces with mean curvature $H = 0$ and the study of those goes back to the 18th century. In 1760, Lagrange studied surfaces in $\mathbb{R}^3$ with given boundary which locally minimize the area and the task of finding such surfaces is nowadays known as the Plateau problem [Lag60]. About 100 years later, Weierstrass and Enneper parameterized all minimal surfaces in $\mathbb{R}^3$ in terms of data of meromorphic functions on simply connected domains [Nit89]. An immediate consequence of this result is that minimal surfaces in $\mathbb{R}^3$ are never compact.

Another subset of CMC surfaces is such where the mean curvature is $H \neq 0$. In the 1840s, Delaunay constructed CMC surfaces of revolution in $\mathbb{R}^3$. General examples of constant mean curvature surfaces were rare. In the 1950s, it was conjectured by Hopf that the only compact immersed CMC surface in $\mathbb{R}^3$ is the round sphere. Alexandrov showed that this is indeed true if we restrict to embedded surfaces [Ale56]. It took another 30 years until Wente showed the existence of infinitely many CMC surfaces of genus one and was finally able to disprove Hopf's conjecture [Wen86]. The Wente tori have beautiful symmetries. Abresch [Abr87] has shown in 1986 that they naturally arise as solutions to the sinh-Gordon equation under the constraint of admitting planar $y =$ const curvature lines. Additionally, a consequence of this constraint is that the $x =$ const curvature lines lie on spheres. Moreover, Walter [Wal87] gave an explicit parameterization of the Wente tori in terms of elliptic integrals. The discovery of the Wente tori and the connection to the underlying integral system was the starting point of many new advances regarding conformally immersed CMC tori not only in $\mathbb{R}^3$ but also the other two space forms S^3 and H^3.

In [PS89], Pinkall and Sterling constructed all conformally immersed CMC tori in $\mathbb{R}^3$. Similarly, Hitchin [Hit90] studied harmonic maps from the torus to the three sphere S^3. These publications were accompanied by [Bob91b, Bob91a], where Bobenko gave explicit parameterizations of doubly periodic solution to the sinh-Gordon equation in terms of Riemann theta functions. In all these works, the task of finding a conformally immersed CMC torus was translated to algebraic data on another compact Riemann surface, called the spectral curve Σ. The genus of Σ is called the spectral genus.

To any such conformally immersed CMC torus T^2 in $\mathbb{R}^3$, one can associate a family of flat connections [Hit90]

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^* \tag{1.1}$$

on the trivial rank two bundle $\underline{\mathbb{C}}^2 \to T^2$, where Φ is a holomorphic $\mathrm{sl}(2, \mathbb{C})$-valued

$(1,0)$-form. The family ∇^λ satisfies the following properties [Hit90]:

1. ∇ and Φ have only diagonal and off-diagonal entries, respectively, and $\det(\Phi) \neq 0$.
2. ∇^λ is unitary along $\lambda \in S^1$.
3. There exists a point $\lambda_1 \in S^1$ such that the monodromy H_p^λ of ∇^λ at $p \in T^2$ along both generators of $\pi_1(T^2, p)$ expands in a neighborhood of $\lambda = \lambda_1$ as

$$H_p^\lambda \sim \pm \mathrm{Id} + \mathcal{O}((\lambda - \lambda_1)^2). \tag{1.2}$$

The first fundamental group of a torus is abelian and thus both monodromies of ∇^λ commute. This allows us to define the spectral curve Σ of a CMC torus as the characteristic polynomial of one of the monodromies of ∇^λ since defining it along the other one yields the same curve. In particular, choosing a different base-point $q \in T^2$ is equivalent to conjugating the monodromy by the parallel transport from p to q, which leaves the spectral curve invariant. The remarkable property in this setup is that there exists a group homomorphism

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma) \tag{1.3}$$

into the Jacobian of the spectral curve. The existence of Ψ implies that the whole construction can be reversed and it is sufficient to study the spectral data in order to draw conclusion of the underlying CMC torus. This is what makes the associated family ∇^λ so interesting: having a family satisfying the three properties listed above, it is possible to reconstruct the immersion (which is, up to isometries, unique for spectral genus $g \leq 2$ [Hit90]). Since the first fundamental group of the torus is abelian, there exists a basis which, generically in λ, simultaneously diagonalizes both monodromies of ∇^λ. The eigenvalues of the monodromies are functions on Σ and they can be characterized explicitly in terms of Riemann theta functions [Bob91b]. Phrased differently, the construction of CMC tori in $\mathbb{R}^3$ boils down to finding functions on hyperelliptic curves satisfying the three conditions listed above. Bobenko further showed that there do not exist spectral curves of compact CMC tori in $\mathbb{R}^3$ of spectral genus $g = 0, 1$ [Bob91b]. In fact, the $g = 2$ case corresponds to the spectral curve of the Wente tori. Moreover, Jaggy [Jag94] showed that for every spectral genus $g \geq 2$, there exists a corresponding CMC torus in $\mathbb{R}^3$.

That the immersion can be reconstructed from given spectral data has further implications. Deformations of CMC tori and doubly periodic CMC cylinders in $\mathbb{R}^3$ can be expressed on the level of deformations of the spectral data. Such deformations, which have the additional property of preserving the intrinsic closing conditions, i.e., the doubly periodicity of the conformal metric, are called Whitham deformations [KS07]. As the subspace of compact CMC tori in $\mathbb{R}^3$ is discrete, we have to open the extrinsic closing conditions, i.e., the immersion has no periods, in a particular way in order to flow to a compact torus. On a dense subset of the time interval, one can find values where the cylinder closes to a compact CMC torus [Kew15]. During the flow, is it further possible to increase the genus of the spectral curve by opening double points.

Having talked so much about constant mean curvature surfaces of genus one, what about higher genus CMC surfaces in $\mathbb{R}^3$? Evidently, the theory that worked so great for CMC tori does not directly translate to higher genus CMC surfaces since the first fundamental group is no longer abelian. It turns out that examples of such surfaces are rare. Although the existence of compact CMC surfaces for arbitrary genus in $\mathbb{R}^3$ has been proven by Kapouleas in [Kap95], the author of this thesis is not aware of further publications which show the existence of other compact higher genus CMC surface in $\mathbb{R}^3$. In this thesis, we want to fill the gap and construct such higher genus surfaces.

We will proceed as follows: The Riemann-Hilbert correspondence states that all representations $\pi_1(\mathbb{C}P^1 \setminus \{p_1, .., p_4\}, *) \to \mathrm{SL}(2, \mathbb{C})$ are realizable as the monodromy representation of a Fuchsian system, i.e., a connection with simple poles at prescribed points on the holomorphically trivial rank two bundle $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus \{p_1, ..., p_4\}$. The eigenlines of the residues of the Fuchsian system define a parabolic structure on the trivial bundle, i.e., a filtration at each fiber at p_i, with prescribed parabolic weight $\rho \in (0, \frac{1}{2})$. The ($\lambda$-independent) local monodromies at the singular points are then conjugated to the matrix

$$\begin{pmatrix} e^{2\pi i\rho} & 0 \\ 0 & e^{-2\pi i\rho} \end{pmatrix}. \tag{1.4}$$

To show that there exists a $\mathbb{C}^*$-family of flat connections ∇^λ on some higher genus Riemann surface $g > 1$, denoted by N_q, inducing a conformal CMC immersion $f : N_q \to \mathbb{R}^3$, we will impose symmetries on the resulting surface. Assume that $\pi_q : N_q \to \mathbb{C}P^1 \setminus \{p_1, ..., p_4\}$ admits a q-fold cover over the 4-punctured sphere branched at four distinct points $p_i \in \mathbb{C}P^1$. The map π_q allows us to study families of Fuchsian systems and then pull them back to N_q. It turns out that on N_q the singular points of the Fuchsian systems become apparent for every $\lambda \in \mathbb{C}^*$, i.e., it is gauge equivalent to a regular connection. In this way, we obtain families of flat connections on higher genus CMC surfaces induced by the underlying Fuchsian systems.

The relationship of the parabolic weight to N_q is obtained by letting $p \in \mathbb{N}$ with $\gcd(p, q) = 1$ and setting $\frac{p}{q} = \rho$. Hence, by varying $\rho \in \mathbb{Q}$, we are continuously changing the genus of the resulting CMC surface. Combined with q, the integer p determines the umbilic and branch order at prescribed points. To ensure the unitarity along $\lambda \in S^1$, we will use the theorem of Mehta-Seshadri: the stability of a parabolic vector bundle is equivalent to the existence of a unique Fuchsian system with unitary monodromy representation [MS80].

In [HH17], useful coordinates parameterizing the underlying Fuchsian system were introduced. We will briefly recall the construction. The difference between two Fuchsian systems inducing the same parabolic structure is a strongly parabolic Higgs field. The eigenlines of such a Higgs field are not well-defined on the 4-punctured sphere but only on a double covering branched over the singular points, which is a torus. The pullback of the Fuchsian system with respect to the eigenlines of the strongly parabolic Higgs field determines a 2:1-correspondence between flat line bundle connections on a torus and Fuchsian systems on $\mathbb{C}P^1 \setminus \{p_1, ..., p_4\}$ with prescribed local monodromies [HH17]. In this way, abelianization coordinates are introduced which give an explicit realization of the 2:1 correspondence. Moreover, using such coordinates, the Mehta-Seshadri section can be realized as mapping holomorphic to anti-holomorphic structures of a flat line bundle connection on a torus such that the underlying Fuchsian system is unitary with respect to some hermitian metric.

Summarized, we consider families of flat connections on higher genus surfaces induced by the underlying Fuchsian system, which in return can be parameterized by flat line bundle connections on a torus. In this setup, the parabolic weight ρ can be used as an additional flow parameter, which determines the genus of the Riemann surface on which the analytic continuation of the CMC surface closes. Hence, we are not only performing deformations on the level of the spectral curve but also change the genus of the CMC surface, thus calling it the generalized Whitham flow [HHS15, HHS18].

In this thesis, we will show the short time existence of the generalized Whitham flow in the case that the initial torus is the 3-lobed Wente torus. As all closing conditions have to be satisfied, we need to characterize the Wente tori's spectral data. The spectral curve is a hyperelliptic curve of genus two. Hence, for a full description of the spectral data, Riemann theta functions have to be studied. However, the symmetries of the Wente tori translate to symmetries on the spectral curve and it turns out that Wente tori can be completely characterized by elliptic data. Having this knowledge helps us to apply implicit function theorem arguments to shows the existence of compact (branched) higher genus CMC surfaces for small weight ρ. The main theorem is the following

Theorem 1.0.1. *Let the conditions of proposition 5.4.6 be satisfied. The surface*

$$f : N_q \to \mathbb{R}^3 \tag{1.5}$$

is a compact and branched CMC surface in $\mathbb{R}^3$*. Over the four branch points* $\{0, 1, m, \infty\}$ *on* $\mathbb{C}P^1$*, the surface has umbilic branch points.*

i. If q is odd, then the genus of f is $g = q-1$. The surface branches with order $2p-1$ at the points over $z = 0$ and $z = \infty$ and with order $q - 2p - 1$ at the points over $z = 1$ and $z = m$. The umbilic order is $2p - 1$ at the points over $z = 1$ and $z = m$ and $q - 2p - 1$ at the points over $z = 0$ and $z = \infty$.

ii. If q is even, then the genus of f is $g = \frac{q}{2} - 1$. The surface branches with order $p - 1$ at the points over $z = 0$ and $z = \infty$ and with order $\frac{q}{2} - p - 1$ at the points over $z = 1$ and $z = m$. The umbilic order is $p - 1$ at the points over $z = 1$ and $z = m$ and $\frac{q}{2} - p - 1$ at the points over $z = 0$ and $z = \infty$.

The main technical lemma needed to prove this lemma 6.2.6. The thesis is structured as follows.

In chapter 2, we recall some basic properties of Riemann surface theory. Starting with the introduction of hyperelliptic curves, we come to the definition of abelian integrals and reciprocity laws. The motivation to view abelian integrals as sections in holomorphic line bundles brings us to the definition of the Jacobian of Riemann surfaces. Finally, we will define elliptic functions and Riemann theta functions and their reduction to Jacobi theta functions.

Chapter 3 recalls the integrable surface theory approach for conformally immersed CMC surfaces in $\mathbb{R}^3$ and closing conditions will be derived. Afterwards, we will restrict solely to the case of the Riemann surface being a torus. In this manner, the spectral curve is defined and we show how CMC tori in $\mathbb{R}^3$ can be completely characterized by data on the spectral curve. In the last pages of this chapter, we discuss the Whitham flow.

Chapter 4 is all about the Wente tori. Starting from a hyperelliptic curve of genus two, spectral data will be defined and closing conditions will be discussed. Via hyperelliptic reduction, the spectral curve admits two branched double coverings of elliptic curves which allow the reduction of Riemann theta to Jacobi elliptic functions. This will allow us to derive some known properties of these tori and restate the closing conditions in terms of data on elliptic curves.

The construction of families of flat connections on higher genus Riemann surfaces is the topic of chapter 5. After introducing the key concepts of parabolic vector bundles, the abelianization of Fuchsian systems on the 4-punctured sphere is performed. Since the families of flat connections on the 4-punctured sphere and on the higher genus Riemann surfaces are related to each other by pullback, we use the abelianization coordinates to parameterize those.

In the final chapter 6, we will introduce suitable Banach function spaces and adjust the Whitham flow to control the pole order of the associated family at $\lambda = 0$. After opening two double points and increase the genus of the spectral curve from 2 to 6, an implicit function theorem argument show the short time existence of the generalized Whitham flow in the case that the initial torus is the 3-lobed Wente torus. This will prove the main theorem of this thesis.

Chapter 2

Fundamentals

In this chapter, we recall some basic properties of Riemann surface theory. The first section introduces hyperelliptic curves, which have various applications in every chapter of this thesis.

Afterwards, we define abelian differential, i.e., holomorphic and meromorphic, differentials on compact Riemann surfaces. Thereby, the Riemann bilinear relations will be proven and important corollaries deduced. Abelian differentials are a key ingredient in Riemann surface theory and they can also be viewed as sections of holomorphic line bundles. These subjects are treated in the second section where the Riemann-Hurwitz theorem and the Jacobian of compact Riemann surfaces are discussed as well.

Finally, the last section completes this chapter with the exposition of Riemann theta and Weierstrass elliptic functions. These functions are indispensable for the analysis of the spectral data of Wente tori. Suitable references for the Riemann surface theory are [Mir95, Bob11, FK92] while we refer to [AS72, Akh90, Law13] for the subject of elliptic functions.

2.1 Hyperelliptic curves

The simplest example of a compact Riemann surface M is the simply connected, i.e., with trivial first fundamental group, complex projective space $M = \mathbb{C}P^1$ of genus $g = 0$. More complicated surfaces are complex tori which are no longer simply connected and have genus $g = 1$. It turns out that a convenient way to construct compact Riemann surfaces of higher genus $g \geq 2$ is to take branched double coverings of $\mathbb{C}P^1$, which are called hyperelliptic curves [Mir95, p. 60].

Definition 2.1.1. *Let $g \in \mathbb{N}$ and $h(\lambda) = \prod_{i=1}^{2g+1}(\lambda - a_i)$ be a polynomial of degree $2g+1$ with only simple zeros $a_i \in \mathbb{C}$. We call the compactification of the set*

$$\Sigma_0 := \{(\lambda, y) \in \mathbb{C}^2 \mid y^2 = h(\lambda)\} \tag{2.1}$$

a hyperelliptic curve and denote it by Σ.

Let $U := \{(\lambda, y) \in \Sigma_0 \mid \lambda \neq 0\}$ be a subset of Σ_0. Set $\mu = 1/\lambda$ and $k(\mu) = \mu^{2g+2} h(1/\mu)$. We define

$$\Sigma_\infty := \{(\mu, w) \in \mathbb{C}^2 \mid w^2 = k(\mu))\}. \tag{2.2}$$

The polynomial k has only simple zeros since h does. For $V := \{(\mu, w) \in \Sigma_\infty \mid \mu \neq 0\} \subset \Sigma_\infty$ we have an isomorphism

$$\phi : U \to V, \quad (\lambda, y) \mapsto (\mu, w) = (1/\lambda, y/\lambda^{g+1}), \tag{2.3}$$

which glues the spaces Σ_0 and Σ_∞. Under this construction, we obtain the hyperelliptic curve Σ. Since the compact sets

$$\{(\lambda, y) \in \Sigma_0 \,|\, |\lambda| \leq 1\}, \quad \{(\mu, w) \in \Sigma_\infty \,|\, |\mu| \leq 1\} \tag{2.4}$$

cover Σ, it is compact. The complex numbers a_i are called the branch points of Σ. The condition $a_i \neq a_j$ for any $i \neq j$ ensures that the hyperelliptic curve is smooth. On Σ, the projection map $(\lambda, y) \mapsto \lambda$ has degree two ramified at the branch points of Σ. By an abuse of notation, we will simply denote this projection map by $\lambda : \Sigma \to \mathbb{C}P^1$.

Definition 2.1.2. *Let Σ be a hyperelliptic curve. Then we call the map*

$$\sigma : \Sigma \to \Sigma, \quad (\lambda, y) \mapsto (\lambda, -y) \tag{2.5}$$

the hyperelliptic involution of Σ.

The fixed points of σ are exactly the branch points of Σ. Hyperelliptic curves play an important role in every chapter of this thesis. We will see that the spectral curve of a compact conformally immersed CMC torus in $\mathbb{R}^3$ is a hyperelliptic curve [Bob91a, Appendix].

One of the benefits of hyperelliptic curves is their favorable local description. Set $f(\lambda, y) = y^2 - h(\lambda)$ where $h(\lambda)$ is the polynomial in definition 2.1.1 with only simple zeros of degree $2g + 1$. Then $\partial f/\partial y = 2y$ vanishes at branch points of Σ. Hence, by the implicit function theorem λ is a chart away from the branch points. Otherwise, whenever $y = 0$, then $\partial f/\partial\lambda \neq 0$ and, as a consequence of the implicit function theorem, y is a chart. As functions on Σ, we can calculate their zero and pole order. Generically, we will study hyperelliptic curves with branch points at $\lambda = 0$.

Proposition 2.1.1. *Let the hyperelliptic curve Σ be defined as in definition 2.1.1 with $a_1 = 0$, i.e., Σ is branched at $\lambda = 0$. Then*

1. *λ has a zero of order 2 at zero and a pole of order 2 at infinity.*
2. *y has simple zeros at $a_i, i = 1, ..., 2g + 1$, and a pole of order $2g + 1$ at infinity.*
3. *the differential $d\lambda$ has simple zeros at $a_i, i = 1, ..., 2g + 1$ and a pole of order 3 at infinity.*

Proof. The hyperelliptic curve is branched at zero. Since Σ admits a 2-fold cover over $\mathbb{C}P^1$ we can take a local chart y near $\lambda = 0$ and write $y^2 = \lambda$. Thus, λ has a zero of order two at the point over $\lambda = 0$. To investigate the points over infinity, we change coordinates $\mu = 1/\lambda$ and $w = y\mu^{g+1}$ and get

$$w^2 = \mu^{2g+2} \prod_{i=1}^{2g+1} \left(\frac{1}{\mu} - a_i\right) = \mu \prod_{i=1}^{2g+1} (1 - a_i\mu). \tag{2.6}$$

By construction we have $\mu \to 0$ as $\lambda \to \infty$. At $\mu = 0$, the map w is a chart. In particular, infinity is a branch point. Similar local analysis as with λ shows that μ has a zero of order two at the point over infinity which implies that λ has a pole of order two at infinity.

For the second statement, as y is a chart at the branch points of Σ we obtain that y has zeros at $\{a_i\}_{i=1,...,2g+1}$. The right hand side of (2.6) vanishes to order two at $w = 0$. Since μ has a zero of oder two at infinity, we obtain from $y^2 = w^2/\mu^{2g+2}$ that y has a pole of over $(2g + 1)$ at infinity.

In order to calculate the zero and pole behavior of $d\lambda$, note that $h'(\lambda)d\lambda = 2ydy$. If $y = 0$ then $h'(\lambda)$ does not vanish and y is a chart. This implies that the zeros of $d\lambda$ are the same as of y. At infinity, $d\lambda = -d\mu/\mu^2$. Since w is a chart at $\mu = 0$, the one-form dw does not have any zeros or poles. Hence,

$$d\lambda = -\frac{1}{\mu^2}d\mu = -\frac{2w}{g'(\mu)\mu^2}dw = -\frac{2y\mu^{g-1}}{g'(\mu)}dw \tag{2.7}$$

where by $g(\mu)$ we denote the polynomial on the right hand side of (2.6). Since y and μ have a pole and zero of order $2g+1$ and two at the point over infinity, respectively, and $g'(\mu)$ is non-vanishing, the right hand side of equation (2.7) has a pole of order three at infinity. This also proves the last statement. □

Proposition 2.1.1 allows us to construct differentials on Σ which prescribed zero and pole behavior. For arbitrary Riemann surfaces, such differentials are also called abelian.

2.2 Abelian differentials and holomorphic line bundles

Generically, there are three different kinds of meromorphic differentials on a compact Riemann surface. Every other meromorphic differential is given as a linear combination of those.

Definition 2.2.1. *Let ω be a differential on a Riemann surface M. Then ω is called an abelian differential of the*

i. first kind, if ω is holomorphic.

ii. second kind, if ω is meromorphic with poles but no residues. If ω has a single pole of order k at a point p we denote the differential by $\omega_p^{(k)}$.

iii. third kind, if ω is meromorphic with a pair of simple poles at two points $p, q \in M$ with residues ± 1, respectively, but no other singularities. These kind of differentials are denoted by ω_{pq}.

Let $\lambda, \lambda_0 \in M$ be arbitrary points. The integral $\int_{\lambda_0}^{\lambda} \omega$ is called an abelian integral.

The nomenclature abelian comes from viewing differential forms on compact Riemann surfaces as sections in holomorphic line bundles, which have rank one. We will come to that matter shortly. But firstly, we give some examples of the differentials introduced in definition 2.2.1. We assume that $M = \Sigma$ is a hyperelliptic curve of odd degree $2g + 1$ with branch points at zero and infinity.

i. The space of holomorphic differential forms on a compact Riemann surface is complex g-dimensional. From proposition 2.1.1 we directly see that

$$\left\{\lambda^{i-1}\frac{d\lambda}{y}\right\}_{i=1,\ldots,g} \tag{2.8}$$

forms a basis on the space of holomorphic differentials on Σ.

ii. As zero and infinity are branch points, the differentials

$$\omega_0^{(2)} = \frac{d\lambda}{2\lambda y}, \quad \omega_\infty^{(2)} = -\frac{\lambda^g d\lambda}{2y} \tag{2.9}$$

have a pole of order two at these points, respectively, and no other singularities. Taking local coordinates $\lambda = z_0^2$ and $\lambda = 1/z_\infty^2$ around the singular points, respectively, yields the asymptotic behavior

$$\omega_0^{(2)} \sim \frac{dz_0}{z_0^2}, \quad \omega_\infty^{(2)} \sim \frac{dz_\infty}{z_\infty^2}. \tag{2.10}$$

iii. For differentials of the third kind ω_{pq}, where p, q are not branch points, we see that

$$\omega_{pq} = \left(\frac{y - y(p)}{\lambda - p} - \frac{y - y(q)}{\lambda - q} \right) \frac{d\lambda}{2y} \tag{2.11}$$

has simple poles at p and q with residues ± 1, respectively.

On a compact Riemann surface M of genus g, the first homology group $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$. Further, it is well-known that there exists an isomorphism between $H_1(M, \mathbb{Z})$ and the abelianization of the first fundamental group

$$H_1(M, \mathbb{Z}) \cong \frac{\pi_1(M, p_0)}{[\pi_1(M, p_0), \pi_1(M, p_0)]}, \tag{2.12}$$

which does not depend on the base-point $p_0 \in M$. A special choice of these $2g$ elements generating $H_1(M, \mathbb{Z})$ is the following.

Definition 2.2.2. *Let M be a compact Riemann surface and assume that the elements $\{A_1, B_1, ..., A_g, B_g\}$ generate $H_1(M, \mathbb{Z})$. We call the basis $\{A_1, B_1, ..., A_g, B_g\}$ a canonical homology basis if their intersection numbers are*

$$A_l \cdot B_j = \delta_{ij}, \quad A_l \cdot A_j = B_l \cdot B_j = 0 \tag{2.13}$$

for $l, j \in \{1, ..., g\}$. The elements $A_l, B_l \in H_1(M, \mathbb{Z})$ are also called cycles. The integral of an abelian differential ω over a cycle is called a period.

Notice that a canonical homology basis is not unique. For hyperelliptic curves, such a canonical basis can be depicted rather explicitly. In this case, it is convenient to let the generators A_l encircle two branch points such that A_l stays on a single sheet of the Riemann surface while the B_l lies in both sheets. A visualization of such a canonical homology basis for a genus two hyperelliptic curve is given in figure 2.1.

Since the space of holomorphic differentials on a compact Riemann surface is g-dimensional we can add an appropriate linear combination of differentials of the first kind to differentials of the second and third kind, such that $\omega_p^{(k)}$ and ω_{pq} have vanishing A_l-periods but the form of the singularity is kept.

Definition 2.2.3. *An abelian differential of the second or third kind is called normalized if it has vanishing A_l-periods, i.e.,*

$$\int_{A_l} \omega_p^{(k)} = 0, \quad \int_{A_l} \omega_{pq} = 0, \tag{2.14}$$

for all $l = 1, .., g$.

Generically, abelian integrals of the form

$$f(p) = \int_{p_0}^{p} \omega \tag{2.15}$$

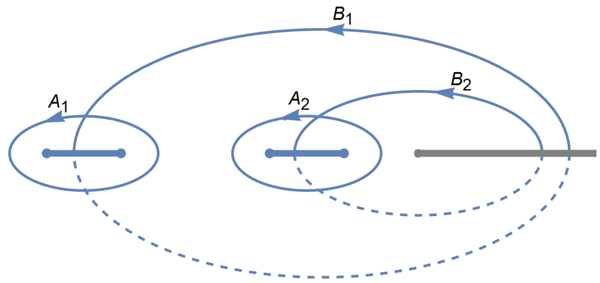

Figure 2.1: Canonical homology basis of a genus two hyperelliptic curve. The thick blue line represents branch cuts connecting pairs of branch points while the thick gray line denotes the branch cut from zero to infinity. Dashed lines denote paths on the other sheet of the hyperelliptic curve.

are not well-defined since the integral might have periods. Let $\{A_1, B_1, ..., A_g, B_g\}$ be a canonical homology basis of M. Via the isomorphism (2.12), we can think of elements in $H_1(M, \mathbb{Z})$ as (modulo homotopy) closed curves on M based at p_0. In particular, in this way we view the compact Riemann surface as a $4g$-polygon with sides bounded by $\{A_1, B_1, ..., A_g, B_g\}$. By removing these $2g$ generators, we obtain a simply connected domain $\mathcal{M}$ where abelian integrals are well-defined.

Definition 2.2.4. *Let M be a compact Riemann surface of genus g with canonical homology basis $\{A_1, B_1, ..., A_g, B_g\} \in H_1(M, \mathbb{Z})$ with common base point $p_0 \in M$. Then we call the simply connected Riemann surface obtained by removing these $2g$ generators the simply connected model of M.*

The following theorem is known as Riemann bilinear relations [GH14, p. 231].

Theorem 2.2.1 (Riemann bilinear relations)**.** *Let $\{A_1, B_1, ..., A_g, B_g\}$ be a canonical homology basis and $\omega, \tilde{\omega}$ two closed differentials on a compact Riemann surface M. Then*

$$\int_M \omega \wedge \tilde{\omega} = \sum_{l=1}^{g} \left[\int_{A_l} \omega \int_{B_l} \tilde{\omega} - \int_{B_l} \omega \int_{A_l} \tilde{\omega} \right]. \tag{2.16}$$

Proof. On the simply connected model $\mathcal{M}$ of M there exists a smooth function f such that $\omega = df$. Then it follows by Stokes theorem that

$$\int_M \omega \wedge \tilde{\omega} = \int_{\mathcal{M}} df \wedge \tilde{\omega} = \int_{\mathcal{M}} d(f\tilde{\omega}) = \int_{\partial\mathcal{M}} f\tilde{\omega}. \tag{2.17}$$

We have $\partial\mathcal{M} = \sum_{l=1}^{g} A_l + B_l + A_l^{-1} + B_l^{-1}$. Let $p_0 \in \mathcal{M}$ be arbitrary and let p and p' be two points on A_l and A_l^{-1}, respectively, which are the same on M. Consequently,

$$\int_{p_0}^{p} \omega - \int_{p_0}^{p'} \omega = \int_{p'}^{p} \omega = -B_l \tag{2.18}$$

where the path of integration stays inside $\mathcal{M}$. Similarly, for two points q, q' on B_l, B_l^{-1}, respectively, we obtain

$$\int_{p_0}^{q} \omega - \int_{p_0}^{q'} \omega = \int_{q'}^{q} \omega = A_l. \tag{2.19}$$

All together we get

$$\begin{aligned}\int_M \omega \wedge \tilde{\omega} &= \sum_{l=1}^{g}\left[\int_{A_l} f\tilde{\omega} + \int_{A_l^{-1}} f\tilde{\omega} + \int_{B_l} f\tilde{\omega} + \int_{B_l^{-1}} f\tilde{\omega}\right] \\ &= \sum_{l=1}^{g}\left[\int_{A_l}\omega\int_{B_l}\tilde{\omega} - \int_{B_l}\omega\int_{A_l}\tilde{\omega}\right].\end{aligned} \tag{2.20}$$

□

Combined with the Residue theorem, we can deduce some strong results from the Riemann bilinear relations. We list a few applications of which will all be used in this thesis.

i. Let ω and $\tilde{\omega}$ be abelian differentials of the first kind. Then it follows from 2.2.1 that

$$0 = \sum_{l=1}^{g}\left[\int_{A_l}\omega\int_{B_l}\tilde{\omega} - \int_{B_l}\omega\int_{A_l}\tilde{\omega}\right]. \tag{2.21}$$

ii. Let $\omega = \omega_p^{(k)}$ and $\tilde{\omega}_j$ be differentials of the second and first kind, respectively, and z a local centered chart around $p \in M$ such that the differentials have expansion

$$\omega = \left(\frac{a_{-k}}{z^k} + \mathcal{O}(1)\right)dz, \quad \tilde{\omega}_j = (b_0 + b_1 z + \mathcal{O}(z^2))dz \tag{2.22}$$

where $k \geq 2$. Additionally, assume that $\int_{A_i}\tilde{\omega}_j = \delta_{ij}$ and ω is normalized. Letting $f(p) = \int_{p_0}^{p}\tilde{\omega}$ for some base point $p_0 \in \mathcal{M}$, we obtain

$$\int_{B_j}\omega_p^{(k)} = 2\pi i\frac{b_{k-2}a_{-k}}{k-1}. \tag{2.23}$$

iii. Let ω and $\tilde{\omega} = \omega_{rq}$ be differentials of the first and third kind, respectively, where $\tilde{\omega}$ has expansion of the form

$$\tilde{\omega} = \left(\frac{1}{z_r} + \mathcal{O}(1)\right)dz, \quad \tilde{\omega} = \left(-\frac{1}{z_q} + \mathcal{O}(1)\right)dz \tag{2.24}$$

for two centered charts z_r and z_q around r and q, respectively. Let $f(p) = \int_{p_0}^{p}\omega$. Then we have

$$\sum_{l=1}^{g}\left[\int_{A_l}\omega\int_{B_l}\tilde{\omega} - \int_{B_l}\omega\int_{A_l}\tilde{\omega}\right] = 2\pi i(f(r) - f(q)). \tag{2.25}$$

Remark: The equations relating integrals of cycles with residues of abelian differentials are also known as reciprocity laws. For example, item two would be called a reciprocity law for abelian differentials of the second and first kind.

2.2.1 Holomorphic line bundles and the Riemann-Hurwitz theorem

As already mentioned above, we can also view abelian differentials on Riemann surfaces as sections of holomorphic line bundles. The main motivation for this lies in our definition of the Jacobian of a compact Riemann surfaces which will be given in the next subsection. A common way to define the Jacobian is to take a basis of differentials of the first kind and calculate their A_l and B_l periods. These values span the Jacobian. Using this approach, one can actually completely forgo the exposure of holomorphic vector bundles. However, we would rather like to view the Jacobian as the moduli space of holomorphic line bundles of degree zero and parameterize it in terms of holomorphic structures. This subsection follows the exposition of [Bob11, Chapter 1.7].

Let $\bigcup_{i\in I} U$ be an open cover of a compact Riemann surface M and

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^* \tag{2.26}$$

holomorphic functions for $i, j \in I$ satisfying the cocycle condition $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. For triples (p, U_i, ξ) with $p \in U_i, i \in I, \xi \in \mathbb{C}$ we define the equivalence relation

$$(p, U_i, \xi) \sim (q, U_j, \eta) \Leftrightarrow p = q \in U_i \cap U_j,\ \xi = g_{ij}(p)\eta. \tag{2.27}$$

Definition 2.2.5. *A holomorphic line bundle $L \to M$ is defined as the union of sets $U_i \times \mathbb{C}$ under the equivalence relation (2.27). The map $\pi : (p, U_i, \xi) \to p$ is called the canonical projection map and its preimage $L_p := \pi^{-1}(p)$ is called the fiber over $p \in M$.*

For two holomorphic line bundles L and $\tilde{L}$ over M, the tensor product bundle $\tilde{L} \otimes L$ is another holomorphic line bundle defined by the pointwise tensor product at each fiber. We denote by L^* the dual of the bundle L. If $f : M \to N$ is a holomorphic map between compact Riemann surfaces and $L \to N$ a holomorphic line bundle then the pull back bundle f^*L is a holomorphic line bundle over M. If a line bundle L has transition functions $g_{ij} = 1$ for all i, j, we call it trivial and denote it by $\underline{\mathbb{C}}$. An important example of a holomorphic line bundle is the canonical bundle.

Definition 2.2.6. *Let z_i and z_j be local charts of M. We call the holomorphic line bundle with transition functions*

$$g_{ij}(p) = \frac{dz_i}{dz_j}(p) \tag{2.28}$$

the canonical bundle $K_M \to M$ over M.

Using the complex structure on M we can identify the canonical bundle with the cotangent bundle $K_M = T^*M$ of M, i.e., the space differential one-forms.

Holomorphic line bundles can be constructed by taking quotients of meromorphic functions on M. Let $\phi_i : U_i \to \mathbb{C}$ be local meromorphic functions on M. Assume that for all i, j, the ratio ϕ_i/ϕ_j is holomorphic. Then the transition functions

$$g_{ij} = \frac{\phi_i}{\phi_j} : U_i \cap U_j \to \mathbb{C}^* \tag{2.29}$$

define a holomorphic line bundle L. Via g_{ij} the locally defined sections glue to a global section ϕ of $L \to M$ and on each subset U_i the section ϕ restricts to ϕ_i. The space of sections of a holomorphic vector bundle $L \to M$ is denoted by $\Gamma(M, L)$. In particular, we see that the zero and pole order of a meromorphic section completely determines the line bundle.

Definition 2.2.7. *Let M be a compact Riemann surface and fix a point $p \in U \subset M$ with centered chart z. We cover M by the two open sets $\{U_1, U_2\}$ where $U_1 = U$ and $U_2 = M \setminus \{p\}$. The line bundle $L(kp) \to M$ with holomorphic transition function $g_{12} = z^k : U \setminus \{p\} \to \mathbb{C}^*$ is called point bundle over M.*

The point bundle $L(kp) \to M$ admits a globally defined holomorphic section with a single zero of order k at p. Its dual bundle is denoted by $L^*(kp) = L(-kp) \to M$ and admits a meromorphic section with a pole of order k at p and everywhere else holomorphic without zeros. If $k = \pm 1$ we will simply write $L(\pm p)$.

Definition 2.2.8. *The degree of a holomorphic line bundle L over a compact Riemann surface M is defined as*

$$\deg(L) := \frac{i}{2\pi} \int_M F^\nabla, \tag{2.30}$$

where F^∇ is the curvature of an arbitrary connection ∇ on M.

It can be shown that this definition is independent of the connection and takes values in $\mathbb{Z}$ [GH14, p. 144]. It follows from equation (2.30) that the degree of the tensor product bundle $\tilde{L} \otimes L$ is given by $\deg(L \otimes \tilde{L}) = \deg(\tilde{L}) + \deg(L)$. The compactness of M and Stokes theorem imply that $\deg(L^*) = -\deg(L)$. The degree of the canonical bundle is $\deg(K) = 2g - 2$ where g is the genus of M.

Definition 2.2.9. *Let M be a compact Riemann surface. A holomorphic line bundle $S \to M$ with the property that $S \otimes S = K$ is called a spin bundle.*

The degree of a spin bundle is $g - 1$. On a compact Riemann surface of genus g, there exist exactly 4^g different, i.e., non-isomorphic, spin bundles [Ati71]. For us, the most relevant cases are spin bundles over Riemann surfaces of genus $g = 0, 1$.

i. Let $M = \mathbb{C}P^1$. We view $\mathbb{C}P^1$ in the usual way with open covering $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{C}^* \cup \{\infty\}$ and charts glued along $\mathbb{C}^*$ via $z \mapsto w = 1/z$. The transition function on the tangent bundle $T\mathbb{C}P^1 \to \mathbb{C}P^1$ is given by

$$\partial_z = \frac{\partial_w}{\partial_z} \partial_w = -\frac{1}{z^2} \partial_w. \tag{2.31}$$

From the point bundle construction (cf. definition 2.2.7) we see that TM admits a global holomorphic section with a single zero of order two at infinity. Hence, we identify $T\mathbb{C}P^1 = L(2\infty)$. As $T^*\mathbb{C}P^1 = K_{\mathbb{C}P^1}$, we obtain $K_{\mathbb{C}P^1} = L(-2\infty)$. Then the bundle $S = L(-\infty)$ admitting a meromorphic section with a simple pole at infinity is a spin bundle of $\mathbb{C}P^1$. Notice that the choice of centered chart z was arbitrary. Via holomorphic isomorphisms from $\mathbb{C}P^1 \to \mathbb{C}P^1$, i.e., Möbius transformations, we see that every holomorphic line bundle of degree minus one is holomorphically isomorph to $L(-\infty)$. Hence, there exists exactly one spin bundle on $\mathbb{C}P^1$. This bundle is also called the tautological bundle and is denoted by $\mathcal{O}(-1) \to \mathbb{C}P^1$.

ii. If $M = T^2 = \mathbb{C}/\Gamma$ is a complex torus, the canonical bundle has degree zero and is isomorphic to the trivial line bundle $K_{T^2} = \underline{\mathbb{C}}$. After fixing a base point $[0] \in \mathbb{C}/\Gamma$, the four spin bundles are given by $S_i = L([\omega_i] - [0])$ where ω_i is one of the four half lattice points of Γ. The trivializing sections of $L([\omega_i] - [0])$ are given by quotients of Jacobi theta functions. Evidently, these four spin bundles all square to the trivial one $\underline{\mathbb{C}} = K_{T^2}$.

We conclude this section by proving the Riemann-Hurwitz theorem. A holomorphic map $f : M \to N$ between Riemann surfaces is called branched at $p \in M$ if its derivative $df : TM \to TN$ vanishes at this point. This leads us to the Riemann-Hurwitz theorem [GH14, pp. 216-219].

Theorem 2.2.2 (Riemann-Hurwitz theorem)**.** *Let $f : M \to N$ be a holomorphic map between compact Riemann surfaces of genus g and g', respectively. Then*

$$2g - 2 = b + n \cdot (2g' - 2) \tag{2.32}$$

where b is the branch order of f and n is its degree.

Proof. We view df as a holomorphic section of $\Gamma(M, K_M \otimes f^*TN)$. The degree of the holomorphic vector bundle $K_M \otimes f^*TN$ is given by

$$\deg(K_M \otimes f^*TN) = \deg(K_M) + \deg(f^*TN). \tag{2.33}$$

But $\deg(f^*TN) = \deg(f)\deg(TN)$. Since $\deg(TM) = \deg(K_M^*) = 2 - 2g$, we obtain the desired result. □

One application of the Riemann-Hurwitz theorem is the calculation of the genus of hyperelliptic curves. Let Σ be defined as in definition 2.1.1. Since λ is a degree two map branched at $2g + 2$ different points we obtain

$$\begin{aligned} 2g(\Sigma) - 2 &= \deg(\lambda)(2g(\mathbb{C}P^1) - 2) + (2g + 2) \\ &= 2(-2) + 2g + 2 = 2g - 2. \end{aligned} \tag{2.34}$$

Hence, the number g indeed was the genus of Σ all along. In the case that $g = 1$, we call a hyperelliptic curve an elliptic curve which is equivalent to Σ being a complex torus after identifying the meromorphic functions (y, λ) with Weierstrass elliptic functions.

2.2.2 The Jacobian

After having established the basic properties of holomorphic line bundles, we can discuss the Jacobian of a compact Riemann surface M. Let $\bar{K}_M \to M$ be the anti-canonical bundle, i.e., the space of $(0, 1)$-forms on M.

Definition 2.2.10. *A holomorphic structure on a holomorphic line bundle $L \to M$ over a compact Riemann surface is an operator $\bar{\partial} : \Gamma(M, L) \to \Omega^{0,1}(L) = \Gamma(M, \bar{K}_M \otimes L)$ satisfying the Leibniz rule*

$$\bar{\partial}(fs) = (\bar{\partial} f)s + f(\bar{\partial} s) \tag{2.35}$$

for all $s \in \Gamma(M, L)$ and $f : M \to \mathbb{C}$.

We say that a section of $L \to M$ is holomorphic if it lies in the kernel of $\bar{\partial}$. Since holomorphic line bundles have holomorphic transition functions, this is well-defined. The space of global holomorphic sections is denoted by $H^0(M, L)$. We denote by

$$\bar{\partial}^{\mathbb{C}} : \Gamma(M, \underline{\mathbb{C}}) \to \Omega^{0,1}(M, \underline{\mathbb{C}}) \tag{2.36}$$

the trivial $\bar{\partial}$-operator acting in the usual way on functions. Let $\bar{\partial}_1$ and $\bar{\partial}_2$ be holomorphic structures on $\underline{\mathbb{C}}$. We say that $\bar{\partial}_1$ and $\bar{\partial}_2$ are holomorphically equivalent if there exists a smooth function f on M such that $\bar{\partial}_1.f = \bar{\partial}_2$ where $\bar{\partial}.f$ is the gauged holomorphic structure defined by

$$\bar{\partial}.f := f^{-1} \circ \bar{\partial} \circ f = \bar{\partial} + \frac{\bar{\partial}^{\mathbb{C}} f}{f}. \tag{2.37}$$

Equivalences between holomorphic structures lead to the definition of the Jacobian.

Definition 2.2.11. *The Jacobian* $\mathrm{Jac}(M)$ *of a compact Riemann surface* M *is defined as the space of holomorphic structures on the trivial rank one bundle* $\underline{\mathbb{C}}$ *modulo holomorphic equivalence.*

There are other, equivalent, definitions of the Jacobian of a compact Riemann surface. One of them is to realize $\mathrm{Jac}(M)$ as a g-dimensional complex torus. For this identification we need Serre duality.

Theorem 2.2.3 (Serre duality)**.** *Let* $L \to M$ *be a holomorphic line bundle over a compact Riemann surface. Then we have*

i. $H^0(M, L)$ *is finite dimensional.*

ii. $H^1(M, L)^* = (\Gamma(M, K_M \otimes L)/\mathrm{Im}(\bar{\partial}))^* \cong H^0(M, K_M \otimes L)$.

For a proof of the Serre duality theorem, we refer to [Huy05, p. 171].

Lemma 2.2.4. *Let* $\underline{\mathbb{C}} \to M$ *be the trivial line bundle over a compact Riemann surface with holomorphic structure* $\bar{\partial}$*. Let* $\alpha \in \Gamma(M, \bar{K}_M)$*. Then there exits a unique holomorphic one-form* $\omega \in H^0(M, K_M)$ *and smooth function* $f \in \Gamma(M, \underline{\mathbb{C}})$ *such that*

$$\bar{\partial} f = \alpha - \bar{\omega}. \tag{2.38}$$

Proof. Let $\bar{\partial} : \Gamma(M, \underline{\mathbb{C}}) \to \Omega^{0,1}(M, \underline{\mathbb{C}})$ be the holomorphic structure. By Serre duality $\bar{\partial}$ is Fredholm and hence realizes the space of sections of $\bar{K}_M$ as $\Gamma(M, \bar{K}_M) = \mathrm{Im}(\bar{\partial}) \oplus \mathrm{coker}(\bar{\partial})$, which is orthogonal with respect to the L^2 scalar product on $\Gamma(M, \bar{K}_M)$. We have $\mathrm{coker}(\bar{\partial}) = H^1(M, \underline{\mathbb{C}}) \cong H^0(M, K_M)^*$ by Serre duality 2.2.3. Furthermore, since the pairing

$$\begin{aligned} H^0(M, K_M) \times \overline{H^0(M, K_M)} \\ (\mu, \bar{\nu}) \mapsto \int_M \mu \wedge \bar{\nu}. \end{aligned} \tag{2.39}$$

is non-degenerate, it induces the isomorphism $H^0(M, K_M)^* \cong \overline{H^0(M, K_M)}$, which proves the lemma. ☐

With respect to a frame, every holomorphic structure on the trivial bundle can be written in the form

$$\bar{\partial} = \bar{\partial}^{\mathbb{C}} + \bar{\mu} \tag{2.40}$$

for a $\mu \in \Gamma(M, K_M)$. Gauging with $g = e^f$ brings $\bar{\partial}$ into the form

$$\bar{\partial}.g = \bar{\partial}^{\mathbb{C}} + \bar{\mu} + \frac{\bar{\partial}^{\mathbb{C}} g}{g} = \bar{\partial}^{\mathbb{C}} + \bar{\mu} + \bar{\partial}^{\mathbb{C}} f. \tag{2.41}$$

By lemma 2.2.4, there exists a unique $\nu \in H^0(M, K_M)$ such that $\bar{\nu} = \bar{\mu} + \bar{\partial}^{\mathbb{C}} f$. Notice that the solution

$$g(z) = \exp\Big(\int_{z_0}^{z} -(\nu - \mu) + (\bar{\nu} - \bar{\mu}) \Big) \tag{2.42}$$

to the equation $\bar{\partial}^{\mathbb{C}} f = \bar{\nu} - \bar{\mu}$ is well-defined if and only if the periods of f are multiples of $2\pi i \mathbb{Z}$. This gives us another representation of the Jacobian.

Proposition 2.2.5. *Let M be a compact Riemann surface and let $\overline{H^0(M, K_M)}$ denote the space of anti-holomorphic one-forms on M. Then we have*

$$\mathrm{Jac}(M) \cong \overline{H^0(M, K_M)}/\Lambda \tag{2.43}$$

where Λ is given by

$$\Lambda = \{\bar{\omega} \in \overline{H^0(M, K)} \mid \int_\gamma (-\omega + \bar{\omega}) \in 2\pi i\mathbb{Z} \text{ for all } \gamma \in H_1(M, \mathbb{Z})\}. \tag{2.44}$$

Since $H_1(M, \mathbb{Z})$ is generated by $2g$ elements, proposition 2.2.5 realizes $\mathrm{Jac}(M)$ as a g-dimensional complex torus. In particular, we will often be interested in the case that M is an elliptic curve with lattice $\Gamma = (X_1 + iY_1)\mathbb{Z} + (X_2 + iY_2)\mathbb{Z}$ for $X_j, Y_j \in \mathbb{R}$. Then the paths $\gamma_j(t) = t(X_j + iYj), j = 1, 2$, with $t \in [0, 1]$ are closed on M. Via the global anti-holomorphic one-form $d\bar{w}$ we can identify the lattice of $\mathrm{Jac}(M)$ with

$$\Lambda = \frac{1}{X_1Y_2 - X_2Y_1}\Big[\pi\mathbb{Z}(X_1 + iY_1) + \pi\mathbb{Z}(X_2 + iY_2)\Big]. \tag{2.45}$$

2.3 Elliptic integrals and Riemann theta functions

In this section we summarize the basic properties of elliptic and Riemann theta functions. These will in particular be important in chapter 4 for the description of spectral data of Wente tori. We start by introducing incomplete elliptic integrals of the first and second kind. Then, Riemann theta functions are defined and their reduction to Jacobi elliptic functions in the genus one case are stated.

2.3.1 Elliptic integrals

Definition 2.3.1. *The incomplete elliptic integral of the first kind $F(\phi, m)$ is defined as*

$$F(\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \tag{2.46}$$

where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le m < 1$. In the case that $\phi = \frac{\pi}{2}$, we call $F(\frac{\pi}{2}, m)$ a complete elliptic integral of the first kind and denote it by $K(m)$.

The real number m is called the elliptic modulus. Notice that for $m = 0$ we have $K(0) = \pi/2$ while $K(m)$ diverges as $m \to 1$. Elliptic integrals can be expressed as inverse functions. The equation

$$\mathrm{sn}^{-1}(\sin\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{2.47}$$

defines the inverse of the Jacobi elliptic functions $\mathrm{sn}(x, m)$. Then we simply have $\mathrm{sn}^{-1}(1, m) = K(m)$. The Jacobi elliptic functions $\mathrm{cn}(u, m)$ and $\mathrm{dn}(u, m)$ are defined by the relations

$$\begin{aligned} \mathrm{sn}^2(u, m) + \mathrm{cn}^2(u, m) &= 1 \\ m\,\mathrm{sn}^2(u, m) + \mathrm{dn}^2(u, m) &= 1. \end{aligned} \tag{2.48}$$

The functions $\mathrm{cn}(u, m)$ and $\mathrm{dn}(u, m)$ satisfy

$$\begin{aligned} \mathrm{cn}(\mathrm{sn}^{-1}(\sin\phi, m), m) &= \cos\phi \\ \mathrm{dn}(\mathrm{sn}^{-1}(\sin\phi, m), m) &= \sqrt{1 - m\,\sin^2\phi}. \end{aligned} \tag{2.49}$$

Definition 2.3.2. *The incomplete elliptic integral of the second kind $E(\phi, m)$ is defined as*

$$E(\phi, m) = \int_0^\phi \sqrt{1 - m\sin^2\theta}\, d\theta \tag{2.50}$$

where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le m \le 1$. In the case that $\phi = \frac{\pi}{2}$, we call $E(\frac{\pi}{2}, m)$ a complete elliptic integral of the second kind and denote it by $E(m)$.

Notice that $E(0) = \pi/2$ and $E(1) = 1$ is finite.

The complementary modulus is $m' = 1 - m$ and we use the notation $F'(\phi, m) = F(\phi, 1-m)$ and $E'(\phi, m) = E(\phi, 1-m)$. Legendre's relation in terms of complete elliptic integrals is

$$K'(m)E(m) + K(m)E'(m) - K(m)K'(m) = \frac{\pi}{2}. \tag{2.51}$$

Taking the derivative of $K(m)$ and $E(m)$ with respect to m, we have the two formulas

$$\frac{dK(m)}{dm} = \frac{E(m)}{2m(1-m)} - \frac{K(m)}{2m}, \qquad \frac{dE(m)}{dm} = \frac{E(m) - K(m)}{2m}. \tag{2.52}$$

Definition 2.3.3. *The incomplete elliptic integral of the third kind $\Pi(n, \phi, m)$ is defined as*

$$\Pi(n, \phi, m) = \int_0^\phi (1 - n\,\sin^2\theta)^{-1}(1 - m\sin^2\theta)^{-1/2}\, d\theta \tag{2.53}$$

where $0 \le \phi \le \frac{\pi}{2}, 0 \le m < 1$ and $n \in \mathbb{R}$. In the case that $\phi = \frac{\pi}{2}$, we call $\Pi(n, \frac{\pi}{2}, m)$ a complete elliptic integral of the third kind and denote it by $\Pi(n, m)$.

Depending on the value of the parameter n every complete elliptic integral of the third kind can be expressed purely in terms of (incomplete) elliptic integrals of the first and second kind [Law13, p. 69].

2.3.2 Riemann theta functions

Every meromorphic function on a compact Riemann surface can be expressed by quotients of Riemann theta functions. In the case that the Riemann surface is an elliptic curve, they reduce to Jacobi's theta functions. Generically, we will not be working with Riemann theta functions since Wente tori can be described explicitly in terms of elliptic data via hyperelliptic reduction of Riemann theta functions. Nonetheless, they will appear in the description of spectral data on a hyperelliptic curve of genus $g \ge 2$ as solutions to the sinh-Gordon equation (cf. equation (4.24)). For further information on Riemann theta functions, we refer to [Fay06, FK92].

Definition 2.3.4. *Let $g \in \mathbb{N}$. The Riemann theta function is defined by*

$$\theta(z, \tau) = \sum_{N \in \mathbb{Z}^g} \exp \pi i(< N, \tau N > + 2 < N, z >) \tag{2.54}$$

where $z \in \mathbb{C}^g$ and τ is a symmetric $g \times g$ matrix with positive definite imaginary part which is called the period matrix.

The condition that τ has positive imaginary part ensures that the theta function converges for any $z \in \mathbb{C}^g$. Usually, we will fix τ and consider θ as a holomorphic map from $\mathbb{C}^g$ to $\mathbb{C}$. The Riemann theta function is even in z and has the transformation property

$$\theta(z + I\mu' + \tau\mu, \tau) = \exp \pi i \left[-\mu^t z - \frac{1}{2}\mu^t \tau \mu \right] \theta(z, \tau) \tag{2.55}$$

where $\mu, \mu' \in \mathbb{Z}^g$.

Definition 2.3.5. *Let $\epsilon, \epsilon' \in \mathbb{Z}^g$. Then the $2 \times g$ matrix*

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \dots & \epsilon_g \\ \epsilon'_1 & \epsilon'_2 & \dots & \epsilon'_g \end{bmatrix} \tag{2.56}$$

is called the theta characteristic. If $<\epsilon, \epsilon'> = 0 \mod 2$, the characteristic is called even. Otherwise it is called odd.

We will also denote theta characteristics by $[\epsilon\,;\,\epsilon']$. Whereas the theta function defined in equation (2.54) above is even, the characteristics give variations such that it is even or odd, depending on the parity of the characteristic. We define

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau) := \exp \pi i \left[\frac{1}{4}\epsilon^t \tau \epsilon + \epsilon^t z + \frac{1}{2}\epsilon^t \epsilon' \right] \theta(z + I\frac{\epsilon'}{2} + \tau\frac{\epsilon}{2}, \tau). \tag{2.57}$$

There are 2^{2g} different types of theta functions where $2^{g-1}(2^g+1)$ are even and $2^{g-1}(2^g-1)$ are odd. For example, for $g = 1$, these are exactly the four Jacobi theta functions (see below). Riemann theta functions with characteristics have further periodic properties. The following proposition is from [FK92, p. 285].

Proposition 2.3.1. *Let $[\epsilon\,;\,\epsilon']$ be a theta characteristic. Then*

$$\begin{aligned} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z + e^{(k)}, \tau) &= \exp 2\pi i \left[\frac{\epsilon_k}{2} \right] \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau), \\ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z + \tau^{(k)}, \tau) &= \exp 2\pi i \left[-z_k - \frac{\tau_{kk}}{2} - \frac{\epsilon'_k}{2} \right] \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau), \\ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-z, \tau) &= \exp 2\pi i \left[\frac{\epsilon^t \epsilon'}{2} \right] \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau), \end{aligned} \tag{2.58}$$

where $e^{(k)}$ and $\tau^{(k)}$ are the k-th column of the identity and the τ-matrix, respectively. ϵ_k and ϵ'_k are the k-th entries of the g-dimensional vectors ϵ and ϵ', respectively.

The proof follows from an application of (2.55). As already mentioned, in the case of $g = 1$ Riemann theta functions reduce to Jacobi elliptic functions. With the notation of [WW20, p. 464] they are given by

$$\begin{aligned} -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) &= \vartheta_1(\pi z, \tau) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \sin[(2n+1)\pi z] \\ \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau) &= \vartheta_2(\pi z, \tau) = 2 \sum_{n=1}^{\infty} q^{(n+1/2)^2} \cos[(2n+1)\pi z] \\ \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) &= \vartheta_3(\pi z, \tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2n\pi z) \\ \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) &= \vartheta_4(\pi z, \tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2n\pi z) \end{aligned} \tag{2.59}$$

where $q = e^{\pi i \tau}$ and τ has positive imaginary part. Finally, the relationship between Jacobi theta functions and Jacobi elliptic functions is given by [WW20, p. 492]

$$\begin{aligned}
\mathrm{sn}(z,m) &= \frac{\vartheta_3(0,\tau)}{\vartheta_2(0,\tau)} \frac{\vartheta_1(z\,\vartheta_3^{-2}(0,\tau),\tau)}{\vartheta_4(z\,\vartheta_3^{-2}(0,\tau),\tau)} \\
\mathrm{cn}(z,m) &= \frac{\vartheta_4(0,\tau)}{\vartheta_2(0,\tau)} \frac{\vartheta_2(z\,\vartheta_3^{-2}(0,\tau),\tau)}{\vartheta_4(z\,\vartheta_3^{-2}(0,\tau),\tau)} \\
\mathrm{dn}(z,m) &= \frac{\vartheta_4(0,\tau)}{\vartheta_3(0,\tau)} \frac{\vartheta_3(z\,\vartheta_3^{-2}(0,\tau),\tau)}{\vartheta_4(z\,\vartheta_3^{-2}(0,\tau),\tau)}
\end{aligned} \tag{2.60}$$

with $\tau = iK'(m)/K(m)$ and $\sqrt{m} = \vartheta_2^2(0,\tau)/\vartheta_3^2(0,\tau)$, $\sqrt{m'} = \vartheta_4^2(0,\tau)/\vartheta_3^2(0,\tau)$.

Chapter 3

Conformal CMC immersions into $\mathbb{R}^3$

In this chapter, we will study conformal immersions into the euclidean space $\mathbb{R}^3$. Their description in terms of 2×2-matrices leads us to the definition of an extended frame and the associated family of flat connections. We derive the Sym-Bobenko formula which allows us to reconstruct the immersion from an extended frame. Afterwards, we will solely restrict our attention to the case that the Riemann surface is a torus. To that effect, the spectral curve is defined and closing conditions will be discussed. In the final section, we will examine the Whitham flow for CMC tori in $\mathbb{R}^3$. This chapter is based on the works of [SKKR07, Bob94] and [KS07].

3.1 The sinh-Gordon equation

Consider the Lie group $\mathrm{SU}(2)$ and let $L_g : \mathrm{SU}(2) \to \mathrm{SU}(2), h \mapsto gh$ be the left translation on $\mathrm{SU}(2)$. Via L_g we can identify the tangent bundle $T\mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathfrak{su}(2)$ where $\mathfrak{su}(2)$ is the Lie algebra of $\mathrm{SU}(2)$. The Maurer-Cartan form on $\mathrm{SU}(2)$ is defined by

$$\theta : T\mathrm{SU}(2) \to \mathfrak{su}(2), \quad v_g \mapsto (dL_{g^{-1}})_g v_g \tag{3.1}$$

which satisfies the Maurer-Cartan equation

$$2d\theta + [\theta \wedge \theta] = 0. \tag{3.2}$$

Let M be a compact Riemann surface. For a map $F : M \to \mathrm{SU}(2)$ the pullback $\alpha = F^*\theta$ satisfies (3.2) as well. Conversely, on a simply connected Riemann surface N, every solution α to (3.2) comes from a smooth map $F : N \to \mathrm{SU}(2)$ such that $\alpha = F^*\theta$ [SKKR07, p. 564].

We now want to discuss conformal immersions into $\mathbb{R}^3$. Identify the euclidean space $\mathbb{R}^3$ with the Lie algebra $\mathfrak{su}(2)$ via

$$X = -i \sum_{i=1}^{3} X_i \sigma_i \in \mathfrak{su}(2) \tag{3.3}$$

for some vector $X = (X_1, X_2, X_3) \in \mathbb{R}^3$, where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.4}$$

The inner product between two vectors $X, Y \in \mathbb{R}^3$ is given by trace

$$< X, Y >= -\frac{1}{2}\mathrm{tr}(X\,Y) \tag{3.5}$$

with the usual matrix multiplication in the brackets. Now assume that $f : M \to \mathfrak{su}(2)$ is a conformally immersed CMC surface and fix a local coordinate $z = x+iy$ on an open and simply connected subset $U \subset M$. Then f being conformal means that there exists a smooth function $u : U \to \mathbb{R}$ such that $< f_z, f_{\bar z} >= \frac{1}{2}e^u$ and $< f_z, f_z >=< f_{\bar z}, f_{\bar z} >= 0$. The Hopf field and mean curvature of the immersion can be calculated via the equations

$$Q =< f_{zz}, N >, \quad \frac{1}{2}He^u =< f_{z\bar z}, N >, \tag{3.6}$$

where $N : U \to S^2$ is the Gauß map normalized by $< N, N >= 1$. The preceding equations then imply

$$f_{zz} = u_z f_{\bar z} + QN, \quad f_{\bar z\bar z} = u_{\bar z} f_z + \bar{Q}N, \quad f_{z\bar z} = \frac{1}{2}e^u HN. \tag{3.7}$$

The triple (f_x, f_y, N) is a moving frame of the surface. We now define on U the unitary matrix $F : U \to \mathrm{SU}(2)$ by the relations

$$f_x = ie^{\frac{u}{2}}F\sigma_2F^{-1}, \quad f_y = -ie^{\frac{u}{2}}F\sigma_1F^{-1}, \quad N = -iF\sigma_3F^{-1}, \tag{3.8}$$

which determine F uniquely up to a sign. The first two equations of (3.8) are equivalent to

$$f_z = e^{\frac{u}{2}}F\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}F^{-1}, \quad f_{\bar z} = e^{\frac{u}{2}}F\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}F^{-1}. \tag{3.9}$$

Let $\alpha = F^{-1}dF$. The integrability condition $f_{z\bar z} = f_{\bar z z}$ and the normalization $\det(F) = 1$ imply that α is of the form

$$\alpha = \frac{1}{4}\begin{pmatrix} -u_zdz + u_{\bar z}d\bar z & 4ie^{-\frac{u}{2}}Qdz + 2ie^{\frac{u}{2}}Hd\bar z \\ 2ie^{\frac{u}{2}}Hdz + 4ie^{-\frac{u}{2}}\bar{Q}d\bar z & u_zdz - u_{\bar z}d\bar z \end{pmatrix}, \tag{3.10}$$

which is $\mathfrak{su}(2)$-valued by construction. From the discussion in the beginning of this section we see that α satisfies the Maurer-Cartan equation (3.2). A calculation reveals that this is equivalent to

$$\begin{aligned} u_{z\bar z} + \frac{1}{2}e^uH^2 - 2e^{-u}Q\bar{Q} &= 0 \\ \frac{1}{2}e^uH_z &= Q_{\bar z} \\ \frac{1}{2}e^uH_{\bar z} &= \bar{Q}_z. \end{aligned} \tag{3.11}$$

If f is a CMC surface then the mean curvature H is constant and equation (3.11) tells us that the Hopf field Q is holomorphic. The zeros of the quadratic holomorphic differential Qdz^2 are called umbilic points of the surface. If $U \subset M$ does not contain an umbilic point, then, by a change of coordinate, we can achieve $H = 1$ and $Q = \frac{1}{2}$. Hence, the first equation of (3.11) takes the form

$$u_{z\bar z} + \sinh(u) = 0 \tag{3.12}$$

which is called the sinh-Gordon equation.

Remark: If $M = T^2$ is a compact CMC torus we can write (3.12) globally. This is possible since the Hopf field is a global holomorphic elliptic function on a compact Riemann surface and hence constant. In particular, it cannot be totally umbilic $Q = 0$ since this case corresponds to the standard sphere. Therefore, compact tori in $\mathbb{R}^3$ have no umbilic points.

It is well-known that there exists the additional freedom to rotate the Hopf field by

$$Q \mapsto Q\lambda^{-1} \tag{3.13}$$

for some $\lambda \in S^1$ which leaves the equations (3.11) invariant. The parameter λ is called spectral parameter. We denote the α from equation (3.10) under the shift of (3.13) by

$$\alpha_\lambda = \frac{1}{4}\begin{pmatrix} -u_z dz + u_{\bar{z}} d\bar{z} & 4ie^{-\frac{u}{2}}\lambda^{-1}Q dz + 2ie^{\frac{u}{2}}H d\bar{z} \\ 2ie^{\frac{u}{2}}H dz + 4ie^{-\frac{u}{2}}\lambda\bar{Q} d\bar{z} & u_z dz - u_{\bar{z}} d\bar{z} \end{pmatrix}. \tag{3.14}$$

Then α_λ is $\mathfrak{su}(2)$-valued for all $\lambda \in S^1$ and we have $\alpha^*_{\bar{\lambda}^{-1}} = -\alpha_\lambda$. Another way to interpret α_λ is to view it as the connection one-form of a family of connections $\nabla^\lambda := d + \alpha_\lambda$ on the trivial rank two bundle which is unitary along $\lambda \in S^1$. As the $(1,0)$ and $(0,1)$-parts α'_λ and α''_λ of α_λ, respectively, extend holomorphically to zero and infinity, respectively, we obtain a $\mathbb{C}^*$-family of connections. Furthermore, by construction, F_λ^{-1} is a parallel frame of ∇^λ for $\lambda \in S^1$.

Definition 3.1.1. *We call*

$$\nabla^\lambda = d + \frac{1}{4}\begin{pmatrix} -u_z & 4ie^{-\frac{u}{2}}\lambda^{-1}Q \\ 2ie^{\frac{u}{2}}H & u_z \end{pmatrix} dz + \frac{1}{4}\begin{pmatrix} u_{\bar{z}} & 2ie^{\frac{u}{2}}H \\ 4ie^{-\frac{u}{2}}\lambda\bar{Q} & -u_{\bar{z}} \end{pmatrix} d\bar{z} \tag{3.15}$$

the associated family of connections on M and F_λ an extended frame of ∇^λ.

The curvature of ∇^λ is $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda]$, which is independent of λ. Moreover, ∇^λ is flat if and only if equations (3.11) are satisfied. We summarize this observation in the following proposition.

Proposition 3.1.1. *Let $f : M \to \mathbb{R}^3$ be a conformally immersed CMC surface in $\mathbb{R}^3$. Let z be a coordinate on a simply connected subset $U \subset M$ such that $Q = \frac{1}{2}, H = 1$ and $u : U \to \mathbb{R}$ is a smooth function. Then there exists a solution to the sinh-Gordon equation (3.12) if and only if ∇^λ is flat for all $\lambda \in \mathbb{C}^*$.*

Conversely, for a given α_λ we can solve the Lax pair

$$(F_\lambda)_z = F_\lambda\alpha'_\lambda, \quad (F_\lambda)_{\bar{z}} = F_\lambda\alpha''_\lambda \tag{3.16}$$

for F_λ with initial value $F_\lambda(0) = \mathrm{Id}$ if and only if the Maurer-Cartan equation is satisfied. Therefore, proposition 3.1.1 also gives rise to an extended frame. This property is well-known and for a proof of this statement we refer the reader to [FKR06, Proposition 3.1.2].

3.2 The Sym-Bobenko formula

The freedom to rotate the Hopf field by the spectral parameter $\lambda = e^{it}$ means that we obtain a one-parameter family of surfaces which are all isometric, i.e., have the same metric, but a Hopf field rotated by λ. For a given extended frame we can reconstruct the immersion from the Sym-Bobenko formula.

Theorem 3.2.1 (Sym-Bobenko formula)**.** *Let M be a simply connected Riemann surface and $F : M \times S^1 \to \mathrm{SU}(2)$ an extended frame. Let $\lambda = e^{it} \in S^1$. Then $f_\lambda : M \to \mathbb{R}^3$, defined by*

$$f_\lambda = -2\frac{1}{H}(\partial_t F_\lambda)F_\lambda^{-1} + i\frac{1}{H}F_\lambda \sigma_3 F^{-1}, \tag{3.17}$$

is a CMC surface with mean curvature H and Hopf field $\lambda^{-1}Q$.

Proof. We write $\alpha_\lambda = \alpha'_\lambda + \alpha''_\lambda$ with type decomposition

$$\alpha'_\lambda = \frac{1}{4}\begin{pmatrix} -u_z & 4i\lambda^{-1}e^{-\frac{u}{2}}Q \\ 2ie^{\frac{u}{2}}H & u_z \end{pmatrix} dz, \quad \alpha''_\lambda = \frac{1}{4}\begin{pmatrix} u_{\bar z} & 2ie^{\frac{u}{2}}H \\ 4i\lambda e^{-\frac{u}{2}}\bar Q & -u_{\bar z} \end{pmatrix} d\bar z \tag{3.18}$$

and calculate

$$\begin{aligned} i[\alpha'_\lambda, \sigma_3] &= \begin{pmatrix} 0 & 2\lambda^{-1}e^{-\frac{u}{2}}Q \\ -e^{\frac{u}{2}}H & 0 \end{pmatrix}, \quad i[\alpha''_\lambda, \sigma_3] = \begin{pmatrix} 0 & e^{\frac{u}{2}}H \\ -2\lambda e^{-\frac{u}{2}}\bar Q & 0 \end{pmatrix} \\ \partial_t \alpha'_\lambda &= \begin{pmatrix} 0 & \lambda^{-1}e^{-\frac{u}{2}}Q \\ 0 & 0 \end{pmatrix}, \quad \partial_t \alpha''_\lambda = \begin{pmatrix} 0 & 0 \\ -\lambda e^{-\frac{u}{2}}\bar Q & 0 \end{pmatrix}. \end{aligned} \tag{3.19}$$

Taking the derivative of (3.17), we obtain

$$\begin{aligned} (f_\lambda)_z &= -H^{-1}F_\lambda(2(\partial_t\alpha') - i[\alpha'_\lambda, \sigma_3])F_\lambda^{-1} \\ &= e^{\frac{u}{2}}F_\lambda \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} F_\lambda^{-1} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} (f_\lambda)_{\bar z} &= -H^{-1}F_\lambda(2(\partial_t\alpha'') - i[\alpha'', \sigma_3])F_\lambda^{-1} \\ &= e^{\frac{u}{2}}F_\lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_\lambda^{-1} \end{aligned} \tag{3.21}$$

which coincides with the two equations in (3.9). Therefore, F_λ is, up to a sign, the extended frame obtained from f_λ. In particular, the surface is a conformal immersion with conformal factor e^u. The normal vector N is given by $N = -iF\sigma_3F^{-1}$ from which we calculate that the mean curvature is H. Taking the derivative of (3.20) with respect to z shows that $< (f_\lambda)_{zz}, N >= \lambda^{-1}Q$. □

Remark: Similar formulas for the reconstruction of CMC surfaces into the other two space forms S^3 and $\mathbb{H}^3$ also exist (see, e.g., [FKR06, section 5]).

As we are interested in non-simply connected Riemann surfaces, the immersion has to satisfy closing conditions. The extrinsic closing condition asserts that the immersion descends to a well-defined immersion to a Riemann surface which is not simply-connected. For this we need to investigate the monodromy of the extended frame.

Definition 3.2.1. *Let $F_\lambda : \tilde M \to \mathrm{SU}(2)$ be an extended frame defined on the universal cover $\tilde M$ of a Riemann surface M. The monodromy of F_λ with respect to $\gamma \in \pi_1(M, p)$ is defined as*

$$\begin{aligned} H_p^\lambda : \pi_1(M, p) &\to \mathrm{SL}(2, \mathbb{C}) \\ \gamma &\mapsto F_\lambda(\tilde\gamma(1))F_\lambda(\tilde\gamma(0))^{-1} \end{aligned} \tag{3.22}$$

where $\tilde\gamma$ is the unique lift of γ to $\tilde M$ such that $\tilde\gamma(0) = \tilde p$ is the point lying over p.

Recall from proposition 3.1.1 that the associated family is flat if and only if the sinh-Gordon equation is satisfied. Since the monodromy of a flat connection only depends on the homotopy class of a closed loop, definition 3.2.1 is well-defined. In particular, as the associated family is unitary along $\lambda \in S^1$ the monodromy is unitary as well, i.e., $(H_p^{\bar{\lambda}^{-1}})^* = (H_p^{\lambda})^{-1}$. Furthermore, the monodromy does not depend on z since both $F_\lambda(\tilde{\gamma}(1))$ and $F_\lambda(\tilde{\gamma}(0))$ give rise to the same unique solution of $dF_\lambda = F_\lambda \alpha_\lambda$ on $\tilde{M}$.

Now consider the parallel frame F_λ undergoing a non-trivial loop γ on M based at p. This changes the Sym-Bobenko formula to

$$\begin{aligned} f(\tilde{\gamma}(1)) =& -2\frac{1}{H}(\partial_t(H_p^\lambda(\gamma)F_\lambda(\tilde{\gamma}(0))))F_\lambda(\tilde{\gamma}(0))^{-1}H_p^\lambda(\gamma)^{-1} \\ &+ i\frac{1}{H}H_p^\lambda(\gamma)F_\lambda(\tilde{\gamma}(0))\sigma_3 F_\lambda^{-1}(\tilde{\gamma}(0))H_p^\lambda(\gamma)^{-1} \\ =& -2\frac{1}{H}\partial_t H_p^\lambda(\gamma)H_p^\lambda(\gamma)^{-1} - 2\frac{1}{H}H_p^\lambda(\gamma)\partial_t F_\lambda(\tilde{\gamma}(0))F_\lambda^{-1}(\tilde{\gamma}(0))H_p^\lambda(\gamma)^{-1} \\ &+ i\frac{1}{H}H_p^\lambda(\gamma)F_\lambda(\tilde{\gamma}(0))\sigma_3 F_\lambda^{-1}(\tilde{\gamma}(0))H_p^\lambda(\gamma)^{-1} \\ =& -2\frac{1}{H}\partial_t H_p^\lambda(\gamma)H_p^\lambda(\gamma)^{-1} + H_p^\lambda(\gamma)f(\tilde{\gamma}(0))H_p^\lambda(\gamma)^{-1}. \end{aligned} \tag{3.23}$$

$f(\tilde{\gamma}(0))$ gives rise to the frame $F_\lambda(\tilde{\gamma}(0))$ whereas $f(\tilde{\gamma}(1))$ gives rise to the frame $F_\lambda(\tilde{\gamma}(1)) = H_p^\lambda(\gamma)F_\lambda(\tilde{\gamma}(0))$. Since the extended frame is uniquely defined up to a sign we must have $H_p^\lambda(\gamma) = \pm\mathrm{Id}$. But from equation (3.23) this implies that $\partial_t H_p^\lambda(\gamma) = 0$. Therefore, we obtain a well-defined surface in $\mathbb{R}^3$ via the Sym-Bobenko formula if and only if

$$\partial_t H_p^\lambda(\gamma) = 0, \quad H_p^\lambda(\gamma) = \pm\mathrm{Id} \tag{3.24}$$

along any generator $\gamma \in \pi_1(M, p)$.

The monodromy depends on the base point p. Let q be another point on M and choose a path $s : [0,1] \to M$ from $s(0) = p$ to $s(1) = q$. Then the monodromy matrices H_p^λ and H_q^λ are related by

$$H_p^\lambda(\gamma) = (P_s^{\nabla^\lambda})^{-1} \circ H_q^\lambda(\gamma) \circ P_s^{\nabla^\lambda} \tag{3.25}$$

where $P_s^{\nabla^\lambda} \in \mathrm{SU}(2)$ for $\lambda \in S^1$ is the parallel transport of ∇^λ along s, i.e., the monodromy matrices are conjugated to each other. A quick calculation reveals that if the monodromy satisfies the equations in (3.24) at p then they are also satisfied at q. Therefore, choosing a different base-point yields the same immersion.

We summarize the conditions needed in order to obtain conformally immersed surface in $\mathbb{R}^3$. Satisfying all these conditions is also known as solving the monodromy problem:

i. $F_\lambda \in \mathrm{SU}(2)$ for all $\lambda \in S^1$.

ii. $\partial_\lambda H^{\lambda_0}(\gamma) = 0$.

iii. $H^{\lambda_0}(\gamma) = \pm\mathrm{Id}$.

Here, $\lambda_0 \in S^1$ and $\gamma \in \pi_1(M, *)$. Generally, we will assume that the point where these conditions are satisfied is $\lambda_0 = 1$ and call this point a Sym-point. If the Riemann surface is a torus we can express these conditions in terms of data on a hyperelliptic curve. This will be our concern in the following section.

3.3 Spectral curve of CMC tori

Given a conformally parametrized CMC surface in $\mathbb{R}^3$, we have seen in the previous section how to obtain an extended frame. Vice versa, the surface can be reconstructed by the Sym-Bobenko formula from such a frame. For surfaces of genus $g \geq 1$, we have to control the monodromy of the extended frame in order to obtain closed surfaces. The simplest non-trivial example is the case that $M = T^2$ is a torus. It turns out that a helpful concept in the description of compact CMC tori in space forms is the spectral curve. Notice that the exposition in this section translates under mild variations to CMC tori in space forms other than $\mathbb{R}^3$ [Bob91b, section 3].

Let $\gamma_j, j = 1, 2$, generate $\pi_1(T^2, p)$. Since the first fundamental group of the torus is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, i.e., it is abelian, the monodromies $H_p^\lambda(\gamma_1)$ and $H_p^\lambda(\gamma_2)$ commute. We use this fact for the following proposition.

Proposition 3.3.1. *Fix $\lambda \in \mathbb{C}^*$ and let $p \in T^2$. Assume that $H_p^\lambda(\gamma_1)$ has only 1-dimensional eigenspaces. Then the eigenvectors of $H_p^\lambda(\gamma_1)$ are also eigenvectors of $H_p^\lambda(\gamma_2)$.*

Proof. Let v be an eigenvector of $H_p^\lambda(\gamma_1)$ to the eigenvalue μ, i.e., $H_p^\lambda(\gamma_1)v = \mu v$. As the monodromy matrices $H_p^\lambda(\gamma_1)$ and $H_p^\lambda(\gamma_2)$ commute, we obtain

$$H_p^\lambda(\gamma_1)H_p^\lambda(\gamma_2)v = H_p^\lambda(\gamma_2)H_p^\lambda(\gamma_1)v = \mu H_p^\lambda(\gamma_2)v \tag{3.26}$$

which implies that $H_p^\lambda(\gamma_2)v$ is also an eigenvector of $H_p^\lambda(\gamma_1)$ to the eigenvalue μ. However, since, by assumption, $H_p^\lambda(\gamma_1)$ has only 1-dimensional eigenspaces, there exists a constant $c \in \mathbb{C}$ with $H_p^\lambda(\gamma_2)v = cv$ which implies that v is an eigenvector of $H_p^\lambda(\gamma_2)$. $\square$

Proposition 3.3.1 implies that to every simple eigenvalue there exists a basis which simultaneously diagonalizes both monodromies and, since the monodromies are $\mathrm{SL}(2, \mathbb{C})$-valued, we can assume that it has the form

$$H_p^\lambda(\gamma_j) = \begin{pmatrix} \mu_j & 0 \\ 0 & \mu_j^{-1} \end{pmatrix} \tag{3.27}$$

with distinct eigenvalues $\mu_j \neq \pm 1$. If q is any other point on T^2 then the monodromies H_p^λ and H_q^λ are related by conjugation of the parallel transport from p to q (cf. (3.25)). Since conjugated matrices have the same set of eigenvalues, we can define the following.

Definition 3.3.1. *The spectral curve of a CMC torus in $\mathbb{R}^3$ is defined as the normalization and compactification of*

$$\Sigma := \left\{(\lambda, \mu_1) \in \mathbb{C}^* \times \mathbb{C}^* \,|\, \det(H_p^\lambda(\gamma_1) - \mu_1 \mathrm{Id}) = 0\right\} \tag{3.28}$$

where μ_1 is the eigenvalue of $H_p^\lambda(\gamma_1)$.

Unraveling the equation in (3.28) we have the two solutions

$$\mu_1 = \frac{1}{2}[h_1(\lambda) \pm \sqrt{h_1(\lambda)^2 - 4}] \tag{3.29}$$

where $h_1(\lambda)$ is the trace of the monodromy with respect to the generator γ_1. Thus, Σ is branched at the odd order zeros of $h_1(\lambda)^2 - 4$ and by [Hit90, Proposition (2.3)] there exist only finitely many of those. Moreover, the odd order zeros of $h_1(\lambda)^2 - 4$

are also odd order zeros of $h_2(\lambda)^2 - 4$, where $h_2(\lambda)$ is the trace of the monodromy with respect to the second generator γ_2 [Hit90, Proposition (2.10)]. Therefore, we equally could have defined (3.28) with respect to $H_p^\lambda(\gamma_2)$ which yields the same spectral curve. To fulfill the property that equation (3.28) can be compactified, one has to add branch points at zero and infinity as described in section three of [Hit90].

The fact that the monodromy is $\mathrm{SL}(2,\mathbb{C})$-valued defines a double covering $\lambda : \Sigma \to \mathbb{C}P^1$. Hitchin has shown that the eigenvalues of the monodromy are well-defined only on a double covering of $\mathbb{C}P^1$ branched at zero and infinity [Hit90, Theorem (8.1)]. Therefore, we associate to the spectral curve the following hyperelliptic curve

$$y^2 = \lambda \prod_{i=1}^{g} (\lambda - a_i)(\lambda - \bar{a}_i^{-1}) \tag{3.30}$$

parameterizing the eigenvalues of the monodromy. The branch points of (3.30) are given by the odd order zeros of $h_1(\lambda)^2 - 4$. For conformally immersed CMC tori in space forms, the curve (3.30) is smooth [Bob91a, Appendix]. With an abuse of notation, we will also denote the hyperelliptic curve (3.30) by Σ and also call it spectral curve.

Σ naturally possesses a hyperelliptic involution $\sigma : \Sigma \to \Sigma$ sending $(y, \lambda) \mapsto (-y, \lambda)$. Furthermore, it comes equipped with an anti-holomorphic involution (real structure) $\rho : \Sigma \to \Sigma$ commuting with the projection map $\lambda : \Sigma \to \mathbb{C}P^1$ and inducing the map $\lambda \mapsto \bar{\lambda}^{-1}$ on $\mathbb{C}P^1$ with S^1 as its fixed point set. To be more precise, with respect to the coordinate λ on $\mathbb{C}P^1$, the anti-holomorphic involution can be written as

$$\rho : (y, \lambda) \mapsto (\bar{y}\bar{\lambda}^{-(g+1)}, \bar{\lambda}^{-1}). \tag{3.31}$$

We summarize the action of the involutions on the eigenvalues of the monodromy and some additional properties in the following proposition [Hit90, Theorem (8.1)].

Proposition 3.3.2. *Let Σ be the spectral curve of an CMC torus in $\mathbb{R}^3$.*

i. The eigenvalues of the monodromies $\mu_j, j = 1, 2$, are holomorphic functions on $\Sigma \backslash \lambda^{-1}(\{0, \infty\})$ and have essential singularities at the points over zero and infinity.

ii. The actions of the involutions σ and ρ on μ_j are

$$\sigma^* \mu_j = \mu_j^{-1}, \quad \rho^* \mu_j = \bar{\mu}_j^{-1}, \quad \tau = (\sigma \circ \rho)^* \mu_j = \bar{\mu}_j. \tag{3.32}$$

iii. The logarithmic derivatives $\theta_j = d \ln \mu_j$ of μ_j are abelian differentials of the second kind with a second order pole at zero and infinity.

iv. The principal parts of θ_j are linearly independent.

The principal parts of the one-forms θ_j determine the conformal type of the complex torus T^2. Moreover, from (3.32) we obtain that θ_j behave under the involutions as

$$\sigma^* \theta_j = -\theta_j, \quad \rho^* \theta_j = -\bar{\theta}_j, \quad \tau^* \theta_j = \bar{\theta}_j. \tag{3.33}$$

Definition 3.3.2. *Fix a point $p \in T^2$. The eigenspace bundle $L_p(\xi)$ on the hyperelliptic curve $\Sigma \setminus \lambda^{-1}(\{0, \infty\})$ is defined by*

$$L_p(\xi) \subseteq \ker(H_p^\lambda(\gamma_1) - \mu_1 Id) \tag{3.34}$$

for all $\xi \in \Sigma \setminus \lambda^{-1}(\{0, \infty\})$ with $\lambda(\xi) = \lambda$.

Since the eigenspaces of the holonomy are generically 1-dimensional, the eigenspace bundles are line bundles. Via parallel transport, they extends to holomorphic line bundles on $L \to T^2 \otimes \Sigma \setminus \lambda^{-1}(\{0, \infty\})$. Investigating the limiting behavior, Hitchin has also shown that the eigenspace bundles extend holomorphically to zero and infinity to become eigenspace bundles of a Higgs field, i.e., a global holomorphic one-form on T^2, and its dual, respectively [Hit90, section 8]. It turns out that the change of $p \in T^2$ yields an eigenline bundle flow in the Jacobian of the spectral curve which is linear in p [Hit90, section 7].

Theorem 3.3.3. *Fix a point $p_0 \in T^2$. The map*

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma), \quad p \mapsto L_p \otimes L_{p_0}^{-1} \tag{3.35}$$

is a group homomorphism.

Theorem 3.3.3 tells us that it is possible to reconstruct the immersion purely in terms of data on the spectral curve [Hit90, section 8]. From proposition 2.2.5 we know that the Jacobian is a complex g-dimensional torus. The holomorphic line bundle $E_p := L_p \otimes L_{p_0}^{-1}$ is real with respect to the fixed point free involution τ on Σ, i.e., there exists an isomorphism $E_p \cong \overline{\tau^* E_p}$ which squares to the identity [Hit90, p. 667]. Therefore, the image of Ψ is mapped into the real g-dimensional torus $\mathrm{Jac}^{\mathbb{R}}(\Sigma)$.

3.3.1 Closing conditions

We have already seen the closing conditions for a compact Riemann surface in section 3.2. Here we will investigate them again in case of $M = T^2$ being a torus. The concept of the spectral curve will allow us to rewrite them completely in terms of conditions on the spectral data.

Intrinsic closing conditions

Recall the map Ψ defined in (3.35). Since the associated family came from a doubly periodic metric, the eigenbundle flow is doubly periodic as well. This asserts the intrinsic closing condition: the existence of a two-dimensional subtorus $T^2 \subset \mathrm{Jac}^{\mathbb{R}}(\Sigma)$ with doubly periodic metric [GPS09, section 2].

We want to give more insight into this statement and show that the intrinsic closing conditions can be expressed purely in terms of properties of spectral data. Let Σ be a hyperelliptic curve of the form (3.30) equipped with an anti-holomorphic involution ρ covering the map $\lambda \mapsto \bar{\lambda}^{-1}$ on $\mathbb{C}P^1$. Denote the hyperelliptic involution by σ. Assume that there exist two abelian differentials θ_j of the second kind with poles of order two at zero and infinity satisfying $\rho^*\theta_j = -\bar{\theta}_j$ and $\sigma^*\theta_j = -\theta_j$. We further impose that the periods of θ_j all are $2\pi i\mathbb{Z}$-valued

$$\int_{A_l} \theta_j \in 2\pi i\mathbb{Z}, \quad \int_{B_l} \theta_j \in 2\pi i\mathbb{Z} \tag{3.36}$$

for $l = 1, ..., g$ where g is the genus of Σ. This implies that there exist globally defined holomorphic functions μ_j on $\Sigma \setminus \lambda^{-1}(\{0, \infty\})$ with essential singularities at zero and infinity such that $\theta_j = d \ln \mu_j$. As θ_j are differentials on a hyperelliptic curve we can write them down explicitly in terms of the coordinates (λ, y). We have seen in section 2.2 that

$$\left\{ \lambda^{i-1} \frac{d\lambda}{\lambda y} \right\}_{i=1,...,g+2} \tag{3.37}$$

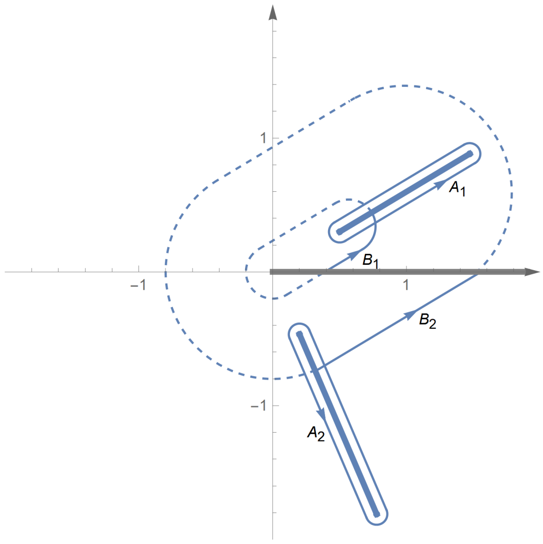

Figure 3.1: Canonical homology basis of a genus two curve Σ with homology basis as in (3.39). The thick blue lines represents branch cuts connecting pairs of branch points reflected along S^1 while the thick gray line denotes the branch cut from zero to infinity. Dashed lines denote paths on the second sheet of Σ.

is a basis for meromorphic differentials with a pole of order two at zero and infinity. For the θ_j this implies

$$\theta_j = \sum_{i=1}^{g+2} d_{j,i} \lambda^{i-1} \frac{d\lambda}{\lambda y} \tag{3.38}$$

where $d_{j,i} \in \mathbb{C}$. By the reality condition $\rho^* \theta_j = -\bar{\theta}_j$, we obtain that the coefficients of θ_j satisfy $d_{j,i} = \bar{d}_{j,g+3-i}$. In particular, the space of differentials of the second kind with reality $\rho^*\theta_j = -\bar{\theta}_j$ is real $(g+2)$-dimensional. Hence, we have $2g - (g+2) = g-2$ constraints for the existence of a θ_j such that the periods are $2\pi i\mathbb{Z}$ valued.

We fix the homology basis on Σ in such a way that

$$\begin{aligned} \rho(A_l) &\equiv -A_l \\ \rho(B_l) &\equiv B_l - A_l + \sum_{i=1}^{g} A_i \end{aligned} \tag{3.39}$$

where the three lines equality sign means modulo curves homological to zero (see figure 3.1). A homology basis satisfying (3.39) forces the A_l-periods of the abelian differentials θ_j

$$-\int_{A_l} \theta_j = \int_{\rho(A_l)} \theta_j = \int_{A_j} \rho^* \theta_j = -\int_{A_j} \bar{\theta}_j = -\overline{\int_{A_j} \theta_j} \tag{3.40}$$

to be real. Hence, the assumption that the periods are $2\pi i\mathbb{Z}$-valued implies that the differentials θ_j are normalized, i.e., have vanishing A_l-periods. Let

$$\omega_i = \sum_{l=1}^{g} v_{il} \frac{\lambda^{l-1} d\lambda}{y} \tag{3.41}$$

with $i = 1, ..., g$ be a basis of abelian differentials of the first kind normalized in such a way that $\int_{A_l} \omega_i = \delta_{il}$, i.e., $(v_{il}) = (C_{il})^{-1}$ where

$$C_{il} = \int_{A_l} \frac{\lambda^{i-1} d\lambda}{y}. \tag{3.42}$$

If the principal parts of θ_j are linearly independent, they define a lattice in $\mathbb{R}^2$. To be more precise, we define the two differentials $d\Omega_0$ and $d\Omega_\infty$ by the relations

$$\begin{aligned} \theta_1 &= 2(d\Omega_\infty \bar{d}_{1,1} - d\Omega_0 d_{1,1}) \\ \theta_2 &= 2(d\Omega_\infty \bar{d}_{2,1} - d\Omega_0 d_{2,1}). \end{aligned} \tag{3.43}$$

Comparing the asymptotic behavior around $\lambda = 0$ and $\lambda = \infty$, we see that $d\Omega_0$ and $d\Omega_\infty$ are normalized abelian differentials of the second kind with a pole of order two at zero and infinity, respectively, and holomorphic everywhere else. Let $z_\infty^2 = 1/\lambda$ be a local coordinate near infinity. Then $d\Omega_\infty$ has the expansion $d\Omega_\infty \sim -dz_\infty / z_\infty^2$ near infinity. With respect to the same coordinate, the local expansion of ω_i near infinity is $\omega_i \sim -2v_{ig} dz_\infty$. A reprocity law for abelian differentials of the second and first kind yields

$$\int_{A_l} d\Omega_\infty \int_{B_l} \omega_i - \int_{A_l} \omega_i \int_{B_l} d\Omega_\infty = -\int_{B_i} \Omega_\infty = -4\pi i v_{ig} \tag{3.44}$$

since ω_i and $d\Omega_\infty$ are normalized. Set

$$d_{1,1} = \frac{iX_1 - Y_1}{4}, \quad d_{2,1} = \frac{iX_2 - Y_2}{4} \tag{3.45}$$

with real numbers X_i, Y_i. Under the assumption that the B_l-periods of θ_j are $2\pi i\mathbb{Z}$-valued we obtain

$$2v_{ig} \in \frac{1}{X_1 Y_2 - X_2 Y_1}((X_1 + iY_1)\mathbb{Z} + (X_2 + iY_2)\mathbb{Z}) \tag{3.46}$$

for all $i = 1, ..., g$ (compare also with (2.45)). Since $d_{1,1}$ and $d_{2,1}$ are $\mathbb{R}$-linearly independent, the lattice

$$\Gamma = (X_1 + iY_1)\mathbb{Z} + (X_2 + iY_2)\mathbb{Z} \tag{3.47}$$

is well-defined. Rewriting (3.46) we get

$$2v_{ig}(X_1 Y_2 - X_2 Y_1) \in (X_1 + iY_1)\mathbb{Z} + (X_2 + iY_2)\mathbb{Z} \tag{3.48}$$

for $i = 1, ..., g$. Let $v_{ig} = \beta_i + i\alpha_i$. Then the left hand side of equation (3.48) is

$$2v_{ig}(X_1 Y_2 - X_2 Y_1) = 2(X_1 + iY_1)(\alpha_i X_2 - \beta_i Y_2) - 2(X_2 + iY_2)(\alpha_i X_1 - \beta_i Y_1) \tag{3.49}$$

and equation (3.48) is equivalent to the matrix

$$\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix} \begin{pmatrix} 2\alpha_1 & ... & 2\alpha_g \\ -2\beta_1 & ... & -2\beta_g \end{pmatrix} \tag{3.50}$$

being an integer matrix (compare also with [Bob91a, p. 225]). If the two horizontal real g-dimensional vectors of (3.50) are linearly independent, they span a lattice plane in $\mathbb{R}^g$ and hence they give rise to a subtorus $T^2 = \mathbb{C}/\Gamma$ with lattice Γ (3.47) in the real g-dimensional Jacobian. These observations allow us to express the intrinsic closing conditions only in terms of spectral data on the hyperelliptic curve.

Definition 3.3.3. *Let $\{A_1, B_1, ..., A_g, B_g\}$ be a canonical homology basis of Σ. Denote by θ_j for $j = 1, 2$, abelian differentials of the second kind with a pole of order two at zero and infinity. We call the condition*

$$\int_{A_l} \theta_j \in 2\pi i\mathbb{Z}, \quad \int_{B_l} \theta_j \in 2\pi i\mathbb{Z} \tag{3.51}$$

for all $j = 1, 2, l = 1, ..., g$ the intrinsic closing condition.

Extrinsic closing conditions

The intrinsic closing condition on its own is not enough to obtain compact CMC tori, we further need periodicity of the immersion. We have already seen in equation (3.24) that the monodromy of the extended frame needs to satisfy

$$\partial_t H_p^\lambda(\gamma) = 0, \quad H_p^\lambda(\gamma) = \pm\mathrm{Id} \tag{3.52}$$

for some $\lambda = e^{it} \in S^1$ where the monodromies were defined with respect to the base point $p \in M$. For $M = T^2$ being a torus, we can rewrite (3.52) to conditions on the eigenvalues of the monodromy on the spectral curve. Notice that the following definition is independent of $p \in T^2$ since the spectral curve is invariant under conjugation of the monodromies.

Definition 3.3.4. *Let $\mu_j(\lambda)$ denote the eigenvalues of the monodromy of the associated family on a torus. Let $\lambda \in S^1$. Then we call the conditions*

$$\mu_j(\lambda) = \pm 1, \quad \partial_\lambda \mu_j(\lambda) = 0 \tag{3.53}$$

the extrinsic closing condition.

A point $\lambda \in S^1$ where both conditions (3.53) are satisfied is called a Sym point. If there are multiple Sym-points, the reconstruction of the surface is no longer uniquely determined by the Sym-Bobenko formula and one can bifurcate to different surfaces (see section 3.4). For conformal CMC tori into $\mathbb{R}^3$ we will fix the Sym-point to be $\lambda = 1$.

The sign in the first equation of (3.53) is meaningful in the sense that it determines the spin class of the immersion [Bob94, Lemma 3]. On a Riemann surface of genus $g = 1$ there exist 4 spin bundles (cf. definition 2.2.9). This is relevant for us since we want to describe spectral data in terms of holomorphic structures on a torus, i.e., maps from the spectral curve into the Jacobian $\mathrm{Jac}(T^2)$, and a spin bundle corresponds to a half-lattice point of the lattice generating $\mathrm{Jac}(T^2)$. However, we will not go into detail for the description of spinors on compact Riemann surfaces. For further reference we advise the reader to [Bob94, KS96].

3.3.2 The associated family in terms of holomorphic structures

Let $f : T^2 \to \mathbb{R}^3$ be a conformally immersed CMC torus. We saw that the immersion can be equipped with the associated family of flat connections ∇^λ (cf. definition 3.1.1). As the eigenline bundle $L_p(\xi)$ of the monodromy along one generator of $\pi_1(T^2, p)$ is also an eigenvector for the monodromy along the other generator (cf. proposition 3.3.1), there exist a gauge which brings the associated family into the form

$$\nabla^\xi = d + \begin{pmatrix} \alpha(\xi)dw - \chi(\xi)d\bar{w} & 0 \\ 0 & -\alpha(\xi)dw + \chi(\xi)d\bar{w} \end{pmatrix} \tag{3.54}$$

on the trivial rank two bundle $\underline{\mathbb{C}}^2 = L_p(\xi) \oplus L_p^*(\xi)$ [Hit90]. Since changing the base-point from p to q amounts to conjugation of the monodromy (cf. equation (3.25)), this is independent of the chosen base-point. By an abuse of notation, we will also call (3.54) the associated family, even though it differs from the original one by a gauge which is singular at the points where the eigenlines coalesce. The maps

$$\chi : \Sigma \setminus \lambda^{-1}(\infty) \to \mathrm{Jac}(T^2), \quad \alpha : \Sigma \setminus \lambda^{-1}(0) \to \overline{\mathrm{Jac}(T^2)} \tag{3.55}$$

parameterize the holomorphic and anti-holomorphic structure of a family of flat connections on T^2, respectively. Here, dw is the global holomorphic one-form on T^2. By definition, the associated family is unitary along S^1 which implies $\alpha(\xi) = \overline{\chi(\xi)}$ for all $\xi \in \lambda^{-1}(S^1)$.

The associated family of holomorphic and anti-holomorphic structures $\bar{\partial}^\lambda = (\nabla^\lambda)''$ and $\partial^\lambda = (\nabla^\lambda)'$ extend holomorphically to infinity and zero, respectively. Under the hyperelliptic involution $\sigma : (\lambda, y) \mapsto (\lambda, -y)$, the diagonal entries of (3.54) are interchanged since $\sigma^* L = L^*$. However, under the $\mathrm{SL}(2, \mathbb{C})$-gauge

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.56}$$

we get back the original connection one-form. Therefore, the gauge class of the associated family is invariant under the hyperelliptic involution σ.

Next we want to write down the explicit form of the holomorphic structures χ. Away from the branch points we fix the coordinate λ on Σ. Comparing the monodromies of the associated family with the lattice in equation (3.47) we see that $d\chi$ is given by

$$d\chi = \frac{i}{2} d\Omega_\infty = \frac{i}{4} \frac{\theta_1 d_{2,1} - \theta_2 d_{1,1}}{\bar{d}_{1,1} d_{2,1} - \bar{d}_{2,1} d_{1,1}} = \frac{i}{2} \frac{\theta_2(X_1 + iY_1) - \theta_1(X_2 + iY_2)}{X_1 Y_2 - X_2 Y_1} \tag{3.57}$$

by equation (3.43). By reality,

$$d\alpha = -\frac{i}{2} d\Omega_0. \tag{3.58}$$

Notice that $d\chi$ and $d\alpha$ have a pole of order 2 at infinity and zero, respectively. As long as the intrinsic closing conditions in definition 3.3.3 are satisfied, the integral

$$\chi(\lambda) = \int_0^\lambda d\chi + c \tag{3.59}$$

is a well-defined map in $\mathrm{Jac}(T^2)$ of $T^2 = \mathbb{C}/\Gamma$ with lattice $\Gamma = (X_1 + iY_1)\mathbb{Z} + (X_2 + iY_2)\mathbb{Z}$ (cf. equation (2.45). Since χ is odd with respect to the hyperelliptic involution on Σ, the integration constant $c \in \mathbb{C}$ must be a half-lattice point of the lattice generating $\mathrm{Jac}(T^2)$. In fact, it determines the spin class of the corresponding CMC immersion [Bob94, p. 96].

3.4 The Whitham flow for CMC tori in $\mathbb{R}^3$

Whitham deformations are deformations on the spectral curve of a CMC torus which keep the intrinsic closing conditions satisfied. Such deformations were first introduced in [KS07] in the study of constant mean curvature cylinders in S^3, which uses the results obtained by Grinevich and Schmidt [GS95]. As the intrinsic closing condition for both CMC immersion of compact tori in S^3 and $\mathbb{R}^3$ are the same, we can use the results from [KS07]. Only for the extrinsic closing conditions the formulation is different.

3.4.1 Deformation of spectral data

We consider a CMC immersion in $\mathbb{R}^3$ with spectral curve Σ of the form

$$\Sigma : \; y^2 = \lambda \prod_{i=1}^{g} (\lambda - a_i)(\lambda - \bar{a}_i^{-1}) =: \lambda a(\lambda) \tag{3.60}$$

where $a(\lambda)$ is a polynomial of degree $2g$. Recall from equation (3.38) that the abelian differentials of the second kind $\theta_j = d \ln \mu_j$ are of the form

$$\theta_j = \sum_{i=1}^{g+2} d_{j,i} \lambda^{i-1} \frac{d\lambda}{\lambda y} =: d_j(\lambda) \frac{d\lambda}{\lambda y} \tag{3.61}$$

where $d_{j,i} \in \mathbb{C}$ are complex numbers. We want to derive the differential equation system of the Whitham flow. In the following we will view the polynomials $a(\lambda)$ and $d_j(\lambda)$ as time dependent variables.

Intrinsic closing conditions

Throughout the deformation we want to preserve the intrinsic closing condition to guarantee that the conformal metric stays doubly periodic and the eigenvalues of the monodromy are globally defined at all times. Let $\{A_1, B_1, ..., A_g, B_g\}$ be a canonical homology basis on Σ. We impose that locally the basis does not depend on the deformation parameter t. Hence,

$$\partial_t \int_{A_l} \theta_j = \partial_t \int_{B_l} \theta_j = 0. \tag{3.62}$$

This implies that the integrals of $\partial_t \theta_j$ over all periods in $H_1(\Sigma, \mathbb{Z})$ vanish. Therefore, the functions

$$\partial_t \ln \mu_j(\lambda) = \partial_t \int \theta_j \tag{3.63}$$

are globally defined. We have to determine the zero and pole behavior of $\partial_t \ln \mu_j$ [HKS12, p. 30]. Let $z_i = \sqrt{\lambda - a_i}, i = 1, ..., 2g+1$, where $a_i \in \mathbb{C}$ denotes a branch point of Σ, be local coordinates around each branch point except infinity. From equation (3.61) we obtain that $\ln \mu_1$ has the following asymptotic in λ near $\lambda = a_i$

$$\ln \mu_1(\lambda) \sim f_i(\lambda) \sqrt{\lambda - a_i} \tag{3.64}$$

where $f_i(\lambda)$ are holomorphic in λ. Taking the derivative of (3.64) with respect to t we obtain

$$\partial_t \ln \mu_1(\lambda) \sim \dot{f}_i(\lambda) \sqrt{\lambda - a_i} - \frac{f_i(\lambda) \dot{a}_i}{\sqrt{\lambda - a_i}} \tag{3.65}$$

which has a first order pole at z_i and the dot denotes the derivative with respect to t. An analogous computation shows that $\partial_t \ln \mu_1$ must also have a first order pole at infinity. Clearly, the same calculation holds for $\partial_t \ln \mu_2$, too. Therefore, we can assume without loss of generality that

$$\partial_t \ln \mu_j(\lambda) = i \frac{c_j(\lambda)}{y} \tag{3.66}$$

where $c_j(\lambda)$ are polynomials of degree $g+1$ in λ satisfying the reality condition $\bar{\lambda}^{-g-1}\bar{c}_j(\lambda) = c_j(\bar{\lambda}^{-1})$. Taking the derivative with respect to λ and imposing the integrability condition $\partial^2_{t\lambda} \ln \mu_j = \partial^2_{\lambda t} \ln \mu_j$, we obtain the following deformation system [KS07, p. 14]

$$2a(\lambda)\dot{d}_j(\lambda) - \dot{a}(\lambda)d_j(\lambda) = 2i\lambda a(\lambda)c_j'(\lambda) - i\lambda a'(\lambda)c_j(\lambda) - ia(\lambda)c_j(\lambda) \tag{3.67}$$

where the dot denotes the derivative with respect to t and the dash denotes the derivative with respect to λ. Each side of the equation is a polynomial of degree $3g+1$, meaning we get $3g+2$ equations for the coefficients of $a(\lambda)$ and $d_j(\lambda)$. As this equation should hold for $j = 1, 2$, we have $6g+4$ equations all together. Notice that by the reality of the spectral data there are $6g+8$ real parameters. Hence, in order to have unique solutions of the deformation system, we have to fix the remaining parameters via the extrinsic closing conditions.

Extrinsic closing conditions

As we are mostly interested in closed CMC surfaces, we also need the extrinsic closing conditions to be satisfied. Assume that the Sym-point is located at $\lambda = 1$. The extrinsic closing conditions are equivalent to the four equations

$$\ln \mu_j(1) \in \pi i \mathbb{Z}, \quad \partial_\lambda \ln \mu_j(1) = 0. \tag{3.68}$$

But now we want to vary t and still find closed surfaces. We make the ansatz

$$\partial_t \ln \mu_j(1) = 0, \quad \partial_\lambda \ln \mu_j(1) = 0 \tag{3.69}$$

while letting the deformation parameter t vary continuously. The equations (3.69) are equivalent to the four conditions

$$c_j(1) = 0, \quad d_j(1) = 0. \tag{3.70}$$

From this, we get $6g+8$ equations for our deformation ODE which match the $6g+8$ parameters from (3.67). However, with these initial conditions the only solution is the constant one. In order to get non-trivial deformations, one could vary the coefficients in such a way that [Kew15, section 4.2]

$$d_j(1) = 0, \quad c_2(1) = 0, \quad c_1(1) = h \tag{3.71}$$

where h is a function on the spectral curve which is not allowed to vanish at any time. The geometric interpretation of the equations (3.71) is that we keep one part of the extrinsic closing conditions while opening the other one. This corresponds to deformations of CMC cylinders in $\mathbb{R}^3$.

In order to obtain closed CMC tori, we extend the intrinsic closing condition in such a way that we allow $\ln \mu_j(\lambda) \in \pi i \mathbb{Q}$ at $\lambda \in S^1$. Then the torus will close on a suitable covering. Using $c_1(1) = h \neq 0$, i.e., $\partial_t \ln \mu_1(1) \neq 0$, allows us to search for values of t where $\ln \mu_j(\lambda)/\pi i \in \mathbb{Q}$. Thus, for a flow which satisfies all other closing condition, we obtain closed surfaces on a dense subset of the time interval [Kew15, Lemma 4.7].

In chapter 4 we will examine the Whitham flow for the family of Wente tori where we will find an explicit solution in terms of elliptic integrals which satisfy (3.71).

3.4.2 Bifurcation of spectral curves

Using the Whitham flow, we deform the spectral data and obtain, possibly non-compact, CMC immersions for fixed spectral genus. It turns out that it is also possible to increase, respectively decrease, the spectral genus at certain times throughout the flow. We call this procedure bifurcation of spectral curves to higher, respectively lower, genus. Points where this is possible are called double points.

Definition 3.4.1. *We call a point $\lambda \in \Sigma$ a double point if $\mu_j(\lambda) = \pm 1$ for $j = 1, 2$ but the spectral curve is not branched.*

Double points naturally occur on any compact CMC torus in $\mathbb{R}^3$ since by definition the Sym-points are such points. However, we will generically be interested in such points which are no Sym-points of the surface. Assume there exists a double point $\lambda_{\mathrm{dp}} \in S^1$ lying on the unit circle and define the following polynomials

$$\tilde{a}(\lambda) = (\lambda - \lambda_{\mathrm{dp}})^2 a(\lambda), \quad \tilde{d}(\lambda) = (\lambda - \lambda_{\mathrm{dp}}) d(\lambda) \tag{3.72}$$

which are of degree $2g + 2$ and $g + 2$, respectively. $\tilde{a}(\lambda)$ defines a singular hyperelliptic curve $\tilde{\Sigma}$ with arithmetic genus $g + 2$. Nevertheless, since the abelian differentials θ_j are invariant under the addition of double points as in (3.72), the closing conditions remain untouched and we have the same solutions to the sinh-Gordon equation. After desingularising $\tilde{\Sigma}$ we would then obtain a new pair of branch points reflected across the unit circle [HKS12, section 9]. This would then defines a smooth hyperelliptic curve of genus $g + 1$ which is the spectral curve of a conformally immersed torus.

Vice versa, starting from a spectral curve Σ of a compact CMC and finding double points satisfying (3.72) allows us to decrease its genus. A way to do this is to use the Whitham flow to move a branch point to the unit circle. As the real structure on Σ implies that the branch points are reflected across the unit circle, they come in pairs and $a(\lambda)$ obtains a zero of order two at this point. In particular, by continuity of the flow, we would have $\mu_j(\lambda) = \pm 1$ at the double point as we started with a branch point. Therefore, we again have obtained a singular hyperelliptic curve and, after normalization, we get a spectral curve of a CMC torus with lower genus than the one we started with.

So far, the discussion above restricts to the case that the double point lies on the unit circle. There is a difference between opening double points on and inside the unit circle. If there exist double points inside the unit circle, which would imply that we also have a double point outside the unit circle by reality of the spectral data, we would have to open both of them to maintain the reality of the spectral curve. Therefore, $a(\lambda)$ would be shifted by a polynomial of order four. Finding double points in this case is more complicated since the condition $\ln \mu_i(\lambda) \in \pi i \mathbb{Q}$ for some $\lambda \in \mathbb{C}^* \setminus S^1$, which is a necessary condition for closed surfaces, is much harder to satisfy.

Chapter 4

The Wente tori

In 1986 Wente showed the existence of compact CMC tori in $\mathbb{R}^3$ [Wen86] and with this, he was able to disprove the conjecture by Hopf claiming the only immersed closed CMC surface in $\mathbb{R}^3$ is the round sphere. We know from chapter 3 that to any such torus, we can associate a spectral curve and vice versa reconstruct the immersion if the spectral data satisfy certain closing conditions. Our aim in this chapter is to present the description of the spectral data for different Wente tori.

In this context, we start this chapter with the definition of a hyperelliptic curve of genus two which has the additional symmetry $i : \lambda \mapsto \lambda^{-1}$ and we write down the two abelian differentials of the second kind $\theta_j, j = 1, 2$, with a pole of order two at zero and infinity (cf. proposition 3.3.2). The assumption that θ_j are symmetric (in the sense that $i^*\theta_j = (-1)^{j-1}\theta_j$) each differential can independently be pushed down to a different elliptic curve. As a byproduct, we will investigate solutions of the sinh-Gordon equation in terms of Riemann theta functions and the hyperelliptic reduction to Jacobi theta functions will be performed. From this, we can deduce that the genus two hyperelliptic curve we started with is indeed the spectral curve of a Wente torus. After that, closing conditions are discussed and we recapture the result that Wente tori are characterized by the rationality of an elliptic integral. In our setup, this result will be expressed purely in terms of data on the spectral curve. Finally, we will close the chapter with the discussion of double points.

4.1 Spectral data

Assume that we want to reconstruct a closed CMC immersion $f : T^2 \to \mathbb{R}^3$ from a given spectral curve Σ with appropriate data. In the case that the spectral genus is $g < 2$, our task comes to a quick end: it turns out that the intrinsic and extrinsic closing conditions cannot simultaneously be satisfied and the immersion would not be compact. In fact, the cases $g = 0$ and $g = 1$ correspond to CMC cylinders and Delaunay surfaces in $\mathbb{R}^3$, respectively [Bob91a, section 13]. For $g \geq 2$, there are enough parameters in the abelian differentials θ_j (cf. equation (3.38)) satisfying the constraints of the closing conditions. Therefore, if we want to study compact CMC tori in $\mathbb{R}^3$, we already have to deal with hyperelliptic curves. In this section, we consider the spectral genus two case which, as we will see, is the appropriate setup for studying Wente tori.

Consider the following smooth hyperelliptic curve

$$\Sigma : y^2 = \lambda(\lambda - a)(\lambda - \bar{a}^{-1})(\lambda - \bar{a})(\lambda - a^{-1}) \tag{4.1}$$

with $a \in \mathbb{C}$ inside the unit circle with non-vanishing imaginary part. We readily see that Σ has genus two and it admits the symmetries

$$\begin{aligned} &\sigma : (\lambda, y) \mapsto (\lambda, -y), &&\rho : (\lambda, y) \mapsto (\bar{\lambda}^{-1}, \bar{y}\bar{\lambda}^{-3}) \\ &i : (\lambda, y) \mapsto (\lambda^{-1}, y\lambda^{-3}), &&\nu := \rho i : (\lambda, y) \mapsto (\bar{\lambda}, \bar{y}). \end{aligned} \tag{4.2}$$

Notice that the branch points inside the unit circle are conjugated to each other. Moreover, y is real for $\lambda \in \mathbb{R}$.

We have seen in chapter 3 that the spectral curve Σ of a conformally parameterized CMC torus in $\mathbb{R}^3$ is defined as the characteristic polynomial of the monodromy of the associated family along a generator of $\pi_1(T^2, p)$. In particular, Σ does not depend on the base point $p \in T^2$ and the choice of generator $\pi_1(T^2, p)$. By proposition 3.3.2, the eigenvalues $\mu_j, j = 1, 2$, of the monodromies have essential singularities at zero and infinity and define abelian differential of the second kind $\theta_j = d \ln \mu_j$ with a pole of order two at zero and infinity. Since Σ has genus two, these can be written as

$$\begin{aligned} \theta_1 = d \ln \mu_1 = (d_{1,1} + d_{1,2}\lambda + d_{1,3}\lambda^2 + d_{1,4}\lambda^3)\frac{d\lambda}{\lambda y} \\ \theta_2 = d \ln \mu_2 = (d_{2,1} + d_{2,2}\lambda + d_{2,3}\lambda^2 + d_{2,4}\lambda^3)\frac{d\lambda}{\lambda y} \end{aligned} \tag{4.3}$$

for $d_{i,j} \in \mathbb{C}$. Assume that θ_j satisfy the reality condition $\rho^*\theta_j = -\bar{\theta}_j$. This implies that $d_{j,1} = \bar{d}_{j,4}$ and $d_{j,2} = \bar{d}_{j,3}$. We fix a canonical homology basis $\{A_1, B_1, A_2, B_2\}$ on Σ with the same properties as in (3.39)

$$\begin{aligned} \rho(A_l) &\equiv -A_l \\ \rho(B_l) &\equiv B_l - A_l + \sum_{l=1}^{2} A_l \end{aligned} \tag{4.4}$$

for $l = 1, 2$. The equality with three lines means modulo closed curves homotopic to zero. Then the involutions ν and i act on the A_l-cycles as

$$\begin{aligned} &i(A_1) \equiv -A_2, &&i(A_2) \equiv -A_1 \\ &\nu(A_1) \equiv A_2, &&\nu(A_2) \equiv A_1. \end{aligned} \tag{4.5}$$

Since ν is orientation reversing, it also reverses the intersection form and therefore acts on the B_l cycles

$$\nu(B_1) \equiv -B_2, \qquad \nu(B_2) \equiv -B_1. \tag{4.6}$$

Generically, arbitrary choices of abelian differential of the second kind as in (4.3) satisfying the reality condition $\rho^*\theta_j = -\bar{\theta}_j$ (and suitable choices of closing conditions) do not yield the symmetric Wente tori as discussed in [Abr87, Wal87]. We enforce a symmetry behavior on θ_j with respect to the involution $i : \Sigma \to \Sigma$. Henceforth, we will assume that

$$i^*\theta_1 = \theta_1, \;\; i^*\theta_2 = -\theta_2. \tag{4.7}$$

These equations force θ_1 and θ_2 to have only imaginary and real coefficients, respectively, and they take the form

$$\begin{aligned} \theta_1 = d \ln \mu_1 = (d_{1,1}(1 - \lambda^3) + d_{1,2}(\lambda - \lambda^2))\frac{d\lambda}{\lambda y} \\ \theta_2 = d \ln \mu_2 = (d_{2,1}(1 + \lambda^3) + d_{2,2}(\lambda + \lambda^2))\frac{d\lambda}{\lambda y}. \end{aligned} \tag{4.8}$$

In particular, since $d_{j,1}$ are $\mathbb{R}$-linear independent, they span a rectangular lattice. We summarize these observations in the following lemma.

Lemma 4.1.1. *Assume that $\theta_j, j = 1, 2$ are two abelian differentials of the second kind on the genus two hyperelliptic curve Σ with a pole of order two at zero and infinity. Assume that*

$$\rho^*\theta_j = -\bar{\theta}_j, \quad i^*\theta_j = (-1)^{j-1}\theta_j. \tag{4.9}$$

If the principal parts of θ_j are non-vanishing, they span a rectangular lattice in $\mathbb{R}^2$.

Notice that $\partial_\lambda \ln \mu_1(1) = 0$ already vanishes and hence one part of the extrinsic closing condition (3.53) is already satisfied.

4.2 Hyperelliptic reduction

The involution i defined in (4.2) has two fixed points over the point $\lambda = 1$. We can compose the hyperelliptic σ involution with i and get

$$\sigma i : (\lambda, y) \mapsto (\lambda^{-1}, -y\lambda^{-3}) \tag{4.10}$$

which now has two fixed points over the two points $\lambda = -1$. Taking the quotient by these two involutions i and σi defines two ramified double coverings

$$\pi_1 : \Sigma \to \Sigma_1, \quad \pi_2 : \Sigma \to \Sigma_2. \tag{4.11}$$

By the Riemann Hurwitz formula we see that Σ_j are Riemann surfaces of genus one, i.e., elliptic curves. The algebraic equations of the curves are

$$\begin{aligned} \Sigma_1 &= {}^{\Sigma}/_{i} &&: t_1^2 = (s+2)(s-E)(s-\bar{E}) \\ \Sigma_2 &= {}^{\Sigma}/_{i\sigma} &&: t_2^2 = (s-2)(s-E)(s-\bar{E}) \end{aligned} \tag{4.12}$$

where the coverings are explicitly given by the formulas [Bob91a, p. 238]

$$\begin{aligned} &E = a + a^{-1}, \ s = \lambda + \lambda^{-1} \\ &t_1 = (\lambda+1)y\lambda^{-2}, \ t_2 = (\lambda-1)y\lambda^{-2} \end{aligned} \tag{4.13}$$

which define new coordinates on the elliptic curves. The complex number a in the definition of E is the branch point of the genus two curve Σ. Equivalently, the hyperelliptic reduction defines the two maps

$$\begin{aligned} &\eta_j : \Sigma \to \Sigma_j \\ &(\lambda, y) \mapsto \left(\lambda + \lambda^{-1}, \frac{\lambda + (-1)^{i+1}}{\lambda^2} y\right). \end{aligned} \tag{4.14}$$

Notice that the coordinate s identifies the points zero and infinity in the covering $\lambda : \Sigma \to \mathbb{C}P^1$. Using (s, t_j) enables us to rewrite differentials on Σ as a linear combination of differentials on Σ_j

$$\begin{aligned} \lambda^{-1}\frac{d\lambda}{y} &= \frac{1}{2}\left[-(s+1)\frac{ds}{t_1} + (s-1)\frac{ds}{t_2}\right], \quad \frac{d\lambda}{y} = \frac{1}{2}\left[-\frac{ds}{t_1} + \frac{ds}{t_2}\right] \\ \lambda^2\frac{d\lambda}{y} &= \frac{1}{2}\left[(s+1)\frac{ds}{t_1} + (s-1)\frac{ds}{t_2}\right], \quad \lambda\frac{d\lambda}{y} = \frac{1}{2}\left[\frac{ds}{t_1} + \frac{ds}{t_2}\right]. \end{aligned} \tag{4.15}$$

Since the differentials θ_j satisfy $i^*\theta_1 = \theta_1$ and $(\sigma i)^*\theta_2 = \theta_2$, they descend to well-defined differentials on the elliptic curves Σ_1 and Σ_2, respectively. Using (4.14) and the formulas (4.15), they take the form

$$\begin{aligned}(\eta_1)_*\theta_1 &= -(d_{1,1}(s+1) + d_{1,2})\frac{ds}{t_1} =: \vartheta_1 \\ (\eta_2)_*\theta_2 &= (d_{2,1}(s-1) + d_{2,2})\frac{ds}{t_2} =: \vartheta_2.\end{aligned} \tag{4.16}$$

Whereas θ_j have a pole of order two at $\lambda = 0$ and $\lambda = \infty$ the newly defined differentials ϑ_j on Σ_j only have poles of order two at $s = \infty$. This is a consequence of $s = \lambda + \lambda^{-1}$ identifying the points over zero and infinity.

4.2.1 Canonical homology basis on Σ_j

The canonical homology basis on the genus two curve Σ defines a canonical homology basis on the elliptic curves. As before, we denote the basis on Σ with the property (4.4) by $\{A_1, B_1, A_2, B_2\}$ and for the elliptic curves Σ_j we denote the basis by $\{a^{(j)}, b^{(j)}\}$. Following the approach of [EKT93], under $s = \lambda+\lambda^{-1}$ the unit circle $\lambda \in S^1$ is mapped to the closed interval $[-2, 2]$. By our choice of homology basis on Σ, the A_l-cycles are defined by closed curves encircling a pair of branch points reflected across the unit circle. Therefore, closed curves encircling the two points E and $\bar{E}$ and going through $[-2, 2]$ determine the homology classes $a^{(j)}$. Assume that ϕ is a differential on Σ_1 which pulls back to a differential Σ via η_1. Then

$$\int_{\eta_1(A_1)} \phi = \int_{A_1} \eta_1^*\phi = \int_{i(A_1)} \eta_1^*\phi = -\int_{A_2} \eta_1^*\phi = \int_{-\eta_1(A_2)} \phi \tag{4.17}$$

where $i^*\phi = \phi$ and (4.5) were used. An analogous consideration for a differential on the other elliptic curve Σ_2 yields

$$\eta_j(A_1) = (-1)^j \eta_j(A_2) = a^{(j)}. \tag{4.18}$$

Similar to the $a^{(j)}$ cycles, we can define the $b^{(j)}$ cycles by pushing them down to Σ_j. As we are particularly interested in the integration of normalized differentials of the second kind we have the following lemma.

Lemma 4.2.1. *Let ω_j be a normalized differential on Σ_j, i.e., vanishing $a^{(j)}$-periods, which pulls back to differentials on Σ via η_j. Then we have*

$$\int_{\eta_j(B_1)} \omega_j = \int_{(-1)^j\eta_j(B_2)} \omega_j = \int_{b^{(j)}} \omega_j. \tag{4.19}$$

Proof. The proof uses similar arguments as in the discussion with the $a^{(j)}$-periods in equation (4.17). Consider a normalized differential ω_1 on Σ_1 which pulls back to Σ. From equation (4.6) we obtain

$$\int_{\eta_1(B_1)} \omega_1 = \int_{i(B_1)} \eta_1^*\omega_1 = \int_{\nu(B_1)} (\rho\eta_1)^*\omega_1 = -\int_{B_2} (\rho\eta_1)^*\omega_1 = \int_{-\eta_1(B_2)} \omega_1 \tag{4.20}$$

where the property that ω_1 is normalized was used in the last step. An analogous computation with a differential ω_2 gives an additional minus sign which already yields the assertion. $\square$

Visualizations of the canonical homology basis of Σ_j are depicted figures 4.1 and 4.2.

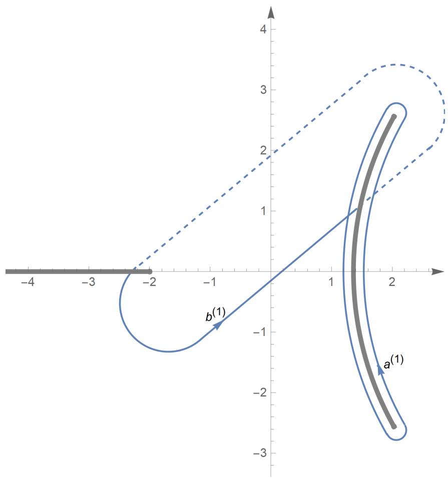

Figure 4.1: Canonical homology basis for the elliptic curve Σ_1. The thick gray lines denote branch cuts. Dashed lines denote paths on the other sheet of the Riemann surface.

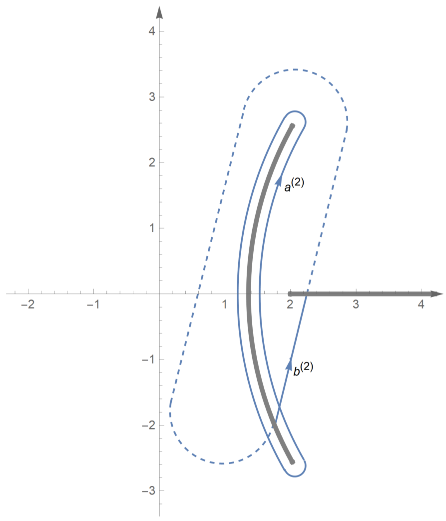

Figure 4.2: Canonical homology basis for the elliptic curve Σ_2. The thick gray lines denote branch cuts. Dashed lines denote paths on the other sheet of the Riemann surface.

4.2.2 Reduction of Riemann theta functions

In [Bob91a], Bobenko showed that a real solution of the sinh-Gordon equation (3.12) can be given explicitly in terms of Riemann theta functions. In the case of the Wente tori, we will see that solutions can be given in terms of Jacobi theta function via hyperelliptic reduction of Riemann theta functions. This allows us to read off the intrinsic closing conditions from the conformal factor in the conformal metric and the lattice of the torus.

Define the two abelian differentials $d\Omega_0$ and $d\Omega_\infty$ by

$$d\Omega_0 = -\frac{1}{4}\left(\frac{\theta_1}{d_{1,1}} + \frac{\theta_2}{d_{2,1}}\right), \quad d\Omega_\infty = \frac{1}{4}\left(\frac{\theta_2}{d_{2,1}} - \frac{\theta_1}{d_{1,1}}\right) \tag{4.21}$$

which we have already encountered in equation (3.43). Let $\sqrt{\lambda} = z_0$ and $\sqrt{\lambda} = 1/z_\infty$ be local coordinates around zero and infinity, respectively. Then (4.21) have second order poles at zero and infinity, respectively, with the following asymptotic

$$\begin{aligned} d\Omega_0 &\sim -\frac{dz_0}{z_0^2}, \quad \lambda \to 0 \\ d\Omega_\infty &\sim -\frac{dz_\infty}{z_\infty^2}, \quad \lambda \to \infty. \end{aligned} \tag{4.22}$$

As $\theta_1/(4a_1)$ has only real coefficients, $\rho^*\theta_1/(4a_1) = \overline{\theta_1/(4a_1)}$ and hence

$$\rho^*\Omega_0 = \overline{d\Omega_\infty}. \tag{4.23}$$

Assume that $d\Omega_\infty$ and $d\Omega_0$ are normalized. For a conformally parameterized CMC immersion $f : T^2 \to \mathbb{R}^3$, the conformal factor $u : \mathbb{R}^2 \to \mathbb{R}$ is of the form

$$u(z, \overline{z}) = 2 \log \frac{\theta(\Psi, \tau)}{\theta(\Psi + \triangle, \tau)} \tag{4.24}$$

where $\Psi = -1/(4\pi)(Uz + V\overline{z}), \triangle = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$ and

$$U_l = \int_{B_l} d\Omega_\infty, \quad V_l = \int_{B_l} d\Omega_0 \tag{4.25}$$

with $l = 1, 2$ [Bob91a, Theorem 4.1]. $\theta(\Psi, \tau)$ denotes the Riemann theta function with respect to the period matrix τ[1] (cf. definition 2.3.4). By the reality condition (4.23) and the normalization, we have $U = \overline{V}$. Using $(\rho i)^* d\Omega_\infty = \overline{d\Omega_\infty}$, we further obtain from (4.6)

$$\int_{B_1} d\Omega_\infty = -\overline{\int_{B_2} d\Omega_\infty} = \alpha + i\beta \tag{4.26}$$

and therefore

$$\Psi = -\frac{1}{4\pi}(Uz + V\overline{z}) = \frac{1}{2\pi}\begin{pmatrix} \beta y - \alpha x \\ \alpha x + \beta y \end{pmatrix} \tag{4.27}$$

where $z = x + iy$.

To reduce the Riemann theta functions, we need to calculate the period matrix τ. For this, consider the space of holomorphic one-forms on Σ which is spanned by $\{d\lambda/y, \lambda d\lambda/y\}$. Define the following integrals

$$\begin{aligned} N &= \int_{A_1} \frac{d\lambda}{y}, \quad M = \int_{A_1} \lambda \frac{d\lambda}{y} \\ N' &= \int_{B_1} \frac{d\lambda}{y}, \quad M' = \int_{B_1} \lambda \frac{d\lambda}{y}. \end{aligned} \tag{4.28}$$

We further set

$$\begin{aligned} du_1 &= \frac{1}{N^2 - M^2}\left(N\frac{d\lambda}{y} - M\frac{\lambda d\lambda}{y}\right) \\ du_2 &= \frac{1}{N^2 - M^2}\left(N\frac{\lambda d\lambda}{y} - M\frac{d\lambda}{y}\right) \end{aligned} \tag{4.29}$$

which are holomorphic one-forms on Σ. Using the identity $i(A_1) \equiv -A_2$, we see that these differentials are normalized in the sense that $\int_{A_l} du_j = \delta_{lj}$. The integral of the

[1] Notice the different normalization of the Riemann theta function here and in [Bob91a]

normalized holomorphic differentials along B_l determine the period matrix τ, which in return defines the Riemann theta functions. We can read off from equation (4.29) that the upper left and upper right entry of the period matrix has the form

$$\tau_{11} = \frac{1}{N^2 - M^2}(NN' - MM'), \quad \tau_{12} = \frac{1}{N^2 - M^2}(NM' - MN'), \tag{4.30}$$

respectively. A reciprocity law for normalized abelian differentials of the first kind similar to (2.21) shows that the period matrix is symmetric, i.e., $\tau_{21} = \tau_{12}$. For the lower right entry τ_{22}, we use equation (4.6) to obtain

$$\tau_{22} = \int_{B_2} du_2 = \frac{1}{N^2 - M^2}(M\overline{N'} - N\overline{M'}). \tag{4.31}$$

On the other hand, by equation (4.4) we have

$$\overline{M'} = -N' - M, \quad \overline{N'} = -M' - N \tag{4.32}$$

which implies that all together the period matrix takes the form

$$\tau = \frac{1}{N^2 - M^2}\begin{pmatrix} NN' - MM' & NM' - MN' \\ NM' - MN' & NN' - MM' \end{pmatrix}. \tag{4.33}$$

From (4.32) we can quickly deduce that the conformal factor is real. Indeed, we have

$$\bar{\tau} = -\tau - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.34}$$

and therefore

$$< N, \bar{\tau}N >= - < N, \tau N > -2\pi i n_1 n_2 \tag{4.35}$$

where $N = (n_1 \; n_2)^t$. This already shows that $\overline{\theta(z,\tau)} = \theta(\bar{z},\tau)$. Moreover, from (4.27) we know that Ψ is real which already implies that $u(z,\bar{z})$ is real. In order to reduce the Riemann theta function to Jacobi elliptic functions, we express the period matrix in terms of data on the elliptic curves Σ_1 and Σ_2. For this, consider the following differentials

$$\begin{aligned} dv_1 = du_1 - du_2 &= \frac{1}{N - M}\left(\frac{d\lambda}{y} - \frac{\lambda d\lambda}{y}\right) = \frac{1}{N - M}\frac{ds}{t_1} \\ dv_2 = du_1 + du_2 &= \frac{1}{N + M}\left(\frac{d\lambda}{y} + \frac{\lambda d\lambda}{y}\right) = \frac{1}{N + M}\frac{ds}{t_2} \end{aligned} \tag{4.36}$$

where we used (4.15). From equation (4.18) we obtain that dv_j are normalized holomorphic differentials on Σ_j satisfying

$$\int_{\eta_1(A_1)} dv_1 = \int_{\eta_2(A_1)} dv_2 = 1. \tag{4.37}$$

These differentials further define period matrices on Σ_j, i.e., complex numbers, via

$$\tau_1 = \int_{\eta_1(B_1)} dv_1 = \frac{N' - M'}{N - M}, \quad \tau_2 = \int_{\eta_2(B_1)} dv_2 = \frac{N' + M'}{N + M}. \tag{4.38}$$

By reality, we have

$$\bar{\tau}_1 = 1 - \tau_1, \quad \bar{\tau}_2 = -1 - \tau_2 \tag{4.39}$$

which highlights the rhombic structure of the elliptic curves Σ_j where τ_1 and τ_2 have real part $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Notice that defining τ_j by the integral along $\eta_j(B_2)$ instead of $\eta_j(B_1)$ does not change τ_2 but multiplies τ_1 with an additional minus sign similar to (4.19). Using equation (4.33) we bring the period matrix τ into the convenient form

$$\tau = \frac{1}{2}\begin{pmatrix} \tau_2 + \tau_1 & \tau_2 - \tau_1 \\ \tau_2 - \tau_1 & \tau_2 + \tau_1 . \end{pmatrix}. \tag{4.40}$$

We can use this result to reduce Riemann theta functions with help of the following proposition [BBMÈ86, p. 11].

Proposition 4.2.2. *Let the period matrix τ of the hyperelliptic curve Σ of genus two be as in (4.40). Let $[\epsilon\,;\epsilon']$ be a theta characteristic on Σ. Then we have the following transformation property of the Riemann theta functions*

$$\begin{aligned}\theta\begin{bmatrix}\epsilon \\ \epsilon'\end{bmatrix}(z,\tau) =&\theta\begin{bmatrix}\frac{1}{2}(\epsilon_1+\epsilon_2) \\ \epsilon_1'+\epsilon_2'\end{bmatrix}(z_1+z_2, 2\tau_2)\theta\begin{bmatrix}\frac{1}{2}(\epsilon_1-\epsilon_2) \\ \epsilon_1'-\epsilon_2'\end{bmatrix}(z_1-z_2, 2\tau_1)+ \\ &\theta\begin{bmatrix}\frac{1}{2}(\epsilon_1+\epsilon_2)+1 \\ \epsilon_1'+\epsilon_2'\end{bmatrix}(z_1+z_2, 2\tau_2)\theta\begin{bmatrix}\frac{1}{2}(\epsilon_1-\epsilon_2)+1 \\ \epsilon_1'-\epsilon_2'\end{bmatrix}(z_1-z_2, 2\tau_1).\end{aligned} \tag{4.41}$$

Using (4.27) and proposition 4.2.2, we rewrite Riemann theta function with zero characteristic, i.e., $\epsilon = \epsilon' = 0$. Further using (2.59) for the identification of Riemann and Jacobi theta functions, we obtain

$$\begin{aligned}\theta(\Psi,\tau) &= \vartheta_3(\beta y, 2\tau_2)\vartheta_3(\alpha x, 2\tau_1) + \vartheta_2(\beta y, 2\tau_2)\vartheta_2(\alpha x, 2\tau_1) \\ \theta(\Psi+\triangle,\tau) &= \vartheta_3(\beta y, 2\tau_2)\vartheta_3(\alpha x, 2\tau_1) - \vartheta_2(\beta y, 2\tau_2)\vartheta_2(\alpha x, 2\tau_1).\end{aligned} \tag{4.42}$$

In the second equation of (4.42) we also used the first equation of proposition 2.3.1 which gives the additional minus sign before ϑ_2. Bringing these results together, we see that the conformal factor takes the form

$$\tanh(u/2) = \frac{\vartheta_2(\alpha x, 2\tau_1)}{\vartheta_3(\alpha x, 2\tau_1)}\frac{\vartheta_2(\beta y, 2\tau_2)}{\vartheta_3(\beta y, 2\tau_2)}. \tag{4.43}$$

Solutions to the sinh-Gordon equation of the form of (4.43) were considered in [Wen86, Wal87, Abr87] for the description of Wente tori $f : T^2 \to \mathbb{R}^3$. Denote the surfaces principle curvatures by k_1 and k_2. Since CMC tori in $\mathbb{R}^3$ have no umbilic points, we either have $k_1 > k_2$ or $k_1 < k_2$. Let the k_1 and k_2-curvature lines be denoted by

$$x \mapsto f(x,y), \quad y \mapsto f(x,y), \tag{4.44}$$

respectively. We summarize the following properties which one can deduce by studying solutions of the sinh-Gordon equation of the form (4.43) [Abr87].

i. With respect to a Frenet frame the torsion of the k_2-curvature line vanishes, i.e., it lies in a plane (cf. figure 4.3).

ii. The k_1-curvature lines lie on spheres (cf. figure 4.3).

iii. If $k_1 < k_2$ then the immersion cannot close up to a compact CMC torus.

In our setup, the geometric interpretation is that the closing conditions 3.3.3 and 3.3.4 for the Wente tori split into closing conditions of the curvature lines described by elliptic integrals on the respective elliptic curves Σ_1 and Σ_2. We will give further insight on

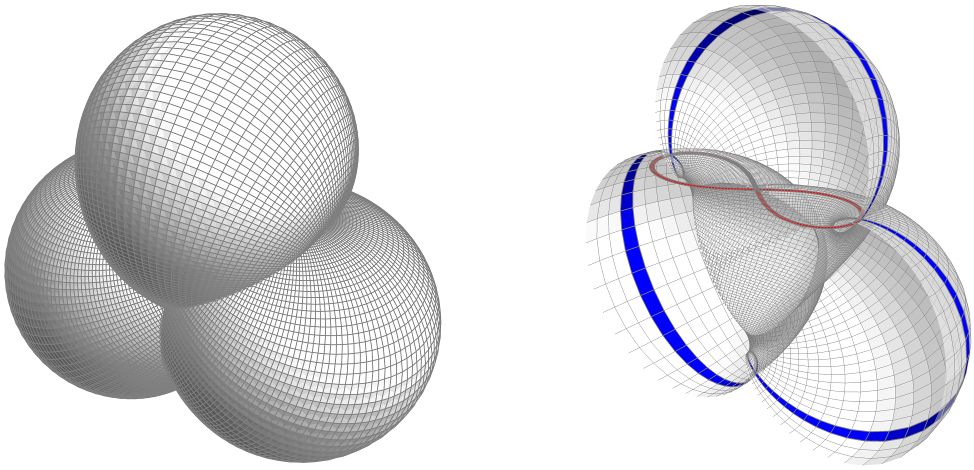

Figure 4.3: The 3-lobed Wente torus and a half-cutaway of it, visualized by using Mathematica. The blue curvature line lies on spheres while the red curvature line lies in a plane. The illustration is based on Mathematica files provided by Wjatscheslaw Kewlin (private communication).

the geometric interpretation of the curvature lines in section 4.3.3 when discussing the lattices generating the Wente tori.

We finish the section by bringing the conformal factor (4.43) in a more convenient form. The reason for this is that, in the following, we want to work with elliptic integrals instead of Jacobi theta functions. Recall the definition of the elliptic curves Σ_j in (4.12). Let $\xi = s - 2$ and $E - 2 = re^{i\delta}$. Then the algebraic equations for the elliptic curves are

$$\begin{aligned} \Sigma_1 : t_1^2 &= (\xi + 4)(\xi - re^{i\delta})(\xi - re^{-i\delta}) \\ \Sigma_2 : t_2^2 &= \xi(\xi - re^{i\delta})(\xi - re^{-i\delta}). \end{aligned} \tag{4.45}$$

We want to rewrite the conformal types $\tau_j, j = 1, 2$, of the respective elliptic curves as defined in (4.38) in terms of complete elliptic integrals, which is equivalent to expressing the $a^{(j)}$ and $b^{(j)}$ periods of $d\xi/t_j$ in terms of complete elliptic integrals. Notice that by reality of the spectral data, the $a^{(j)}$-periods of the one-forms $d\xi/t_j$ are purely imaginary and real, respectively.

Consider a simply closed curve γ on $\mathbb{C}P^1$ around the two points -4 and $-\infty$. Recall that the homology basis of Σ_1 is denoted by $\{a^{(1)}, b^{(1)}\}$. Since the lift of γ to Σ_1, which is denoted by the same symbol, is a non-trivial loop, we have

$$\gamma = ma^{(1)} + nb^{(1)} \tag{4.46}$$

with $m, n \in \{-1, 0, 1\}$. As γ does not intersect $a^{(1)}$ but $b^{(1)}$ exactly once, we can conclude that $n = 0$ and $m = \pm 1$. After choosing a sign, we use [BF13, eq. 243.00] to obtain

$$\int_{a^{(1)}} \frac{d\xi}{t_1} = 2\int_{-\infty}^{-4} \frac{d\xi}{t_1} = -4i\frac{K(m_1)}{s^{1/4}} \tag{4.47}$$

where $K(m_1)$ is the complete elliptic integral of the first kind and m_1 and s have the form

$$m_1 = \frac{-4 - r\cos\delta + \sqrt{s}}{2\sqrt{s}}, \quad s = 16 + r^2 + 8r\cos\delta. \tag{4.48}$$

Similar arguments also work for $j = 2$ and in this case we use [BF13, eq. 241.00] to get

$$\int_{a^{(2)}} \frac{d\xi}{t_2} = 2\int_0^\infty \frac{d\xi}{t_1} = 4\frac{K(m_2)}{\sqrt{r}} \tag{4.49}$$

where

$$m_2 = \cos^2\left(\frac{\delta}{2}\right). \tag{4.50}$$

With the help of (4.39), we further get

$$\begin{aligned} \int_{b^{(1)}} \frac{d\xi}{t_1} &= \frac{2}{s^{1/4}}(-K'(m_1) + iK(m_1)) \\ \int_{b^{(2)}} \frac{d\xi}{t_2} &= \frac{2}{\sqrt{r}}(K'(m_2) + iK(m_2)), \end{aligned} \tag{4.51}$$

i.e.,

$$\tau_1 = -\frac{1}{2} + i\frac{K'(m_1)}{2K(m_1)}, \quad \tau_2 = \frac{1}{2} + i\frac{K'(m_2)}{2K(m_2)} \tag{4.52}$$

where $K'(m_j) = K(1-m_j)$. We use these forms of τ_j to simplify Jacobi theta functions. From the definition of the Jacobi theta functions, we have

$$\vartheta_2(z, 2\tau_j) = \vartheta_2\left(z, i\frac{K'(m_j)}{K(m_j)}\right), \quad \vartheta_3(z, 2\tau_j) = \vartheta_4\left(z, i\frac{K'(m_j)}{K(m_j)}\right). \tag{4.53}$$

After replacing the Jacobi theta constant by [AS72, eq. 16.36.6]

$$\pi\vartheta_3^2\left(0, i\frac{K'(m_j)}{K(m_j)}\right) = 2K(m_j) \tag{4.54}$$

and using the equations of (2.60), we see that the conformal factor (4.43) can be rewritten as

$$u(x,y) = 4\operatorname{arctanh}\left[\left(\frac{m_1 m_2}{m_1' m_2'}\right)^{1/4} \operatorname{cn}\left(\frac{2\alpha x}{\pi}K(m_1), m_1\right)\operatorname{cn}\left(\frac{2\beta y}{\pi}K(m_2), m_2\right)\right]. \tag{4.55}$$

Here $\operatorname{cn}(z,m)$ is the Jacobi elliptic function which is related to the Jacobi theta functions by equation (2.60). Notice that the conformal factor already is doubly periodic with respect to some lattice in $\mathbb{R}^2$ since $\operatorname{cn}(z,m)$ is a doubly periodic elliptic function. Hence, the intrinsic closing condition 3.3.3, which guarantees the doubly periodicity of the conformal metric, is already satisfied.

4.3 Closing conditions

In order to obtain a compact CMC immersion $f : T^2 \to \mathbb{R}^3$, closing conditions have to be satisfied. As we have seen in chapter 3, these are expressible in terms of data of the spectral curve. The conditions we need are the intrinsic 3.3.3 and extrinsic 3.3.4

closing conditions. For the Wente tori, the closing conditions are conditions on the spectral data of a hyperelliptic curve of genus two. In particular, this gives equations involving hyperelliptic integrals, which are generally hard to deal with.

In the present case, we assumed that the logarithmic derivative of the eigenvalues of the monodromy $\theta_j = d \ln \mu_j, j = 1, 2$, satisfy $i^*\theta_j = (-1)^{j-1}\theta_j$ where $i : \Sigma \to \Sigma$ is the involution (4.2) covering the map $\lambda \to \lambda^{-1}$ on $\mathbb{C}P^1$. This additional symmetry allows us to view each θ_j as a differential on the following respective elliptic curves

$$\begin{aligned} \Sigma_1 \ &: \ t_1^2 = (\xi + 4)(\xi - re^{i\delta})(\xi - re^{-i\delta}) \\ \Sigma_2 \ &: \ t_2^2 = \xi(\xi - re^{i\delta})(\xi - re^{-i\delta}). \end{aligned} \tag{4.56}$$

It would be desirable to also express the closing conditions in terms of elliptic integrals. As we will see in the following, this is indeed possible and we will formulate the closing conditions on each elliptic curve independently. In this way, we gather more information on the spectral data, which will be useful in the subsequent sections.

Throughout the section, we let $\{A_1, B_1, A_2, B_2\}$ denote the canonical homology basis on Σ satisfying equations (4.4) and inducing a canonical homology basis $\{a^{(j)}, b^{(j)}\}$ on Σ_j, as described in subsection 4.2.1.

4.3.1 On the elliptic curve Σ_2

Intrinsic closing conditions

The intrinsic closing conditions 3.3.3 ensure the doubly periodicity of the conformal metric. We have seen in equation (3.40) that the intrinsic closing conditions imply that the integral of θ_2 along the A_l-cycles vanishes

$$\int_{A_1} \theta_2 = \int_{\eta_2(A_1)} (\eta_2)_* \theta_2 = \int_{a^{(2)}} \vartheta_2 = 0. \tag{4.57}$$

Here we used equation (4.18) and ϑ_2 is given by (4.16). After the substitution $\xi = s - 2$ in (4.16) and further setting

$$-\kappa = \frac{d_{2,1} + d_{2,2}}{d_{2,1}} \tag{4.58}$$

equation (4.57) can be reformulated as

$$\int_{re^{i\delta}}^{re^{-i\delta}} \frac{(\xi - \kappa) d\xi}{\sqrt{\xi(\xi - re^{i\delta})(\xi - re^{-i\delta})}} = 0 \tag{4.59}$$

which defines $\kappa = \kappa(r, \delta)$ as a function of r and δ. After the substitution $u = \xi / r$, the first term of equation (4.59) takes the form

$$\sqrt{r} \int_{e^{i\delta}}^{e^{-i\delta}} \frac{u du}{\sqrt{u(u - e^{i\delta})(u - e^{-i\delta})}}. \tag{4.60}$$

Proceeding as in [Bob91a, p. 239], we substitute $u = e^{i\gamma}$ and choose the integration contour from $e^{i\delta}$ to -1 and then from -1 to $e^{-i\delta}$. Hence, equation (4.60) takes the form

$$\sqrt{\frac{r}{2}} \int_{\delta}^{\pi} \frac{\cos\gamma d\gamma}{\sqrt{\cos\delta - \cos\gamma}} + \sqrt{\frac{r}{2}} \int_{\pi}^{2\pi - \delta} \frac{\cos\gamma d\gamma}{\sqrt{\cos\delta - \cos\gamma}}. \tag{4.61}$$

Replacing γ by $2\pi-\gamma$ in the second term of (4.61), we bring them together and obtain

$$\sqrt{8r}\int_{\delta}^{\pi}\frac{\cos\gamma d\gamma}{\sqrt{\cos\delta-\cos\gamma}}. \tag{4.62}$$

There exists a unique solution $\delta_0 \in (0,\frac{\pi}{2})$ such that this integral vanishes [Bob91a, p. 239]. Therefore,

$$\kappa(r,\delta_0)=0. \tag{4.63}$$

Applying similar transformations as above for the second term in equation (4.59) it is also easy to see that $\kappa(r,\delta)=r\kappa(1,\delta)$. Summarized, we have the following lemma.

Lemma 4.3.1. *Let θ_2 on Σ be given by equation (4.8) and $\kappa=-(d_{2,1}+d_{2,2})/d_{2,1}$. Then the intrinsic closing conditions (4.57) imply that $\kappa(r,\delta_0)=r\kappa(1,\delta_0)=0$ for a $\delta_0\in(0,\frac{\pi}{2})$ and all $r\in\mathbb{R}$.*

We can also formulate the intrinsic closing condition in terms of complete elliptic integrals. Using [BF13, eq. 289.03], we see that the vanishing of equation (4.62) is equivalent to

$$2E(m_2)-K(m_2)=0 \tag{4.64}$$

where $m_2=\cos^2(\frac{\delta}{2})$.

Extrinsic closing conditions

The extrinsic closing conditions 3.3.4 ensure the closedness of the immersion. Assume that the Sym point, i.e., the point were the immersion is reconstructed via the Sym-Bobenko formula, is located at $\lambda=1$. For $\theta_2=d\ln\mu_2$, these conditions consist of the two equations

$$\partial_\lambda \ln\mu_2(1)=0,\quad \ln\mu_2(1)\in\pi i\mathbb{Z}. \tag{4.65}$$

From (4.8), one immediately sees that the first equation implies $d_{2,1}+d_{2,2}=0$, i.e., $\kappa=0$ (4.58), for all $(r,\delta)\in\mathbb{R}\times(0,\frac{\pi}{2})$. In view of lemma 4.3.1, we see that the extrinsic closing conditions force us to take $\delta=\delta_0$ if the intrinsic closing conditions are satisfied as well.

For the second part in (4.65), consider the following differential on Σ

$$\phi=\frac{y(1)\lambda}{\lambda-1}\frac{d\lambda}{y} \tag{4.66}$$

which has a pole of order one at the points lying over $\lambda=1$ with residue $+1$ and -1, i.e., it is an abelian differential of the third kind. Furthermore, it has a zero of order two at $\lambda=0$ and $\lambda=\infty$

Let $f(\lambda)=\int_{\lambda_0}^{\lambda}\theta_2$ with respect to some base point $\lambda_0\in\Sigma$ be an abelian integral. As in section 2.2, $f(\lambda)$ is well-defined on the simply connected model of Σ (cf. definition 2.2.4). Assume that θ_2 is normalized. By the reciprocity law for abelian differentials of the third and second kind (2.25), we have

$$\sum_{l=1}^{2}\left[\int_{A_l}\theta_2\int_{B_l}\phi-\int_{B_l}\theta_2\int_{A_l}\phi\right]=2\pi i\sum_p \operatorname{res}_p(f\phi). \tag{4.67}$$

Since θ_2 has vanishing A_l-periods, we can apply lemma 4.2.1

$$\int_{B_1}\theta_2=\int_{B_2}\theta_2 \tag{4.68}$$

and the left hand side of equation (4.67) is

$$-\int_{B_1} \theta_2 \Big(\int_{A_1} \phi + \int_{A_2} \phi \Big). \tag{4.69}$$

But we also have $i^* \phi = \phi$ and hence

$$\int_{A_1} \phi = -\int_{A_2} \phi. \tag{4.70}$$

which implies that (4.69) is zero. The only residues of $f\phi$ are the two points over $\lambda = 1$ and as $\sigma^* \ln \mu_j = -\ln \mu_j$, we get

$$\sum_p \operatorname{res}_p(f\phi) = \ln \mu_2(1) - \sigma^* \ln \mu_2(1) = 2 \ln \mu_1(1). \tag{4.71}$$

We have obtained the following lemma.

Lemma 4.3.2. *Let $\theta_2 = d \ln \mu_2$ on Σ be given by equation (4.8). If θ_2 has vanishing A_l-periods, then*

$$\ln \mu_2(1) = 0. \tag{4.72}$$

As Σ_2 contains the information about the k_2-curvature lines (cf. subsection 4.2.2), the extrinsic closing conditions are equivalent to the closedness of these curves. For the Wente tori, the planar closed curve is a figure eight lying perpendicular to the symmetry plane which cuts the k-spheres, $k \geq 3$, of the Wente tori in half [Abr87].

4.3.2 On the elliptic curve Σ_1

Now we want to reformulate the closing conditions for the differential $\theta_1 = d \ln \mu_1$ on Σ in terms of elliptic data on the curve Σ_1. Our aim is to identify more conditions the spectral data has to satisfy in order to get closed surfaces.

From the discussion about the elliptic curve Σ_2 in the above subsection 4.3.1 we saw that the closing conditions fix the angle δ of the branch point $re^{i\delta}$ of Σ_j. Therefore, we are left with a real one dimensional parameter $r \in \mathbb{R}^{\geq 0}$. We expect that the closing conditions on the differential $\theta_1 = d \ln \mu_1$ give rise to equations which determine r. Indeed, after reducing differentials on Σ to elliptic ones on Σ_1, we will see that the parameter r is determined by the rationality of an elliptic integral.

Intrinsic and extrinsic closing conditions

The intrinsic closing conditions, which determine the lattice where the immersion is well-defined, are given by the equations

$$\int_{A_l} \theta_1 = 0, \quad \int_{B_l} \theta_1 \in 2\pi i \mathbb{Z} \tag{4.73}$$

where $l = 1, 2$ and $n \in \mathbb{Z}$. Again, assume that the Sym-point is $\lambda = 1$. For the differential $\theta_1 = d \ln \mu_1$ the extrinsic closing conditions are given by the two equations

$$\partial_\lambda \ln \mu_1(1) = 0, \ \ \ln \mu_1(1) \in \pi i \mathbb{Z}. \tag{4.74}$$

The condition $\partial_\lambda \ln \mu_1(1) = 0$ is already trivially satisfied as we see from equation (4.8). However, opposed to equation (4.72), the second part of (4.74) give non-trivial closing conditions. Combined with the intrinsic closing conditions, it determines the parameter r in the elliptic curves Σ_j. For this, we take a closer look at the differential

$$\phi = \frac{y(1)\lambda}{\lambda - 1}\frac{d\lambda}{y} \tag{4.75}$$

that we have already used in (4.67). As $\rho^*\phi = \bar{\phi}$,

$$\int_{A_l} \phi = -\int_{A_l} \bar{\phi} = -\overline{\int_{A_l} \phi} \tag{4.76}$$

and therefore the A_l-periods of ϕ are imaginary. Moreover, since $i^*\phi = \phi$, this differential descends to a well-defined differential on Σ_1 where we have $t_1(\xi = 0) = 2r$ and therefore

$$2(\eta_1)_*\phi = \frac{2r}{\xi}\frac{d\xi}{t_1}. \tag{4.77}$$

$2(\eta_1)_*\phi$ has simple poles at the two points over $\xi = 0$ with residues ± 1, respectively. Assume that θ_1 is normalized. As in equation (4.67), we use the reciprocity law for the differentials θ_1 and ϕ. Again, denote by $f(\lambda) = \int_{\lambda_0}^{\lambda} \theta_1$ the single valued functions on the simply connected model of Σ. After applying lemma 4.2.1 for the B_l-periods of θ_1, we have

$$\int_{B_1} \theta_1 = -\int_{B_2} \theta_1. \tag{4.78}$$

Let $\{a^{(1)}, b^{(1)}\}$ be the canonical homology basis on Σ_1 as described in subsection 4.2.1. With the help of equation (4.70), we obtain

$$\sum_{k=1}^{2}\left[\int_{A_l}\theta_1\int_{B_l}\phi - \int_{B_l}\theta_1\int_{A_l}\phi\right] = -2\int_{B_1}\theta_1\int_{A_1}\phi = -\int_{b^{(1)}}\vartheta_1\int_{a^{(1)}}\frac{2r}{\xi}\frac{d\xi}{t_1} \tag{4.79}$$

where lemma 4.2.1 and equation (4.77) were used in the last step. On the other hand, on the right hand side of equation (4.79), we have

$$2\pi i\sum_p \mathrm{res}_p(f\phi) = 2\pi i(\ln\mu_1(1) - \sigma^*\ln\mu_1(1)) = 4\pi i\ln\mu_1(1). \tag{4.80}$$

In order to derive (4.79) and (4.80), we only used the assumption that θ_1 has vanishing A_l-periods. Now assume that the full set of equations of the intrinsic and extrinsic closing conditions are satisfied, i.e.,

$$\ln\mu_1(1) = \pi i m, \quad \int_{b^{(1)}}\vartheta_1 = 2\pi i n \tag{4.81}$$

for some $m, n \in \mathbb{Z}$. Bringing (4.79) and (4.80) under the conditions of equations (4.81) together, we obtain

$$\frac{1}{2\pi i}\int_{a^{(1)}}\frac{2r d\xi}{\xi\sqrt{(\xi+4)(\xi - re^{i\delta})(\xi - re^{-i\delta})}} = -\frac{m}{n}. \tag{4.82}$$

As $\rho^*\phi = \overline{\phi}$ and $\rho(A_l) = -A_l$, the integral is purely imaginary by equation (4.76) and the right hand side of (4.82) is indeed real. Hence, we have found an equation which relates the radial coordinate r of the branch point $re^{i\delta}$ in the elliptic curves Σ_j to the closing conditions on Σ_1.

While the angle $\delta = \delta_0 \in (0, \frac{\pi}{2})$ is fixed by the closing conditions on Σ_2 (cf. lemma 4.3.1), the radius r is determined by satisfying equation (4.82). Let us denote the integrand of (4.82) by $\Phi(r, \delta)$. We have obtained the following result.

Proposition 4.3.3. *Let Σ be the hyperelliptic curve of genus two as defined in (4.1) and let θ_1 be a normalized abelian differential of the second kind on Σ. Assume that $i^*\theta_1 = \theta_1, \int_{B_1} \theta_1 = -\int_{B_2} \theta_1 = 2\pi in$ and $\ln \mu_1(1) = \pi im$, where $i : \Sigma \to \Sigma$ is the involution defined in (4.2) and $n, m \in \mathbb{Z}$. A neccessary condition for the existence of closed Wente tori is that $\frac{1}{2\pi i}\int_{a^{(1)}} \Phi(r, \delta)$ is the rational number $-\frac{m}{n}$.*

We will now study the values of $\Phi(r, \delta)$ in the limit cases $r \to 0$ and $r \to \infty$. This will give constraints which possible values the ratio $\frac{m}{n}$ attains. While varying r, we will fix $\delta = \delta_0 \in (0, \frac{\pi}{2})$.

Proposition 4.3.4. *Let $\Phi(r, \delta)$ and $m, n \in \mathbb{Z}$ be as in proposition 4.3.3.*

i. In the limit $r \to 0$, the integral $\int_{a^{(1)}} \Phi(r, \delta_0)$ converges to $2\pi i$.

ii. In the limit $r \to \infty$, the integral $\int_{a^{(1)}} \Phi(r, \delta_0)$ converges to $4\pi i$.

In particular, a neccessary condition for closed Wente tori is $1 < \frac{m}{n} < 2$. Furthermore, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ must have opposite signs.

Proof. The proof consists of calculating contour integrals for the different limiting behaviors. Notice that, by construction, the integral does in fact converge for any $r \in \mathbb{R}^{\geq 0}$. Without loss of generality, we will parameterize the $a^{(1)}$-period enclosing a pair of branch points on Σ_1 by the path going twice from $re^{i\delta}$ to $re^{-i\delta}$ and intersecting the horizontal segment $[-4, 0)$ (see also subsection 4.2.1). Since $\Phi(r, \delta_0)$ is holomorphic on $[-4, 0)$, this is indeed well-defined.

We begin with the limit $r \to 0$. With the substitution $u = \xi/r$ we rewrite (4.82) as

$$\int_{e^{i\delta_0}}^{e^{-i\delta_0}} \frac{2r du}{u\sqrt{(ur + 4)(ur - re^{i\delta_0})(ur - re^{-i\delta_0})}} \tag{4.83}$$

where the path of integration goes trough the interval $[-4, 0)$. Pulling out a $4r^2$ term in the square root cancels the factor $2r$ in the numerator and letting $r \to 0$, the integral simplifies to

$$\int_{e^{i\delta_0}}^{e^{-i\delta_0}} \frac{du}{u\sqrt{(u - e^{i\delta_0})(u - e^{-i\delta_0})}}. \tag{4.84}$$

Now let C_1 be the closed upper semi circle excluding the pole at $u = 0$ and going through the point $e^{i\delta_0}$ on S^1 as shown in figure 4.4. By the residue theorem

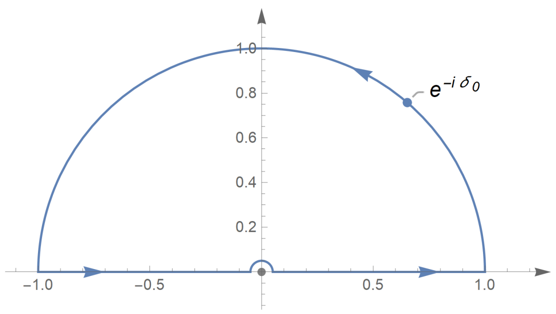

Figure 4.4: The path C_1 used for calculating the integral (4.86).

$$\int_{C_1} \frac{du}{u\sqrt{(u - e^{i\delta_0})(u - e^{-i\delta_0})}} = 0. \tag{4.85}$$

We can evaluate this integral by splitting it into different paths in the following way

$$\begin{aligned}
&\int_{C_1} \frac{du}{u\sqrt{(u-e^{i\delta_0})(u-e^{-i\delta_0})}} = i\int_0^{\delta_0} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}} + \\
&i\int_{\delta_0}^{\pi} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}} + \lim_{\epsilon\to 0}\int_{-1}^{-\epsilon} \frac{du}{u\sqrt{(u-e^{i\delta_0})(u-e^{-i\delta_0})}} + \\
&\lim_{\epsilon\to 0} i\int_{\pi}^{0} \frac{d\varphi}{\sqrt{(\epsilon e^{i\varphi}-e^{i\delta_0})(\epsilon e^{i\varphi}-e^{-i\delta_0})}} + \lim_{\epsilon\to 0}\int_{\epsilon}^{1} \frac{du}{u\sqrt{(u-e^{i\delta_0})(u-e^{-i\delta_0})}}.
\end{aligned} \tag{4.86}$$

In equation (4.86), we start the integration at $u = 1$ and go along the path C_1 counter-clockwise. In the limit $\epsilon \to 0$, the fourth term is $-\pi i$. The third and fifth term are purely real since the square root is positive for real u. This implies that the imaginary part of the first plus the second term equals πi. Let us take a closer look at the first term which can be written as

$$\begin{aligned}
&i\int_0^{\delta_0} \frac{d\varphi}{\sqrt{e^{i\varphi}(2\cos\varphi - 2\cos\delta_0)}} \\
&\quad = \frac{i}{\sqrt{2}}\int_0^{\delta_0} \frac{\cos\frac{\varphi}{2}d\varphi}{\sqrt{\cos\varphi-\cos\delta_0}} + \frac{1}{\sqrt{2}}\int_0^{\delta_0} \frac{\sin\frac{\varphi}{2}d\varphi}{\sqrt{\cos\varphi-\cos\delta_0}}.
\end{aligned} \tag{4.87}$$

Since $\cos\varphi - \cos\delta_0 > 0$ for $\varphi \in [0, \delta_0)$, the first and second term of equation (4.87) are imaginary and real, respectively. Substituting

$$x = \frac{\sqrt{2}\sin\frac{\varphi}{2}}{\sqrt{\cos\varphi-\cos\delta_0}}, \quad dx = \frac{\sqrt{2}\cos\frac{x}{2}\sin^2\frac{\delta_0}{2}}{(\cos\varphi-\cos\delta_0)^{\frac{3}{2}}}d\varphi \tag{4.88}$$

and using the identity $\sin^2\frac{\delta_0}{2} = \frac{1-\cos\delta_0}{2}$, the imaginary part of (4.87) takes a simple form and we can explicitly integrate

$$\frac{i}{\sqrt{2}}\int \frac{\cos\frac{\varphi}{2}d\varphi}{\sqrt{\cos\varphi-\cos\delta_0}} = i\int\frac{dx}{1+x^2} = i\cdot\arctan x. \tag{4.89}$$

Hence, in the limit $\varphi \to 0$, the integral (4.89) vanishes while for $\varphi \to \delta_0$, it converges to $i\frac{\pi}{2}$. Since the imaginary part of the first plus second term in (4.86) equal πi, each term independently is $i\frac{\pi}{2}$. To finally obtain the first statement of the proposition, notice that (4.84) can be written as

$$i\int_{\delta_0}^{\pi} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}} + i\int_{\pi}^{2\pi-\delta_0} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}} = \tag{4.90}$$

$$i\int_{\delta_0}^{\pi} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}} - \overline{i\int_{\delta_0}^{\pi} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}}} = \tag{4.91}$$

$$2i\,\mathrm{Im}\left(i\int_{\delta_0}^{\pi} \frac{d\varphi}{\sqrt{(e^{i\varphi}-e^{i\delta_0})(e^{i\varphi}-e^{-i\delta_0})}}\right) = \pi i \tag{4.92}$$

where the substitution $\varphi \to 2\pi - \varphi$ was used in the second line. This proves the first statement.

For the second case $r \to \infty$, the argument is similar. Again, consider the integral (4.82) where we use for $a^{(1)}$ the same integration path as before. Pulling out an r in the denominator cancels the r in the numerator. In the limit $r \to \infty$, the exponential functions cancel each other out and we are left with the integral

$$\int_{re^{i\delta_0}}^{re^{-i\delta_0}} \frac{2d\xi}{\xi\sqrt{(\xi+4)}}. \tag{4.93}$$

The path of integration again goes trough the interval $[-4, 0)$ and we choose the path as shown in figure 4.5.

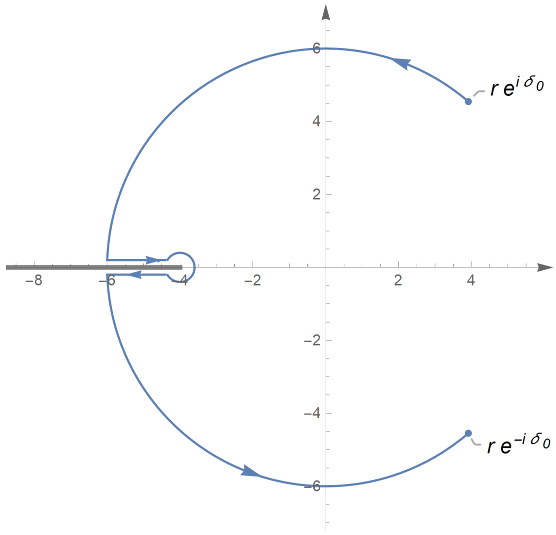

Figure 4.5: Schematic visualization of the integration path for equation (4.93). The gray bar denotes the branch cut from -4 to $-\infty$.

$$\begin{aligned} \int_{re^{i\delta_0}}^{re^{-i\delta_0}} \frac{2d\xi}{\xi\sqrt{(\xi+4)}} &= i\int_{\delta_0}^{\pi} \frac{2d\varphi}{\sqrt{re^{i\varphi}+4}} + \int_{-r}^{-4} \frac{2d\xi}{\xi\sqrt{(\xi+4)}} \\ \lim_{\epsilon\to 0} i\int_{\pi}^{-\pi} &\frac{\epsilon e^{i\varphi}d\varphi}{(-4+\epsilon e^{i\varphi})\sqrt{\epsilon e^{i\varphi}}} - \int_{-4}^{-r} \frac{2d\xi}{\xi\sqrt{(\xi+4)}} + i\int_{\pi}^{2\pi-\delta_0} \frac{2d\varphi}{\sqrt{re^{i\varphi}+4}}. \end{aligned} \tag{4.94}$$

In the limit $\epsilon \to 0$, the third term on the right hand side of equation (4.94) vanishes. Furthermore, we also see that in the limit $r \to \infty$, the first and fifth terms of (4.94) vanish. The second and fourth term of equation (4.94) add up and it is a known result from complex analysis that

$$\int_{-\infty}^{-4} \frac{4d\xi}{\xi\sqrt{(\xi+4)}} = 2\pi i \tag{4.95}$$

which proves the second claim of the statement. □

By proposition 4.3.4, the integers m, n in (4.82) must have opposite signs. Hence, without loss of generality, we can from now assume that $n \in \mathbb{N}$ and $-m \in \mathbb{N}$. We want to show that the imaginary part of $\ln \mu_1(1)$ is monotonically decreasing in r. This would imply that for every ratio $m/n \in (1, 2)$ there exists exactly one r which satisfies equation (4.82). For this, we will first gather some information about the coefficient $d_{1,1}$ in the differential

$$\vartheta_1 = -(d_{1,1}(\xi+3) + d_{1,2})\frac{d\xi}{t_1} \tag{4.96}$$

which was the pushforward of $\theta_1 = d\ln \mu_1$ on Σ to the elliptic curve Σ_1 (cf. equation (4.16)).

Proposition 4.3.5. *Let $d_{1,1}$ be a coefficient of the normalized abelian differential ϑ_1 in equation (4.96). Assume that*

$$\int_{b^{(1)}} \vartheta_1 = 2\pi i n \tag{4.97}$$

where $n \in \mathbb{N}$. Then we have $d_{1,1} = niK(m_1)/s^{1/4}$ where $s = 16+8r\cos\delta+r^2$ and $m_1 = \frac{-4-r\cos\delta+\sqrt{s}}{2\sqrt{s}}$. Moreover, for fixed $\delta \in (0, \frac{\pi}{2})$, the imaginary part of $d_{1,1}$ is monotonically decreasing in r.

Proof. Using the assumption that ϑ_1 is normalized and (4.97), a straight forward calculation implies that $d_{1,1}$ is given by

$$d_{1,1} = -2\pi i n \frac{\int_{a^{(1)}} \frac{d\xi}{t_1}}{\int_{a^{(1)}} \frac{d\xi}{t_1} \int_{b^{(1)}} \frac{(\xi+4)d\xi}{t_1} - \int_{b^{(1)}} \frac{d\xi}{t_1} \int_{a^{(1)}} \frac{(\xi+4)d\xi}{t_1}}. \tag{4.98}$$

The differential $\frac{(\xi+4)d\xi}{t_1}$ has a pole of order two at infinity. If we take a local coordinate $\xi + 4 = \frac{1}{z_\infty^2}$ such that $z_\infty = 0$ at infinity, it expands as $\frac{(\xi+4)d\xi}{t_1} \sim \frac{-2dz_\infty}{z_\infty^2}$. Similarly, $\frac{d\xi}{t_1} \sim -2dz_\infty$. We define the abelian integral $f(\xi) = \int_{\xi_0}^{\xi} \frac{d\hat{\xi}}{t_1}$ which is well defined on the simply connected model of Σ_1. By the reciprocity law for abelian differential of the first and second kind on elliptic curves (which is essentially the Legendre relation)

$$\begin{aligned}\int_{a^{(1)}} \frac{d\xi}{t_1} \int_{b^{(1)}} \frac{(\xi+4)d\xi}{t_1} - \int_{b^{(1)}} \frac{d\xi}{t_1} \int_{a^{(1)}} \frac{(\xi+4)d\xi}{t_1} &= 2\pi i \sum_p \mathrm{Res}_p \left(f \frac{(\xi+4)d\xi}{t_1} \right) \\ &= 8\pi i.\end{aligned} \tag{4.99}$$

For the numerator in (4.98), we use the first equation of (4.47) and obtain

$$d_{1,1} = niK(m_1)/s^{1/4} \tag{4.100}$$

where m_1 and s are as in (4.48). To show that $d_{1,1}$ is monotonically decreasing in r, we take the derivative

$$\partial_r d_{1,1} = -\frac{in}{2rs^{3/4}}((4+\sqrt{s})K(m_1) - 8E(m_1)). \tag{4.101}$$

From [OLBC10, p. 494] we have the inequality $E(m_1)/K(m_1) \leq 1$ and therefore $\partial_r d_{1,1} \leq 0$ is equivalent to $8/(4+\sqrt{s}) \leq 1$. The term $8/(4+\sqrt{s})$ attains its maximum value at $r = 0$ where it equals 1, which shows that the imaginary part of $d_{1,1}$ is monotonically decreasing in r. □

Using proposition 4.3.5, we now show that the imaginary part of $\ln\mu_1(1)$ is, for fixed $\delta \in (0, \frac{\pi}{2})$, monotonically decreasing in r.

Proposition 4.3.6. *Assume that the conditions of proposition 4.3.5 are satisfied and let $\ln\mu_1(1)$ be defined by equation (4.82). Then $\partial_r \ln\mu_1(1) \neq 0$ for all $r \in \mathbb{R}^{\geq 0}$ and $\delta \in (0, \frac{\pi}{2})$.*

Proof. $\ln\mu_1(1)$ is given by equation (4.82). Taking the derivative of $\ln\mu_1(1)$ with respect to r we obtain

$$\begin{aligned}\partial_r \ln\mu_1(1) &= -2n \int_{e^{i\delta}}^{e^{-i\delta}} \partial_r \frac{du}{u\sqrt{(u\cdot r+4)(u-e^{i\delta})(u-e^{-i\delta})}} \\ &= n \int_{e^{i\delta}}^{e^{-i\delta}} \frac{du}{(u\cdot r+4)\sqrt{(u\cdot r+4)(u-e^{i\delta})(u-e^{-i\delta})}}.\end{aligned} \tag{4.102}$$

To show that this is non-vanishing, we express $\partial_r \ln \mu_1(1)$ in terms of $d_{1,1}$ and its derivative $\partial_r d_{1,1}$. We have from (4.98)

$$\begin{aligned}\partial_r d_{1,1} &= \frac{n}{4}\int_{e^{i\delta}}^{e^{-i\delta}} \frac{u\,du}{(u\cdot r+4)\sqrt{(u\cdot r+4)(u-e^{i\delta})(u-e^{-i\delta})}} \\ &= \frac{n}{4r}\Bigg[\int_{e^{i\delta}}^{e^{-i\delta}} \frac{du}{\sqrt{(u\cdot r+4)(u-e^{i\delta})(u-e^{-i\delta})}} \\ &\quad -\int_{e^{i\delta}}^{e^{-i\delta}} \frac{4\,du}{(u\cdot r+4)\sqrt{(u\cdot r+4)(u-e^{i\delta})(u-e^{-i\delta})}}\Bigg] \\ &= -\frac{1}{2r}d_{1,1} - \frac{1}{r}\partial_r \ln \mu_1(1).\end{aligned} \tag{4.103}$$

After using equations (4.100) and (4.101), we see that $\partial_r \ln \mu_1(1) = 0$ is equivalent to

$$2\frac{E(m_1)}{K(m_1)} = 1. \tag{4.104}$$

We make use of the inequality [OLBC10, p. 494]

$$\sqrt{m_1'} < \frac{E(m_1)}{K(m_1)} \tag{4.105}$$

where $m_1' = 1 - m_1$, but

$$2\sqrt{m_1'} > \sqrt{2 + \frac{8}{\sqrt{16+r^2}}} > 1 \tag{4.106}$$

and hence $\partial_r \ln \mu_1(1)$ cannot vanish for $(r,\delta) \in \mathbb{R}^{\geq 0} \times (0, \frac{\pi}{2})$. □

Using both propositions 4.3.4 and 4.3.6, we obtain the following corollary.

Corollary 1. *Assume that the conditions of proposition 4.3.5 are satisfied. For fixed $\delta \in (0, \frac{\pi}{2})$, the imaginary part of $\ln \mu_1(1)$ is monotonically decreasing in r.*

Remark: Notice the similarity of proposition 4.3.6 with equation (3.71) in section 3.4 about the Whitham flow. In particular, we now see that if the equations (3.70) are satisfied, we indeed only have trivial Whitham deformations for the Wente tori since in this case the ratio m/n is constant, i.e., we start and end at the same torus.

4.3.3 Lattices and new notation

So far, we have studied the intrinsic and extrinsic closing conditions. Instead of working on the genus two hyperelliptic curve Σ, we saw that the closing conditions were expressible in terms or elliptic data on the elliptic curves

$$\Sigma_j : \; t_j^2 = (\xi + 4(2-j))(\xi - re^{i\delta})(\xi - re^{-i\delta}) \tag{4.107}$$

for $j = 1, 2$. In particular, we saw that

i. The closing conditions on Σ_2 fix the angle $\delta = \delta_0 \in (0, \frac{\pi}{2})$ via equation (4.64).

ii. The closing conditions on Σ_1 determine the radial coordinate $r \in \mathbb{R}^{\geq 0}$ by satisfying (4.82). For every rational $m/n \in (1, 2)$ there exists exactly one $r \in \mathbb{R}^{>0}$ such that (4.82) is satisfied.

We will now introduce a new notation for the Wente tori. In order to differentiate between different Wente tori, we will attach the index (m, n) to them where $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$. The subscript (m, n) comes from the dependency of the radial coordinate r in the branch points of the elliptic curves Σ_j via equation (4.82). Then the torus with rectangular lattice spanned by the principal parts of the differentials $\theta_j = d \ln \mu_j$ is denoted by $\hat{T}^2_{(m,n)} = \mathbb{C}/\hat{\Gamma}_{(m,n)}$. The reason for the hats becomes apparent shortly. It follows from the calculations in equation (3.47) that the lattice $\hat{\Gamma}_{(m,n)}$ is given by

$$\hat{\Gamma}_{(m,n)} = 4|d_{1,1}|\mathbb{Z} + 4id_{2,1}\mathbb{Z}. \tag{4.108}$$

where $d_{j,1}$ are the coefficients of the principal parts of θ_j.

In proposition 4.3.5, we have studied $d_{1,1}$ and wrote it down in terms of complete elliptic integrals. On the other hand, the coefficient $d_{2,1}$ is still undetermined since we have not considered the B_l-periods, $l = 1, 2$, of the differential θ_2 yet. The intrinsic closing conditions imply that these must be integer multiples of $2\pi i$. In particular, if we assume that the extrinsic closing condition is satisfied, i.e., κ in (4.58) vanishes at $\delta = \delta_0$, then $\int_{B_l} \theta_2 = 0$ for $l = 1, 2$ (cf. lemma 4.2.1) cannot be possible since a reciprocity law for abelian differentials of the first and second kind would show that this contradicts Legendre's relation. Therefore, assume from now on that

$$\int_{B_l} \theta_2 = 2\pi i \tag{4.109}$$

for $l = 1, 2$.

Definition 4.3.1. *Let $l = 1, 2$ and $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Let $\theta_j = d \ln \mu_j$ for $j = 1, 2$ be normalized meromorphic differentials given by equation (4.8). We denote the Wente torus with spectral data*

$$\begin{aligned} \int_{B_l} \theta_1 &= 2\pi i n(-1)^{l-1}, & \int_{B_l} \theta_2 &= 2\pi i \\ \ln \mu_1(1) &= -\pi i m, & \ln \mu_2(1) &= 0 \end{aligned} \tag{4.110}$$

and lattice

$$\hat{\Gamma}_{(m,n)} = 4|d_{1,1}|\mathbb{Z} + 4id_{2,1}\mathbb{Z}. \tag{4.111}$$

by $\hat{T}^2_{(m,n)}$.

It turns out that $\hat{T}^2_{(m,n)}$ is actually a double cover of a compact CMC torus $T^2_{(m,n)} = \mathbb{C}/\Gamma_{(m,n)}$ since the immersion $f : \hat{T}^2_{(m,n)} \to \mathbb{R}^3$ already closes on a smaller lattice. The reason for this is the following: The symmetry group of the $\hat{T}^2_{(m,n)}$ Wente tori is $D_n \times \mathbb{Z}_2$ if m is even and $D_{2n} \times \mathbb{Z}_2$ if m is odd, where D_k is the dihedral group of order k [Abr87, p. 169]. The figure eight k_2-curvature lines (cf. (4.44)) lie in planes orthogonal to the symmetry plane which generates the $\mathbb{Z}_2$ symmetry. In particular, their vertices lie on [Abr87, p. 169]

i. the

$$x \to f(x, 0) \tag{4.112}$$

curvature line if m is even.

ii. the

$$\begin{aligned} x &\to f(x,0) \\ x &\to f(x,2d_{2,1}) \end{aligned} \tag{4.113}$$

curvature lines if m is odd.

Therefore, the immersion already closes on a smaller lattice. We now want to determine the lattice $\Gamma_{(m,n)}$ of the smaller torus $T^2_{(m,n)}$. Recall the expression of the conformal factor of the conformal metric given by equation (4.55). The Jacobi elliptic function $\mathrm{cn}(z)$ has periods $\mathrm{cn}(z+2K(m),m) = -\mathrm{cn}(z,m)$ and $\mathrm{cn}(z+4K(m),m) = \mathrm{cn}(z,m)$. Thus, we see from the definition of the conformal factor (4.55) that $u(x,y)$ is doubly periodic if we shift x and y by

$$x \mapsto x + \frac{2\pi}{\alpha}, \quad y \mapsto y + \frac{2\pi}{\beta}, \tag{4.114}$$

respectively, where α and β are given by the relation

$$\int_{B_1} d\Omega_\infty = \alpha + i\beta = \int_{B_1} \left(\frac{\theta_2}{4d_{2,1}} - \frac{\theta_1}{4d_{1,1}} \right) = \frac{2\pi i}{4d_{2,1}} - \frac{2\pi i n}{4d_{1,1}} \tag{4.115}$$

(cf. equation (4.21)). Since $d_{1,1}$ and $d_{2,1}$ are imaginary and real, respectively, we obtain that the $T^2_{(m,n)}$ Wente tori are spanned by the following lattice.

Lemma 4.3.7. *Let $f : T^2_{(m,n)} \to \mathbb{R}^3$ denote the conformal immersion of an (m,n) CMC Wente tori. Assume that* $\gcd(m,n) = 1$. *Then the lattice of $T^2_{(m,n)}$ is spanned by*

$$\Gamma_{(m,n)} = \{(2|d_{1,1}|, 0), (0, 4d_{2,1})\} \tag{4.116}$$

if m is odd and by

$$\Gamma_{(m,n)} = \{(2|d_{1,1}|, 2d_{2,1}), (0, 4d_{2,1})\} \tag{4.117}$$

if m is even.

Hence, if m is even the lattice is of rhombic type while for odd m the lattice remains rectangular.

4.4 Double points on Wente tori

In the last section of this chapter, we will investigate the existence of double points for $\hat{T}^2_{(m,n)}$ Wente tori, i.e., points where the eigenvalues of the monodromy of the associated family on $\hat{T}^2_{(m,n)}$ satisfy $\mu_j(\lambda)^2 = 1$. At such points, the holomorphic structure (3.57) is a half-lattice point in the lattice $\hat{\Lambda}_{(m,n)}$ generating $\mathrm{Jac}(\hat{T}^2_{(m,n)})$. However, we are actually interested on the existence of double points on the smaller torus $T^2_{(m,n)}$. Since $\hat{T}^2_{(m,n)}$ doubly covers $T^2_{(m,n)}$, half (and quarter) lattice points of $\Lambda_{(m,n)}$ correspond to half-lattice points of $\hat{\Lambda}_{(m,n)}$. In the next chapter, we will see that double points are related to the stability of the underlying parabolic structure. For example, it is not allowed to have unstable parabolic structures along S^1 since the associated family is unitary along S^1, which implies that it is at least semi stable.

To verify the existence of double points, we first need to further investigate some properties of the coefficients of θ_1. It is convenient to define

$$-\kappa = \frac{d_{2,1}+d_{2,2}}{d_{2,1}}, \quad \nu = \frac{d_{1,1}-d_{1,2}}{d_{1,1}} \tag{4.118}$$

where $d_{j,i}$ are the coefficients of the respective differentials θ_j in equation (4.8).

Lemma 4.4.1. *Let ν be given by (4.118). Assume that the conditions of proposition 4.3.5 are satisfied. Then we have*

$$\nu = \sqrt{s}\left(2\frac{E(m_1)}{K(m_1)} - 1\right) \tag{4.119}$$

where $s = 16 + 8r\cos\delta + r^2$ and $m_1 = \frac{-4-r\cos\delta+\sqrt{s}}{2\sqrt{s}}$. Moreover, $\nu \geq 4$ with equality if and only if $r = 0$.

Proof. To show that ν has the desired form we use a reciprocity law for the following two abelian differentials

$$\frac{d\xi}{(\xi+4)t_1}, \quad \frac{(\xi+4-\nu)d\xi}{t_1} = -\frac{\vartheta_1}{d_{1,1}}, \tag{4.120}$$

which have a pole of order two at -4 and ∞, respectively. As ϑ_1 is normalized, a reciprocity law yields

$$\int_{a^{(1)}} \frac{d\xi}{(\xi+4)t_1} \int_{b^{(1)}} \frac{(\xi+4-\nu)d\xi}{t_1} = -\frac{2\pi i n}{d_{1,1}} \int_{a^{(1)}} \frac{d\xi}{(\xi+4)t_1} = \frac{8\pi i \nu}{s}. \tag{4.121}$$

To treat the integral concerning the $a^{(1)}$-period in (4.121), we use equation (4.102) to obtain

$$\int_{a^{(1)}} \frac{d\xi}{(\xi+4)t_1} = \frac{2}{n}\partial_r \ln \mu_1(1) \tag{4.122}$$

and hence

$$d_{1,1} + 2r\partial_r d_{1,1} = \frac{4\nu d_{1,1}}{s} \tag{4.123}$$

by equation (4.103). From the formulas in the proof of proposition 4.3.5, we obtain

$$\nu = \sqrt{s}\left(2\frac{E(m_1)}{K(m_1)} - 1\right) \tag{4.124}$$

which shows the first claim of the proposition.

To validate the last assertion, set $m_1' = 1 - m_1$ and make use of the inequality [OLBC10, p. 494]

$$\sqrt{m_1'} \leq \frac{E(m_1)}{K(m_1)} \tag{4.125}$$

with equality at $m_1' = 1$, i.e., $\nu \geq \sqrt{s}(2\sqrt{m_1'} - 1)$. Clearly, $s > 0$ for all $(r, \delta) \in \mathbb{R}^{>0} \times (0, \frac{\pi}{2})$ and $\sqrt{m_1'} = 1$ for $r = 0$. Moreover, $\sqrt{s}(2\sqrt{m_1'} - 1)$ is monotonically increasing in r and has its minimum at $r = 0$, where it equals 4. Since $E(m_1) = K(m_1)$ at $r = 0$, this also proves the last claim. $\square$

It turns out that there are always points where the eigenvalue of the monodromy μ_1 equals ± 1.

Proposition 4.4.2. *Let $\int_{b(1)} \vartheta_1 = 2\pi i n$ with $n \in \mathbb{N}$. Then $\ln \mu_1(-1) = -2n\pi i$ independent of r.*

Proof. Consider the following differential of a third kind

$$\phi = \frac{\lambda d\lambda}{(\lambda+1)y} \tag{4.126}$$

on the genus two spectral curve Σ which has simple poles at the two points over $\lambda = -1$ with residues ± 1, respectively. Similar considerations as in equation (4.70) yield

$$\int_{A_1} \phi = \int_{A_2} \phi. \tag{4.127}$$

Therefore, we obtain

$$\int_{B_1} \theta_1 \Big(\int_{A_1} \phi - \int_{A_2} \phi \Big) = 0. \tag{4.128}$$

On the one hand, a reciprocity law for abelian differentials of the second and third kind yields

$$\ln \mu_1(-1) = 0. \tag{4.129}$$

On the other hand, by definition we have

$$\ln \mu_1(-1) = \ln \mu_1(1) - d_{1,1} \int_0^{-4} (\xi + 4 - \nu) \frac{d\xi}{t_1}. \tag{4.130}$$

Proposition (4.119) implies that $\nu \geq 4$ and as the imaginary part of $d_{1,1}$ is positive (cf. (4.100)), both terms in equation (4.130) have the same sign. In particular, we obtain that $|\ln \mu_1(-1)| > |\ln \mu_1(1)|$. Notice that this does not contradict (4.129) since equation (4.130) equals (4.129) up to a B_l-period of θ_1. We confirm this in the following calculation. Consider the limit $r \to \infty$ on the right hand side of equation (4.130). From proposition 4.3.4 we know that $\ln \mu_1(1) = -2\pi i n$ in this limit. However, the second term in (4.130) goes to zero as r tends to infinity. Hence, $\ln \mu_1(-1) = -2\pi i n$ in the limit. But the whole calculation is independent of the chosen r by (4.129). Therefore, $\ln \mu_1(-1) = -2\pi i n$ for all $r \in \mathbb{R}^{>0}$. □

We also see that $\int_0^{\xi} \vartheta_1$ is purely imaginary along $\xi \in [-4, 0]$. In particular, its imaginary part is monotonically increasing in ξ. As $\lambda + \lambda^{-1} - 2 = \xi \in (-4, 0)$ has two solutions in $\lambda \in S^1$, we can conclude the following.

Corollary 2. *Let $\int_{b(1)} \vartheta_1 = 2\pi i n$ and $\ln \mu_1(1) = -\pi i m$ for some fixed values of $(r, \delta_0) \in \mathbb{R}^{>0} \times (0, \frac{\pi}{2})$ with $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$. Then $\operatorname{Im}[\ln \mu_1(\lambda)] \in [-\pi m, -2\pi n\pi]$ for $\lambda \in S^1$. Since $2n > m$ by (4.82), there exist exactly $2(2n - m)$ points $\lambda_i \in S^1$ where $\ln \mu_1(\lambda_i) \in -\pi i \mathbb{N}$. In particular, two of those points lie on $\pm 1 \in S^1$.*

To finalize the discussion about double points, we must also look at the other eigenvalue $\ln \mu_2$. Since the coefficient $d_{2,1}$ in the abelian differential ϑ_2 diverges with order $\sqrt{r}$ as $r \to 0$, it is generally hard to rule out the existence of points where $\ln \mu_2 \in \pi i \mathbb{Z}$ throughout variations of r. Nevertheless, assuming that r stays large enough, we can still argue that there cannot exist points where both eigenvalues of the monodromy $\mu_j = \pm 1$ for $j = 1, 2$.

Lemma 4.4.3. *Let $\int_{b^{(2)}} \vartheta_2 = 2\pi i$. If $r \geq 4$ and $\delta = \delta_0$ (cf. equation(4.64)), then $|\ln \mu_2(\lambda)| < \pi$ for all $\lambda \in S^1$. In particular, $\ln \mu_2(\lambda) = 0$ if and only if $\lambda = 1$.*

Proof. We have

$$\ln \mu_2(\xi) = \ln \mu_2(1) + d_{2,1} \int_0^{-\xi} (\hat{\xi} - \kappa) \frac{d\hat{\xi}}{t_2} \tag{4.131}$$

where $\xi \in [0,4]$ for $\lambda \in S^1$. Recalling that $\ln \mu_2(1) = 0$ by lemma 4.3.2 and $\kappa(r, \delta_0) = 0$ for all r, equation (4.131) simplifies to

$$\ln \mu_2(\xi) = d_{2,1} \int_0^{-\xi} \hat{\xi} \frac{d\hat{\xi}}{t_2}. \tag{4.132}$$

Assume that $r \geq 4$. Since $|\ln \mu_2(\xi)|$ is monotonically increasing in ξ, consider the integral

$$\left| d_{2,1} \int_0^{-r} \frac{\xi d\xi}{t_2} \right| \tag{4.133}$$

which gives an upper bound of (4.132) for $\xi \in [0,4]$. With the help of equation (4.49), the coefficient $d_{2,1}$ is easy to determine. The vanishing of the integral of $\xi d\xi / t_2$ along $a^{(2)}$ at $\delta = \delta_0$ shows that $d_{2,1} = K(m_2)/\sqrt{r}$, where $m_2 = \cos^2(\frac{\delta_0}{2})$. After substituting $u = \xi / r$, equation (4.133) takes the form

$$d_{2,1} \int_0^{-r} \frac{\xi d\xi}{t_2} = K(m_2) \int_0^{-1} \frac{u\, du}{\sqrt{u(u - e^{i\delta_0})(u - e^{-i\delta_0})}} \tag{4.134}$$

which is independent of r. We use [BF13, eq. 243.03] to rewrite

$$\int_0^{-1} \frac{u\, du}{\sqrt{u(u - e^{i\delta_0})(u - e^{-i\delta_0})}} = i \left(2E(m_2') - K(m_2') - 2\cos\frac{\delta_0}{2} \right) \tag{4.135}$$

where $m_2' + m_2 = 1$. Using the Legendre relation (2.51) and the fact that $K(m_2) = 2E(m_2)$ at $\delta = \delta_0$ (cf. (4.64)), we get $2E(m_2') - K(m_2') = \pi/(2E(m_2))$ and the upper bound (4.133) is

$$\left| d_{2,1} \int_0^{-r} \frac{\xi d\xi}{t_2} \right| = \left| \pi - 2K(m_2) \cos\left(\frac{\delta_0}{2}\right) \right|. \tag{4.136}$$

If we show that $K(m_2)\cos(\frac{\delta_0}{2}) < \pi$ we are done. But this inequality is satisfied since $K(m_2) = 2E(m_2) \leq \pi$ with equality if and only if $m_2 = 0$, which cannot happen for $\delta \in (0, \frac{\pi}{2})$. $\square$

Remark: Another way to analyse $\ln \mu_2(-1)$ is to define the abelian differential of the third kind $\phi = \lambda d\lambda / (\lambda + 1) y$ which has simple poles at the two points over $\lambda = -1$ with residues ± 1, respectively, and descends to a differential on Σ_2. A reciprocity law similar to (4.79) expresses $\ln \mu_2(-1)$ in terms of complete elliptic integrals of the third and first kind similar to (8.18).

Even tough lemma 4.4.3 is only true for $r \geq 4$, it already captures the case of the 3-lobed Wente torus $\hat{T}^2_{(4,3)}$ by the estimation of r in Appendix 8.1.2. Hence, it also true for every other Wente torus with $4/3 \leq m/n < 2$.

Chapter 5

Irreducible flat connections on compact Riemann surfaces

We have seen in chapter 3 that to any conformal CMC immersion $f : M \to \mathbb{R}^3$ of genus g, we can associate a family of flat $\mathrm{SL}(2,\mathbb{C})$-connections (cf. definition 3.1.1). On the other hand, given a $\mathbb{C}^*$-family of flat connections ∇^λ with parallel frame $F_\lambda \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and monodromy that expands around a $\lambda_1 \in S^1$ as $\pm\mathrm{Id} + \mathcal{O}((\lambda - \lambda_1)^2)$ along all generators of $\pi_1(M, *)$, we obtain closed CMC surfaces via the Sym-Bobenko formula 3.2.1. The point $\lambda_1 \in S^1$ is called the Sym-point. Hence, in principle, we can construct higher genus CMC surfaces from families of flat connections satisfying these conditions. However, finding such families is highly non-trivial.

In the case of $M = T^2$ being a torus we have seen in section 3.3 that we can parameterize the monodromies of ∇^λ in terms of data on a smooth hyperelliptic curve, which is called the spectral curve. Closing conditions on the torus, e.g., trivial monodromy at the Sym point, can be restated as conditions on the spectral data. However, it is clear from the very definition of the spectral curve that this construction heavily relies on the fact that the first fundamental group of a torus is abelian. Thus, a naive generalization of this setup for genus $g > 1$ surfaces does not work and we need a different approach.

The idea we are going to follow is to study logarithmic connections, i.e., connections with simple poles at prescribed points, on the 4-punctured spheres. As punctured surfaces are not simply connected, we will have a non-trivial first fundamental. This allows us to study logarithmic connections with non-trivial local monodromy representations. We will then pull these connections back to a suitable covering $N \to \mathbb{C}P^1$ of a Riemann surface N with genus $g > 1$. After desingularising them, we obtain families of flat connections on a higher genus Riemann surface with controlled monodromies. Finally, in order to get closed (and possibly branched) CMC surfaces in $\mathbb{R}^3$, the pulled back connections have to satisfy the closing conditions listed above. Since the families of connections are related to each other via pullback, we can adjust the covering and monodromy such that these conditions are satisfied.

5.1 Logarithmic connections and parabolic structures

In this section, the basic concepts of logarithmic connections on Riemann surfaces and parabolic structures are introduced. For now, we let M be an arbitrary compact Riemann surface. Throughout this section, $E \to M$ denotes a holomorphic rank two vector bundle with trivial determinant $\Lambda^2 E = \underline{\mathbb{C}}$. The holomorphic structure on E will be denoted by $\bar{\partial}_E$. We will also set

$$D = \{p_1, ..., p_n\} \tag{5.1}$$

as a finite set of distinct points on M. For further references of logarithmic connections and parabolic structures, we refer the reader to [MS80, Kon93].

Firstly, we show how we can introduce poles for sections of vector bundles. Recall the definition of the point bundle construction 2.2.7. The holomorphic line bundle $L(-kp) \to M$ with $k \in \mathbb{N}$ and $p \in M$ admits a nowhere vanishing section with a pole of order k at p and holomorphic everywhere else. We define

$$L(D) := L(-p_1) \otimes ... \otimes L(-p_n) \tag{5.2}$$

which means that a section $s \in \Gamma(L(D))$ has n poles which are all of order one. In this way, poles at certain points are introduced. In the following, we will use the notation $E(D) := E \otimes L(D)$.

Definition 5.1.1. *A logarithmic* $\mathrm{SL}(2,\mathbb{C})$*-connection* $\nabla = \bar{\partial}_E + \partial^\nabla$ *on* $E \to M$ *is a* $\mathbb{C}$*-linear map*

$$\partial^\nabla : H^0(U, E) \to H^0(U, K_M \otimes E(D)) \tag{5.3}$$

on some neighborhood $U \subset M$ *such that*

i. it satisfies the Leibniz rule $\nabla(fs) = s\,df + f\nabla s$ *for a local holomorphic map* f *and local holomorphic section* s.

ii. the induced differential operator on $\Lambda^2 E = \underline{\mathbb{C}}$ *is the trivial connection.*

Logarithmic connections generalize the notion of connections on vector bundles as they are allowed to have poles at certain points. In particular, every logarithmic connection ∇ on $M \setminus D$ is already flat as the curvature of ∇ is

$$F^\nabla = \bar{\partial}_E \circ \partial^\nabla + \partial^\nabla \circ \bar{\partial}_E, \tag{5.4}$$

which is zero on holomorphic sections. The singularities of a logarithmic connection are, by definition, contained in its $(1,0)$-part. At each singular point $p_i \in D, i = 1, ..., n$, we have well-defined traceless residues

$$\mathrm{Res}_{p_i}(\nabla) \in \mathrm{End}(E_{p_i}). \tag{5.5}$$

A logarithmic connection induces a parabolic structure on $E \to M$, i.e., weight filtrations at each fiber E_{p_i} of $p_i \in D$. Assume that each residue (5.5) has only real eigenvalues $\pm\hat{\rho}_i$ where $\hat{\rho}_i \geq 0$. The eigenlines of each residue $\mathrm{Res}_{p_i}(\nabla)$ to the positive eigenvalue $\hat{\rho}_i$ define subspaces of the fibers E_{p_i}

$$L_{p_i} = \ker(\mathrm{Res}_{p_i}\nabla - \hat{\rho}_i \,\mathrm{Id}). \tag{5.6}$$

Thus, we obtain a weight filtration of each fiber E_{p_i} by

$$\begin{aligned} &E_{p_i} \supset L_{p_i} \supset \{0\} \\ &-\hat{\rho}_i < \hat{\rho}_i. \end{aligned} \tag{5.7}$$

The notation of (5.7) means that we equip each L_{p_i} with the positive weight $\hat{\rho}_i$ and $E_{p_i} \setminus L_{p_i}$ with the weight $-\hat{\rho}_i$. For stability reasons (cf. subsection 5.2.2), we will further assume that the eigenvalues lie in the interval $\hat{\rho}_i \in (0, \frac{1}{2})$ and the local monodromies of the logarithmic connections at each singular point are conjugated to [Del06, Theorem 1.17]

$$\begin{pmatrix} \exp(2\pi i\hat{\rho}_i) & 0 \\ 0 & \exp(-2\pi i\hat{\rho}_i) \end{pmatrix}. \tag{5.8}$$

Definition 5.1.2. *Let $D \subset M$ be a finite set of distinct points. A parabolic structure $\mathcal{P}$ on $E \to M$ is given by a filtration of each fiber E_{p_i} at the singular points $p_i \in D$ and a collection of weights $\hat{\rho}_i \in (0, \frac{1}{2})$ as in (5.7). A vector bundle with a parabolic structure is called a parabolic vector bundle.*

Subbundles $V \subset E$ of parabolic vector bundles have induced parabolic structures. We equip each complex vector space $V_{p_i} \subset E_{p_i}$ with the positive weight $\hat{\rho}_i$ if it equals the eigenline L_{p_i} of the logarithmic connection to the positive eigenvalue $\hat{\rho}_i$ and with the weight $-\hat{\rho}_i$ if it does not.

Definition 5.1.3. *Let $V \subset E$ be a holomorphic subbundle of the parabolic vector bundle $E \to M$. The parabolic degree of V is defined as*

$$\operatorname{pardeg}(V) = \deg(V) + \sum_{i=1}^{n} \gamma_i \tag{5.9}$$

where $\gamma_i = \hat{\rho}_i$ if V_{p_i} is the eigenline of the logarithmic connection to the positive eigenvalue $\hat{\rho}_i$ and $\gamma_i = -\hat{\rho}_i$ otherwise.

Definition 5.1.4. *A holomorphic parabolic vector bundle E is called stable (respectively semi-stable) if for every holomorphic subbundle $V \subset E$ we have*

$$\operatorname{pardeg}(V) < 0 \ (\textit{respectively} \leq). \tag{5.10}$$

Otherwise, it is called unstable.

The importance of stability in our setup comes from the famous theorem of Mehta-Seshadri, which manifests the relation between stable parabolic vector bundles of degree zero and irreducible unitary representations on the first fundamental group, i.e., unitarity monodromy of the underlying flat logarithmic connection. A proof of the following theorem can be found in [MS80, p. 238].

Theorem 5.1.1. *There is a* 1:1 *correspondence between*

i. stable parabolic vector bundles $E \to M$ and logarithmic connections with irreducible $\mathrm{SU}(2)$*-monodromy representations.*

ii. semi-stable parabolic vector bundles $E \to M$ and logarithmic connections with reducible $\mathrm{SU}(2)$*-monodromy representations.*

Let $\tilde{\nabla}$ and ∇ be two logarithmic connections on $E \to M$ with the same underlying holomorphic structure. A natural question that arises is: when do $\tilde{\nabla}$ and ∇ induce the same parabolic structure $\mathcal{P}$? Since $(\tilde{\nabla})'' = (\nabla)''$, the difference $\tilde{\nabla} - \nabla =: \Phi$ is an endomorphism valued $(1,0)$-form with at most simple poles at $p_i \in D$. The flatness of the connections implies that Φ is holomorphic. Generically, we cannot expect that ∇ and $\nabla + \Phi$ induce the same parabolic structure.

However, assume that the eigenlines L_{p_i} at each residue of the logarithmic connection ∇ lie in the kernel of the residues of Φ at the respective poles. Then L_{p_i} is also an eigenline of $\nabla + \Phi$ to the same eigenvalue $\hat{\rho}_i$. Therefore, they give rise to the same parabolic structure $\mathcal{P}$. On the other hand, if $\tilde{\nabla}$ and ∇ induce the same parabolic structure then they give rise to the same weight filtration of the bundle $E \to M$ and share the same set of eigenvalues. In particular, both connections have the same local monodromies at each residue. But this implies that the eigenlines of ∇ (or $\tilde{\nabla}$) must lie the kernel of Φ.

The deeper reasoning of these arguments is that the tangent space to the moduli space of stable parabolic bundles of degree zero is isomorphic to the space of strongly parabolic Higgs fields [BR94, section 6].

Definition 5.1.5. *Let ∇ be a logarithmic connection on $E \to M$ which induces a parabolic structure as in (5.6). Let $\Phi \in H^0(M, \mathrm{End}(E) \otimes K_M(D))$ be traceless such that*

$$\mathrm{Res}_{p_i}(\Phi)(L_{p_i}) = 0, \tag{5.11}$$

where L_{p_i} are the eigenlines of the logarithmic connections at each residue to the positive eigenvalue $\hat{\rho}_i$. Then Φ is called a strongly parabolic Higgs field to the logarithmic connection ∇.

A parabolic vector bundle equipped with a strongly parabolic Higgs field will also be called a parabolic Higgs bundle. By the Mehta-Seshadri theorem 5.1.1, if $E \to M$ is a stable parabolic vector bundle then there exists a unique strongly parabolic Higgs field Φ such that $\nabla + \Phi$ has irreducible unitary monodromy representation.

5.2 Fuchsian systems on the 4-punctured sphere

We have already seen in section 5.1 that a logarithmic connection on a rank two vector bundle equips it with a parabolic structure. In this section, we will discuss the case where $E = \underline{\mathbb{C}}^2$ is the trivial bundle and the underlying Riemann surface the 4-punctured sphere in detail. Logarithmic connections on the trivial bundle over the 4-punctured sphere are also called Fuchsian systems. For our purpose, the study of Fuchsian systems is the foundation for the construction of families of flat connections on higher genus Riemann surfaces. We will see in the subsequent sections that every such family that we construct comes from the pullback of a Fuchsian system to the higher genus surface. This section follows the lines of [LS15, HH17, HH22].

5.2.1 Parabolic structures and Higgs fields

Throughout this section, let $E = \underline{\mathbb{C}}^2 \to \mathbb{C}P^1$ denote the trivial bundle over the complex projective line equipped with the trivial holomorphic structure. Let $D = \{p_1, ..., p_4\} \subset \mathbb{C}P^1$. By a suitable choice of Möbius transformation, we can always accomplish that $p_1 = 0, p_2 = 1, p_3 = m$ and $p_4 = \infty$ where $m \in \mathbb{C} \setminus \{0, 1\}$. The open Riemann surface $\mathbb{C}P^1 \setminus D$ is called the 4-punctured sphere. Consider smooth loops $\gamma_i(t), i = 1, .., 4$, based at some point $x \in \mathbb{C}P^1 \setminus D$ encircling the four singular points $p_i \in \mathbb{C}P^1$. These curves generate the first fundamental group $\pi_1(\mathbb{C}P^1 \setminus D, x)$ and satisfy the relation

$$\prod_{i=1}^{4} \gamma_i = \mathrm{Id}. \tag{5.12}$$

In the case that the structure group is $\mathrm{SL}(2, \mathbb{C})$, the Riemann Hilbert problem has been solved for the n-punctured sphere [Dek79]. This means that any representation of $\pi_1(\mathbb{C}P^1 \setminus D, x) \to \mathrm{SL}(2,\mathbb{C})$ can be realized as the monodromy representation of a Fuchsian system.

Definition 5.2.1. *A Fuchsian system on the 4-punctured sphere is a logarithmic* $\mathrm{SL}(2,\mathbb{C})$*-connection on the trivial* $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus D$ *bundle with local non-zero residues at all points in* D*.*

Let z be a coordinate on $\mathbb{C} \subset \mathbb{C}P^1$. We equip $\underline{\mathbb{C}}^2$ with the trivial holomorphic structure and a Fuchsian system is of the form

$$\nabla^u = d + A_1^u \frac{dz}{z} + A_2^u \frac{dz}{z-1} + A_3^u \frac{dz}{z-m} \tag{5.13}$$

where the $A_i^u \neq 0$ are trace-free endomorphisms. Note that the connection (5.13) also has a simple pole at infinity with residue $A_4^u := -A_1^u - A_2^u - A_3^u$.

The eigenlines of the residues of the Fuchsian system determine a parabolic structure on the trivial bundle. Without loss of generality, we can assume that the parabolic lines L_{p_i} with respect to some frame are given by

$$L_{p_1} = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}, \quad L_{p_2} = \mathbb{C}\begin{pmatrix}1\\1\end{pmatrix}, \quad L_{p_3} = \mathbb{C}\begin{pmatrix}u\\1\end{pmatrix}, \quad L_{p_4} = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} \tag{5.14}$$

where $u \in \mathbb{C}$. The eigenlines are equipped with the weights $\hat{\rho}_i$ while $E_{p_i} \setminus L_{p_i}$ has the weight $-\hat{\rho}_i$. Notice that u is given by the cross section of the four lines (5.14) and hence it is invariant under Möbius transformation on $\mathbb{C}P^1$. In this way, the parabolic structure $\mathcal{P}$ on $\underline{\mathbb{C}}^2$ is parametrized by the complex number u.

For our purpose, i.e., the construction of higher genus CMC surfaces, it is not necessary to have four different eigenvalues at each residue of ∇^u and we restrict to the following convention.

Convention I: For the following, we will restrict to the case that all eigenvalues $\hat{\rho}_i$ are the same, i.e., $\hat{\rho}_i = \hat{\rho} \in (0, \frac{1}{2})$.

Proposition 5.2.1. *Let* ∇^u *be a Fuchsian system on the* 4*-punctured sphere of the form (5.13) such that the eigenlines of* A_i^u *to the positive eigenvalue* $\hat{\rho}$ *at the respective poles are given by (5.14). Then the* A_i^u *are gauge equivalent to*

$$A_1^u = \begin{pmatrix}-\hat{\rho} & 0\\ -2\hat{\rho} & \hat{\rho}\end{pmatrix}, \; A_2^u = \begin{pmatrix}\hat{\rho} & 0\\ 2\hat{\rho} & -\hat{\rho}\end{pmatrix}, \; A_3^u = \begin{pmatrix}-\hat{\rho} & 2\hat{\rho}u\\ 0 & \hat{\rho}\end{pmatrix}, \; A_4^u = \begin{pmatrix}\hat{\rho} & -2\hat{\rho}u\\ 0 & -\hat{\rho}\end{pmatrix}. \tag{5.15}$$

Proof. The proof follows from a straightforward calculation by using (5.14) and noticing that in the limiting case $\hat{\rho} \to 0$ the local monodromies (5.8) are all the identity. Therefore, the singular points become apparent, i.e., there exists a gauge such that the connection one-form of ∇^u extends smoothly to all of $\mathbb{C}P^1$. □

We have seen in section 5.1 that two Fuchsian systems induce the same parabolic structure if and only if their difference is a strongly parabolic Higgs field. With the choice of eigenlines in (5.14), we can write down parabolic Higgs fields on the 4-punctured sphere rather explicitly. With respect to the coordinate z on $\mathbb{C} \subset \mathbb{C}P^1$, the most general form of a Higgs field with simple poles at $p_i \in D$ is

$$\Phi^u = \phi_1^u \frac{dz}{z} + \phi_2^u \frac{dz}{z-1} + \phi_3^u \frac{dz}{z-m} \tag{5.16}$$

where the ϕ_i^u are holomorphic and $\operatorname{tr}(\phi_i^u) = 0$. Notice that Φ^u also has a simple pole at infinity with residue $\phi_4^u := -\phi_1^u - \phi_2^u - \phi_3^u$. As the bundle is trivial, the entries in ϕ_i are holomorphic functions on $\mathbb{C}P^1$, i.e., constants. The definition of strongly parabolic Higgs fields implies that the eigenlines L_{p_i} of the residues A_i^u must lie in the kernel of ϕ_i^u. The representation of the Fuchsian system given in (5.15) and the properties of the strongly parabolic Higgs field determine ϕ_i^u uniquely

$$\phi_1^u = \begin{pmatrix} 0 & 0 \\ 1-u & 0 \end{pmatrix}, \quad \phi_2^u = \begin{pmatrix} u & -u \\ u & -u \end{pmatrix}, \quad \phi_3^u = \begin{pmatrix} -u & u^2 \\ -1 & u \end{pmatrix}, \quad \phi_4^u = \begin{pmatrix} 0 & u-u^2 \\ 0 & 0 \end{pmatrix}. \tag{5.17}$$

Here, $u \in \mathbb{C}$ is the same as in (5.15) parameterizing the vector bundle's parabolic structure. Altogether, we obtain that every Fuchsian system with strongly parabolic Higgs field and local monodromies (5.8) on the trivial bundle $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus D$ is gauge equivalent to

$$\nabla^{u,v} = \nabla^u + v\Phi^u \tag{5.18}$$

where ∇^u is the Fuchsian system with connection one-form given by (5.15) and $v \in \mathbb{C}$. The complex numbers (u, v) parameterize the space of parabolic structures and strongly parabolic Higgs fields, respectively, as long as $u \notin \{0, 1, m, \infty\}$ since in this case the residues of Φ^u are non-vanishing.

5.2.2 Stability

We now determine under which conditions the parabolic structure induced by a Fuchsian system is stable, semi-stable or unstable. By the Mehta-Seshadri theorem 5.1.1, stability of parabolic vector bundles correspond to irreducible and unitary Fuchsian systems. In particular, semi-stable parabolic structures correspond to unitary reducible Fuchsian systems. In the later pages of this thesis, we want to have a family of Fuchsian systems parameterizing the associated family of flat connections (cf. definition 3.1.1) of a higher genus CMC surface. The unitarity along the unit circle is vital for the existence of such surfaces.

Again, let $E = \underline{\mathbb{C}}^2$ be the trivial holomorphic bundle over the 4-punctured sphere with induced parabolic structure from a Fuchsian system. By definition 5.1.4, a rank two holomorphic vector bundle E is parabolic stable if $\operatorname{pardeg}(V) < 0$ for any holomorphic line subbundle $V \subset E$. Hence, we need to determine the possible holomorphic line subbundles of the trivial bundle $\underline{\mathbb{C}}^2$. By the Birkhoff–Grothendieck theorem, every holomorphic line bundle on $\mathbb{C}P^1$ is of the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$ where $\mathcal{O}(k)$ is the bundle which admits a unique section with a zero, respectively pole, of order k if $k > 0$, respectively of order $-k$ if $k < 0$ [Gro57, Theorem 2.1].

Proposition 5.2.2. *Let $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1$ be the trivial holomorphic bundle and $V \subset \underline{\mathbb{C}}^2$ a holomorphic subbundle. Then* $\deg(V) \leq 0$.

Proof. Assume that there exists a holomorphic line subbundle of positive degree, which must be of the form $\mathcal{O}(k)$ with $k > 0$. We realize the inclusion map $i : \mathcal{O}(k) \hookrightarrow \underline{\mathbb{C}}^2$ as

$$i = \begin{pmatrix} a \\ b \end{pmatrix} \tag{5.19}$$

where $a, b \in \Gamma(\mathbb{C}P^1, \mathcal{O}(-k))$. Since $\mathcal{O}(k) \subset \underline{\mathbb{C}}^2$ is a holomorphic line subbundle, the sections a, b have to satisfy

$$\bar{\partial} a = \bar{\partial} b = 0 \tag{5.20}$$

with respect to the holomorphic structure $\bar{\partial}$ on $\mathcal{O}(-k)$. However, this is a contradiction as $\mathcal{O}(-k)$ is a line bundle of negative degree and hence does not admit holomorphic sections. $\square$

Let $\mathcal{P}$ be the parabolic structure on $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus D$ induced by a Fuchsian system as described in subsection 5.2.1. With the convention of equal weights, the parabolic degree is

$$\operatorname{pardeg}(V) = \deg(V) + \sum_{i=1}^{4} \gamma_i \tag{5.21}$$

where $\gamma_i = \hat{\rho}$ if V_{p_i} is the eigenline L_{p_i} (cf. 5.14) of the Fuchsian system to the positive eigenvalue $\hat{\rho}$ of A_i^u and $\gamma_i = -\hat{\rho}$ otherwise. Since $\hat{\rho} \in (0, \frac{1}{2})$, we see that for any subbundle $V \subset E$ with $\deg(V) < -1$ we already have $\operatorname{pardeg}(V) < 0$. Therefore, only the two cases $V = \underline{\mathbb{C}}$ and $V = \mathcal{O}(-1)$ have to be further studied [HH17, p. 6].

i. Assume that $V = \underline{\mathbb{C}}$. The fiber over any point $z \in \mathbb{C}P^1$ is constant. If $u \neq \{0, 1, \infty\}$ then we either have $\gamma_i < 0$ for all $i = 1, ..., 4$ or $\gamma_i > 0$ for one i if the fiber is one of the eigenspaces of the residues of the Fuchsian system. In either case, the parabolic structure is stable. For $u \in \{0, 1, \infty\}$, there exist exactly two i such that $\gamma_i > 0$ and we have semi-stability.

ii. Assume that $V = \mathcal{O}(-1)$, i.e., it is the tautological bundle on $\mathbb{C}P^1$. The fiber over any point $z \in \mathbb{C}P^1$ is given by

$$V_z = \mathbb{C} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{5.22}$$

If $u \notin \{0, 1, \infty\}$, we have $\gamma_i > 0$ for three i. If $u \in \{0, 1, \infty\}$ then we also have $\gamma_i > 0$ for three i. Hence, in both cases the parabolic structure is stable. Finally, if $u = m$ then we have $\gamma_i > 0$ for all $i = 1, ..., 4$, and the parabolic structure is stable if and only if $\hat{\rho} < \frac{1}{4}$.

We summarize these results in the following proposition.

Proposition 5.2.3. *For $u \notin \{0, 1, \infty, m\}$, the parabolic structure $\mathcal{P}$ on $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus D$ induced by the Fuchsian system $\nabla^{u,v}$ is stable. If*

- *$u \in \{0, 1, \infty\}$ then the parabolic structure is semi-stable.*
- *$u = m$ then the parabolic structure is stable if and only if $\hat{\rho} < \frac{1}{4}$. For $\hat{\rho} = \frac{1}{4}$ it is semi-stable. Otherwise, it is unstable.*

Note that stability of the parabolic bundle only imposes conditions on the weight $\hat{\rho}$ and hence (u, v) in equation (5.18) remain viable coordinates to parameterize the moduli space of stable parabolic structures and strongly Higgs fields.

5.3 Abelianization of Fuchsian systems

In the previous section, we have studied Fuchsian system on the trivial rank two bundle $E = \underline{\mathbb{C}}^2$ over the 4-punctured sphere. The eigenlines of the residues of a Fuchsian system give a filtration of the vector bundle's fibers at the punctures and induce a parabolic structure $\mathcal{P}$. In our convention, each eigenline is equipped with the same weight $\hat{\rho} \in (0, \frac{1}{2})$, which is related to the stability of a parabolic bundle by proposition 5.2.3. In particular, in the present case here, the parabolic structures are parametrized by a single complex number $u \in \mathbb{C}$ (cf. (5.14)). Since the space of strongly parabolic Higgs fields with stable parabolic structure on $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1 \setminus D$ is complex one-dimensional as well, we concluded that any Fuchsian system inducing a stable parabolic vector bundle is, up to conjugation, of the form

$$\nabla^{u,v} = \nabla^u + v\Phi^u \tag{5.23}$$

with prescribed local monodromies (5.8) and ∇^u and Φ^u are of the form (5.15) and (5.17), respectively. Hence, for fixed weight $\hat{\rho}$, the coordinates $(u, v) \in \mathbb{C}^2$ parameterize the space of stable parabolic structures and strongly parabolic Higgs fields, respectively.

In this section, we will pullback the Fuchsian system to a double cover $T^2 \to \mathbb{C}P^1$ branched over the punctures and gauge it with respect to the eigenline frame of a strongly parabolic Higgs field. Then the connection takes a convenient form and we will see that Fuchsian systems can be parameterized by flat line bundle connections on T^2. This procedure is called the abelianization of Fuchsian systems which we want to develop here. The section follows the lines of [HH17].

5.3.1 Eigenlines of the strongly parabolic Higgs field

Fix the four points $p_1 = 0, p_2 = 1, p_3 = m$ and $p_4 = \infty$ on $\mathbb{C}P^1$. Without loss of generality, we have $m \neq \{0, 1, \infty\}$. For the time being, assume that $u \neq \{0, 1, m, \infty\}$. Then the parabolic structure is stable (cf. proposition 5.2.3) and the determinant of the strongly parabolic Higgs field (5.17) is

$$\det \Phi^u = u(u-1)(m-u)\frac{dz^2}{z(z-1)(z-m)}. \tag{5.24}$$

The eigenvalues of Φ^u are given by

$$\mp\sqrt{u(u-1)(u-m)}\frac{dz}{\sqrt{z(z-1)(z-m)}} \tag{5.25}$$

which implies that the eigenlines of Φ^u are not well-defined on $\mathbb{C}P^1$. We define a double cover of $\mathbb{C}P^1$ branched over the points p_i by the algebraic equation

$$T^2 \,:\, y^2 = z(z-1)(z-m). \tag{5.26}$$

Via $z : T^2 \to \mathbb{C}P^1$, we can pullback the strongly parabolic Higgs field to T^2 where its eigenlines are well-defined. We will denote the lattice of $T^2 = \mathbb{C}/\Gamma$ by $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. Denote by $w_1, ..., w_4$ the preimages of $p_1, ..., p_4$ under z such that

$$w_1 = [0]\,,\; w_2 = \left[\frac{\tau}{2}\right],\; w_3 = \left[\frac{1+\tau}{2}\right],\; w_4 = \left[\frac{1}{2}\right] \tag{5.27}$$

are the half-lattice points of Γ. A straightforward calculation reveals that the eigenlines of the strongly parabolic Higgs field are generated by

$$\begin{pmatrix} (-1+mu)uz \mp \sqrt{u(u-1)(u-m)z(z-1)(z-m)} \\ -uz + m(-1+u+z) \end{pmatrix} =: s^{\pm}. \tag{5.28}$$

The eigenline bundles will be denoted by $L^{\pm} \to T^2$.

Proposition 5.3.1. *The eigenline bundles $L^{\pm} \to T^2$ are holomorphic line subbundles of the trivial bundle $\underline{\mathbb{C}}^2 \to T^2$ of degree -2. In particular, they are isomorphic to the point bundles (cf. definition 2.2.7)*

$$L^{\pm} = L(-3w_4) \otimes L(P^{\pm}) \to T^2 \tag{5.29}$$

where $P^{\pm} = (z^+, y^{\pm}) \in T^2$ are given by

$$z^+ = \frac{m - um}{m - u}, \quad y^{\pm} = \pm \frac{m(m-1)}{(u-m)^2} k \tag{5.30}$$

and

$$k^2 = u(u-1)(u-m). \tag{5.31}$$

Proof. The pullback of the eigenlines of Φ^u via z are line subbundles of the trivial bundle $\underline{\mathbb{C}}^2 \to T^2$. The sections $s^{\pm}$ in (5.28) span these eigenlines. Since $z : T^2 \to \mathbb{C}P^1$ has a pole of order two at w_4 and a zero of order two at w_1, the first entry of $s^{\pm}$ has a pole of order three at w_4. On the other hand, $s^{\pm}$ have a simple zero at $P^{\pm}$, respectively. As $s^{\pm}$ do not have any other poles or zeros, this shows the assertion. □

5.3.2 Abelianization coordinates

We will now compute the pullback of a Fuchsian system with respect to the eigenline frame of its strongly parabolic Higgs fields. Following the description of [HH17, section 3], the abelianization coordinates $(\chi, \alpha) \in \mathbb{C}^2$ will be derived. To avoid repetition, we sketch the computational part only at the branch point $z = 0$. For the other branch points the calculations are very similar.

Let $F = (s^+, s^-)$ be the eigenline frame of the strongly parabolic Higgs field (cf. equation (5.28)). Consider the Fuchsian system

$$\nabla^u = d + \sum_{i=1}^{3} A_i^u \frac{dz}{z - p_i} =: d + \omega \tag{5.32}$$

with connection one-form given by (5.15). With respect to the frame F, the connection one-form of ∇^u is given by

$$F^{-1}\omega F + F^{-1}dF. \tag{5.33}$$

Near the branch point $z = 0$, take a local coordinate w on T^2 such that $w^2 = z$. Then $y \sim w\sqrt{m}$. Expanding the frame F and its inverse at $w = 0$ in w yields

$$F \sim \begin{pmatrix} -w\sqrt{m}k & w\sqrt{m}k \\ m(u-1) & m(u-1) \end{pmatrix}, \quad F^{-1} \sim \begin{pmatrix} -\frac{1}{2\sqrt{m}kw} & \frac{1}{2m(u-1)} \\ \frac{1}{2\sqrt{m}kw} & \frac{1}{2m(u-1)} \end{pmatrix}. \tag{5.34}$$

Using (5.34) shows that near $z = 0$ the term $F^{-1}\omega F$ is given by

$$F^{-1}\omega F \sim \begin{pmatrix} 0 & 2\hat{\rho} \\ 2\hat{\rho} & 0 \end{pmatrix} \frac{dw}{w}. \tag{5.35}$$

For the derivative dF, we expand up the lowest order and obtain

$$dF \sim \begin{pmatrix} -t\sqrt{m} & t\sqrt{m} \\ 0 & 0 \end{pmatrix} dw \tag{5.36}$$

which implies that

$$F^{-1}dF \sim \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} dw. \tag{5.37}$$

For the other branch points the calculations are analogous. Hence, the pullback of the Fuchsian system (5.18) to T^2 via the degree two map $z : T^2 \to \mathbb{C}P^1$ has local residues at the branch points given by

$$\mathrm{Res}_{w_i}(z^*\nabla^{u,v}) = \begin{pmatrix} \frac{1}{2} & 2\hat{\rho} - \frac{1}{2} \\ 2\hat{\rho} - \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{5.38}$$

where $w_i, i = 1, ..., 4$, are the half-lattice points of the torus. Equation (5.38) suggests a new convention for the parabolic weights.

Convention II: Define the shifted parabolic weight $\rho := 2\hat{\rho} - \frac{1}{2}$. Then the condition $\hat{\rho} \in (0, \frac{1}{2})$ implies that $\rho \in (-\frac{1}{2}, \frac{1}{2})$.

To get rid of the poles at the diagonals in equation (5.38), we will tensor the pullback connection with a meromorphic line bundle connection that we introduce in the following proposition. Denote by $\wp'$ the derivative of the Weierstrass $\wp$-function.

Proposition 5.3.2. *Let $T^2 = \mathbb{C}/\Gamma$ with $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ and $w_4 = [\frac{1}{2}]$. There exists a unique flat meromorphic connection ∇^S on the point bundle $S := L(-2w_4) \to T^2$ such that*

$$\nabla^S s_{-2w_4} = \frac{1}{2}\frac{d\wp'(w - w_4)}{\wp'(w - w_4)} s_{-2w_4} \tag{5.39}$$

where s_{-2w_4} is the unique global section of the holomorphic line bundle $L(-2w_4) \to T^2$ with a pole of order two at w_4 and everywhere else holomorphic without zeros. In particular, the monodromies of ∇^S are all $-\mathrm{Id}$.

Proof. Consider the point bundle $L(-w_1 - w_2 - w_3 - w_4) \to T^2$ which admits a unique meromorphic section $s_{-w_1-w_2-w_3-w_4}$ with four simple poles at the half-lattice points. Denote by ∇ the unique connection such that $\nabla s_{-w_1-w_2-w_3-w_4} = 0$. The Weierstrass $\wp'$-function has simple zeros at the half-lattice points $w_2, ..., w_4$ and a pole of order 3 at w_1. Via $\wp'(w - w_4)$ we identify $L(-w_1 - w_2 - w_3 - w_4)$ with the point bundle $L(-4w_4)$. In particular, via this identification, there is an induced logarithmic connection on $L(-4w_4)$ such that $\nabla(\frac{1}{\wp'} s_{-4w_4}) = 0$. Now let $\tilde{\nabla}$ be a connection on $L(-2w_4)$ without poles. Notice that $L(-2w_4) \otimes L(-2w_4) = L(-4w_4)$. The difference $\nabla - \tilde{\nabla} \otimes \tilde{\nabla} = \theta$ is an endomorphism valued one-form with simple poles at the four singular points $w_1, ..., w_4$. Defining $\nabla^S = \tilde{\nabla} + \frac{\theta}{2}$, we have by construction

$$\nabla^S s_{-2w_4} = \frac{1}{2}\frac{d\wp'(w - w_4)}{\wp'(w - w_4)} s_{-2w_4} \tag{5.40}$$

which gives the first part of the proposition.

For the second part, notice that $\sqrt{\frac{1}{\wp'} s_{2w_4}}$ is a parallel frame of ∇^S. Since s_{-2w_4} is globally defined, one only needs to calculate the monodromy of $\sqrt{\wp'}$. Then the assertion follows from an application of the residue theorem. □

We now continue to analyse the pole behavior of the pullback of the Fuchsian system. We have seen that with respect to the frame $F = (s^+, s^-)$, the connection $z^*\nabla^{u,v}$ on the holomorphic rank two bundle

$$L^+ \oplus L^- \to T^2, \tag{5.41}$$

where $L^\pm$ are the eigenlines of the strongly parabolic Higgs field, has poles at the diagonals by equations (5.38). Consider the dual connection $(\nabla^S)^*$ of the connection ∇^S we introduced in proposition 5.3.2. Then $z^*\nabla^{u,v} \otimes (\nabla^S)^*$ is a meromorphic connection on

$$S^* \otimes (L^+ \oplus L^-) =: E^+ \otimes E^- \to T^2 \tag{5.42}$$

where $E^\pm \in \mathrm{Jac}(T^2)$ are holomorphic line bundles of degree zero. In fact, $E^\pm$ are the point bundles which admit a section with a simple zero at $P^\pm$ and a simple pole at w_4, respectively. Denote by σ the elliptic involution on T^2. Since $\sigma(P^+) = P^-$ (cf. (5.30)) and the half-lattice points are fixed by σ, we have $\sigma^*E^+ = E^-$. We now define a frame where the one-form of $z^*\nabla^{u,v} \otimes (\nabla^S)^*$ has smooth diagonal entries. For this, define the following function on T^2

$$t_x(w) := \pi \frac{\vartheta_1(\pi w - \pi x, \tau)}{\vartheta_1(\pi w - \pi w_4, \tau)} e^{\frac{2\pi i}{\bar\tau - \tau}(x - w_4)(w - \bar w)}. \tag{5.43}$$

Here $\vartheta_1(w, \tau)$ is the odd Jacobi theta function (cf. equation (2.59)) having a simple zero at $w = 0$ and satisfying the periodic relations (cf. proposition 2.3.1)

$$\vartheta_1(\pi w + \pi, \tau) = -\vartheta_1(\pi w, \tau), \quad \vartheta_1(\pi w + \pi\tau, \tau) = -\vartheta_1(\pi w, \tau)e^{-\pi i\tau}e^{-2\pi i w}. \tag{5.44}$$

Hence, as long as $x \notin w_4 + \Gamma$, the function $t_x(w)$ is doubly periodic in w and has a pole of order one at w_4 and a simple zero at $w = x$. Consider the new frame defined by

$$\hat F := (\frac{1}{t_x} s_{2w_4} \otimes s^+, \frac{1}{t_{-x}} s_{2w_4} \otimes s^-) \tag{5.45}$$

where $P^\pm = [\pm x]$. Notice that $\hat F$ has neither poles nor zeros. A straight forward calculation reveals that $z^*\nabla^{u,v} \otimes (\nabla^S)^*$ on $E^+ \oplus E^- \to T^2$ with respect to the frame (5.45) has smooth diagonal entries [HH17, section 3]. Therefore, after tensoring the pullback of the Fuchsian system with $(\nabla^S)^*$, it is gauge equivalent to

$${}^\rho\nabla^{\chi,\alpha} = d + \begin{pmatrix} \alpha dw - \chi d\bar w & \beta^- \\ \beta^+ & -\alpha dw + \chi d\bar w \end{pmatrix} \tag{5.46}$$

for some $\alpha \in \mathbb{C}$ and holomorphic structure χ given by

$$\chi = \frac{2\pi i}{\tau - \bar\tau}\left(x - \frac{1}{2}\right). \tag{5.47}$$

The superscript ρ in ${}^\rho\nabla^{\chi,\alpha}$ highlights its dependence on the parabolic weight of the underlying Fuchsian system.

Let $\mathcal{A}^1(T^2)$ denote the moduli space of flat line bundle connections on T^2 and $\mathcal{A}^2_\rho(\mathbb{C}P^1 \setminus \{p_1, ..., p_4\})$ the moduli space of Fuchsian systems on the 4-punctured sphere with local monodromies conjugated to

$$\begin{pmatrix} e^{2\pi i\frac{2\rho+1}{4}} & 0 \\ 0 & e^{-2\pi i\frac{2\rho+1}{4}} \end{pmatrix}. \tag{5.48}$$

The above calculations show the following theorem [HH17, Theorem 1].

Theorem 5.3.3. *Let $\rho \in (-\frac{1}{2}, \frac{1}{2})$. Away from the punctures $\{p_1, ..., p_4\}$, there is a 2:1-correspondence between an open dense set of $\mathcal{A}^1(T^2)$ and an open dense set of $\mathcal{A}^2_\rho(\mathbb{C}P^1 \setminus \{p_1, ..., p_4\})$.*

The function t_x defined in (5.43) satisfies

$$(\bar\partial + \frac{2\pi i}{\bar\tau - \tau}\left(x - \frac{1}{2}\right) d\bar w)t_x = (\bar\partial - \chi d\bar w)t_x = 0 \tag{5.49}$$

and thus identifies $E^\pm = L(P^\pm) \otimes L(-w_4)$ with

$$E^\pm \cong L(\bar\partial - \chi d\bar w). \tag{5.50}$$

Under this identification, we view the off-diagonals $\beta^\pm$ as meromorphic sections of the line bundle $L(\bar\partial \pm 2\chi d\bar w) \to T^2$. Furthermore, $\beta^\pm$ can be written down explicitly in terms of theta functions. By (5.38), the expansion of $\beta^+\beta^-$ in w at a half-lattice w_i point is

$$\beta^+\beta^- \sim \frac{\rho^2}{(w - w_i)^2} + \mathcal{O}(w^0). \tag{5.51}$$

Since $\beta^\pm \in L(\bar\partial \pm 2\chi d\bar w)$ and because of equation (5.51), a calculation reveals that they are given by [HH17, p. 11]

$$\beta^\pm(w) = \sum_{i=1}^{4} \alpha_i^\pm(x) t_{\mp 2x}(w - w_i) dw \tag{5.52}$$

where

$$\alpha_i^\pm(x) = e^{\pm\frac{4\pi i}{\bar\tau - \tau}(x - w_4)(w_i - \bar w_i)} \frac{\vartheta_1(\pi w_i \pm \pi x, \tau)}{\vartheta_1(\pi w_i \mp \pi x, \tau)} \frac{\vartheta_1'(0, \tau)}{\vartheta_1(\pm 2\pi x, \tau)}\rho. \tag{5.53}$$

Notice that if $\rho = 0$, the connection (5.46) is totally reducible.

Throughout the calculations above we have assumed that the parabolic structure u is not equal to one of the exceptional values $u \in \{0, 1, m, \infty\}$, as in this case the eigenlines of the strongly parabolic Higgs field coalesce. After identifying T^2 with its Jacobian $\mathrm{Jac}(T^2)$ via equation (5.47)

$$x \in T^2 \mapsto L(\bar\partial - \chi d\bar w), \tag{5.54}$$

we see from equation (5.30) that at these values x is a half-lattice point of T^2, i.e., the right hand side of (5.54) is a spin bundle. An asymptotic analysis shows that it is possible to extend the 2:1-correspondence of theorem 5.3.3 to the points $u \in \{0, 1, m, \infty\}$ and gives the following result [HH17, Theorem 2].

Theorem 5.3.4. *Let $\mathrm{Jac}(T^2) = \mathbb{C}/\Lambda$. The 2:1-correspondence of theorem 5.3.3 induced by the flat line bundle connection*

$$d + \alpha(\chi)dw - \chi d\bar w \tag{5.55}$$

extends to the spin bundles $\chi \in \frac{1}{2}\Lambda$ if and only if $\alpha(\chi)$ expands in a neighborhood of $\chi = \gamma \in \frac{1}{2}\Lambda$ as

$$\alpha(\chi) \sim \pm\frac{4\pi i}{\tau - \bar\tau}\frac{\mu_\gamma}{\chi - \gamma} + \bar\gamma + \textit{higher order terms in } \chi \tag{5.56}$$

where

$$\mu_\gamma = \begin{cases} \rho, & \text{if } \gamma \in \Lambda \ (\Leftrightarrow u = m) \\ 0, & \text{if } \gamma \in \frac{1}{2}\Lambda \setminus \Lambda \ (\Leftrightarrow u \in \{0,1,\infty\}). \end{cases} \tag{5.57}$$

The sign of the residue in equation (5.56) determines the stability of the parabolic structure: for positive residue, the parabolic structure is stable, while it is unstable for negative residue. For $\mu_\gamma = 0$, the parabolic structure is semi-stable.

For the final part of this subsection, we consider Fuchsian systems with unitary monodromy representation and the role of (χ, α) in this setup. The Mehta-Seshadri theorem 5.1.1 states that for every stable parabolic structure, there exists a Fuchsian system with irreducible unitary monodromy representation. The coordinate $u \in \mathbb{C}P^1$ introduced in subsection 5.2.1 parameterizes the moduli space of parabolic structures on the trivial rank two bundle $\underline{\mathbb{C}}^2$ over the 4-punctured sphere. We see from equation (5.14), that this moduli space is, via u, identified as $\mathbb{C}P^1$ with three double points at $u = 0, 1, \infty$ (see also [LS15, p. 1011]) which is doubly covered by the elliptic curve

$$k^2 = u(u-1)(u-m), \tag{5.58}$$

we had already defined in (5.31). In fact, the discussion following theorem 5.3.3 shows that the elliptic curve (5.58) is actually the Jacobian $\mathrm{Jac}(T^2)$ parameterizing the space of holomorphic structures on T^2. Therefore, theorem 5.3.3 also induces a 2:1-correspondence between the Jacobian $\mathrm{Jac}(T^2)$ and the moduli space of parabolic structures on the 4-punctured sphere. Then the Mehta-Seshadri theorem implies the existence of a real analytic section α_ρ^{MS} in the affine bundle $\mathcal{A}^1(T^2) \to \mathrm{Jac}(T^2)$ [HH17, p. 19]

$$\begin{aligned} \alpha_\rho^{MS} : \mathrm{Jac}(T^2) &\to \mathcal{A}^1(T^2) \\ [\bar\partial - \chi d\bar w] &\mapsto [d + \alpha_\rho^u(\chi) dw - \chi d\bar w] \end{aligned} \tag{5.59}$$

as long as $\chi \notin \frac{1}{2}\Lambda$, where Λ is the lattice generating $\mathrm{Jac}(T^2) \cong \mathbb{C}/\Lambda$. Here, $\alpha_\rho^u(\chi) \in \mathbb{C}$ is determined such that ${}^\rho\nabla^{\chi, \alpha_\rho^u(\chi)}$ has irreducible unitary monodromy representation.

The following lemma summarizes certain properties of the map (5.59) as shown in [HHS18, Lemma 3.1] and [HH17, Proof of theorem 4].

Lemma 5.3.5. *Let $\alpha_\rho^{MS} \in \Gamma(\mathrm{Jac}(T^2), \mathcal{A}^1(T^2))$ be the unitarizing section of (5.59) and $\chi \notin \frac{1}{2}\Lambda$ where Λ is the lattice generating the Jacobian $\mathrm{Jac}(T^2) = \mathbb{C}/\Lambda$. Let $\rho \in (-\frac{1}{2}, \frac{1}{2})$.*

i. α_ρ^u satisfies the functional properties

$$\begin{aligned} \alpha_\rho^u(\chi + \frac{2\pi i}{\tau - \bar\tau}\tau) &= \alpha_\rho^u(\chi) + \frac{2\pi i}{\tau - \bar\tau}\bar\tau \\ \alpha_\rho^u(\chi + \frac{2\pi i}{\tau - \bar\tau}) &= \alpha_\rho^u(\chi) + \frac{2\pi i}{\tau - \bar\tau}. \end{aligned} \tag{5.60}$$

ii. The section α_ρ^{MS} is odd with respect to the involution on $\mathrm{Jac}(T^2)$ sending the holomorphic structure to its dual, i.e.,

$$\alpha_\rho^u(-\chi) = -\alpha_\rho^u(\chi). \tag{5.61}$$

iii. If T^2 is a rectangular or rhombic torus, then we further have

$$\overline{\alpha_\rho^u(\chi)} = \alpha_\rho^u(\bar\chi). \tag{5.62}$$

Proof. i. Consider the flat line bundle connection given by the upper left entry of the connection ${}^{\rho}\nabla^{\chi,\alpha}$ in equation (5.46)

$$d^{\chi,\alpha} = d + \alpha dw - \chi d\bar{w}. \tag{5.63}$$

The following $\mathbb{C}^*$-gauge transformations

$$g_1 = e^{\frac{2\pi i}{\tau-\bar{\tau}}(w-\bar{w})}, \quad g_2 = e^{\frac{2\pi i}{\tau-\bar{\tau}}(\bar{\tau}w-\tau\bar{w})} \tag{5.64}$$

are well-defined on $T^2 = \mathbb{C}/\Gamma$ with lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. Gauging $d^{\chi,\alpha}$ with respect to g_1 and g_2, we obtain

$$\begin{aligned} d^{\chi,\alpha}.g_1 &= d + (\alpha + \frac{2\pi i}{\tau-\bar{\tau}})dw - (\chi + \frac{2\pi i}{\tau-\bar{\tau}})d\bar{w} \\ d^{\chi,\alpha}.g_2 &= d + (\alpha + \frac{2\pi i}{\tau-\bar{\tau}}\bar{\tau})dw - (\chi + \frac{2\pi i}{\tau-\bar{\tau}}\tau)d\bar{w}. \end{aligned} \tag{5.65}$$

Combined with the uniqueness of α^u_ρ, these calculations give the first assertion.

ii. For the second statement, notice that ${}^{\rho}\nabla^{\chi,\alpha}$ and ${}^{\rho}\nabla^{-\chi,-\alpha}$ give rise to the same underlying Fuchsian system on the 4-punctured sphere. Therefore, α^{MS}_ρ must be odd with respect to the involution which sends the holomorphic structure to its dual. On the universal covering of $\mathrm{Jac}(T^2)$, this is equivalent to $\alpha^u_\rho(-\chi) = -\alpha^u_\rho(\chi)$.

iii. Symmetries on the torus T^2 exhibit symmetries of the map (5.59). On the complex conjugated torus, $\partial - \alpha^u_\rho(\chi)dw$ is a holomorphic structure. If T^2 is rectangular or rhombic then there exists an isomorphism $T^2 \cong \bar{T}^2$ by sending $w \mapsto \bar{w}$. The uniqueness of α^u_ρ implies $\overline{\alpha^u_\rho(\chi)} = \alpha^u_\rho(\bar{\chi})$.

□

5.4 Pullback of Fuchsian systems to higher genus Riemann surfaces

Our aim in this section is the construction of families of flat connections on higher genus Riemann surfaces. These families will be constructed as the associated family of the Gauß map of a conformally immersed CMC surface $f : M \to \mathbb{R}^3$ [Hit90, Proposition 1.9]. Viewing $S^2 \subset S^3$ as the fixed-point set of the involution $g \mapsto -g^{-1}$ on $\mathrm{SU}(2)$, such a family is of the form

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*. \tag{5.66}$$

Under the constant gauge

$$g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{5.67}$$

we have

$$\nabla.g = \nabla, \quad \Phi.g = -\Phi \tag{5.68}$$

and consequently $\nabla^\lambda.g = \nabla^{-\lambda}$. Thus, ∇ is reducible and splits into the direct sum of flat line bundle connections. The diagonal entries of the Higgs field are zero but $\det(\Phi) \neq 0$. Taking a local coordinate on M, it can be shown that ∇^λ is gauge equivalent to the associated family in definition 3.1.1 [CLP13, section 4] and the immersion is reobtained via the Sym-Bobenko formula (cf. theorem 3.2.1).

In this section, we will study Fuchsian systems and strongly parabolic Higgs fields which have the symmetry (5.68). After pulling such families back to a suitable cover of the 4-punctured sphere, the singularities become apparent, i.e., there exists a gauge transformation such that they vanish. In this way, well-defined families of flat connections on a higher genus Riemann surface are constructed. We show that the corresponding higher genus CMC surface in $\mathbb{R}^3$ coincides with the analytic continuitation of the surface obtained from ${}^{\rho}\nabla^{\chi,\alpha}$ (cf. equation (5.46)).

It should be noted that our aim here is to find out properties, i.e., the branch and umbilic orders, of such surfaces in the case that they exist. The existence of these surfaces where the initial torus is the 3-lobed Wente torus will be proven in the next chapter.

5.4.1 Semi-stable Fuchsian systems

Let g be the gauge in (5.67). We want to study Fuchsian systems and strongly parabolic Higgs fields on $\underline{\mathbb{C}}^2 \to \mathbb{C}P^1\backslash\{0,1,m,\infty\}$ which satisfy $\nabla.g = \nabla$ and $\Phi.g = -\Phi$. The condition $\nabla.g = \nabla$ is very restrictive and such Fuchsian systems have a simple form.

Lemma 5.4.1. *Let ∇ be a Fuchsian system such that $\nabla.g = \nabla$ where g is the gauge (5.67). Then we must have $u \in \{0,1,\infty\}$ and ∇ is gauge equivalent to*

$$\nabla = d + \frac{2\rho+1}{4}\begin{pmatrix} \frac{1}{z} + \frac{\epsilon_1}{z-1} + \frac{\epsilon_2}{z-m} & 0 \\ 0 & -\frac{1}{z} - \frac{\epsilon_1}{z-1} - \frac{\epsilon_2}{z-m} \end{pmatrix} dz \tag{5.69}$$

where $(\epsilon_1,\epsilon_2) \in \{(-1,1),(-1,-1),(1,-1)\}$ for $u \in \{0,1,\infty\}$, respectively.

Proof. By proposition 5.2.3 we know that for $u \in \{0,1,\infty\}$ the parabolic structure is semi-stable and a holomorphic line subbundle meets exactly two eigenlines. Comparing the residues and as $\nabla.g = \nabla$, the connection must gauge equivalent to the stated one. $\square$

A direct consequence of lemma 5.4.1 is the following.

Lemma 5.4.2. *Let ∇ be a Fuchsian system satisfying $\nabla.g = \nabla$ where g is the gauge (5.67). The space of strongly parabolic Higgs fields to such Fuchsian systems is complex two-dimensional. In particular, the strongly parabolic Higgs fields satisfy $\Phi.g = -\Phi$.*

Proof. Consider a Fuchsian system ∇ of the form (5.69). Since the eigenlines of the residues of ∇ lie in the kernel at the respective residues of Φ, the upper-right entry of Φ has the form $a(\frac{dz}{z} - \frac{dz}{z-p_1})$ for some $a \in \mathbb{C}^*$. Here, $p_1 \in \{1,m,\infty\}$ depends on the values of ϵ_1 and ϵ_2 in lemma 5.4.1. But this implies that the lower-left entry of Φ is of the form $b(\frac{dz}{z-p_2} - \frac{dz}{z-p_3})$ with $b \in \mathbb{C}^*$, where $p_2, p_3 \in \{1,m,\infty\}, p_2 \neq p_3$ and both p_2 and p_3 are different from p_1. Taking into account that the residues of Φ are nilpotent, we further conclude that Φ is zero at the diagonal, i.e., $\Phi.g = -\Phi$. Therefore, the space of strongly parabolic Higgs fields to a Fuchsian system satisfying $\nabla.g = \nabla$ is complex two-dimensional. $\square$

5.4.2 Pullback to higher genus Riemann surfaces

We define

$$N_q \ : \ Y^q = \frac{Z}{(Z-1)(Z-m)} \tag{5.70}$$

of genus $g = q-1$, which admits a q-fold covering

$$\pi_q : N_q \to \mathbb{C}P^1 \tag{5.71}$$

branched at four points. In terms of the coordinate z on $\mathbb{C}P^1$, we have $\pi_q^* z = Z$. We further set

$$\overset{\circ}{N}_q := N_q \setminus \pi_q^{-1}(\{0, 1, m, \infty\}). \tag{5.72}$$

The following theorem shows that the pullback of families of Fuchsian systems to N_q can be desingularised. In view of chapter 6 we will study the case $u = 1$ which is equivalent to $\chi \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$.

Theorem 5.4.3. *Let g be the gauge in (5.67). Let $\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of Fuchsian systems on the 4-punctured sphere with asymptotic in λ at $\lambda = 0$ of the form*

$$\tilde{\nabla}^\lambda = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + ... \tag{5.73}$$

where $\tilde{\nabla}$ and $\tilde{\Phi}$ are as in lemma 5.4.1 and 5.4.2 with $u = 1$, respectively. Let N_q be defined as in (5.70) and $\frac{p}{q} = \frac{2\rho+1}{4}$ with $\gcd(p, q) = 1$. Then the pullback connection of $\tilde{\nabla}^\lambda$ to N_q via π_q is gauge equivalent to a non-singular family of flat connections

$$\nabla^\lambda = \lambda^{-1}\Phi + \nabla + ... \tag{5.74}$$

such that $\nabla.g = \nabla, \Phi.g = -\Phi$ and $\det(\Phi) \neq 0$. Moreover,

i. if q is odd, then the upper right entry of Φ vanishes to order

- *$2p-1$ at the points over $z = 0$ and $z = \infty$*
- *$q - 2p - 1$ at the points over $z = 1$ and $z = m$*

while the lower left entry of Φ vanishes to order

- *$2p-1$ at the points over $z = 1$ and $z = m$.*
- *$q - 2p - 1$ at the points over $z = 0$ and $z = \infty$.*

ii. if q is even, then the upper right entry of Φ vanishes to order

- *$p-1$ at the points over $z = 0$ and $z = \infty$*
- *$\frac{q}{2} - p - 1$ at the points over $z = 1$ and $z = m$*

while the lower left entry of Φ vanishes to order

- *$p-1$ at the points over $z = 1$ and $z = m$*
- *$\frac{q}{2} - p - 1$ at the points over $z = 0$ and $z = \infty$.*

Proof. By lemma 5.4.1, the Fuchsian system $\tilde{\nabla}$ at $u = 1$ is gauge equivalent to

$$\tilde{\nabla} = d + \frac{p}{q}\begin{pmatrix} \frac{1}{z} - \frac{1}{z-1} - \frac{1}{z-m} & 0 \\ 0 & -\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-m} \end{pmatrix} dz \tag{5.75}$$

while the strongly parabolic Higgs field takes the form (cf. lemma 5.4.2)

$$\tilde{\Phi} = \begin{pmatrix} 0 & a\frac{1}{z} \\ b\left(\frac{1}{z-1} - \frac{1}{z-m}\right) & 0 \end{pmatrix} dz \tag{5.76}$$

where $a, b \in \mathbb{C}^*$. We have to differentiate between the q odd and q even case.

i. Let q be odd. Let $U \subset N_q$ be a neighborhood around $\pi_q^{-1}(0)$ with centered chart y satisfying $y^q = z$. Then the pullback of $\tilde{\nabla}^\lambda$ near $\pi_q^{-1}(0)$ is given by

$$\pi_q^*\tilde{\nabla}^\lambda = d + \begin{pmatrix} p & \lambda^{-1}\frac{a\,q}{y} \\ \lambda^{-1}\frac{b(1-m)qy^{q-1}}{m} & -p \end{pmatrix} dy + \eta(y) + ... \tag{5.77}$$

where $\eta(y)$ is a λ-independent holomorphic one-form with zero off-diagonal entries. Define the following matrix

$$h_1 = \begin{pmatrix} y^{-p} & 0 \\ 0 & y^p \end{pmatrix}. \tag{5.78}$$

On $U \setminus \pi_q^{-1}(0)$, gauging with h is well-defined and we obtain

$$\pi_q^*\tilde{\nabla}^\lambda.h = d + \lambda^{-1}\begin{pmatrix} 0 & aqy^{2p-1} \\ \frac{b(1-m)qy^{q-2p-1}}{m} & 0 \end{pmatrix} dy + h^{-1}\eta(y)h + ... \tag{5.79}$$

which is well-defined on U. Notice that $h^{-1}\eta(y)h$ is diagonal. Equation (5.79) shows the vanishing order at $z = 0$. As the upper-left entry of the Fuchsian system (5.75) has a pole of order one at $z = \infty$ with residue $\frac{p}{q}$, we can use the same gauge (5.78) at infinity to desingularize $\pi_q^*\tilde{\nabla}^\lambda$. This shows that in a local coordinate near $z = 0$ and $z = \infty$ the upper-right entry of the λ^{-1}-term in (5.79) vanishes to order $2p - 1$, while the lower-left entry vanishes to order $q - 2p - 1$. At $z = 1$ and $z = m$, the upper-left entry of the Fuchsian system has simple poles with residue $-\frac{p}{q}$. Near these points, the gauge which desingularizes $\pi_q^*\tilde{\nabla}^\lambda$ is given by h^{-1}. Hence, the vanishing orders of $z = 1$ and $z = m$ are reversed to the $z = 0$ and $z = \infty$ case.

ii. Let q be even. Consider $N_{\frac{q}{2}}$ which admits a $\frac{q}{2}$-fold covering $\pi_{\frac{q}{2}} : N_{\frac{q}{2}} \to \mathbb{C}P^1$. As in the odd case, we take a neighborhood $U \subset N_{\frac{q}{2}}$ around $\pi_{\frac{q}{2}}^{-1}(0)$ with centered chart y satisfying $y^{\frac{q}{2}} = z$. A problem that arises is that gauging with

$$h_2 = \begin{pmatrix} y^{-\frac{p}{2}} & 0 \\ 0 & y^{\frac{p}{2}} \end{pmatrix} \tag{5.80}$$

is not well-defined since p is odd. To overcome this obstacle, consider the following logarithmic line bundle connection

$$d^S = d + \frac{1}{q}\left(\frac{dz}{z} - \frac{dz}{z-1} - \frac{dz}{z-m}\right) \tag{5.81}$$

on the 4-punctured sphere. The pullback of d^S to $N_{\frac{q}{2}}$ has local monodromies -1 around each marked point $\pi_{\frac{q}{2}}^{-1}(\{0,1,m,\infty\})$. Therefore, gauging the pullback of the connection $\hat{\nabla}^\lambda \otimes d^S$ to $N_{\frac{q}{2}}$ with (5.80) is well-defined. A straightforward calculation as in the odd case shows the vanishing order of the Higgs field.

For odd q, there exists a globally defined gauge transformation H on $\overset{\circ}{N}_q$ such that near $z=0$ and $z=\infty$ (respectively $z=1$ and $z=m$), it restricts to the local gauge h defined in (5.78) (respectively h^{-1}) [Hel14, p. 646]. After tensoring the Fuchsian system with d^S as defined in (5.81), the same applies for even q but with local gauges defined by (5.80) and its inverse. By construction, we obtain a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on N_q (respectively $N_{\frac{q}{2}}$) for odd q (respectively even q) which has the properties stated in the theorem. □

Notice that the vanishing order of Φ comes in pairs at the points over $z=0, z=\infty$ and $z=1, z=m$. The reason for this is that the parabolic directions induced by the underlying Fuchsian system are the same at $z=0,\infty$ and $z=1,m$.

A necessary condition for the existence of closed CMC surfaces $f: M \to \mathbb{R}^3$ is that at the Sym-point, the monodromy of the associated family takes values in $\pm\mathrm{Id}$. The following proposition is an extension of [HHS18, p. 33].

Proposition 5.4.4. *Let ∇ be as in lemma 5.4.1 with $u=1$. Let N_q be the Riemann surface defined in (5.70) and set $\frac{p}{q} = \frac{2\rho+1}{4}$ where* $\gcd(p,q)=1$.

i. If q is odd, then the monodromy of $\pi^*\nabla$ is trivial.

ii. If q is even, then the monodromy of $\pi^*(\nabla \otimes d^S)$ is trivial, where d^S is the flat line bundle connection on the 4-punctured sphere defined in (5.81).

Proof. Again, we have to distinguish between the odd and even case.

i. Assume that q is odd. Notice that

$$\frac{dY}{Y} = \frac{1}{q}\pi^*\left(\frac{dz}{z} - \frac{dz}{z-1} - \frac{dz}{z-m}\right) \tag{5.82}$$

where Y is the global meromorphic function on N_q defined by equation (5.70). We see that

$$F_1 = \begin{pmatrix} Y^{-p} & 0 \\ 0 & Y^p \end{pmatrix} \tag{5.83}$$

is a global meromorphic parallel frame of $\pi^*\nabla$. Therefore, the monodromy of $\pi^*\nabla$ is Id.

ii. Now let q be even and consider the Riemann surface $N_{\frac{q}{2}}$. As in theorem 5.4.3, the difference to the odd case is that

$$F_2 = \begin{pmatrix} Y^{-\frac{p}{2}} & 0 \\ 0 & Y^{\frac{p}{2}} \end{pmatrix} \tag{5.84}$$

is no longer single-valued and hence does not give rise to a globally defined parallel frame. Let d^S be the connection (5.81) and consider the pullback of $\nabla \otimes d^S$ to $N_{\frac{q}{2}}$. A straightforward calculation shows that $F_2 \otimes Y^{-\frac{1}{2}}$ is a globally well-defined parallel frame of $\pi_{\frac{q}{2}}^*(\nabla \otimes d^S)$.

□

5.4.3 Relation to the spectral data of a CMC torus in $\mathbb{R}^3$

In the final part of this chapter, we will relate the spectral data of CMC tori in $\mathbb{R}^3$ with certain initial conditions to the abelianization coordinates (χ, α) which parameterize an open dense set of the moduli space of Fuchsian systems on the 4-punctured sphere.

Consider the connection ${}^\rho\nabla^{\chi,\alpha}$ given in (5.46), which was derived from the pullback of a Fuchsian system. For $\rho = 0$, this connection is totally reducible. We want to show that in a neighborhood of $\rho = 0$ we still obtain a $\mathbb{C}^*$-family of flat connections on T^2 and relate the analytic continuation of this surface to N_q defined in (5.70). This would allow us to parameterize families of flat rank two connections on higher genus Riemann surfaces by holomorphic structures of flat line bundle connections on a torus. The following theorem is an adjustment of [HHS18, Theorem 3.2] and parts of our proof are motivated by it. Since the lattice of the 3-lobed Wente torus is rhombic (cf. equation (4.117), we will restrict to such cases.

Theorem 5.4.5. *Let $\rho \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$ and let $\lambda : \Sigma \to D_{1+\epsilon}$ be a double covering of a disk of radius $(1+\epsilon)$ branched at finitely many points. Let $T^2 = \mathbb{C}/\Gamma$ with rhombic lattice spanned by $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ where $\mathrm{Re}(\tau) = \frac{1}{2}$. Define the paths*

$$l_1 = \{\left[t\frac{\tau}{2}\right] \mid t \in [0,1]\}, \quad l_2 = \{\left[\frac{1}{2} + t\frac{\tau}{2}\right] \mid t \in [0,1]\}. \tag{5.85}$$

Let $\chi : \Sigma \to Jac(T^2) \cong \mathbb{C}/\Lambda$, where $\Lambda = \frac{2\pi i}{\tau - \bar{\tau}}(\mathbb{Z} + \tau\mathbb{Z})$, be an odd map with respect to the holomorphic involution σ on Σ. Consider the lift

$$d^{\chi,\alpha} = d - \chi d\bar{w} + \alpha dw \tag{5.86}$$

of χ to the moduli space of flat line bundle connections on T^2 satisfying the following properties:

1. *$\chi(0) \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$ is a half-lattice point of $\mathrm{Jac}(T^2)$.*
2. *$\alpha(\xi)$ has a first order pole at $\xi = \lambda^{-1}(0)$.*
3. *$\alpha(\xi)$ has a first order pole satisfying the condition (5.56) at every $\xi_s \in \Sigma$ with $\chi(\xi_s) \in \Lambda$ and no further singularities.*
4. *for all $\xi \in \lambda^{-1}(S^1)$ we have $\alpha(\xi) = \alpha^u_\rho(\chi(\xi))$ where $\alpha^u_\rho(\chi(\xi)) \in \mathbb{C}$ is determined by the map in (5.59)*

Then there exists a $\mathbb{C}^*$-family of flat connections

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^* \tag{5.87}$$

on $\underline{\mathbb{C}}^2 \to T^2 \setminus (l_1 \cup l_2)$ which is unitary along $\lambda \in S^1$ and satisfies $\nabla.g = \nabla$ and $\Phi.g = -\Phi$ where $g = \mathrm{diag}(i, -i)$. Moreover, if

5. *there exists a point ξ_{sym} over $\lambda_{\mathrm{sym}} \in S^1$ such that $\chi(\xi_{\mathrm{sym}}) \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$ and, with respect to the spectral parameter λ, we have $\partial_\lambda \chi(\lambda_{\mathrm{sym}}) = 0$*

then there exists a well-defined CMC immersion

$$f : T^2 \setminus (l_1 \cup l_2) \to \mathbb{R}^3 \tag{5.88}$$

with spectral data (Σ, χ, α).

Proof. Let $z : T^2 \to \mathbb{C}P^1$ be the projection map branched over the four points $\{0, 1, m, \infty\}$. We denote the preimages of these points by $w_1, ..., w_4$ in the same way as in equation (5.27). Let σ be the hyperelliptic involution on Σ. As χ and α are odd with respect to σ, the family ${}^{\rho}\nabla^{\chi(\xi),\alpha(\xi)}$ defines a $\mathbb{C}^*$-family $\tilde{\nabla}^{\lambda}$ of flat connections on $\mathbb{C}P^1 \setminus \{p_1, ..., p_4\}$. By item (1), we have $\chi(0) \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$, which corresponds to the semi-stable parabolic structure $u = 1$ (cf. equation (5.57)). By lemma 5.4.1 and 5.4.2 we can assume without loss of generality that

$$\tilde{\nabla}^{\lambda} = \hat{\Phi}^{\lambda} + \hat{\nabla} \tag{5.89}$$

where $\hat{\nabla}$ is as in lemma 5.4.1 with $u = 1$. By lemma 5.4.2, the family of strongly parabolic Higgs fields $\hat{\Phi}^{\lambda}$ is given by

$$\hat{\Phi}^{\lambda} = \begin{pmatrix} 0 & a(\lambda)\frac{dz}{z} \\ b(\lambda)\left(\frac{dz}{z-1} - \frac{dz}{z-m}\right) & \end{pmatrix}. \tag{5.90}$$

Recall that the connection ${}^{\rho}\nabla^{\chi,\alpha}$ was derived in section 5.3 by gauging the Fuchsian system with the eigenlines of the strongly parabolic Higgs fields. In the present case, the eigenline frame of $\hat{\Phi}^{\lambda}$ is

$$F = \begin{pmatrix} \sqrt{a(\lambda)(m-1)z(z-1)(z-m)} & -\sqrt{a(\lambda)(m-1)z(z-1)(z-m)} \\ \sqrt{b(\lambda)}z(m-1) & \sqrt{b(\lambda)}z(m-1) \end{pmatrix}. \tag{5.91}$$

This implies that the diagonal terms of $\tilde{\nabla}^{\lambda}.F$ in (5.89) are completely determined by the strongly parabolic Higgs fields $\hat{\Phi}^{\lambda}$. Since $\alpha(\xi)$ has a first order pole at $\xi = 0$ by item (2), we see that both $a(\lambda)$ and $b(\lambda)$ in (5.90) must have a first order pole at $\lambda = 0$. In particular, since χ and α are odd in ξ, the Higgs fields $\hat{\Phi}^{\lambda}$ are odd in λ. Therefore, the asymptotic expansion of $\tilde{\nabla}^{\lambda}$ in λ at $\lambda = 0$ is of the form

$$\tilde{\nabla}^{\lambda} = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + ... \tag{5.92}$$

where $\tilde{\nabla}$ and $\tilde{\Phi}$ are as in lemma 5.4.1 and 5.4.2, respectively. Pulling this family back to T^2, we obtain a $\mathbb{C}^*$-family ∇^{λ} on $T^2 \setminus \{w_1, ..., w_4\}$ with the same asymptotic as in (5.92). By item (3), there exist gauge transformations such that ${}^{\rho}\nabla^{\chi,\alpha}$ extends to the points where $\chi \in \frac{1}{2}\Lambda$. Furthermore, for $\rho \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$, the connection ${}^{\rho}\nabla^{\chi(\xi),\alpha(\xi)}$ is generically irreducible and reducible if and only if $L(\bar{\partial} - \chi d\bar{w}) \in \mathrm{Jac}(T^2)$ is a spin bundle. The fourth condition implies that the family is unitary along $\lambda \in S^1$ and by the arguments of [Hel14, p. 641] we obtain a well-defined family of flat connections on $T^2 \setminus \{w_1, ..., w_4\}$ of the form

$$\nabla^{\lambda} = \lambda^{-1}\Phi + \nabla - \lambda\Phi^* \tag{5.93}$$

which is unitary along $\lambda \in S^1$ with respect to a λ-independent hermitian metric and satisfies $\nabla.g = \nabla$, $\Phi.g = -\Phi$ where $g = \mathrm{diag}(i, -i)$. Since $T^2 \setminus (l_1 \cup l_2) \subset T^2 \setminus \{w_1, ..., w_4\}$, we also obtain a family on $T^2 \setminus (l_1 \cup l_2)$ by restriction.

Item (5) ensures that ∇^{λ} has trivial monodromy at the Sym-point and the right asymptotic behavior. To validate this, we need to show that at $\lambda_{\mathrm{sym}} \in S^1$ the underlying Fuchsian system has trivial monodromy. At the points where χ is a half-lattice point of the lattice generating $\mathrm{Jac}(T^2)$, the connection ${}^{\rho}\nabla^{\chi,\alpha}$ is gauge equivalent to a reducible. In particular, at $\lambda_{\mathrm{sym}} \in S^1$ where $\chi(\lambda_{\mathrm{sym}}) \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$, we have, as above, a semi-stable parabolic structure with $u(\lambda_{\mathrm{sym}}) = 1$. A straight forward calculation using the eigenline

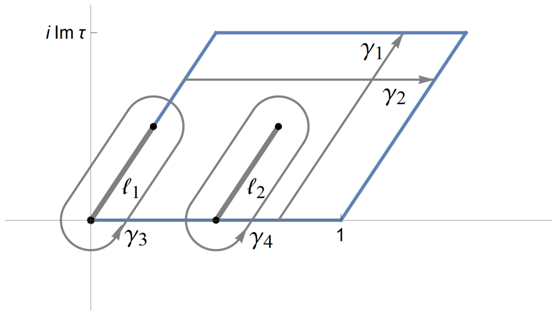

Figure 5.1: Lattice of T^2 with cuts l_1 and l_2. The paths $\gamma_i, i = 1, ..., 4$, generating the homology basis of $T^2 \setminus (l_1 \cup l_2)$ are also shown.

frame of the parabolic Higgs field as shown in [HHS18, p. 25] yields that the underlying Fuchsian system is gauge equivalent to

$$\hat{\nabla} = d + \frac{p}{q}\begin{pmatrix} \frac{1}{z} - \frac{1}{z-1} - \frac{1}{z-m} & 0 \\ 0 & -\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-m} \end{pmatrix} dz. \tag{5.94}$$

Now it remains to prove that the pullback of (5.94) to T^2 has trivial monodromy on $T^2 \setminus (l_1 \cup l_2)$. By the Seifert-Van Kampen theorem, the first fundamental group $\pi_1(T^2 \setminus (l_1 \cup l_2))$ is generated by the generators of $\pi_1(T^2)$ and paths encircling l_1 and l_2. Consider the paths $\gamma_i, i = 1, ..., 4$, as indicated in figure 5.1. Let s be parallel with respect to the line bundle connection $d - \frac{p}{q}z^*(\frac{dz}{z} - \frac{dz}{z-1} - \frac{dz}{z-m}))$. For example, we have for the path γ_3

$$\int_{\gamma_3} \frac{ds}{s} = \int_{z(\gamma_3)} \frac{p}{q}\left(\frac{dz}{z} - \frac{dz}{z-1} - \frac{dz}{z-m}\right) = 0 \tag{5.95}$$

since the closed path $z(\gamma_3)$ encircles the two points $z = 0$ and $z = 1$ on $\mathbb{C}P^1$. A similar consideration for the other 3 curves shows that the pullback of the Fuchsian system has trivial monodromy.

Lastly, the parabolic structure $u : \mathrm{Jac}(T^2) \to \mathbb{C}$ (cf. equation (5.58)) composed with χ is a well-defined map on the spectral curve, which is also well-defined on $\mathbb{C}$ since χ is odd with respect to the hyperelliptic involution. Taking the derivative with respect to λ at $\lambda = 1$ implies that $\partial_\lambda u(1) = 0$. Moreover, since the connection is unitary at $\lambda = 1$, the derivative of the anti-holomorphic structure at $\lambda = 1$ vanishes as well. Hence, the derivative of the monodromy of ∇^λ along any generator γ_i at $\lambda = 1$ vanishes. □

Remark: Notice that we actually only need a double cover of the disc $D_{1+\epsilon}$ of radius $(1 + \epsilon)$ in theorem 5.4.5. This is because as long as we can ensure unitarity along S^1 with a simple pole at $\lambda = 0$, then the family is gauge equivalent to one which extends holomorphically to $\lambda = \infty$ with a simply pole by the Schwarzian reflection lemma [Hel14, proof of Theorem 6].

Recall that we pulled back a family of Fuchsian systems $\tilde{\nabla}^\lambda$ on $\mathbb{C}P^1 \setminus \{0, 1, m, \infty\}$ to a $\mathbb{C}^*$-family of flat connections $\pi_q^*\tilde{\nabla}^\lambda$ to N_q in theorem 5.4.3. We can apply the same arguments as in theorem 5.4.5, which shows that the family is gauge equivalent to one of the form

$$\nabla^\lambda = \lambda^{-1}\Phi + \nabla - \lambda\Phi^* \tag{5.96}$$

which is unitary along $\lambda \in S^1$ with respect to a fixed hermitian metric. Then we have two $\mathbb{C}^*$-families of flat connections parameterized by the underlying family of Fuchsian systems $\tilde{\nabla}^\lambda$:

i. The pullback of $\tilde{\nabla}^\lambda$ to $T^2 \setminus (l_1 \cup l_2)$ as described theorem 5.4.5.

ii. The pullback of $\tilde{\nabla}^\lambda$ to N_q as described in theorem 5.4.3.

The link between the two families on $T^2 \setminus (l_1 \cup l_2)$ and N_q is given in the following proposition.

Proposition 5.4.6. *Assume that* $\frac{2\rho+1}{4} = \frac{p}{q} \in (0, \frac{1}{2})$ *with* $\gcd(p,q) = 1$ *and assume that there exists a point* $\lambda_{\mathrm{sym}} \in S^1$ *such that* $\chi(\lambda_{sym}) \in \frac{\pi i}{\tau - \bar{\tau}} + \Lambda$ *and* $\partial_\lambda \chi(\lambda_{\mathrm{sym}}) = 0$. *The analytic continuation of the surface*

$$\hat{f} : T^2 \setminus (l_1 \cup l_2) \to \mathbb{R}^3 \tag{5.97}$$

in theorem 5.4.5, parameterized by the spectral data (Σ, χ, α), *coincides with the surface*

$$f : N_q \to \mathbb{R}^3 \tag{5.98}$$

obtained from 5.4.3.

Proof. Consider the associated families $\nabla^\lambda_{T^2}$ and $\nabla^\lambda_{N_q}$ on $T^2 \setminus (l_1 \cup l_2)$ and N_q, respectively. The pushforward of these families to $\mathbb{C}P^1 \setminus z(l_1 \cup l_2)$, where $z : T^2 \to \mathbb{C}P^1$ is the double covering, is gauge equivalent to a family of Fuchsian systems $\tilde{\nabla}^\lambda$. Without loss of generality, we can assume that all three families are unitary along $\lambda \in S^1$ with respect to the same hermitian metric. Moreover, as they have the same asymptotic in λ at $\lambda = 0$, the λ-dependent gauges between the associated families extend holomorphically to $\lambda = 0$. By unitarity along $\lambda \in S^1$, they also extend holomorphically to $\lambda = \infty$, which implies that they must be constant in λ. Hence, the analytic continuation of the surface $\hat{f}$ agrees with f. □

It is well known that the entires of the λ^{-1} part of the family of flat connections determine the umbilic and branch order of the corresponding surface [FKR06, section 4.3]. Hence, we obtain from theorem 5.4.3 the following result.

Theorem 5.4.7. *Let the conditions of proposition 5.4.6 be satisfied. The surface*

$$f : N_q \to \mathbb{R}^3 \tag{5.99}$$

is a compact and branched CMC surface in $\mathbb{R}^3$. *Over the four branch points* $\{0, 1, m, \infty\}$ *on* $\mathbb{C}P^1$, *the surface has umbilic branch points.*

i. If q is odd, then the genus of f is $g = q - 1$. The surface branches with order $2p - 1$ at the points over $z = 0$ and $z = \infty$ and with order $q - 2p - 1$ at the points over $z = 1$ and $z = m$. The umbilic order is $2p - 1$ at the points over $z = 1$ and $z = m$ and $q - 2p - 1$ at the points over $z = 0$ and $z = \infty$.

ii. If q is even, then the genus of f is $g = \frac{q}{2} - 1$. The surface branches with order $p - 1$ at the points over $z = 0$ and $z = \infty$ and with order $\frac{q}{2} - p - 1$ at the points over $z = 1$ and $z = m$. The umbilic order is $p - 1$ at the points over $z = 1$ and $z = m$ and $\frac{q}{2} - p - 1$ at the points over $z = 0$ and $z = \infty$.

Similar results have been obtained in [HHS18, HHS15], where the initial surfaces admit an additional $\mathbb{Z}_l$-symmetry, $l \geq 2$, induced by the hyperelliptic involution. The surfaces constructed in these papers are unbranched at certain values of ρ and have some of the symmetries of the Lawson (k,l)-surfaces.

The surfaces constructed here are branched in any case. At $\rho = \frac{g-1}{2g+2} \in (0, \frac{1}{2})$, we have $q - 2p - 1 = 0$ (respectively $\frac{q}{2} - p - 1 = 0$) and therefore, we can accomplish that the surface is no longer branched at $z = 1$ and $z = m$. However, the points $z = 0$ and $z = \infty$ remain branched nonetheless.

It should be mentioned that theorem 5.4.7 does not imply the existence of the map $f : N_q \to \mathbb{R}^3$. All we have done so far is to prove properties of such surfaces in the case that they exist. Their existence will be proven in the next chapter via the generalized Whitham flow.

Chapter 6

The generalized Whitham flow

Let $f : T^2 \to \mathbb{R}^3$ be a conformally immersed CMC torus. To such an immersion, we can associate a spectral curve Σ which parameterizes the eigenvalues of the monodromy of the corresponding associated family (cf. chapter 3). The existence of Σ allows to us to deform CMC tori in $\mathbb{R}^3$ by deformations on the level of the spectral data. Such deformations are called Whitham deformations, which we have discussed in section 3.4. They preserve the intrinsic closing conditions, i.e., the unitary of the associated family along the unit circle, while opening the extrinsic closing conditions. Throughout the flow it is possible to search for values where the extrinsic closing conditions are satisfied as well and we obtain compact CMC tori in $\mathbb{R}^3$ on a dense time interval. Nevertheless, the Whitham flow only allows us to flow between CMC tori, i.e., the genus of the immersion is fixed, while possibly changing the genus of the corresponding spectral curve (cf. subsection 3.4.2).

We have seen in the previous chapter that by pulling back Fuchsian systems on the 4-punctured sphere to a q-fold cover N_q of $\mathbb{C}P^1$ via $\pi_q : N_q \to \mathbb{C}P^1$ (cf. subsection 5.4.2) we obtain families of flat connections on higher genus Riemann surfaces $g \geq 2$. The eigenvalue at the residues of the underlying Fuchsian system is a real number $\rho \in (-\frac{1}{2}, \frac{1}{2})$ and changing ρ is equivalent to changing q and hence the genus of N_q. In this sense, we extend the Whitham flow by introducing an additional flow parameter ρ which changes the genus of the Riemann surface. Starting from a CMC torus T^2, i.e., $\rho = 0$, which satisfies the intrinsic and extrinsic closing conditions, we want to apply the implicit function theorem. This would show the existence of the flow for ρ in a neighborhood of zero such that the associated family satisfies all closing conditions.

The 2:1-correspondence of theorem 5.3.3 allows us to parameterize Fuchsian systems via holomorphic structures of flat line bundle connections on T^2. Consequently, we can translate the deformation of families of flat rank two connections on N_q to deformations of holomorphic structures of flat line bundle connections. Such deformations are functions on the spectral curve of the initial torus and we need to introduce suitable Banach function spaces.

6.1 Banach function spaces

In this section, we will introduce Banach function spaces of holomorphic functions on the unit circle which extend holomorphically to an open annulus of S^1. We follow the description and notation of [Tra20, section 3.6]. Let $f : S^1 \to \mathbb{C}$ be smooth function with Fourier series

$$f(\lambda) = \sum_{n=-\infty}^{\infty} a_n \lambda^n. \tag{6.1}$$

Let $\eta > 1$. We denote by $\mathbb{D}_\eta = \{\lambda \in \mathbb{C} \,:\, 0 \leq \lambda \leq \eta\}$ the closed disc of radius η and $\mathbb{A}_\eta = \{\lambda \in \mathbb{C} \,:\, \frac{1}{\eta} < \lambda < \eta\}$ the open annulus. Define the following norm

$$||f||_\eta := \sum_{n=-\infty}^{\infty} |a_n| \eta^{|n|}. \tag{6.2}$$

Let

$$\mathcal{W} := \{f : S^1 \to \mathbb{C} \,:\, ||f||_\eta < \infty\}. \tag{6.3}$$

Functions in $\mathcal{W}$ extend holomorphically to $\mathbb{A}_\eta$. One can show that $\mathcal{W}$ is a Banach algebra.

We can further decompose $\mathcal{W}$ into the following sets

$$\begin{aligned} \mathcal{W}^{<0} &= \{f = \sum_{n\in\mathbb{Z}} a_n \lambda^n \in \mathcal{W} \,:\, a_n = 0 \text{ for } n \geq 0\} \\ \mathcal{W}^{>0} &= \{f = \sum_{n\in\mathbb{Z}} a_n \lambda^n \in \mathcal{W} \,:\, a_n = 0 \text{ for } n \leq 0\}. \end{aligned} \tag{6.4}$$

Functions in $\mathcal{W}^{>0}$ and $\mathcal{W}^{<0}$ extend holomorphically to $\mathbb{D}_\eta$ and $\mathbb{C}P^1 \setminus \mathbb{D}_{\frac{1}{\eta}}$, respectively. It is useful to further define

$$\mathcal{W}^0 = \{f = \sum_{n\in\mathbb{Z}} a_n \lambda^n \in \mathcal{W} \,:\, a_n = 0 \text{ for } n \neq 0\}, \tag{6.5}$$

i.e., functions constant in λ. Then any element $f \in \mathcal{W} = \mathcal{W}^{<0} \oplus \mathcal{W}^0 \oplus \mathcal{W}^{>0}$ admits the unique decomposition of functions

$$f = f^{<0} + f^0 + f^{>0} \tag{6.6}$$

lying in the respective subsets. We will also write $\mathcal{W}^{\geq 0} = \mathcal{W}^{>0} \oplus \mathcal{W}^0$ and $\mathcal{W}^{\leq 0} = \mathcal{W}^{<0} \oplus \mathcal{W}^0$.

The Banach space $\mathcal{W}$ admits an involution

$$\begin{aligned} &* : \mathcal{W} \to \mathcal{W} \\ &f(\lambda) \mapsto f^*(\lambda) := \overline{f(\bar{\lambda}^{-1})}, \quad \sum_{n\in\mathbb{Z}} a_n \lambda^n \mapsto \sum_{n\in\mathbb{Z}} \overline{a_{-n}} \lambda^n \end{aligned} \tag{6.7}$$

for $\lambda \in S^1$. For our purpose, we make the Banach space $\mathcal{W}$ smaller and consider the subspace

$$\mathcal{W}_\mathbb{R} := \{f \in \mathcal{W} \mid f(\lambda) = f^*(\lambda^{-1})\}. \tag{6.8}$$

equipped with the same norm (6.2). Hence, functions lying in $\mathcal{W}$ are real in the sense that $\overline{f(\lambda)} = f(\overline{\lambda})$. The respective subsets $\mathcal{W}_\mathbb{R}^{<0}, \mathcal{W}_\mathbb{R}^0, \mathcal{W}_\mathbb{R}^{>0}$ are defined analogously. In particular, via the involution (6.7) we obtain an isomorphism between $\mathcal{W}_\mathbb{R}^{<0}$ and $\mathcal{W}_\mathbb{R}^{>0}$.

6.2 Flowing from the 3-lobed Wente torus

Having defined suitable Banach function spaces, we can now consider deformations of the holomorphic structure χ at the initial 3-lobed Wente torus. We will apply the implicit function theorem and show the short time existence of the generalized Whitham flow. Firstly, let us recall some definitions and properties of the 3-lobed Wente torus.

We have seen in chapter 4 that the spectral curve of the Wente tori is a hyperelliptic curve of genus 2 given by

$$\Sigma : y^2 = \lambda(\lambda - a)(\lambda - \bar{a}^{-1})(\lambda - \bar{a})(\lambda - a^{-1}), \tag{6.9}$$

where $a \in \mathbb{C}$, which admits the real structure

$$(y, \lambda) \mapsto (\bar{y}\bar{\lambda}^{-3}, \bar{\lambda}^{-1}). \tag{6.10}$$

The fact that the hyperelliptic curve Σ doubly covers two elliptic curves (cf. section 4.2)

$$\begin{aligned} \Sigma_1 &: t_1^2 = (\xi + 4)(\xi - re^{i\delta})(\xi - re^{-i\delta}) \\ \Sigma_2 &: t_2^2 = \xi(\xi - re^{i\delta})(\xi - re^{-i\delta}) \end{aligned} \tag{6.11}$$

helped us to characterize the Wente tori in terms of elliptic functions. The branch points of Σ and Σ_j are related by

$$a + a^{-1} - 2 = re^{i\delta} \tag{6.12}$$

and the closing conditions determine δ and r via the equations (4.64) and (4.82), respectively. In the following, we will use the notation of lemma 4.3.7 and denote the 3-lobed Wente torus by $T^2_{(4,3)} = \mathbb{C}/\Gamma_{(4,3)}$, where the numbers $(4, 3)$ are as in definition 4.3.1 and uniquely determine the parameters r and δ. The lattice of $T^2_{(4,3)}$ is rhombic and denoted by $\Gamma_{(4,3)} = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau = \frac{1}{2} + \frac{d_{1,1}}{2d_{2,1}}$ and $d_{i,1}$ are the first coefficients of the abelian differentials θ_i from equation (4.3). Numerically, the conformal type approximately has the value

$$\tau = \frac{1}{2} + 0.77579i. \tag{6.13}$$

As we have seen in subsection 3.3.2, the associated family of flat connections is gauge equivalent to the totally reducible connection (3.54) and is parameterized by the maps

$$\chi_0 : \Sigma \setminus \lambda^{-1}(\infty) \to \mathrm{Jac}(T^2), \quad \alpha_0 : \Sigma \setminus \lambda^{-1}(0) \to \overline{\mathrm{Jac}(T^2)} \tag{6.14}$$

where $\alpha_0(\xi) = \overline{\chi_0(\xi)}$ for all $\xi \in \lambda^{-1}(S^1)$. We will at first gather some information on the holomorphic structure χ_0.

6.2.1 χ_0 for the 3-lobed Wente torus

The general form of the holomorphic structure as the $(0, 1)$-part of the upper left entry of the associated family of flat connections has already been derived in equation (3.57). After rotating the coordinate w on $T^2_{(4,3)}$, the holomorphic structure is given by

$$d\chi_0 = \frac{1}{2(\tau - \bar{\tau})}(\theta_1 - \theta_2(\tau - \bar{\tau})) = \frac{d_{2,1}}{2}(2 - 2\lambda^2 + \lambda(\kappa + \nu) + \kappa - \nu)\frac{d\lambda}{y} \tag{6.15}$$

where θ_i are given by (4.3) and κ and ν defined by the equations (4.118).

Proposition 6.2.1. *The spin structure of the $T^2_{(4,3)}$ Wente torus is the trivial one, i.e., $c = 0$ in (3.59).*

Proof. We have $\ln \mu_1(1) = -4\pi i$ and $\ln \mu_2(1) = 0$ with respect to the rectangular lattice of the double covering $\hat{T}^2_{(4,3)}$. It follows from (4.117) that on the smaller torus $T^2_{(4,3)}$ the monodromy of the associated family is the identity matrix along both generators of $T^2_{(4,3)}$. Hence, the spin structure is trivial. □

From proposition 2.2.5, we see that the lattice $\Lambda_{(4,3)}$ for the Jacobian $\mathrm{Jac}(T^2_{(4,3)}) \cong \mathbb{C}/\Lambda_{(4,3)}$ is given by

$$\Lambda_{(4,3)} = \frac{2\pi i}{\tau - \bar{\tau}}\mathbb{Z} + \frac{2\pi i}{\tau - \bar{\tau}}\tau\mathbb{Z}. \tag{6.16}$$

At the Sym-point $\lambda = 1$, the extrinsic closing conditions imply that the holomorphic structure $\chi(1)$ should represent the trivial one 6.2.1. Indeed, since $\ln \mu_1(1) = -4\pi i$ and $\ln \mu_2(1) = 0$, we obtain that

$$\chi_0(1) = -\frac{2\pi i}{\tau - \bar{\tau}} \in \Lambda_{(4,3)} \tag{6.17}$$

which is a lattice point.

The value of χ_0 at the branch points can also easily be calculated. Recall that the hyperelliptic curve Σ has, by definition, two branch points within the unit circle which are conjugated to each other. Let λ_1 and $\lambda_2 = \bar{\lambda}_1$ denote these points. Since $\overline{\chi_0(\lambda)} = \chi_0(\bar{\lambda})$ we get

$$\begin{aligned} \chi_0(\lambda_1) &= \frac{1}{2}\left(\frac{3\pi i}{\tau - \bar{\tau}} - \pi i\right) \\ \chi_0(\lambda_2) &= \frac{1}{2}\left(\frac{3\pi i}{\tau - \bar{\tau}} + \pi i\right) \end{aligned} \tag{6.18}$$

which are different half lattice points of $\Lambda_{(4,3)}$. The image of $\mathbb{D}_1$ under a lift of χ_0 to $\mathbb{C}$ is depicted in figure 6.1.

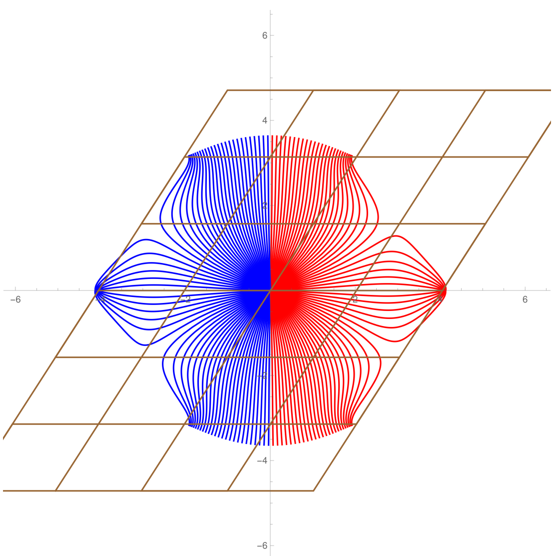

Figure 6.1: The image of $\mathbb{D}_1$ under a lift of χ_0 to $\mathbb{C}$. Every red line is the image of a line $l(t) = te^{i\theta}$ under χ_0 with $0 < t \leq 1$ and fixed angles $\theta \in (0, 2\pi]$ in steps of 0.1. The blue lines are the negative of such images. Every rectangle is bounded by half lattice points of the Jacobian of $T^2_{(4,3)}$ as defined by (6.16)

Remark: As we want the holomorphic structure in (6.15) to coincide with the one defined in (5.47), we shift χ_0 in (6.15) by adding the half-lattice point $\frac{\pi i}{\tau-\bar{\tau}} \in \frac{1}{2}\Lambda_{(4,3)}$ to it. This corresponds to tensoring the associated family of the 3-lobed Wente torus with a flat $\mathbb{Z}_2$-bundle. By unitarity, the anti-holomorphic structure attains a shift as well. With abuse of notation, we will denote the shifted χ_0 by the same symbol.

6.2.2 Deformations of the holomorphic structure

The holomorphic structure χ_0 is odd with respect to the hyperelliptic involution on Σ. Fix the coordinate λ on Σ. Since $\chi_0(\bar{\lambda}) = \overline{\chi_0(\lambda)}$ and $\chi_0(0) \neq 0$ after the shift we applied, we view χ_0 as an element in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$. We will now consider deformations of the holomorphic structure which are odd with respect to the hyperelliptic involution. By [Mir95, Proposition 1.10], every odd holomorphic function on Σ is of the form yf where f is a rational function in λ. With abuse of notation, we will write for locally defined functions $f = f \circ \lambda : U \subset \Sigma \to \mathbb{C}$ with fixed coordinate λ. Consider deformations of the form

$$\chi = \chi_0 + yf : \lambda^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda| < 1 + \eta\}) \to \mathbb{C} \tag{6.19}$$

where $\eta > 0$ and $f \in \mathcal{W}_{\mathbb{R}}^{\geq 0}$. Then χ has the same translational periods as χ_0. Moreover, it extends holomorphically to zero.

On the other hand, consider the real-analytic unitarizing section

$$\begin{aligned} \alpha^{MS} : \left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathrm{Jac}(T^2_{(4,3)}) \setminus \frac{1}{2}\Lambda_{(4,3)} &\mapsto \mathcal{A}^1(T^2_{(4,3)}) \\ (\rho, [\bar{\partial} - \chi_0 d\bar{w}]) &\mapsto [d + \alpha^u(\rho, \chi_0)dw - \chi_0 d\bar{w}] \end{aligned} \tag{6.20}$$

we had already defined (for fixed ρ) in equation (5.59). Recall from lemma 5.3.5 that $\alpha^u(\rho, \chi_0)$ satisfies

$$\begin{aligned} \alpha^u(\rho, \chi_0 + \frac{2\pi i}{\tau - \bar{\tau}}\tau) &= \alpha^u(\rho, \chi_0) + \frac{2\pi i}{\tau - \bar{\tau}}\bar{\tau} \\ \alpha^u(\rho, \chi_0 + \frac{2\pi i}{\tau - \bar{\tau}}) &= \alpha^u(\rho, \chi_0) + \frac{2\pi i}{\tau - \bar{\tau}}. \end{aligned} \tag{6.21}$$

for all $\chi_0 \notin \frac{1}{2}\Lambda_{(4,3)}$ and $\rho \in (-\frac{1}{2}, \frac{1}{2})$. Therefore, we obtain that

$$\mathfrak{a}(\rho, \chi_0) := \alpha^u(\rho, \chi_0) - \alpha_0 : U \to \mathbb{C}, \tag{6.22}$$

where α_0 is as defined in 3.3.2, is a single-valued function in an open neighborhood U of $\lambda^{-1}(S^1)$ for $\chi_0 \notin \frac{1}{2}\Lambda_{(4,3)}$. We have to show that under deformations of the holomorphic structure of the form (6.19) the function $\mathfrak{a}(\rho, \chi)$ is still well-defined. The following proof is an adjustment of [HHS18, Lemma 4.1] to our setup.

Lemma 6.2.2. *Let $f : T^2_{(4,3)} \to \mathbb{R}^3$ be the 3-lobed Wente torus with χ_0 defined by (6.15). Let $\mathcal{U} \subset \mathcal{W}_{\mathbb{R}}^{\geq 0}$ be an open neighborhood of the zero function. Let $\epsilon, \eta > 0$. Then the function $\mathfrak{a}(\rho, \chi)$, where $\chi = \chi_0 + yf$, extends to a well-defined bounded map on $\lambda^{-1}(\mathbb{A}_{1+\eta})$ for $\rho \in (-\epsilon, \epsilon)$ and $f \in \mathcal{U}$. Moreover, $\mathfrak{a}(\rho, \chi)$ is odd with respect to the hyperelliptic involution on Σ and satisfies $\overline{\mathfrak{a}(\rho, \chi)(\xi)} = \mathfrak{a}(\rho, \chi)(\bar{\xi})$.*

Proof. For the 3-lobed Wente torus we have $\chi_0(\xi) \notin \frac{1}{2}\Lambda_{(4,3)}$ for all $\xi \in \lambda^{-1}(S^1)$ by lemma 4.4.3. Hence, the underlying parabolic structure is stable and the unitarizing section (6.20) is well-defined. Define neighborhoods $I \times V \subset \mathbb{C} \times \mathbb{C}^2$ such that

$$\{0\} \times \{\chi_0(\xi) \mid \xi \in \lambda^{-1}(S^1)\} \subset I \times V. \tag{6.23}$$

Since the map (6.20) depends real-analytically on (ρ, χ), there exists a well-defined bounded map

$$\begin{aligned} \mathfrak{h} : I \times V &\to \mathbb{C} \\ (\rho, \chi_0(\xi)) &\mapsto \alpha^u(\rho, \chi_0(\xi)) \end{aligned} \tag{6.24}$$

for $\rho \in (-\epsilon, \epsilon) \subset I$ with $\epsilon > 0$. Let $f \in \mathcal{U} \subset \mathcal{W}_{\mathbb{R}}^{\geq 0}$ where $\mathcal{U}$ is a neighborhood around the zero function. Since f is sufficiently close to zero, there exists a $c > 0$ such that

$$\{(\chi_0 + yf)(\xi) \mid \xi \in \lambda^{-1}(\mathbb{A}_{1+\eta})\} \subset V. \tag{6.25}$$

and $\chi_0 + yf \notin \frac{1}{2}\Lambda_{(4,3)}$ for all $\xi \in \lambda^{-1}(\mathbb{A}_{1+\eta})$. Let $\alpha_0(\xi)$ be defined by $\alpha_0(\xi) = \overline{\chi_0(\xi)}$ for $\xi \in \lambda^{-1}(S^1)$. Then

$$\mathfrak{h}(\rho, (\chi_0 + yf)(\xi)) - \alpha_0(\xi) =: \mathfrak{a}(\rho, \chi_0 + yf)(\xi) \tag{6.26}$$

is, for all $f \in \mathcal{U}$ and $\rho \in (-\epsilon, \epsilon)$, a holomorphic map for $\xi \in \lambda^{-1}(\mathbb{A}_{1+\eta})$. Moreover, $\mathfrak{a}(\rho, \chi_0 + yf)$ is odd with respect to the hyperelliptic involution on Σ as χ_0 is. Since $\overline{\chi_0(\xi)} = \chi_0(\bar{\xi})$ and because of the uniqueness of $\alpha^u(\rho, \chi)$ (cf. item *(ii)* in lemma (5.3.5), we also have $\overline{\mathfrak{a}(\rho, \chi_0 + yf)(\xi)} = \mathfrak{a}(\rho, \chi_0 + yf)(\bar{\xi})$. □

As we also want to apply the implicit function theorem to the parameters $(r, \delta) \in \mathbb{R}^{>0} \times (0, \frac{\pi}{2})$, we will encode them as additional arguments and write $\mathfrak{a}(\rho, \mathbf{x}, \chi)$ where $\mathbf{x} = (r, \delta)$ to highlight its dependence. The values of (r, δ), where the initial closing conditions of the 3-lobed Wente torus are satisfied, are denoted by $\mathbf{x}_0 = (r_0, \delta_0)$. Let $V \subset \mathbb{R}^2$ be a neighborhood of $\mathbf{x}_0$ and $\mathcal{U}$ as above. By lemma 6.2.2 there exists a well-defined map

$$J(\rho, \mathbf{x}, f) := \mathfrak{a}(\rho, \mathbf{x}, \chi_0 + yf) : \lambda^{-1}(\mathbb{A}_{1+\nu}) \to \mathbb{C} \tag{6.27}$$

for all $(\rho, \mathbf{x}, f) \in (-\epsilon, \epsilon) \times V \times \mathcal{U}$. Since the Mehta-Seshadri section depends real-analytically on χ, we consider the derivative of J at $(\rho, \mathbf{x}, f)$ in the direction $h \in \mathcal{W}_{\mathbb{R}}^{\geq 0}$

$$d_{(\rho,\mathbf{x},f)} J(0, 0, h) = \lim_{t \to 0} \frac{\mathfrak{a}(\rho, \mathbf{x}, \chi_0 + yf + tyh) - \mathfrak{a}(\rho, \mathbf{x}, \chi_0 + yf)}{t}. \tag{6.28}$$

for all $(\rho, \mathbf{x}, f) \in (-\epsilon, \epsilon) \times V \times \mathcal{U}$. In particular, at $(\rho, \mathbf{x}, f) = (0, \mathbf{x}_0, 0)$ equation (6.28) takes the simple form

$$d_{(0,\mathbf{x}_0,0)} J(0, 0, h) = y\lambda^{-3} h(\lambda^{-1}) \tag{6.29}$$

by (6.10) and since $\overline{h(\lambda)} = h(\bar{\lambda})$. Equation (6.29) already shows an obstacle that occurs if we want to construct families of flat connections which are unitary along $\lambda \in S^1$ and have a pole of order one at $\lambda = 0$: the principle part must be controlled. Define the linear map

$$M(yf) = f^- \in \mathcal{W}_{\mathbb{R}}^{<0} \tag{6.30}$$

where f^- denotes the principal part of f. We further define

$$\mathcal{D}(\rho, \mathbf{x}, f) := M \circ d_{(\rho,\mathbf{x},f)} J(0, 0) : \mathcal{W}_{\mathbb{R}}^{\geq 0} \to \mathcal{W}_{\mathbb{R}}^{<0}. \tag{6.31}$$

At the point $(\rho, \mathbf{x}, f) = (0, \mathbf{x}_0, 0)$ we simply write

$$\mathcal{D}_0 = \mathcal{D}(0, \mathbf{x}_0, 0). \tag{6.32}$$

Lemma 6.2.3. *The operator $\mathcal{D}_0$ defined in (6.32) is injective with cokernel dimension 2.*

Proof. By equation (6.29) we have

$$\mathcal{D}_0\left(\sum_{n=0}^{\infty} a_n \lambda^n\right) = \sum_{n=0}^{\infty} a_n \lambda^{-(n+3)}. \tag{6.33}$$

Hence, we can view $\mathcal{D}_0$ as the composition of two right shift operators sending

$$\{a_0, a_1, ...\} \mapsto \{0, 0, a_0, a_1, ...\} \tag{6.34}$$

with respect to the basis $\{\lambda^i\}_{i\in\mathbb{N}}$. It is well-known that such an operator is injective and the numbers of zeros in (6.34) is the dimension of its cokernel. □

In particular, by lemma 6.2.3 the operator $\mathcal{D}_0$ fails to be an isomorphism. However, $\mathcal{W}_{\mathbb{R}}^{<0}$ admits the decomposition $\mathcal{W}_{\mathbb{R}}^{<0} = \mathrm{Im}(\mathcal{D}_0) \oplus \mathrm{coker}(\mathcal{D}_0)$ into its image and cokernel. Then $\mathcal{D}_0$ is an isomorphism when we restrict it to its image.

Proposition 6.2.4. *Let $f : T^2_{(4,3)} \to \mathbb{R}^3$ be the 3-lobed Wente torus with χ_0 defined by (6.15) and let $\mathbf{x} = (r, \delta)$. Then there exist open neighborhoods $\mathcal{V} \subset \mathbb{R}^3$ and $\mathcal{U} \subset \mathcal{W}_{\mathbb{R}}^{\geq 0}$ around $(0, \mathbf{x}_0)$ and the zero function, respectively, with a unique function*

$$f : \mathcal{V} \to \mathcal{U}, \quad (\rho, \mathbf{x}) \mapsto f(\rho, \mathbf{x}) \tag{6.35}$$

such that the anti-holomorphic structure $\mathfrak{a}(\rho, \mathbf{x}, \chi_0 + yf(\rho, \mathbf{x}))$ extends holomorphically to $\lambda = 0$ with a pole of order 3 there.

Proof. The operator $\mathcal{D}_0$ is an isomorphism when we restrict it to its image. Since the cokernel of $\mathcal{D}_0$ is spanned by $< \lambda^{-1}, \lambda^{-2} >$, we obtain from the implicit function theorem a unique smooth function

$$f : \mathcal{V} \to \mathcal{U}, \quad (\rho, \mathbf{x}) \mapsto f(\rho, \mathbf{x}) \tag{6.36}$$

in some neighborhoods $\mathcal{V} \subset (-\epsilon, \epsilon) \times \mathbb{R}^2$ of $(0, \mathbf{x}_0)$ and $\mathcal{U} \subset \mathcal{W}_{\mathbb{R}}^{\geq 0}$ of the zero function such that the map $\mathfrak{a}(\rho, \mathbf{x}, \chi_0 + yf(\rho, \mathbf{x}))$ extends holomorphically to zero with at most a pole of order 3 at $\lambda = 0$. □

6.2.3 Adjusting the Whitham equations

In order to have a well-defined family of flat connection, we need to control the pole of order three at $\lambda = 0$ of the anti-holomorphic structure in proposition 6.2.4. The terms involved in the usual Whitham flow equations (cf. section 3.4) have only simple poles at $\lambda = 0$. Therefore, we will adjust the holomorphic structure and the Whitham flow equations in order to allow for higher order poles.

We redefine $d\chi_0$ and consider the following differential on Σ with a pole of order 4 at $\lambda = \infty$

$$d\chi_0 = -\frac{d_{2,1}}{2}[2\gamma\lambda^3 + 2\lambda^2 - 2 + \nu(1-\lambda) - \kappa(1+\lambda)]\frac{d\lambda}{y} \tag{6.37}$$

such that $(\gamma, \nu, \kappa) = (0, \nu_0, \kappa_0)$ at the initial data of the 3-lobed Wente. We assume that $\gamma \in \mathbb{R}$. Moreover, we similarly redefine

$$\begin{aligned}
\theta_1 &= \frac{d\lambda}{y\lambda^2}(\gamma d_{1,1}(1-\lambda^5) + d_{1,1}(\lambda - \lambda^4) + (1-\nu)(\lambda^2 - \lambda^3)) =: \frac{d\lambda}{y\lambda^2} d_1(\lambda) \\
\theta_2 &= \frac{d\lambda}{y\lambda^2}(\gamma d_{2,1}(1+\lambda^5) + d_{2,1}(\lambda + \lambda^4) - (1+\kappa)(\lambda^2 + \lambda^3)) =: \frac{d\lambda}{y\lambda^2} d_2(\lambda).
\end{aligned} \tag{6.38}$$

Imposing the integrability conditions as in section 3.4 defines the following deformation ODE

$$2a(\lambda)\dot{d}_j(\lambda) - \dot{a}(\lambda)d_j(\lambda) = 2i\lambda a(\lambda)c_j'(\lambda) - i\lambda a'(\lambda)c_j(\lambda) - 3ia(\lambda)c_j(\lambda) \tag{6.39}$$

which differs only slightly from equation (3.67). The dot denotes the derivative with respect to ρ and the dash the derivative with respect the spectral parameter λ. The polynomials

$$\begin{aligned} c_1 &= c_{1,1}(\lambda^5+1) + c_{1,2}(\lambda^4+\lambda) + c_{1,3}(\lambda^3+\lambda) \\ c_2 &= c_{2,1}(\lambda^5-1) + c_{2,2}(\lambda^4-\lambda) + c_{2,3}(\lambda^3-\lambda) \end{aligned} \tag{6.40}$$

are defined analogously to equation (3.66).

Definition 6.2.1. *We call (6.39) the adjusted Whitham flow equations.*

The left and right hand sides of (6.39) are polynomials of degree 9. Therefore, we have 20 equations which determine the coefficients of the polynomials. The involution $\lambda \mapsto \lambda^{-1}$ on Σ reduces this to 10 equations. On the other hand, we have 13 real parameters. The extrinsic closing conditions

$$\partial_\lambda \chi_0(1) = 0, \quad \chi_0(1) \in \frac{1}{2}\Lambda_{(4,3)} \tag{6.41}$$

give two more real equation, i.e.,

$$d_2(1) = 0, \quad c_1(1) = 0. \tag{6.42}$$

Hence, the remaining parameter can be used to control the pole of order three from proposition 6.2.4. It should be noted that there is a substantial difference concerning the extrinsic closing conditions in the Whitham flow and the adjusted one. While we have seen in section 3.4 that both equations of (6.42) cannot be simultaneously satisfied throughout the flow, this is no longer true for the adjusted Whitham flow. In fact, if both of these equations are satisfied, then the non-vanishing of $c_{1,1}$ throughout the flow is a necessary conditions to have non-trivial deformations. We want to elaborate more on this matter.

Taking the derivative of χ_0 with respect to ρ at $\rho = 0$ we obtain

$$\left.\frac{d}{d\rho}\right|_{\rho=0} \chi_0 = \frac{1}{2y\lambda}\left(c_1\frac{1}{\tau-\bar{\tau}} - c_2\right) - \frac{\dot{\tau}-\dot{\bar{\tau}}}{(\tau-\bar{\tau})^2}\int \theta_1. \tag{6.43}$$

The first term of (6.43) has a pole of order 3 at infinity given by

$$\frac{1}{2y}\left(c_{1,1}\frac{1}{\tau-\bar{\tau}} - c_{2,1}\right). \tag{6.44}$$

Moreover, the right hand side of (6.43) has a pole of order 3 at $\lambda = 0$ given by

$$\frac{1}{2y}\left(c_{1,1}\frac{1}{\tau-\bar{\tau}} + c_{2,1}\right). \tag{6.45}$$

By construction, χ is holomorphic on the unit disc. Therefore, equation (6.45) must vanish. Hence, to control the pole of order 3 at $\lambda = 0$ we require

$$c_{1,1} \neq 0 \tag{6.46}$$

which is the final equation for our deformation ODE and the solution to the adjusted Whitham flow is, if it exists, unique.

Proposition 6.2.5. *There exists an unique non-trivial solution to the adjusted Whitham flow equations (6.39) accompanied to by the three equations*

$$d_2(1) = 0, \quad c_1(1) = 0, \quad c_{1,1} = h, \tag{6.47}$$

where h is a non-vanishing function.

Proof. Together with the extrinsic closing conditions, the (usual) Whitham flow equations involve 10 equations with 10 parameters. We denote the vector of parameters by

$$Y^{Wf} = (\dot{r}, \dot{\delta}, \dot{d}_{1,1}, \dot{d}_{2,1}, \dot{\nu}, \dot{\kappa}, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}). \tag{6.48}$$

As shown in [Kew15, section 4.3], after unraveling (3.67) the Whitham flow equations can be brought into the form $M^{Wf}Y^{Wf} = V^{Wf}$ where M^{Wf} is a 10×10-matrix and $V^{Wf} = (0, ..., 0, h)$, where h was a non-vanishing function as in (3.71). The first 8 rows of M^{Wf} come from the deformation ODE, i.e., the intrinsic closing condition, and the last 2 rows ensure that the extrinsic closing conditions are satisfied as well. It is not hard to see that we can similarly bring the adjusted Whitham flow equations into the form $M^{aWf}Y^{aWf} = V^{aWf}$ where

$$Y^{aWf} = (\dot{r}_1, \dot{\delta}_1, \dot{d}_{1,1}, \dot{d}_{2,1}, \dot{\nu}, \dot{\kappa}, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{1,3}, c_{2,3}, \dot{\gamma}) \tag{6.49}$$

where M^{aWf} is a 13×13-matrix and $V^{aWf} = (0, ..., 0, h)$. The first 10 rows of M^{aWf} come from the deformation ODEs (6.39) while the last 3 rows ensure that $d_2(1) = 0, c_1(1) = 0$ and $c_{1,3} \neq 0$. We want to show that $\det(M^{aWf}) \neq 0$. At the initial value we have $\gamma = 0$ and we can bring M^{aWf} into the form

$$M^{aWf} = \begin{pmatrix} M^{Wf} & B \\ 0 & C \end{pmatrix} \tag{6.50}$$

where B is a 10×3 and C a 3×3-matrix. In (6.50) the last 3 rows now represent the two equations in (6.39) coming from the highest order terms in λ and $c_{1,3} \neq 0$. A straightforward calculation shows that

$$C = \begin{pmatrix} -3 & 0 & -2d_{1,1} \\ 0 & -3 & -2d_{2,1} \\ 1 & 0 & 0 \end{pmatrix} \tag{6.51}$$

with respect to (6.49). Hence, the determinant of M^{aWf} equals

$$\det(M^{aWf}) = \det(M^{Wf})\det(C) = -\det(M^{Wf})6d_{1,1} \tag{6.52}$$

which is non-vanishing at the initial data. □

6.2.4 Opening two χ-double points

We finally have to investigate the points $\lambda \in \mathbb{D}_{1+\eta}$ where $\chi(\lambda) \in \Lambda_{(4,3)}$. From theorem 5.3.4 we know that χ has to satisfy a particular asymptotic expansion at such points. Since $|\chi_0|$ (at $\rho = 0$) is monotonically increasing from 0 to ± 1, there are two different points $\lambda_{\mathrm{dp}}^{(1)}, \lambda_{\mathrm{dp}}^{(2)} \in \mathbb{R}$ where $\chi_0(\lambda_{\mathrm{dp}}^{(i)}) \in \Lambda_{(4,3)}$. Without loss of generality we set $\lambda_{\mathrm{dp}}^{(1)} \in \mathbb{R}^{>0}$ and $\lambda_{\mathrm{dp}}^{(2)} \in \mathbb{R}^{<0}$. Opposed to χ_0, the anti-holomorphic structure α_0 is not a half-lattice point of Λ at those two points. The reason for this is that χ_0 and α_0 being lattice points in Λ implies that both eigenvalues of the monodromy of the associated family take values in ± 1. But this is impossible by the proof of lemma 4.4.3.

Definition 6.2.2. *Let $T^2_{(4,3)}$ be the 3-lobed Wente torus and $\mathrm{Jac}(T^2_{(4,3)})$ its Jacobian with lattice $\Lambda_{(4,3)}$. We call a point $\lambda \in \mathbb{C}$ where $\chi_0(\lambda) \in \Lambda_{(4,3)}$ and $\alpha_0(\lambda) \notin \Lambda_{(4,3)}$ a χ-double point.*

For $\rho \neq 0$ the asymptotics of the abelianization coordinates (χ, α) are governed by theorem 5.3.4. In particular, the constant order term of the anti-holomorphic structure in the expansion of (5.56) must be a lattice point whenever χ is. We accomplish this by opening the two χ-double points on the spectral curve which allows us to control the residue term. After a χ-double point is being opened, the genus of the spectral curve increases by 2. Moreover, we will open both χ-double points in such a way that the spectral curve still admits the involution $\lambda \mapsto \lambda^{-1}$, i.e., the coefficients of χ are all real.

Denote by

$$(y^{[2]})^2 = \lambda a^{[2]}(\lambda) \tag{6.53}$$

the genus two spectral curve of the 3-lobed Wente torus with coefficients determined by the adjusted Whitham flow equations. Define

$$\begin{aligned} y^2 = \lambda a(\lambda) = \lambda((\lambda - u_1)^2 + v_1^2)\left(\lambda^2 + \frac{1 - 2u_1\lambda}{u_1^2 + v_1^2}\right) \\ ((\lambda - u_2)^2 + v_2^2)\left(\lambda^2 + \frac{1 - 2u_2\lambda}{u_2^2 + v_2^2}\right) a^{[2]}(\lambda) \end{aligned} \tag{6.54}$$

where we assume that $u_i, v_i \in \mathbb{R}$ which implies that the spectral curve admits the involution $i : \lambda \to \lambda^{-1}$ for $\rho \neq 0$ as well. At $\rho = 0$ we have

$$u_i = \lambda^{(i)}_{\mathrm{dp}}, \quad v_i = 0. \tag{6.55}$$

and hence the right hand side of (6.54) has double roots at $\lambda^{(i)}_{\mathrm{dp}}$ and $(\lambda^{(i)}_{\mathrm{dp}})^{-1}$. Since the holomorphic structure χ should be well-defined on the genus 6 hyperelliptic curve defined by equation (6.54), we redefine the holomorphic structure as

$$d\chi_0 = \prod_{i=1}^{2} (\lambda - \beta_i)(\lambda - \zeta_i)\chi_0^{[2]}(\lambda)\frac{d\lambda}{y} \tag{6.56}$$

where $\chi_0^{[2]}(\lambda)$ is the polynomial of degree 3 given in equation (6.37). At $\rho = 0$ the new parameters ζ_i and β_i satisfy

$$\beta_i = \lambda^{(i)}_{\mathrm{dp}}, \quad \zeta_i = (\lambda^{(i)}_{\mathrm{dp}})^{-1}. \tag{6.57}$$

We investigate the space of solutions to the adjusted Whitham flow with double points. Taking the derivative of (6.39) with respect to λ at $(\rho, \lambda) = (0, \lambda^{(i)}_{\mathrm{dp}})$ and solving for c_j we obtain $c_j = 0$, i.e., the deformation vector field has a zero at the χ-double points. Moreover, we can solve for $\dot{\beta}_i$ and $\dot{\zeta}_i$ by considering the second order derivative of (6.39) with respect to λ at $(\rho, \lambda) = (0, \lambda^{(i)}_{\mathrm{dp}})$. A straightforward calculation shows that

$$\dot{\beta}_i = \dot{u}_i, \quad \dot{\zeta}_i = -\frac{\dot{u}_i}{u_i^2}. \tag{6.58}$$

We can show that the space of solutions is at least 3-dimensional.

i. With

$$\begin{aligned} \dot{u}_1 = 1, \quad \dot{a}^{[2]} = 0, \quad \dot{\beta}_1 = 1, \quad \dot{\zeta}_1 = -\frac{1}{(\lambda_{\mathrm{dp}}^{(1)})^2}, \\ \dot{\chi}_0^{[2]} = 0, \quad c_j = 0, \quad \dot{u}_2 = 0, \quad \dot{\beta}_2 = \dot{\zeta}_2 = 0 \end{aligned} \tag{6.59}$$

we directly see that both sides of equation (6.39) are identically zero. However, condition (6.46) cannot be satisfied.

ii. Similarly,

$$\begin{aligned} \dot{u}_2 = 1, \quad \dot{a}^{[2]} = 0, \quad \dot{\beta}_2 = 1, \quad \dot{\zeta}_2 = -\frac{1}{(\lambda_{\mathrm{dp}}^{(2)})^2}, \\ \dot{\chi}_0^{[2]} = 0, \quad c_j = 0, \quad \dot{u}_1 = 0, \quad \dot{\beta}_1 = \dot{\zeta}_1 = 0 \end{aligned} \tag{6.60}$$

gives another solution which does not satisfy condition (6.46).

iii. Setting

$$\begin{aligned} \dot{u}_j = 0, \quad \dot{a}^{[2]} = \dot{\hat{a}}, \quad \dot{\chi}_0^{[2]} = \dot{\hat{\chi}}_0, \quad \dot{\beta}_j = \dot{\zeta}_j = 0, \\ c_j = \prod_{i=1}^{2} (\lambda - \lambda_{\mathrm{dp}}^{(i)})(\lambda - (\lambda_{\mathrm{dp}}^{(i)})^{-1})\hat{c}_j \end{aligned} \tag{6.61}$$

implies that

$$\dot{d}_j = \prod_{i=1}^{2} (\lambda - \lambda_{\mathrm{dp}}^{(i)})(\lambda - (\lambda_{\mathrm{dp}}^{(i)})^{-1})\dot{\hat{d}}_j \tag{6.62}$$

and hence equations (6.39) are equivalent to

$$\begin{aligned} \prod_{i=1}^{2} (\lambda - \lambda_{\mathrm{dp}}^{(i)})^3(\lambda - (\lambda_{\mathrm{dp}}^{(i)})^{-1})^3(2a^{[2]}\lambda\dot{\hat{d}}_j(\lambda) - \dot{\hat{a}}(\lambda)d_j(\lambda) \\ - 2i\lambda a^{[2]}(\lambda)\hat{c}_j'(\lambda) - i\lambda(a^{[2]})'(\lambda)\hat{c}_j(\lambda) - 3ia^{[2]}(\lambda)\hat{c}_j(\lambda)) = 0, \end{aligned} \tag{6.63}$$

which is a solution of the adjusted Whitham flow equations without the χ-double point. Moreover, notice that we can guarantee that equations (6.42) and (6.46) are satisfied as well.

The third solution from above looks promising but it does not open the double points on its own. To adequately open both χ-double points, we have to introduce poles to the anti-holomorphic structure and coefficients which control the residues. For this, we introduce two additional parameters $R_i \in \mathbb{R}$ and consider the following functions

$$g_i(\lambda) := -\frac{2R_i y}{(\lambda - u_i)^2 + v_i^2} \tag{6.64}$$

on the genus 6 spectral curve which has simple poles at $u_i \pm iv_i$. Partial fraction decomposition yields

$$-\frac{2R_i y}{(\lambda - u_i)^2 + v_i^2} = \frac{iyR_i}{v_i}\left(\frac{1}{\lambda - (u_i + iv_i)} - \frac{1}{\lambda - (u_i - iv_i)}\right). \tag{6.65}$$

We now further investigate the asymptotic and ensure that it has the right asymptotics behavior at $\lambda = u_i \pm iv_i =: z_i^\pm$. y has a single zero of order 1 at $z_i^\pm$ and we write

$$y \sim y_i^\pm z_i^\pm + \mathcal{O}((z_i^\pm)^3). \tag{6.66}$$

Notice that $\underline{y_i^\pm}$ vanishes to order $\sqrt{v_i}$ at $\rho = 0$. For the moment, we will only consider z_i^+ since $z_i^- = \overline{z_i^+}$. By theorem 5.3.4, the residue terms of the anti-holomorphic structure at z_i^+ should satisfy

$$\frac{iy_i^+ R_i}{v_i}\frac{1}{z_i^+} \overset{!}{=} \pm\frac{4\pi i}{\tau - \bar{\tau}}\frac{\rho}{\chi_i^+} \tag{6.67}$$

where τ spans the rhombic lattice of the Wente torus and χ_i^+ denotes the order one term in the expansion of χ at $\lambda = z_i^+$. The $\pm$-signs on the right hand side of equation (6.67) determine the stability of the underlying parabolic structure. We calculate $\chi_i^+ = (\chi_0^+)_i + (yf)_i^+$ explicitly. We have

$$(\chi_0^+)_i = \frac{2(z_i^+ - \beta_1)(z_i^+ - \zeta_1)(z_i^+ - \beta_2)(z_i^+ - \zeta_2)\chi_0^{[2]}(z^+)}{y_i^+} z^+ \tag{6.68}$$

which vanishes to order $\sqrt{v_i}$ at $\rho = 0$. The term yf has a zero of order one at z_i^+. As $(yf)_i^+$ vanishes to order $\sqrt{v_i}\rho$ at $\rho = 0$, which is greater than 1, we will neglect it for the moment. Rewriting (6.67) implies (for the positive sign) the following equations

$$2iR_i(z_i^+ - \beta_1)(z_i^+ - \zeta_1)(z_i^+ - \beta_2)(z_i^+ - \zeta_2)\chi_0^{[2]}(z_i^+) = \frac{4\pi i}{\tau - \bar{\tau}}\rho v_i. \tag{6.69}$$

The right hand side of (6.69) is real while the left hand side has real and imaginary terms. Taking the derivative of (6.69) with respect to ρ at the initial condition $R_i = v_i = 0$ and recalling that $\chi_0^{[2]}$ is real along the real axis, we can solve for $\dot{R}_i$ and obtain

$$\begin{aligned}
\dot{R}_1 &= -\frac{4\pi i}{\tau - \bar{\tau}}\frac{\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)}}{2\,\mathrm{Im}(\chi_0^{[2]}(\lambda_{\mathrm{dp}}^{(1)}))(\lambda_{\mathrm{dp}}^{(1)} - 1)(\lambda_{\mathrm{dp}}^{(1)} + 1)(\lambda_{\mathrm{dp}}^{(1)} - \lambda_{\mathrm{dp}}^{(2)})(\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)} - 1)} \\
\dot{R}_2 &= \frac{4\pi i}{\tau - \bar{\tau}}\frac{\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)}}{2\,\mathrm{Im}(\chi_0^{[2]}(\lambda_{\mathrm{dp}}^{(2)}))(\lambda_{\mathrm{dp}}^{(2)} - 1)(\lambda_{\mathrm{dp}}^{(2)} + 1)(\lambda_{\mathrm{dp}}^{(1)} - \lambda_{\mathrm{dp}}^{(2)})(\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)} - 1)}.
\end{aligned} \tag{6.70}$$

Finally, we need an additional equation involving v_i. We require $R_i = w_i v_i$ for some non-zero constants $w_i \in \mathbb{R}$, which forces v_i to have the same vanishing order as R_i. A suitable choice of w_i allows us to control the constant order terms of the anti-holomorphic structure in the limit where the χ-double points come together. Evaluating (6.64) at $\lambda = \lambda_{\mathrm{dp}}^{(i)}$ and letting $\rho \to 0$ we obtain with $\dot{u}_i = 0$

$$g_i(\lambda_{\mathrm{dp}}^{(i)}) \overset{\rho\to 0}{=} 2iy^{[2]}(\lambda_{\mathrm{dp}}^{(i)})\left(\frac{1}{\lambda_{\mathrm{dp}}^{(2)}} - \frac{1}{\lambda_{\mathrm{dp}}^{(1)}}\right)\left(1 - \frac{1}{\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)}}\right)((\lambda_{\mathrm{dp}}^{(i)})^2 - 1)w_i \tag{6.71}$$

and we can control the value of α in the limit $\rho \to 0$ at $\lambda = \lambda_{\mathrm{dp}}^{(i)}$. Let $\alpha_0(\lambda_{\mathrm{dp}}^{(i)})$ denote the value of the anti-holomorphic structure at $\lambda_{\mathrm{dp}}^{(i)}$ for $\rho = 0$. Since the constant order term in the expansion of theorem 5.3.4 is $\overline{\chi_0(\lambda_{\mathrm{dp}}^{(i)})}$, we set

$$w_i = \frac{\alpha_0(\lambda_{\mathrm{dp}}^{(i)}) - \overline{\chi_0(\lambda_{\mathrm{dp}}^{(i)})}}{2iy^{[2]}(\lambda_{\mathrm{dp}}^{(i)})\left(\frac{1}{\lambda_{\mathrm{dp}}^{(2)}} - \frac{1}{\lambda_{\mathrm{dp}}^{(1)}}\right)\left(1 - \frac{1}{\lambda_{\mathrm{dp}}^{(1)}\lambda_{\mathrm{dp}}^{(2)}}\right)((\lambda_{\mathrm{dp}}^{(i)})^2 - 1)} \tag{6.72}$$

and obtain the original value of the anti-holomorphic structure in the limit $\rho \to 0$.

Finally, we have to investigate the stability of the underlying parabolic structure for the complex conjugated points z_i^-. Assume that equation (6.67) is already satisfied. Complex conjugation yields

$$\frac{-iy_i^- R_i}{v_i}\frac{1}{z_i^-} = \pm\overline{\frac{4\pi i}{\tau-\bar{\tau}}\frac{\rho}{\chi_i^+}} = \pm\frac{4\pi i}{\tau-\bar{\tau}}\frac{\rho}{\chi_i^-} \tag{6.73}$$

where χ_i^- is (6.68) with z_i^+ replaced by z_i^-. Therefore, the asymptotic conditions are satisfied for z_i^- as well and we see from equation (6.65) that z_i^+ is parabolically stable (respectively unstable) if and only if z_i^- is unstable (respectively stable). Bringing these results together we obtain the main lemma of the thesis.

Lemma 6.2.6. *Let $\Sigma^{[2]}$ be the genus two spectral curve (4.1) of the 3-lobed Wente torus $T^2_{(4,3)}$ and (χ_0, α_0) defined by (6.14). After opening two χ-double points as described in subsection 6.2.4, there exists a hyperelliptic curve Σ of genus 6 defined by (6.54), real numbers $\eta, \epsilon > 0$, points $z_i^+, z_i^- \in \mathbb{D}_{1+\eta}, i = 1, 2$, inside the disc of radius $1 + \eta$ and unique maps*

$$(\chi, \alpha) : \lambda^{-1}(\mathbb{D}_{1+\eta}) \subset \Sigma \to (\mathrm{Jac}(T^2_{(4,3)}), \overline{\mathrm{Jac}(T^2_{(4,3)})}) \tag{6.74}$$

satisfying the conditions of theorem 5.4.5 for every $\rho \in (-\epsilon, \epsilon)$ such that

i. the parabolic structure is stable for all $\lambda \in \mathbb{D}_{1+\eta} \setminus \{z_1^-, z_2^-\}$ and unstable at z_1^-, z_2^-.

ii. the parabolic structure is stable for all $\lambda \in \mathbb{D}_{1+\eta} \setminus \{z_1^+, z_2^-\}$ and unstable at z_1^+, z_2^-.

iii. the parabolic structure is stable for all $\lambda \in \mathbb{D}_{1+\eta} \setminus \{z_1^-, z_2^+\}$ and unstable at z_1^-, z_2^+.

iv. the parabolic structure is stable for all $\lambda \in \mathbb{D}_{1+\eta} \setminus \{z_1^+, z_2^+\}$ and unstable at z_1^+, z_2^+.

This implies the existence of the higher genus CMC surfaces from theorem 5.4.7. Moreover, for $\rho = 0$, the tuple (Σ, χ, α) is the spectral data of the 3-lobed Wente torus.

Proof. We will recall the steps to prove this theorem. It was shown in proposition 6.2.4 that we can deform the holomorphic structure in such a way that the anti-holomorphic structure has at most a pole of order 3 at $\lambda = 0$. Since the associated family of flat connection needs to have at most a simple pole at $\lambda = 0$, we further need to control the order 3 pole. We do this by using the remaining parameters on the spectral curve and adjusted the Whitham flow equation in subsection 6.2.3. By proposition 6.2.5 a solution to such deformation exist and we can ensure that the anti-holomorphic structure extends to $\lambda = 0$ with a most a pole of order 1 there.

Finally, by theorem 5.3.4 we need to control the asymptotic of the anti-holomorphic structure at the two points $\lambda^{(i)}_{\mathrm{dp}} \in \mathbb{D}_{1+\eta} \setminus S^1, i = 1, 2$ where $\chi(\lambda) \in \Lambda_{(4,3)}$. After opening both of them, the genus of the spectral curve increased from 2 to 6 (cf. subsection 6.2.4). This procedure allows us to control the residue theorem of α in such a way that (χ, α) coincide with the spectral data for the 3-lobed Wente torus at $\rho = 0$. Hence, all conditions of theorem 5.4.5 are satisfied and the corresponding surface has branch and umbilic order as described in theorem 5.4.7. □

6.3 Conclusion

A similar approach, on which some of the results obtained in this thesis were based on and motivated by, is the use of the generalized Whitham flow by L. Heller, S. Heller and N. Schmitt in [HHS18]. In this publication, the authors constructed symmetric higher genus CMC surfaces in the 3-sphere where the initial CMC tori at $\rho = 0$ are the homogenous and 2-lobed Delaunay tori. In this last section of this thesis, we would like to point out the similarities and differences to the work presented here.

Starting with the similarities, in both cases the Ansatz to the construction of higher genus CMC surfaces in S^3 and $\mathbb{R}^3$ is the same. Starting from a family of Fuchsian systems, where all eigenvalues of the connection one-forms are the same, we obtain the flow parameter ρ. The resulting higher genus CMC surface is constructed by considering the same q-fold cover $N_q \to \mathbb{C}P^1$, where the singular points of the pullback of the logarithmic connections can by gauged away. Deformations of the holomorphic structure χ_0 are of the same form $\chi = \chi_0 + yf$ (cf. (6.19)) and implicit function theorem arguments as in 6.2.4 ensure (some of) the pole orders of the anti-holomorphic structure at $\lambda = 0$, yielding the existence of such higher surfaces in the respective space-forms S^3 and $\mathbb{R}^3$.

As for the differences, the first to mention is that CMC surface in S^3 and $\mathbb{R}^3$ have different extrinsic closing conditions. For CMC surfaces in S^3, there must exist two points $\lambda_i \in S^1, i = 1, 2$, where the monodromy of the associated family is trivial, while in the $\mathbb{R}^3$ case the translational invariance of the immersion must be ensured by imposing the asymptotic expansion condition of the monodromy as shown at the end of section 3.2.

As the genus of the Wente tori's spectral curve is 2, the study of the spectral data is in general more complicated than for the homogenous and Delaunay tori in [HHS18], which have spectral genus $g = 0$ and $g = 1$, respectively. Fortunately, the symmetries on the genus two spectral curve allowed us to express hyperelliptic integrals in terms of elliptic ones, which simplified the study of closing conditions.

The surfaces constructed in [HHS18] are unbranched at all points over $\{0, 1, m, \infty\}$ if $\rho = \frac{g-1}{2g+2}$, while in the setup of this thesis we could only accomplish that the surfaces branch at at most 2 points. Essentially, this is due to the fact that $\chi \in \frac{1}{2}\Lambda$ at $\lambda = 0$ in [HHS18], where Λ is the lattice generating the Jacobian of the torus, corresponds to the case $u = m$, which implies that the underlying parabolic structure is either stable or unstable by theorem 5.3.4. Therefore, the upper right entry of the pullback of the strongly parabolic Higgs field, as shown in section 3.3 of [HHS18], vanishes to the same order at the points over $\{0, 1, m, \infty\}$. For the 3-lobed Wente torus, $\chi \in \frac{1}{2}\Lambda$ at $\lambda = 0$ corresponds to the case that $u = 1$. Hence, as the underlying parabolic structure is semi-stable and of the form (5.69), parabolic Higgs fields are of the form 5.4.2 and the branch and umbilic order come in pairs. It seems like this could be the case for any $T^2_{(m,n)}$ Wente tori (cf. section 4.3.3 for the notation) with m even, i.e., odd number of lobs, as in this case χ is the same half-lattice point of $\frac{1}{2}\Lambda_{(m,n)}$ at $\lambda = 0$ and at the Sym-point $\lambda = 1$. As the parabolic structure at the Sym-point should be semi-stable, is must also be semi-stable at $\lambda = 0$.

The most important differences lie in controlling the pole order of the anti-holomorphic structure at $\lambda = 0$ and the existence of χ-double points. The real structure on a spectral curve of genus g is given by the mapping $(y, \lambda) \mapsto (\bar{y}\bar{\lambda}^{-(g+1)}, \bar{\lambda}^{-1})$. Hence, deformations of the holomorphic structure of the form $\chi_0 \mapsto \chi_0 + yf(\lambda)$ (cf. (6.19)) map under the real structure and complex conjugation to $\alpha_0 + y\lambda^{-(g+1)}f(\lambda^{-1})$ (where we still assume that $\overline{f(\lambda)} = f(\bar{\lambda})$). Since the homogenous and Delaunay tori of [HHS18] have spectral genus 0 and 1, respectively, ensuring that α has at most a pole of order 1 at $\lambda = 0$ is, via an implicit function theorem argument in the sense of 6.2.4, trivially satisfied. On the other hand, for spectral genus $g = 2$ the term $y\lambda^{-3}$ has a pole of order 5 at $\lambda = 0$

and the same implicit theorem argument only ensures that α has at most a pole of 3 at $\lambda = 0$. Therefore, for the 3-lobed Wente torus we needed to adjust the Whitham flow equations as in subsection 6.2.3, i.e., introduce additional parameters, to give us control of the pole of order 3.

As for the existence of double points, for the homogenous and 2-lobed Delaunay tori in [HHS18], there neither exist double points in the sense of definition 3.4.1 nor χ-double points 6.2.2. However, since the 2-lobed Delaunay tori have spectral genus 1, there exists a branch point λ_{br} inside the unit circle and only at that point (and $\lambda = 0$) we have $\chi(\lambda_{\text{br}}) \in \Lambda$ (cf. lemma 2.1 in [HHS18]). This yields two possible flow directions in the generalized Whitham flow, where the sign of the residue in the asymptotic expansion of the anti-holomorphic structure yields either a stable or unstable parabolic structure [HHS18, Theorem 4.2]. An important fact to note is that since λ_{br} is a branch point of the initial genus 1 spectral curve, we also have $\alpha(\lambda_{\text{br}}) \in \Lambda$ and there is no need to increase the genus of the spectral curve by opening double points. On the other hand, as we have seen in subsection 6.2.4, for the 3-lobed Wente torus there do exist points inside the unit circle where $\chi \in \Lambda_{(4,3)}$ but $\alpha \notin \Lambda_{(4,3)}$, which is called a χ-double point. This forces us to open such points in order to control the asymptotic expansion of the anti-holomorphic structure and view such points as branch points coming from a spectral curve whose genus is higher than the initial one.

Chapter 7

Outlook

Using the generalized Whitham flow, the present work shows the existence of higher genus $g > 1$ compact CMC surfaces in $\mathbb{R}^3$ with $\mathbb{Z}_q$-symmetry and 4 branch points. The initial surface to begin flow with is the 3-lobed Wente torus $f : T^2_{(4,3)} \to \mathbb{R}^3$ and via hyperelliptic reduction, the spectral data could be characterized rather explicitly in terms of data on elliptic curves. The key properties to proof the main theorem lie in the study of χ-double points and controlling the pole order at $\lambda = 0$. Nevertheless, open questions still remain.

It would be interesting to know how we can make the surface unbranched at every point. If $\rho = \frac{g-1}{2g+2}$, the surfaces branches at 2 instead of 4 points. In order to cure the remaining two branch points, a possible Ansatz is to introduce an additional parabolic weight, i.e., not all eigenvalues of the underlying Fuchsian system are the same, which could control the branch order at the other two branch points.

Moreover, it is further interesting to study Wente tori $f : T^2_{(m,n)} \to \mathbb{R}^3$ with lobe count > 3. It seems that this would make the study ob χ-double points noticeably more complicated as, in general, higher lobe counts would mean that χ meets more half-lattice points of the lattice generating the Jacobian of $T^2_{(m,n)}$. One would need to investigate if it is possible to open multiple χ-double points similar to subsection 6.2.4.

Finally, one could study CMC tori in $\mathbb{R}^3$ with spectral genus $g > 2$. Classifying the spectral data would be more complicated and by lemma 6.2.3 we would need to control pole orders higher than 3 at $\lambda = 0$. It does not seem to be so obvious that adjusting the Whitham flow equations gives solution that could control higher order poles. However, if would be interesting to know if χ-double points still occur or if this is just a phenomena for the Wente tori.

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Chapter 8

Appendix

8.1 Estimating the parameters δ_0 and r_0

In this section we want to estimate the parameters δ_0 and r_0 using formulas from the arithmetic geometric mean. The arithmetic geometric mean is a procedure to calculate diverse complete elliptic integrals very quickly. Hereby series are defined recursively and approximates the elliptic integrals. Let $a_0 = a$ and $b_0 = b$. For the further definitions and applications we direct the reader to [Coh93, OLBC10]. Define

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \quad b_n = \sqrt{a_{n-1} b_{n-1}}. \tag{8.1}$$

We have [Was08, p. 289]

Proposition 8.1.1. *Let $a \geq b > 0$. Then*

$$\begin{aligned} b_{n-1} \leq b_n \leq a_n \leq a_{n-1} \\ 0 \leq a_n - b_n \leq \frac{1}{2}(a_{n-1} - b_{n-1}). \end{aligned} \tag{8.2}$$

The series converges and its limit, denoted by $\operatorname{agm}(a, b)$*, is*

$$\operatorname{agm}(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{8.3}$$

Notice that the arithmetic geometric mean is symmetric in its arguments $\operatorname{agm}(a, b) = \operatorname{agm}(b, a)$ since a_n and b_n have a common limit. The relationship between the arithmetic geometric mean and its relevance to complete elliptic integrals is established in the following equality

$$\frac{\pi}{2 \operatorname{agm}(a, b)} = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{a^2 \cos^2 t + b^2 \sin^2 t}}. \tag{8.4}$$

The relationship of equation (8.4) to elliptic integrals of the first kind is well know [OLBC10, p. 493]

$$K(m) = \frac{\pi}{2 \operatorname{agm}(1, \sqrt{m'})}, \quad K'(m) = \frac{\pi}{2 \operatorname{agm}(1, \sqrt{m})}. \tag{8.5}$$

Formulas for the second and third complete elliptic integral are given by [OLBC10, p. 493]

$$\begin{aligned} E(m) &= K(m)\left(a_1^2 - \sum_{n=2}^{\infty} 2^{n-1} c_n^2\right) \\ \Pi(\alpha^2, m) &= \frac{K(m)}{2}\left(2 + \frac{\alpha^2}{1-\alpha^2} \sum_{n=0}^{\infty} Q_n\right), \end{aligned} \tag{8.6}$$

respectively, where $a_0 = 1, b_0 = \sqrt{m'}, c_n = \sqrt{a_n^2 - b_n^2}, Q_0 = 1, p_0^2 = 1 - \alpha^2$ and

$$\begin{aligned} p_{n+1} &= \frac{p_n^2 + a_n b_n}{2p_n}, \quad \epsilon_n = \frac{p_n^2 - a_n b_n}{p_n^2 + a_n b_n} \\ Q_{n+1} &= \frac{1}{2} Q_n \epsilon_n. \end{aligned} \tag{8.7}$$

8.1.1 Estimating δ_0

Recall from equation (4.64) that δ_0 was determined by the equality

$$2E(m_2) = K(m_2). \tag{8.8}$$

where m_2 is as in (4.50). Using the series expansion of the complete elliptic integrals from (8.6), equation (8.8) is equivalent to

$$\frac{1}{a_1^2}\left(a_1 - \sum_{n=2}^{\infty} 2^{n-1} c_n^2\right) = \frac{1}{2}. \tag{8.9}$$

Omitting terms of order $n \geq 2$ in equation (8.9) we get the estimate

$$\delta_0 = 2 \arcsin(\sqrt{2} - 1). \tag{8.10}$$

Continuing with the series shows that the relative error for this δ_0 is about 0.7%. Note that δ_0 is, compared to r_0, fixed for all symmetric Wente tori independently of the lobe count.

8.1.2 Estimating r_0

Generically, we will only be interested in the value of r for closed doubly periodic Wente tori. Hence assume that $\ln \mu_1(1) = -\pi i m$ and $\int_{b(1)} \vartheta_1 = 2\pi i n$ for $m, n \in \mathbb{N}$. With these assumptions we have seen that the radial coordinate is implicitly given by the extrinsic closing condition via equation (4.82).

Estimating r for the 3-lobed Wente torus

Here we want to use the previously established results to estimate the value of r for the 3-lobed Wente torus which means that we set $m = -4$ and $n = 3$ in equation (4.82). First we bring Φ defined in (4.82) in a more convenient form such that it is easier to express it in terms of incomplete elliptic integrals. For the following we will fix $\delta = \delta_0$. Let γ_1 denote the path from $re^{i\delta_0}$ to $re^{-i\delta_0}$ as in figure 4.5 and γ_2 the vertical path from $re^{-i\delta_0}$ to $re^{i\delta_0}$. By the residue theorem we get

$$\int_{\gamma_1} \Phi + \int_{\gamma_2} \Phi = 2\pi i. \tag{8.11}$$

Since the first term of (8.11) equals $-\pi i \frac{m}{n}$ we obtain

$$\int_{\gamma_2} \Phi = 2\pi i (1 + \frac{m}{2n}). \tag{8.12}$$

Additionally, denote by γ_3 the path from $re^{i\delta_0}$ to $\mathrm{Re}(re^{i\delta_0})$, by γ_4 the path from $\mathrm{Re}(re^{i\delta_0})$ to infinity and by γ_5 the path from infinity to $re^{i\delta_0}$. Notice that Φ is purely real along the path γ_4. Then it follows from equation (8.12) and after using the residue theorem again that

$$\mathrm{Im}\Big[\int_{\gamma_3}\Phi\Big] = \pi(-\frac{m}{2n}-1) = -\mathrm{Im}\Big[\int_{\gamma_5}\Phi\Big]. \tag{8.13}$$

However, one quickly confirms by symmetric reasons that

$$\int_{\gamma_5}\Phi = \int_{re^{i\delta_0}}^{-\infty}\Phi. \tag{8.14}$$

Also,

$$\mathrm{Im}\Big[\int_{re^{i\delta}}^{-4}\Phi\Big] = \mathrm{Im}\Big[\int_{re^{i\delta}}^{-\infty}\Phi + \int_{-\infty}^{-4}\Phi\Big] = -\pi\frac{m}{2n}. \tag{8.15}$$

Notice that Φ is purely imaginary along the path from minus infinity to -4. Collecting all these equations we can conclude

$$\begin{aligned}\int_{\gamma_1}\Phi = 2\pi i - \int_{\gamma_2}\Phi &= 2\pi i + 2i\mathrm{Im}\Big[\int_{\gamma_3}\Phi\Big]\\ &= 2\pi i - 2i\mathrm{Im}\Big[\int_{\gamma_5}\Phi\Big]\\ &= 2\pi i - 2i\mathrm{Im}\Big[\int_{re^{i\delta_0}}^{-4}\Phi - \int_{-\infty}^{-4}\Phi\Big]\\ &= 2i\pi(1+\frac{m}{2n}) + 2\int_{-\infty}^{-4}\Phi.\end{aligned} \tag{8.16}$$

Hence we can write

$$\int_{-\infty}^{-4}\Phi = -i\int_{-\infty}^{-4}\frac{2rd\xi}{\xi\sqrt{(-4-\xi)(\xi-re^{i\delta_0})(\xi-re^{-i\delta_0})}}. \tag{8.17}$$

Let us denote $s = 16 + r^2 + 8r\cos\delta_0$. Using equations 245.06 and 361.60 of [BF13], we can express (8.17) via complete elliptic integrals of the first and third kind

$$\int_{-\infty}^{-4}\Phi = i\cdot\frac{2r}{-4s^{1/4}+s^{3/4}}\Big[-2K(m_1) + \frac{2}{1-\alpha}\Pi\Big(\frac{\alpha^2}{\alpha^2-1}, m_1\Big)\Big] \tag{8.18}$$

where α and m_1 are given by

$$\alpha = \frac{-4+\sqrt{s}}{4+\sqrt{s}}, \quad m_1 = \frac{-4-r\cos\delta_0+\sqrt{s}}{2\sqrt{s}}. \tag{8.19}$$

By equations (8.16) and (8.18) we have

$$i\cdot\frac{2r}{-4s^{1/4}+s^{3/4}}\Big[-2+\frac{2}{1-\alpha}\frac{\Pi(\frac{\alpha^2}{\alpha^2-1}, m_1)}{K(m_1)}\Big] = \frac{\pi i}{3K(m_1)}. \tag{8.20}$$

Using the arithmetic geometric mean (8.6), the left term in the brackets of the left hand side of equation (8.20) can be written as

$$-2+\frac{2}{1-\alpha}\frac{\Pi(\frac{\alpha^2}{\alpha^2-1}, m_1)}{K(m_1)} = -2+\frac{1}{1-\alpha}(2-\alpha^2\sum_{n=0}^{\infty}Q_n) \tag{8.21}$$

First we show that r_0 must be greater than four. Using the estimate for δ_0 from equation (8.10) and substituting $r = 4$ into the formulas for s, α one sees that

$$\left| \frac{2r}{-4s^{1/4} + s^{3/4}} \left[\frac{\alpha^2}{1-\alpha} \sum_{n=1}^{\infty} Q_n \right] \right| < 4 \cdot 10^{-3}. \tag{8.22}$$

The right hand side of equation (8.20) can be further estimated using the arithmetic geometric mean. Since $K(m_1) = \frac{\pi}{2\mathrm{agm}(1,m_1')}$ where $m_1 + m_1' = 1$ and $\frac{\pi}{2} < K(m_1) < \frac{\pi}{2m_1'}$ we can calculate that for $r = 4$

$$\frac{\pi}{3K(m_1)} > \frac{\sqrt{2}}{3}\sqrt{\cos\frac{\delta_0}{2} + 1}. \tag{8.23}$$

On the other hand, using the right hand side of equation (8.21) to order $n = 0$ with $r = 4$ and also (8.23) we obtain the inequality

$$\frac{3 + 2\cos\frac{\delta_0}{2}}{2\sqrt{2}\sqrt{\cos\frac{\delta_0}{2}}(1 + 2\cos\frac{\delta_0}{2})} + \mathcal{O}(10^{-3}) > \frac{\sqrt{2}}{3}\sqrt{\cos\frac{\delta_0}{2} + 1}. \tag{8.24}$$

Using the approximation (8.10) one quickly sees that the inequality is wrong and hence we must have $r > 4$ as $|\ln \mu_1(1)|$ is monotonically increasing in r by corollary 1. Next we find an upper bound for r. Using $r = \frac{9}{2}$ one obtains that the left hand side of equation (8.21) is less than $5 \cdot 10^{-3}$. A similar calculation as before but with $\frac{2}{3} > \frac{\pi}{3K(m_1)}$ then shows that $r_0 < \frac{9}{2}$. We summarize these results in the following proposition.

Proposition 8.1.2. *Equation (4.82) with $m = -4, n = 3$ and $\delta = \delta_0$ given by (8.10) is satisfied for an $r \in (4, \frac{9}{2})$.*

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