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The Mullins–Sekerka problem via the method of potentials

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Abstract

It is shown that the two-dimensional Mullins–Sekerka problem is well-posed in all subcritical Sobolev spaces $H^r(\mathbb{R})$ with $r \in (3/2, 2)$. This is the first result, where this issue is established in an unbounded geometry. The novelty of our approach is the use of the potential theory to formulate the model as an evolution problem with nonlinearities expressed by singular integral operators.

KEYWORDS

Mullins–Sekerka, parabolic smoothing, singular integrals, well-posedness

1 | INTRODUCTION

The Mullins–Sekerka problem in a bounded geometry is a moving boundary problem which appears as the gradient flow of the area functional with respect to a suitable metric on the tangent space of all oriented hypersurfaces which enclose a fixed volume [23, 38]. It describes the evolution of two domains $\Omega^+(t)$ and $\Omega^-(t)$ together with the sharp interfaces $\Gamma(t)$ that separates them in such a way that the volumes of $\Omega^\pm(t)$ are preserved and the area of $\Gamma(t)$ is decreased [15, 23, 25]. The Mullins–Sekerka problem may also be derived as a singular limit of the Cahn–Hilliard problem when the thickness of the transition layer between the phases vanishes [3, 43]. This model has been introduced by Mullins and Sekerka in [37] to study the solidification and liquidation of materials of negligible specific heat.

Most of the mathematical studies regarding this two-phase problem consider a bounded geometry with $\Omega^\pm(t)$ being open subsets of a larger domain Ω and either $\Gamma(t)$ is a compact manifold without boundary [8, 19, 21, 36] or $\Gamma(t)$ intersects the boundary $\partial\Omega$ of Ω orthogonally [2, 4, 24]. Existence results in the setting of classical solutions have been established almost simultaneously in [12, 19, 21] under the assumption that $\Gamma(0)$ is a compact $C^{k+\beta}$-hypersurface without boundary in $\mathbb{R}^n$, $\beta \in (0,1)$ and $n \geq 2$, with $k = 3$ in [12] and $k = 2$ in [19, 21]. Subsequently, the well-posedness of the Mullins–Sekerka problem for $W_p^{1+3\mu-4/p}$ initial geometries, where $1/3 + (n+3)/3p < \mu \leq 1$, was proven in the recent monograph [41], see also [28]. The existence theory in the situation with a contact angle condition of $\pi/2$ was established only recently in [2, 24]. We also refer to [21, 24, 41] where stability issues are investigated and to [8, 22, 39, 45] for numerical studies pertaining to this problem. Finally, we mention the papers [10, 11, 26, 27, 42] where weak solutions to the Mullins–Sekerka problem are studied.

In this paper, we consider the situation when the two phases are both unbounded and we restrict to the two-dimensional case. To be more precise, we assume that at each time instant $t \geq 0$ we have

$$\Omega^\pm(t) = \left\{(x,y) \in \mathbb{R}^2 \,:\, y \gtrless f(t,x)\right\} \qquad \text{and} \qquad \Gamma(t) := \{(x, f(t,x)) \,:\, x \in \mathbb{R}\},$$

where $f(t) : \mathbb{R} \to \mathbb{R}$, $t \geq 0$, is an unknown function. The same setting has been also considered in [13], where the authors establish convergence rates to a planar interface for global solutions (assuming they exist). Our goal is to establish the well-posedness of the Mullins–Sekerka problem in this unbounded regime for initial data whose regularity is close of being optimal. To be more precise, the equations of motion are described by the following system of equations

$$\left.\begin{aligned} \Delta u^{\pm}(t) &= 0 && \text{in } \Omega^{\pm}(t), \\ u^{\pm}(t) &= \kappa_{\Gamma(t)} && \text{on } \Gamma(t), \\ \nabla u^{\pm}(t) &\to 0 && \text{for } \|(x,y)\| \to \infty, \\ V(t) &= -[\partial_{\nu_{\Gamma(t)}} u(t)] && \text{on } \Gamma(t) \end{aligned}\right\} \tag{1.1a}$$

for $t > 0$. Above, $\nu_{\Gamma(t)}$, $V(t)$, and $\kappa_{\Gamma(t)}$ are the unit normal which points into $\Omega^+(t)$, the normal velocity, and the curvature of $\Gamma(t)$. Moreover,

$$[\partial_{\nu_{\Gamma(t)}} u(t)] := \partial_{\nu_{\Gamma(t)}} u^+(t) - \partial_{\nu_{\Gamma(t)}} u^-(t), \qquad t > 0,$$

represents the jump of $\nabla u(t)$ across $\Gamma(t)$ in the normal direction. The system (1.1a) is supplemented by the initial condition

$$f(0) = f_0. \tag{1.1b}$$

Before presenting our main result, we emphasize that, under suitable conditions, the interface $f(t)$ identifies at each time instant $t \geq 0$ the functions $u^{\pm}(t)$ uniquely, see Proposition 2.4. Therefore, from now on, we shall only refer to f as being a solution to Equation (1.1). A further observation is that if f is a solution to Equation (1.1) then, given $\lambda > 0$, also the function f_λ with

$$f_\lambda(t,x) := \lambda^{-1} f(\lambda^3 t, \lambda x),$$

is a solution to Equation (1.1). Since

$$\left\| \frac{d}{dx} f_\lambda(t) \right\|_\infty = \left\| \frac{d}{dx} f(\lambda^3 t) \right\|_\infty \qquad \text{and} \qquad \|f_\lambda(t)\|_{\dot H^{3/2}} = \|f(\lambda^3 t)\|_{\dot H^{3/2}},$$

where $\|\cdot\|_{\dot H^{3/2}}$ is the homogeneous Sobolev norm, we identify $\mathrm{BUC}^1(\mathbb{R})$ and $H^{3/2}(\mathbb{R})$ as critical spaces for Equation (1.1). In Theorem 1.1, we establish the well-posedness of Equation (1.1) together with a parabolic smoothing property in all subcritical Sobolev spaces $H^r(\mathbb{R})$ with $r \in (3/2, 2)$. With respect to this point, we mention that all previous existence results in the setting of classical solutions [2, 12, 19, 21, 24, 41] consider initial data with at least C^2-regularity.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $r \in (3/2, 2)$ and choose $\bar r \in (3/2, r)$. Then, given $f_0 \in H^r(\mathbb{R})$, there exists a unique maximal solution $f := f(\cdot\,; f_0)$ to (1.1) such that*

$$f \in \mathrm{C}([0,T^+), H^r(\mathbb{R})) \cap \mathrm{C}((0,T^+), H^{\bar r+1}(\mathbb{R})) \cap \mathrm{C}^1((0,T^+), H^{\bar r-2}(\mathbb{R})),$$

$$f(t) \in H^4(\mathbb{R}) \quad \text{for } t \in (0,T^+),$$

$$u^{\pm}(t) \in \mathrm{C}^2(\Omega^{\pm}(t)) \cap \mathrm{C}^1(\overline{\Omega^{\pm}(t)}) \quad \text{for } t \in (0,T^+),$$

$$\partial_{\nu_{\Gamma(t)}} u^{\pm}(t) \circ \Xi_{\Gamma(t)} = (1 + (f(t))'^2)^{-1/2} (\phi^{\pm}(t))' \quad \text{for } t \in (0,T^+) \text{ and some } \phi^{\pm}(t) \in H^2(\mathbb{R}),$$

where $T^+ = T^+(f_0) \in (0,\infty]$ is the maximal existence time and $\Xi_{\Gamma(t)} : \mathbb{R} \to \Gamma(t)$ is defined by $\Xi_{\Gamma(t)}(x) = (x, f(t,x))$. Moreover, $[(t,f_0) \mapsto f(t;f_0)]$ defines a semiflow on $H^r(\mathbb{R})$ which is smooth in the open set

$$\{(t,f_0) \,:\, f_0 \in H^r(\mathbb{R}),\ 0 < t < T^+(f_0)\} \subset \mathbb{R} \times H^r(\mathbb{R})$$

and

$$f \in C^\infty((0,T^+)\times\mathbb{R},\mathbb{R}) \cap C^\infty((0,T^+),H^k(\mathbb{R})) \quad \textit{for all } k \in \mathbb{N}. \tag{1.2}$$

In Theorem 1.1, we let $(\cdot)'$ denote the spatial derivative d/dx.

The strategy to prove Theorem 1.1 consists in several steps. To begin with, we first prove that if $f(t)$ is known and belongs to $H^4(\mathbb{R})$, then the first three equations of Equation (1.1a) identify the functions $u^\pm(t)$ uniquely, see Proposition 2.4. Furthermore, we can also represent the right side of Equation $(1.1a)_4$ in terms of certain singular integral operators which involve only the function $f(t,\cdot)$. In this way, we reformulate the problem as an evolution problem with only f as unknown, see Equation (3.1). In the proof of Proposition 2.4, we rely on potential theory and some formulas, see Lemma 2.2 (iv), that relate the derivatives of certain singular integral operator evaluated at some density β to the L_2-adjoints of these operators evaluated at β', which have been used already in the context of the Muskat problem in [14, 30]. Thanks to these formulas, we may formulate Equatin (1.1), see Equation (3.1) in Section 3.1, as an evolution problem in $H^{r-2}(\mathbb{R})$, $r \in (3/2, 2)$, with nonlinearities which are expresses as a derivative. Then, using a direct localization argument, we show in Section 3.2 that the problem is of the parabolic type by identifying the right side of Equation (3.1) as the generator of an analytic semigroup. The proof of the main result is established in Section 3.3 and relies on the quasilinear parabolic theory presented in [5, 35].

1.1 | Notation

Given Banach spaces E_1 and E_0, we define $\mathcal{L}(E_1,E_0)$ as the space of bounded linear operators from E_1 to E_0 and $\mathcal{L}(E_0) := \mathcal{L}(E_0,E_0)$. Moreover, $\mathrm{Isom}(E_1,E_0)$ is the open subset of $\mathcal{L}(E_1,E_0)$ which consists of isomorphisms and $\mathrm{Isom}(E_0) := \mathrm{Isom}(E_0,E_0)$. Furthermore, $\mathcal{L}^k_{\mathrm{sym}}(E_1,E_0)$, $k \geq 1$, is the space of k-linear, bounded, and symmetric operators $T : E_1^k \to E_0$. The set of all locally Lipschitz continuous mappings from E_1 to E_0 is denoted by $C^{1-}(E_1,E_0)$ and $C^\infty(\mathcal{O},E_0)$ is the set which consists only of smooth mappings from an open set $\mathcal{O} \subset E_1$ to E_0.

If E_1 is additionally densely embedded in E_0, we set (following [6])

$$\mathcal{H}(E_1,E_0) := \{A \in \mathcal{L}(E_1,E_0) \ : \ -A \text{ generates an analytic semigroup in } \mathcal{L}(E_0)\}.$$

Given a Banach space E, an interval $I \subset \mathbb{R}$, $n \in \mathbb{N}$, and $\gamma \in (0,1)$, we define $C^n(I,E)$ as the set of all n-times continuously differentiable functions and $C^{n+\gamma}(I,E)$ is its subset consisting of those functions which possess a locally γ-Hölder continuous nth derivative. Moreover, $\mathrm{BUC}^n(I,E)$ is the Banach space of functions with bounded and uniformly continuous derivatives up to order n and $\mathrm{BUC}^{n+\gamma}(I,E)$ denotes its subspace which consists of those functions which have a uniformly γ-Hölder continuous nth derivative. We also set $\mathrm{BUC}^\infty(I,E) = \cap_{n\in\mathbb{N}}\mathrm{BUC}^n(I,E)$. Finally, if $\Omega \subset \mathbb{R}^2$ is open and $n \in \mathbb{N}$, then $C^n(\overline{\Omega},E)$ is the set of functions defined on Ω which possess uniformly continuous derivatives up to order n.

2 | SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS

Our strategy to solve Equation (1.1) is to reformulate this model as an evolution problem for the function f only. To this end, we first solve via the method of potentials, for each given function $f \in H^4(\mathbb{R})$, the (decoupled) boundary value problems for u^+ and u^- given by the systems

$$\left.\begin{aligned} \Delta u^\pm &= 0 && \text{in } \Omega^\pm, \\ u^\pm &= \kappa_\Gamma && \text{on } \Gamma, \\ \nabla u^\pm &\to 0 && \text{for } \|(x,y)| \to \infty, \end{aligned}\right\} \tag{2.1}$$

where

$$\Omega^\pm = \{(x,y) \in \mathbb{R}^2 \ : \ y \gtrless f(x)\} \qquad \text{and} \qquad \Gamma := \partial\Omega^\pm = \{(x,f(x)) \ : \ x \in \mathbb{R}\}.$$

Below ν_Γ is the outward unit normal at Γ which points into Ω^+. The corresponding existence and uniqueness result is provided in Proposition 2.4 below. Before stating this result we first introduce some notation. Observe that Γ is the image of the diffeomorphism $\Xi_\Gamma : \mathbb{R} \to \Gamma$ defined by $\Xi(x) := (x, f(x))$ for $x \in \mathbb{R}$. Then, the pulled-back curvature $\kappa(f) := \kappa_\Gamma \circ \Xi_\Gamma$ is given by the relation

$$\kappa(f) := \kappa_\Gamma \circ \Xi_\Gamma := \left(\frac{f'}{(1+f'^2)^{1/2}} \right)' \quad \text{on } \mathbb{R}. \tag{2.2}$$

Moreover, given functions $w^\pm \in \mathrm{C}(\overline{\Omega^\pm})$, we set

$$[w](x, f(x)) := w^+(x, f(x)) - w^-(x, f(x)). \tag{2.3}$$

2.1 | Some singular integral operators

We now introduce some singular integral operators which are used when solving Equation (2.1). Given $f \in W^1_\infty(\mathbb{R})$, we set

$$\begin{aligned} \mathbb{A}(f)[\alpha](x) &:= \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{f'(x) - (\delta_{[x,s]}f)/s}{1 + \big[(\delta_{[x,s]}f)/s\big]^2}\,\frac{\alpha(x-s)}{s}\,ds, \\ \mathbb{B}(f)[\alpha](x) &:= \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{1 + f'(x)(\delta_{[x,s]}f)/s}{1 + \big[(\delta_{[x,s]}f)/s\big]^2}\,\frac{\alpha(x-s)}{s}\,ds \end{aligned} \tag{2.4}$$

for $\alpha \in L_2(\mathbb{R})$, where PV is the principal value and

$$\delta_{[x,s]}f := f(x) - f(x-s), \qquad x,\, s \in \mathbb{R}.$$

Lemma 2.1 (i) below ensures that these singular integral operators belong to $\mathcal{L}(L_2(\mathbb{R}))$. Their L_2-adjoints are given by the relations

$$\begin{aligned} \mathbb{A}(f)^*[\alpha](x) &= \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{(\delta_{[x,s]}f)/s - f'(x-s)}{1 + \big[(\delta_{[x,s]}f)/s\big]^2}\,\frac{\alpha(x-s)}{s}\,ds, \\ \mathbb{B}(f)^*[\alpha](x) &= -\frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{1 + f'(x-s)\big(\delta_{[x,s]}f\big)/s}{1 + \big[(\delta_{[x,s]}f)/s\big]^2}\,\frac{\alpha(x-s)}{s}\,ds. \end{aligned} \tag{2.5}$$

An important observation is that the operators defined in Equations (2.4) and (2.5) can be represented in terms of a family of singular integral operators $\{B^0_{n,m}(f) \,:\, n,\, m \in \mathbb{N}\}$ which we now introduce. Given $n,\, m \in \mathbb{N}$ and Lipschitz continuous mappings $a_1, \dots, a_m,\, b_1, \dots, b_n : \mathbb{R} \to \mathbb{R}$, we set

$$B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \alpha](x) := \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{\prod_{i=1}^n \big(\delta_{[x,s]}b_i\big)/s}{\prod_{i=1}^m \Big[1 + \big[\big(\delta_{[x,s]}a_i\big)/s\big]^2\Big]}\,\frac{\alpha(x-s)}{s}\,ds \tag{2.6}$$

for $\alpha \in L_2(\mathbb{R})$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous we use the short notation

$$B^0_{n,m}(f) := B_{n,m}(f, \dots, f)[f, \dots, f, \cdot]. \tag{2.7}$$

These operators have been defined in the context of the Muskat problem in [31]. It is now a straightforward consequence of Equations (2.4)–(2.7) to observe that

$$\begin{aligned}\mathbb{A}(f)[\alpha] &= f' B^0_{0,1}(f)[\alpha] - B^0_{1,1}(f)[\alpha], & \mathbb{A}(f)^*[\alpha] &= B^0_{1,1}(f)[\alpha] - B^0_{0,1}(f)[f'\alpha],\\ \mathbb{B}(f)[\alpha] &= B^0_{0,1}(f)[\alpha] + f' B^0_{1,1}(f)[\alpha], & \mathbb{B}(f)^*[\alpha] &= -B^0_{0,1}(f)[\alpha] - B^0_{1,1}(f)[f'\alpha].\end{aligned} \tag{2.8}$$

In view of the representation (2.8), several mapping properties for the operators introduced in Equations (2.4) and (2.5) can be derived from the following result.

Lemma 2.1. *Let $n, m \in \mathbb{N}$.*

(i) Let $a_1, \dots, a_m : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous mappings. Then, there exists a positive constant $C = C(n, m, \max_{i=1,\dots,m} \|a_i'\|_\infty)$ such that for all Lipschitz continuous functions $b_1, \dots, b_n : \mathbb{R} \to \mathbb{R}$ we have

$$\|B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \cdot\,]\|_{\mathcal{L}(L_2(\mathbb{R}))} \le C \prod_{i=1}^{n} \|b_i'\|_\infty.$$

Moreover, $B_{n,m} \in \mathrm{C}^{1-}(W^1_\infty(\mathbb{R})^m, \mathcal{L}^n_{\mathrm{sym}}(W^1_\infty(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))))$.

(ii) Given $k \ge 2$, it holds that $B_{n,m} \in \mathrm{C}^{1-}(H^k(\mathbb{R})^m, \mathcal{L}^n_{\mathrm{sym}}(H^k(\mathbb{R}), \mathcal{L}(H^{k-1}(\mathbb{R}))))$.

(iii) Given $r \in (3/2, 2)$, it holds that $[f \mapsto B^0_{n,m}(f)] \in \mathrm{C}^\infty(H^r(\mathbb{R}), \mathcal{L}(H^{r-1}(\mathbb{R})))$.

(iv) Let $r \in (3/2, 2)$ and $a_1, \dots, a_m \in H^r(\mathbb{R})$ be given. Then, there exists a positive constant $C = C(n, m, \max_{i=1,\dots,m} \|a_i\|_{H^r})$ such that for all $b_1, \dots, b_n \in H^r(\mathbb{R})$ we have

$$\|B_{n,m}(a_1, \dots, a_m)[b_1, \dots, b_n, \cdot\,]\|_{\mathcal{L}(H^{r-2}(\mathbb{R}))} \le C \prod_{i=1}^{n} \|b_i\|_{H^r}.$$

Proof. The property (i) is established in [31, Lemma 3.1]. The claim (ii) is proven for $k = 2$ in [34, Lemma 4.3] and the case $k \ge 3$ follows from this result via induction. Moreover, (iii) is established in [33, Appendix C] and (iv) in [29, Lemma 2.5]. □

The next lemma collects some important properties of the operators defined in Equations (2.4) and (2.5).

Lemma 2.2. *Let $\lambda \in \mathbb{R} \setminus (-1, 1)$.*

(i) If $f \in \mathrm{BUC}^1(\mathbb{R})$, then $\lambda - \mathbb{A}(f), \lambda - \mathbb{A}(f)^* \in \mathrm{Isom}(L_2(\mathbb{R}))$.

(ii) If $f \in H^r(\mathbb{R})$, $r \in (3/2, 2)$, then $\lambda - \mathbb{A}(f), \lambda - \mathbb{A}(f)^* \in \mathrm{Isom}(H^{r-1}(\mathbb{R}))$.

(iii) If $f \in H^2(\mathbb{R})$, then $\lambda - \mathbb{A}(f), \lambda - \mathbb{A}(f)^* \in \mathrm{Isom}(H^1(\mathbb{R}))$.

(iv) If $f \in H^2(\mathbb{R})$ and $\beta \in H^1(\mathbb{R})$, then $\mathbb{A}(f)^*[\beta]$ and $\mathbb{B}(f)^*[\beta]$ belong to $H^1(\mathbb{R})$ with

$$(\mathbb{A}(f)^*[\beta])' = -\mathbb{A}(f)[\beta'] \qquad \text{and} \qquad (\mathbb{B}(f)^*[\beta])' = -\mathbb{B}(f)[\beta'].$$

(v) If $f \in H^3(\mathbb{R})$, then $\lambda - \mathbb{A}(f)^* \in \mathrm{Isom}(H^2(\mathbb{R}))$.

Proof. The property (i) follows from [31, Theorem 3.5] and (ii) is established in [1, Theorem 5] and [30, Propositin 3.4]. Moreover, the claim (iii) is proven in [31, Proposition 3.6 and Lemma 3.8] and (iv) in [30, Proposition 2.3]. The assertion (v) is a consequence of (iii) and (iv). Indeed, given $f \in H^3(\mathbb{R})$, $\lambda \in \mathbb{R} \setminus (-1, 1)$, and $\alpha \in H^2(\mathbb{R})$, the properties (iii) and (iv) imply that $(\lambda - \mathbb{A}(f)^*)[\alpha] \in H^1(\mathbb{R})$ with

$$((\lambda - \mathbb{A}(f)^*)[\alpha])' = (\lambda + \mathbb{A}(f))[\alpha'] \in H^1(\mathbb{R}).$$

Hence, $(\lambda - \mathbb{A}(f)^*)[\alpha] \in H^2(\mathbb{R})$ and

$$\begin{aligned} 2\|(\lambda - \mathbb{A}(f)^*)[\alpha]\|_{H^2}^2 &\geq \|(\lambda - \mathbb{A}(f)^*)[\alpha]\|_{H^1}^2 + \|((\lambda - \mathbb{A}(f)^*)[\alpha])'\|_{H^1}^2 \\ &= \|(\lambda - \mathbb{A}(f)^*)[\alpha]\|_{H^1}^2 + \|(\lambda + \mathbb{A}(f))[\alpha']\|_{H^1}^2 \\ &\geq C\Big(\|\alpha\|_{H^1}^2 + \|\alpha'\|_{H^1}^2\Big) \\ &\geq C\|\alpha\|_{H^2}^2, \end{aligned}$$

the inequalities in the second last line of the formula (with a sufficiently small constant C independent of λ and α) being a straightforward consequence of (iii). The assertion (v) follows now from this estimate via the method of continuity [6, Proposition I.1.1.1]. □

2.2 | The solvability of the boundary value problems (2.1)

As a preliminary result, we provide in Proposition 2.3 the unique solvability of a transmission-type boundary value problem which is used to establish the uniqueness claim in Proposition 2.4.

Proposition 2.3. *Given $f \in H^3(\mathbb{R})$ and $\phi \in H^2(\mathbb{R})$, the boundary value problem*

$$\left.\begin{aligned} \Delta U^\pm &= 0 && \text{in } \Omega^\pm, \\ [U] &= 0 && \text{on } \Gamma, \\ [\partial_{\nu_\Gamma} U] &= \big((1+f'^2)^{-1/2}\phi'\big)\circ\Xi_\Gamma^{-1} && \text{on } \Gamma, \\ \nabla U^\pm &\to 0 && \text{for } \|(x,y)| \to \infty, \end{aligned}\right\} \tag{2.9}$$

has a solution (U^+, U^-) such that $U^\pm \in \mathrm{C}^2(\Omega^\pm) \cap \mathrm{C}^1(\overline{\Omega^\pm})$. Moreover, the solution is, up to an additive constant, unique.

Proof. We first prove uniqueness of solutions in the class described above. Let therefore U be a solution to the homogeneous problem associated with (2.9) (that is with $\phi = 0$). Setting $U := U_+\mathbf{1}_{\Omega^+} + U_-\mathbf{1}_{\Omega^-}$, Stokes' theorem leads us to the equation

$$\Delta U = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2).$$

Hence, U is the real part of a holomorphic function $h : \mathbb{C} \to \mathbb{C}$. Since h' is also holomorphic and $h' = \nabla U$ is bounded and vanishes for $|(x, y)| \to \infty$, it follows that $h' = 0$, meaning that U is constant in $\mathbb{R}^2$.

In order to establish the existence of solutions, we set

$$r = (r_1, r_2) = (x - s, y - f(s)) \quad \text{for } s \in \mathbb{R} \text{ and } (x, y) \in \mathbb{R}^2 \setminus \Gamma. \tag{2.10}$$

Defining $U : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ by the formula

$$U(x, y) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{r_1 + f'(s)r_2}{|r|^2}\phi(s)\,\mathrm{d}s \tag{2.11}$$

and setting $U^\pm := U|_{\Omega^\pm}$, we next show that (U^+, U^-) is a solution to (2.9) with the required properties. To start, we note that

$$U(x, y) = \int_{\mathbb{R}} K(x, y, s)\phi(s)\,\mathrm{d}s, \qquad (x, y) \in \mathbb{R}^2 \setminus \Gamma,$$

and, for every $\alpha \in \mathbb{N}^2$, we have $\partial^\alpha_{(x,y)} K(x,y,s) = O(s^{-1})$ for $|s| \to \infty$ and locally uniformly in $(x,y) \in \mathbb{R}^2 \setminus \Gamma$. This shows that U is well-defined and that integration and differentiation with respect to x and y may be commuted.

Furthermore, Equation (2.11) and [9, Lemma A.1, Lemma A.4], imply that $U^\pm \in \mathrm{C}^\infty(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm})$ with $[U] = 0$, the gradient $\nabla U = (\partial_x U, \partial_y U)$ being given by the formula

$$\nabla U(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|r|^4} (-f'(s) \;\; 1) \begin{pmatrix} 2r_1 r_2 & r_2^2 - r_1^2 \\ r_2^2 - r_1^2 & -2r_1 r_2 \end{pmatrix} \phi(s)\, ds, \qquad (x,y) \in \mathbb{R}^2 \setminus \Gamma. \tag{2.12}$$

Using the matrix identity

$$\frac{1}{|r|^4} (-f'(s) \;\; 1) \begin{pmatrix} 2r_1 r_2 & r_2^2 - r_1^2 \\ r_2^2 - r_1^2 & -2r_1 r_2 \end{pmatrix} = -\partial_s \frac{(r_1, r_2)}{|r|^2}$$

together with integration by parts we obtain that

$$\nabla U(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(r_1, r_2)}{|r|^2} \phi'(s)\, ds, \qquad (x,y) \in \mathbb{R}^2 \setminus \Gamma. \tag{2.13}$$

Combining Equation (2.13) and [9, Lemma A.1, Lemma A 4], we obtain that $U^\pm \in \mathrm{C}^1(\overline{\Omega^\pm})$ satisfies also Equation $(2.9)_4$ and

$$[\partial_{\nu_\Gamma} U] = \Big(\big(1 + f'^2\big)^{-1/2} \phi' \Big) \circ \Xi_\Gamma^{-1}.$$

It is now easy to infer from Equation (2.13) that also Equation $(2.9)_1$ holds true, and therewith we established the existence of a solution. □

We are now in a position to solve the boundary value problems (2.1) for u^+ and u^-. To this end, we first motivate heuristically the explicit formula (2.15) for the gradient $v^- := \nabla u^-$ of the solution, which is the building block in the proof of Proposition 2.4 (the formula for v^+ is motivated similarly). The starting point is the observation that $\operatorname{div} v^- = 0$ in Ω^-, which implies there is a stream function $\psi^- : \Omega_- \to \mathbb{R}$ such that $\nabla \psi^- = (\partial_2 u^-, -\partial_1 u^-)$. Set $\psi := \psi_- \mathbf{1}_{\Omega^-}$. Taking into account that $\Delta \psi^- = 0$ in Ω^- and using Stokes's formula, we deduce that the distribution $\Delta \psi \in \mathcal{D}'(\mathbb{R}^2)$ is supported on the interface Γ, and we presume that

$$\Delta \psi = ((1 + f'^2)^{-1/2} \alpha^-) \circ \Xi_\Gamma^{-1} \delta_\Gamma \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{2.14}$$

with some unknown density function α^-, that is

$$\langle \Delta \psi, \varphi \rangle = \int_{\mathbb{R}} \alpha^- \varphi \circ \Xi_\Gamma \, \mathrm{d}s \qquad \text{for } \varphi \in \mathcal{D}(\mathbb{R}^2).$$

We now formally obtain ψ by taking the convolution of the right side of Equation (2.14) with the fundamental solution G of the Laplacian given by

$$G(x,y) = \frac{1}{2\pi} \ln(|(x,y)|), \qquad 0 \neq (x,y) \in \mathbb{R}^2,$$

hence

$$\psi^-(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \ln(|(x-s, y-f(s))|) \alpha^-(s)\, \mathrm{d}s, \qquad (x,y) \in \Omega^-.$$

Formally computing $\nabla \psi^-$ we arrive, in view of the relation $v^- = (-\partial_2 \psi^-, \partial_1 \psi^-)$, at the integral formula (2.15). In the proof of Proposition 2.4, we show, under suitable assumptions, there exists a unique density α such that the formula for v^- identifies, via the relation $v^- = \nabla u^-$, the unique solution u^- to Equation (2.1).

Proposition 2.4. *Given $f \in H^4(\mathbb{R})$, there exist unique solutions $u^\pm \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ to Equation (2.1) such that $\partial_{\nu_\Gamma} u^\pm \circ \Xi_\Gamma = (1 + f'^2)^{-1/2}(\phi^\pm)'$ for some functions $\phi^\pm \in H^2(\mathbb{R})$. Furthermore, $\nabla u^\pm = v^\pm$ in $\Omega^\pm$, where*

$$v^\pm(x, y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(f(s) - y, x - s)}{(x - s)^2 + (y - f(s))^2} \alpha^\pm(s)\, ds, \qquad (x, y) \in \Omega^\pm, \tag{2.15}$$

and with density functions $\alpha^\pm \in H^1(\mathbb{R})$ given by the relation

$$\alpha^\pm = 2(\mp 1 + \mathbb{A}(f))^{-1}[(\kappa(f))'] \in H^1(\mathbb{R}). \tag{2.16}$$

Proof.

(i) *Existence*. According to Lemma 2.2 (iii), we have $\mp 1 + \mathbb{A}(f) \in \mathrm{Isom}(H^1(\mathbb{R}))$ and, since $(\kappa(f))' \in H^1(\mathbb{R})$, the density functions $\alpha^\pm$ defined in Equation (2.16) are well-defined and belong to $H^1(\mathbb{R})$. We next infer from [9, Lemmas A.1, A.4] that the vector fields $v^\pm$ defined in Equation (2.15) belong to $C^\infty(\Omega^\pm) \cap C(\overline{\Omega^\pm})$ and

$$v^\pm \circ \Xi_\Gamma(x) = \frac{1}{2\pi} \,\mathrm{PV} \int_{\mathbb{R}} \frac{(f(s) - f(x), x - s)}{(x - s)^2 + (f(x) - f(s))^2} \alpha^\pm(s)\, ds \mp \frac{1}{2} \frac{\alpha^\pm(1, f')}{1 + f'^2}(x), \quad x \in \mathbb{R}. \tag{2.17}$$

Moreover, $v^\pm$ satisfies the asymptotic boundary condition $v^\pm(x, y) \to 0$ for $|(x, y)| \to \infty$ and

$$\mathrm{div}\, v^\pm = \mathrm{rot}\, v^\pm = 0 \qquad \text{in } \Omega^\pm,$$

see [9, Lemma A.4]. Setting $v^\pm = (v_1^\pm, v_2^\pm)$, the relation $\mathrm{rot}\, v^\pm = 0$ in $\Omega^\pm$ ensures that the functions

$$u^\pm(x, y) := c^\pm + \int_0^x v_1^\pm(s, \pm d)\, ds + \int_{\pm d}^y v_2^\pm(x, s)\, ds, \qquad (x, y) \in \Omega^\pm,$$

where $c^\pm \in \mathbb{R}$ and $d > \|f\|_\infty$, satisfy $\nabla u^\pm = v^\pm$ in $\Omega^\pm$. Moreover, $u^\pm \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ and, since $v^\pm$ are divergence free, Equation $(2.1)_1$ is satisfied. It is clear that also the asymptotic boundary conditions $(2.1)_2$ hold. Combining Equations (2.4), (2.17), and the relation $\nabla u^\pm = v^\pm$ on Γ, we further have

$$\partial_{\nu_\Gamma} u^\pm \circ \Xi_\Gamma = \frac{(1 + f'^2)^{-1/2}}{2} \mathbb{B}(f)[\alpha^\pm]. \tag{2.18}$$

In order to show that $\mathbb{B}(f)[\alpha^\pm]$ are derivatives of functions in $H^2(\mathbb{R})$ we define $\beta^\pm \in H^2(\mathbb{R})$ by the relations

$$\beta^\pm = 2(\mp 1 - \mathbb{A}(f)^*)^{-1}[\kappa(f)], \tag{2.19}$$

see Lemma 2.2 (v). We next differentiate Equation (2.19) with respect to x and infer then from Lemma 2.2 (iii)–(iv) that $(\beta^\pm)' = \alpha^\pm$ and

$$\mathbb{B}(f)[\alpha^\pm] = \mathbb{B}(f)[(\beta^\pm)'] = -(\mathbb{B}(f)^*[\beta^\pm])'.$$

Setting $\phi^\pm := -\mathbb{B}(f)^*[\beta^\pm/2]$, it follows from Equation (2.8) and Lemma 2.1 (ii) that $\phi^\pm \in H^2(\mathbb{R})$. Moreover, Equation (2.18) lead to $\partial_{\nu_\Gamma} u^\pm \circ \Xi_\Gamma = (1 + f'^2)^{-1/2}(\phi^\pm)'$. As a final step, we show that the additive constants $c^\pm$ can be chosen such that also Equation $(2.1)_2$ are satisfied. Indeed, in view of Equations (2.16) and (2.17), we have

$$\frac{d}{dx}(u^\pm|_\Gamma \circ \Xi_\Gamma) = (1, f') \cdot v^\pm|_\Gamma \circ \Xi_\Gamma = \frac{1}{2}(\mp 1 + \mathbb{A}(f))[\alpha^\pm] = (\kappa(f))',$$

so that $u^\pm|_\Gamma \circ \Xi_\Gamma - \kappa(f)$ is a constant function. Therewith, we established the existence of a solution to Equation (2.1).

(ii) *Uniqueness*. It suffices to show that the homogeneous problems

$$\left.\begin{aligned} \Delta u^{\pm} &= 0 \quad \text{in } \Omega^{\pm}, \\ u^{\pm} &= 0 \quad \text{on } \Gamma, \\ \nabla u^{\pm} &\to 0 \quad \text{for } \|(x,y)\| \to \infty, \end{aligned}\right\} \tag{2.20}$$

have unique solutions $u^{\pm}$ with the required properties. We establish only the uniqueness of u^{+} (that of u^{-} follows by similar arguments). Let thus $\phi^{+} \in H^2(\mathbb{R})$ be the function which satisfies the relation $\partial_{\nu_\Gamma} u^{+} \circ \Xi_\Gamma = (1+f'^2)^{-1/2}(\phi^{+})'$. Setting $U^- := 0$ and $U^+ := u^+$, we note that (U^+, U^-) solves the boundary value problem (2.9) (with $\phi = \phi^+$) and it is thus given by formula (2.11). In particular, it follows from Equation (2.11) and [9, Lemma A.1] that

$$0 = U^-|_\Gamma \circ \Xi_\Gamma = U^+|_\Gamma \circ \Xi_\Gamma = -\frac{1}{2}\mathbb{B}(f)^*[\phi'],$$

and together with Lemma 2.2 (iv) we get

$$0 = -(\mathbb{B}(f)^*[\phi'])' = \mathbb{B}(f)[\phi''].$$

However, as shown in [31, Equations (3.22) and (3.25)], there exits a positive constant C such that $\|\mathbb{B}(f)[\alpha]\|_2 \geq C\|\alpha\|_2$ for all $\alpha \in L_2(\mathbb{R})$. Therefore $\phi'' = 0$, hence also $\phi = 0$. We now infer from Equation (2.11) that $U^+ = u^+ = 0$, and the uniqueness claim is proven. □

3 | THE EVOLUTION PROBLEM AND THE PROOF OF THE MAIN RESULT

In this section, we first formulate the original problem (1.1) as an evolution problem for f, see Equation (3.1). Subsequently, we prove that the linearization of the right side of Equation (3.1) is the generator of an analytic semigroup, see Theorem 3.1, and we conclude the section with the proof of the main result stated in Theorem 1.1.

3.1 | The evolution problem

In order to formulate the system (1.1) as an evolution problem for f we first infer from Proposition 2.4 that if $(f, u^{\pm})$ is a solution to Equation (1.1) as stated in Theorem 1.1, then, for each $t > 0$, we have

$$\partial_{\nu_{\Gamma(t)}} u^{\pm}(t) \circ \Xi_{\Gamma(t)} = -(1+f'^2(t))^{-1/2}(\mathbb{B}(f(t))^*[(\mp 1 - \mathbb{A}(f(t))^*)^{-1}[\kappa(f(t))]])'.$$

Together with Equation $(1.1)_4$ we arrive at the following evolution equation:

$$\frac{df}{dt}(t) = (\mathbb{B}(f(t))^*[((-1-\mathbb{A}(f(t))^*)^{-1} - (1-\mathbb{A}(f(t))^*)^{-1})[\kappa(f(t))]])' \qquad \text{for } t > 0.$$

As we want to solve the latter equation in the phase space $H^r(\mathbb{R})$ with $r \in (3/2, 2)$, we encounter the problem that the curvature $\kappa(f)$ is in general not a function, but a distribution. However, taking full advantage of the quasilinear character of the curvature operator we can formulate the system (1.1) as the following quasilinear evolution problem:

$$\frac{df}{dt}(t) = \Phi(f(t))[f(t)], \quad t > 0, \qquad f(0) = f_0, \tag{3.1}$$

where $\Phi : H^r(\mathbb{R}) \to \mathcal{L}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R}))$ is defined by the following formula:

$$\Phi(f)[h] := (\mathbb{B}(f)^*[((-1-\mathbb{A}(f)^*)^{-1} - (1-\mathbb{A}(f)^*)^{-1})[\kappa(f)[h]]])', \tag{3.2}$$

with $\varkappa : H^r(\mathbb{R}) \to \mathcal{L}(H^{r+1}(\mathbb{R}), H^{r-1}(\mathbb{R}))$ given by

$$\varkappa(f)[h] := \frac{h''}{(1+f'^2)^{3/2}}. \tag{3.3}$$

We point out that, if $f \in H^2(\mathbb{R})$, then $\varkappa(f)[f]$ is exactly the pulled-back curvature $\varkappa(f)$. Moreover, arguing as in [33, Appendix C], it is not difficult to prove that

$$\varkappa \in \mathrm{C}^\infty(H^r(\mathbb{R}), \mathcal{L}(H^{r+1}(\mathbb{R}), H^{r-1}(\mathbb{R}))). \tag{3.4}$$

Recalling Equation (2.8), it follows from Lemmas 2.1 (iii) and 2.2 (ii), by also using the smoothness of the map which associate to an isomorphism its inverse, that

$$\mathbb{B}(f)^*,\ (\pm 1 - \mathbb{A}(f)^*)^{-1} \in \mathrm{C}^\infty(H^r(\mathbb{R}), \mathcal{L}(H^{r-1}(\mathbb{R}))). \tag{3.5}$$

Gathering Equations (3.2)–(3.5), we obtain in view of $d/dx \in \mathcal{L}(H^{r-1}(\mathbb{R}), H^{r-2}(\mathbb{R}))$ that

$$\Phi \in \mathrm{C}^\infty(H^r(\mathbb{R}), \mathcal{L}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R}))). \tag{3.6}$$

3.2 | The parabolicity property

Our next goal is to prove that the problem (3.1) is of parabolic type in the sense that, for each $f \in H^r(\mathbb{R})$, $r \in (3/2, 2)$, the operator $\Phi(f)$ is the generator of an analytic semigroup in $\mathcal{L}(H^{r-2}(\mathbb{R}))$. This is the content of the next result.

Theorem 3.1. *Given $f \in H^r(\mathbb{R})$, $r \in (3/2, 2)$, it holds that $-\Phi(f) \in \mathcal{H}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R}))$.*

In the proof of Theorem 3.1, we exploit the fact that, given $h \in H^{r+1}(\mathbb{R})$, the action $\Phi(f)[h]$ is the derivative of a function which lies in $H^{r-1}(\mathbb{R})$. The proof of Theorem 3.1 is postponed to the end of this subsection and it relies on a strategy inspired by [16, 17, 20].

As a first step, we associate with $\Phi(f)$ the continuous path

$$[\tau \mapsto \Phi(\tau f)] : [0,1] \to \mathcal{L}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R})),$$

and we note that

$$\Phi(0) = -2\frac{d}{dx}B(0)^*\frac{d^2}{dx^2} = 2H\frac{d^3}{dx^3},$$

where H is the Hilbert transform. In particular, $\Phi(0)$ is the Fourier multiplier defined by the symbol $[\xi \mapsto 2|\xi|^3]$. As a second step, we locally approximate in Proposition 3.2 the operator $\Phi(\tau f)$ by Fourier multipliers which coincide, up to some positive multiplicative constants, with $\Phi(0)$. As a final third step, we establish for these Fourier multipliers suitable (uniform) resolvent estimates, see Equations (3.14) and (3.15). The proof of Theorem 3.1 follows then by combining the results established in these three steps.

Before presenting Proposition 3.2, we choose for each $\varepsilon \in (0,1)$, a finite ε-localization family, that is a family

$$\{(\pi_j^\varepsilon, x_j^\varepsilon) : -N+1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1]) \times \mathbb{R},$$

with $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large, such that $x_j^\varepsilon \in \operatorname{supp} \pi_j^\varepsilon$, $-N+1 \le j \le N$, and

- $\operatorname{supp} \pi_j^\varepsilon \subset \{|x| \le \varepsilon + 1/\varepsilon\}$ is an interval of length ε for $\||j| \le N-1$;
- $\operatorname{supp} \pi_N^\varepsilon \subset \{|x| \ge 1/\varepsilon\}$;
- $\pi_j^\varepsilon \cdot \pi_l^\varepsilon = 0$ if $[|j-l| \ge 2, \max\{|j|, |l|\} \le N-1]$ or $[|l| \le N-2, j = N]$;

- $\sum_{j=-N+1}^{N}(\pi_j^\varepsilon)^2 = 1$;
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \le C\varepsilon^{-k}$ for all $k \in \mathbb{N}, -N+1 \le j \le N$.

To each finite ε-localization family, we associate a second family

$$\{\chi_j^\varepsilon \,:\, -N+1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1])$$

with the following properties:

- $\chi_j^\varepsilon = 1$ on $\operatorname{supp} \pi_j^\varepsilon$ for $-N+1 \le j \le N$ and $\operatorname{supp} \chi_N^\varepsilon \subset \{|x| \ge 1/\varepsilon - \varepsilon\}$;
- $\operatorname{supp} \chi_j^\varepsilon$ is an interval of length 3ε and with the same midpoint as $\operatorname{supp} \pi_j^\varepsilon$, $|j| \le N-1$.

It is not difficult to prove that, given $r \in \mathbb{R}$ and $\varepsilon \in (0,1)$, there exists $c = c(\varepsilon, r) \in (0,1)$ such that for all $h \in H^r(\mathbb{R})$ we have

$$c\|h\|_{H^r} \le \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon h\|_{H^r} \le c^{-1}\|h\|_{H^r}. \tag{3.7}$$

We are now in a position to establish the aforementioned localization result.

Proposition 3.2. *Let $3/2 < r' < r < 2$, $f \in H^r(\mathbb{R})$, and $\nu > 0$ be given. Then, there exist $\varepsilon \in (0,1)$, a ε-localization family $\{(\pi_j^\varepsilon, x_j^\varepsilon) \,:\, -N+1 \le j \le N\}$, and a constant $K = K(\varepsilon)$ such that*

$$\|\pi_j^\varepsilon \Phi(\tau f)[h] - 2a_{\tau,j}\Phi(0)[\pi_j^\varepsilon h]\|_{H^{r-2}} \le \nu\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}} \tag{3.8}$$

for all $-N+1 \le j \le N$, $\tau \in [0,1]$, and $h \in H^{r+1}(\mathbb{R})$, where, letting $a_\tau := (1+\tau^2 f'^2)^{-3/2}$, we set

$$a_{\tau,N} := \lim_{|x|\to\infty} a_\tau(x) = 1 \qquad \text{and} \qquad a_{\tau,j} := a_\tau(x_j^\varepsilon), \quad |j| \le N-1.$$

Proof. In the following, C and C_0 are constants that do not depend on ε, while constants denoted by K may depend on ε.

Given $-N+1 \le j \le N$, $\tau \in [0,1]$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\begin{aligned}
&\|\pi_j^\varepsilon \Phi(\tau f)[h] - a_{\tau,j}\Phi(0)[\pi_j^\varepsilon h]\|_{H^{r-2}} \\
&= \|\pi_j^\varepsilon \Phi(\tau f)[h] - a_{\tau,j}(H[(\pi_j^\varepsilon h)''])'\|_{H^{r-2}} \\
&\le \|(\pi_j^\varepsilon \mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(\tau f)^*)^{-1})[\kappa(\tau f)[h]]])' - a_{\tau,j}(H[(\pi_j^\varepsilon h)''])'\|_{H^{r-2}} \\
&\quad + \|(\pi_j^\varepsilon)'\mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(f)^*)^{-1})[\kappa(\tau f)[h]]]\|_{H^{r-2}},
\end{aligned}$$

where, in view of Equations (2.8), (3.3), Lemmas 2.1 (i), and 2.2 (i) we have

$$\begin{aligned}
&\|(\pi_j^\varepsilon)'\mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(f)^*)^{-1})[\kappa(\tau f)[h]]]\|_{H^{r-2}} \\
&\le K\|\mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(f)^*)^{-1})[\kappa(\tau f)[h]]]\|_{H^{r-2}} \\
&\le K\|\mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(f)^*)^{-1})[\kappa(\tau f)[h]]]\|_2 \\
&\le K\|\kappa(\tau f)[h]\|_2 \\
&\le K\|h\|_{H^{r'+1}}.
\end{aligned}$$

Since $d/dx \in \mathcal{L}(H^{r-1}(\mathbb{R}), H^{r-2}(\mathbb{R}))$ is a contraction, we have shown that

$$\begin{aligned}&\|\pi_j^\varepsilon \Phi(\tau f)[h] - 2a_{\tau,j}\Phi(0)[\pi_j^\varepsilon h]\|_{H^{r-2}}\\ &\le \|\pi_j^\varepsilon \mathbb{B}(\tau f)^*[((-1-\mathbb{A}(\tau f)^*)^{-1} - (1-\mathbb{A}(\tau f)^*)^{-1})[\kappa(\tau f)[h]]] - 2a_{\tau,j}H[(\pi_j^\varepsilon h)'']\|_{H^{r-1}}\\ &\quad + K\|h\|_{H^{r'+1}}.\end{aligned} \tag{3.9}$$

In remains to estimate the first term on the right side of Equation (3.9). To this end, several steps are needed.

Step 1. Given $\tau \in [0,1]$ and $h \in H^{1+r}(\mathbb{R})$ we define $\vartheta^\pm(\tau)[h] \in H^{r-1}(\mathbb{R})$ as the unique solutions to

$$(\pm 1 - \mathbb{A}(\tau f)^*)[\vartheta^\pm(\tau)[h]] = \kappa(\tau f)[h], \tag{3.10}$$

see Equation (3.4) and Lemma 2.1 (ii). In this step, we prove that there exists a constant $C_0 > 0$ such that for all $\varepsilon \in (0,1)$, $\tau \in [0,1]$, $-N+1 \le j \le N$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\|\vartheta^\pm(\tau)[h]\|_{H^{r-1}} \le C_0\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}. \tag{3.11}$$

Indeed, after multiplying Equation (3.10) by π_j^ε, we arrive at

$$(\pm 1 - \mathbb{A}(\tau f)^*)[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]] = \pi_j^\varepsilon \kappa(\tau f)[h] - (\mathbb{A}(\tau f)^*[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]] - \pi_j^\varepsilon \mathbb{A}(\tau f)^*[\vartheta^\pm(\tau)[h]]),$$

and it can be easily shown that

$$\|\pi_j^\varepsilon \kappa(\tau f)[h]\|_{H^{r-1}} \le C\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}.$$

Moreover, since $r-1<1$, the commutator estimate in Lemma A.1 together with Equation (2.8) yields

$$\|\mathbb{A}(\tau f)^*[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]] - \pi_j^\varepsilon \mathbb{A}(\tau f)^*[\vartheta(\tau)[h]]\|_{H^{r-1}} \le K\|\vartheta(\tau)[h]\|_2 \le K\|h\|_{H^{r'+1}}.$$

The estimates (3.11) follow now from Lemma 2.2 (ii).

Step 2. Recalling Equation (2.8), we infer from Lemma A.2 if $|j| \le N-1$, respectively from Lemma A.3 if $j = N$, that, if $\varepsilon \in (0,1)$ is sufficiently small, then for all $\tau \in [0,1]$, $-N+1 \le j \le N$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\|\pi_j^\varepsilon \mathbb{B}(\tau f)^*[\vartheta^\pm(\tau)[h]] + H[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]]\|_{H^{r-1}} \le \frac{\nu}{4C_0}\|\pi_j^\varepsilon \vartheta^\pm(\tau)[h]\|_{H^{r-1}} + K\|\vartheta^\pm(\tau)[h]\|_{H^{r'-1}}.$$

The estimates (3.11) and the property (3.4) (with $r = r'$) enable us to conclude that for all $\tau \in [0,1]$, $-N+1 \le j \le N$, and $h \in H^{r+1}(\mathbb{R})$ it holds that

$$\|\pi_j^\varepsilon \mathbb{B}(\tau f)^*[\vartheta^\pm(\tau)[h]] + H[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]]\|_{H^{r-1}} \le \frac{\nu}{4}\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}},$$

provided that ε is sufficiently small, and therefore

$$\begin{aligned}&\|\pi_j^\varepsilon \mathbb{B}(\tau f)^*[\vartheta^-(\tau)[h] - \vartheta^+(\tau)[h]] + H[\pi_j^\varepsilon(\vartheta^-(\tau)[h] - \vartheta^+(\tau)[h])]\|_{H^{r-1}}\\ &\le \frac{\nu}{2}\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}.\end{aligned} \tag{3.12}$$

Step 3. We show that, if $\varepsilon \in (0,1)$ is sufficiently small, then for all $\tau \in [0,1]$, $-N+1 \le j \le N$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\|\pm H[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]] - a_{\tau,j}H[(\pi_j^\varepsilon h)'']\|_{H^{r-1}} \le \frac{\nu}{4}\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}. \tag{3.13}$$

To start with, we note that since $H \in \mathcal{L}(H^{r-1}(\mathbb{R}))$ is an isometry we have

$$\| \pm H[\pi_j^\varepsilon \vartheta^\pm(\tau)[h]] - a_{\tau,j} H[(\pi_j^\varepsilon h)'']\|_{H^{r-1}} \leq \| \pm \pi_j^\varepsilon \vartheta^\pm(\tau)[h] - a_{\tau,j}(\pi_j^\varepsilon h)''\|_{H^{r-1}},$$

and it remains to estimate the right side of the latter inequality. To this end, we first infer from Equation (3.10) that

$$\pm\pi_j^\varepsilon \vartheta^\pm(\tau)[h] - a_{\tau,j}(\pi_j^\varepsilon h)'' = \pi_j^\varepsilon \mathbb{A}(\tau f)^*[\vartheta^\pm(\tau)[h]] + \pi_j^\varepsilon \kappa(\tau f)[h] - a_{\tau,j}(\pi_j^\varepsilon h)''.$$

Noticing that $\|a_\tau\|_{\mathrm{BUC}^{s-3/2}} \leq 3\|f'\|_{\mathrm{BUC}^{s-3/2}}$ and $a_\tau(x) \to 1$ for $|x| \to \infty$ uniformly with respect to $\tau \in [0,1]$ and using the estimate

$$\|g_1 g_2\|_{H^{r-1}} \leq C(\|g_1\|_\infty \|g_2\|_{H^{r-1}} + \|g_2\|_\infty \|g_1\|_{H^{r-1}}) \qquad \text{for } g_1,\, g_2 \in H^{r-1}(\mathbb{R}),$$

we have in view of $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$ that

$$\begin{aligned}\|\pi_j^\varepsilon \kappa(\tau f)[h] - a_{\tau,j}(\pi_j^\varepsilon h)''\|_{H^{r-1}} &\leq \|(a_\tau - a_\tau(x_j^\varepsilon))(\pi_j^\varepsilon h)''\|_{H^{r-1}} + K\|h\|_{H^{r'+1}} \\ &\leq C\|\chi_j^\varepsilon(a_\tau - a_\tau(x_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}} \\ &\leq \frac{\nu}{8}\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}\end{aligned}$$

for all $\tau \in [0,1]$, $|j| \leq N-1$, and $h \in H^{r+1}(\mathbb{R})$, provided that ε is sufficiently small. Similarly, for $j = N$ we have

$$\begin{aligned}\|\pi_N^\varepsilon \kappa(\tau f)[h] - a_{\tau,N}(\pi_N^\varepsilon h)''\|_{H^{r-1}} &\leq \|(a_\tau - 1)(\pi_N^\varepsilon h)''\|_{H^{r-1}} + K\|h\|_{H^{r'+1}} \\ &\leq C\|\chi_N^\varepsilon(a_\tau - 1)\|_\infty \|\pi_N^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}} \\ &\leq \frac{\nu}{8}\|\pi_N^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}.\end{aligned}$$

Furthermore, appealing to Lemma A.2 if $|j| \leq N-1$, respectively to Lemma A.3 if $j = N$, we find together with the representation (2.8) of $\mathbb{A}(\tau f)^*$ that, if ε is sufficiently small, then

$$\|\pi_j^\varepsilon \mathbb{A}(\tau f)^*[\vartheta^\pm(\tau)[h]]\|_{H^{r-1}} \leq \frac{\nu}{8C_0}\|\pi_j^\varepsilon \vartheta^\pm(\tau)[h]\|_{H^{r-1}} + K\|\vartheta^\pm(\tau)[h]\|_{H^{r'-1}},$$

and, together with Equation (3.11) and the property (3.4) (with $r = r'$), we get

$$\|\pi_j^\varepsilon \mathbb{A}(\tau f)^*[\vartheta^\pm(\tau)[h]]\|_{H^{r-1}} \leq \frac{\nu}{8}\|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}$$

for all $\tau \in [0,1]$, $-N+1 \leq j \leq N$, and $h \in H^{r+1}(\mathbb{R})$. This proves Equation (3.13).

Combining the estimates (3.9), (3.12), and (3.13), we conclude that Equation (3.8) holds true and this completes the proof. □

In Proposition 3.2, we have locally approximated $\Phi(\tau f)$ by Fourier multipliers $2a_{\tau,j}\Phi(0)$, and, since f' is a bounded function, there exists a constant $\eta = \eta(\|f'\|_\infty) \in (0,1)$ such that $2a_{\tau,j} \in [\eta, \eta^{-1}]$. Elementary Fourier analysis arguments enable us to conclude there exists a constant $\kappa_0 = \kappa_0(\eta) \geq 1$ such that for all $\delta \in [\eta, \eta^{-1}]$ and all $\operatorname{Re}\lambda \geq 1$ we have

$$\bullet\ \lambda - \delta\Phi(0) \in \mathrm{Isom}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R})), \tag{3.14}$$

$$\bullet\ \kappa_0\|(\lambda - \delta\Phi(0))[h]\|_{H^{r-2}} \geq |\lambda| \cdot \|h\|_{H^{r-2}} + \|h\|_{H^{r+1}} \quad \text{for all } h \in H^{r+1}(\mathbb{R}). \tag{3.15}$$

We are now in a position to establish Theorem 3.1.

Proof of Theorem 3.1. Let $\kappa_0 \geq 1$ be as identified in Equation (3.15). Setting $\nu := (2\kappa_0)^{-1}$, Proposition 3.2 ensures that there exist $\varepsilon \in (0,1)$, a ε-localization family $\{(\pi_j^\varepsilon, x_j^\varepsilon) : -N+1 \leq j \leq N\}$, and a constant $K = K(\varepsilon)$ such that for all $\tau \in [0,1]$, $-N+1 \leq j \leq N$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\|\pi_j^\varepsilon \Phi(\tau f)[h] - 2a_{\tau,j}\Phi(0)[\pi_j^\varepsilon h]\|_{H^{r-2}} \leq \nu \|\pi_j^\varepsilon h\|_{H^{r+1}} + K\|h\|_{H^{r'+1}}.$$

Recalling Equation (3.15), we also have

$$\kappa_0 \|(\lambda - 2a_{\tau,j}\Phi(0))[\pi_j^\varepsilon h]\|_{H^{r-2}} \geq |\lambda| \cdot \|\pi_j^\varepsilon h\|_{H^{r-2}} + \|\pi_j^\varepsilon h\|_{H^{r+1}}$$

for all $\tau \in [0,1]$, $-N+1 \leq j \leq N$, $\operatorname{Re}\lambda \geq 1$, and $h \in H^{r+1}(\mathbb{R})$. Combining these estimates, we get

$$\begin{aligned} 2\kappa_0 \|\pi_j^\varepsilon(\lambda - \Phi(\tau f))[h]\|_{H^{r-2}} \geq& 2\kappa_0 \|(\lambda - 2a_{\tau,j}\Phi(0))[\pi_j^\varepsilon h]\|_{H^{r-2}} \\ &- 2\kappa_0 \|\pi_j^\varepsilon \Phi(\tau f)[h] - 2a_{\tau,j}\Phi(0)[\pi_j^\varepsilon h]\|_{H^{r-2}} \\ \geq& 2|\lambda| \cdot \|\pi_j^\varepsilon h\|_{H^{r-2}} + \|\pi_j^\varepsilon h\|_{H^{r+1}} - 2\kappa_0 K\|h\|_{H^{r'+1}}. \end{aligned}$$

Summing up over j, the estimates (3.7), Young's inequality, and the interpolation property

$$[H^{s_0}(\mathbb{R}), H^{s_1}(\mathbb{R})]_\theta = H^{(1-\theta)s_0+\theta s_1}(\mathbb{R}), \qquad \theta \in (0,1),\ -\infty < s_0 \leq s_1 < \infty, \tag{3.16}$$

cf., for example, [44, Section 2.4.2/Remark 2], where $[\cdot,\cdot]_\theta$ is the complex interpolation functor, imply there exist constants $\kappa \geq 1$ and $\omega \geq 1$ such that for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq \omega$, and $h \in H^{r+1}(\mathbb{R})$ we have

$$\kappa \|(\lambda - \Phi(\tau f))[h]\|_{H^{r-2}} \geq |\lambda| \cdot \|h\|_{H^{r-2}} + \|h\|_{H^{r+1}}. \tag{3.17}$$

The property (3.14) together with the method of continuity [6, Proposition I.1.1.1] and Equation (3.17) now yield

$$\omega - \Phi(f) \in \operatorname{Isom}(H^{r+1}(\mathbb{R}), H^{r-2}(\mathbb{R})). \tag{3.18}$$

The desired generator property follows now directly from Equation (3.17) (with $\tau = 1$) and Equation (3.18), see [6, Chapter I]. □

3.3 | The proof of the main result

We complete this section with the proof of the main result which exploits the abstract quasilinear parabolic theory presented in [5] (see also [35, Theorem 1.1]).

Proof of Theorem 1.1. Let $E_1 := H^{\bar{r}+1}(\mathbb{R})$, $E_0 := H^{\bar{r}-2}(\mathbb{R})$, and $E_\theta := [E_0, E_1]_\theta$, $\theta \in (0,1)$. Defining $\beta := 2/3$ and $\alpha := (r - \bar{r} + 2)/3$, it holds that $0 < \beta < \alpha < 1$, $E_\beta = H^{\bar{r}}(\mathbb{R})$, and $E_\alpha = H^r(\mathbb{R})$. Theorem 3.1 together with the regularity property (3.6) (both with $r = \bar{r}$) ensure that $-\Phi \in \mathrm{C}^\infty(E_\beta, \mathcal{H}(E_1, E_0))$. This enables us to apply [35, Theorem 1.1] in the context of the quasilinear parabolic evolution problem (3.1). Consequently, given $f_0 \in H^r(\mathbb{R})$, there exists a unique maximal classical solution $f = f(\cdot\,; f_0)$ to Equation (3.1) such that

$$f \in \mathrm{C}([0,T^+), H^r(\mathbb{R})) \cap \mathrm{C}((0,T^+), H^{\bar{r}+1}(\mathbb{R})) \cap \mathrm{C}^1((0,T^+), H^{\bar{r}-2}(\mathbb{R})) \tag{3.19}$$

and

$$f \in C^{\zeta}([0,T^+), H^{\bar{r}}(\mathbb{R}), \tag{3.20}$$

where $T^+ = T^+(f_0) \in (0,\infty]$ is the maximal existence time and $\zeta \in (0, \alpha - \beta]$ can be chosen arbitrary small, cf. [35, Remark 1.2 (ii)]. Moreover, the mapping $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $H^r(\mathbb{R})$ which is smooth in the open set

$$\{(t, f_0) \,:\, f_0 \in H^r(\mathbb{R}),\, 0 < t < T^+(f_0)\} \subset \mathbb{R} \times H^r(\mathbb{R}).$$

We next prove that the uniqueness claim holds in the class of classical solutions; that is, of solutions which satisfy merely Equation (3.19). To this end, prove that each such solution with the property (3.19) satisfies Equation (3.20) for some small ζ. Let therefore $T \in (0, T^+)$ be arbitrary but fixed. Then, there exists a positive constant C such that for all $t \in [0, T]$ we have

$$\|\kappa(f(t))\|_{H^{r-2}} = \left\| \left(\frac{(f(t))'}{(1+(f(t))')^{1/2}} \right)' \right\|_{H^{r-2}} \leq \left\| \frac{(f(t))'}{(1+(f(t))'^2)^{1/2}} \right\|_{H^{r-1}} \leq C. \tag{3.21}$$

Moreover, in virtue of Lemma 2.2 (i) and (ii) $\pm 1 - \mathbb{A}(f(t)) \in \mathrm{Isom}(L_2(\mathbb{R})) \cap \mathrm{Isom}(H^{r-1}(\mathbb{R}))$ for all $t \in [0,T]$, and together with Equation (3.16) and the observation that $0 < 2 - r < r - 1$ we get that $\pm 1 - \mathbb{A}(f(t)) \in \mathrm{Isom}(H^{2-r}(\mathbb{R}))$. Since by Lemma 2.1 (i) and (ii) and Equation (2.8) the mapping

$$[t \mapsto \mathbb{A}(f(t))] \,:\, [0, T] \to \mathcal{L}(L_2(\mathbb{R})) \cap \mathcal{L}(H^{r-1}(\mathbb{R}))$$

is in particular continuous, we may chose $C > 0$ sufficiently large to guarantee that for all $t \in [0, T]$ it holds that

$$\|(\pm 1 - \mathbb{A}(f(t))^{-1}\|_{\mathcal{L}(H^{2-r}(\mathbb{R}))} \leq C. \tag{3.22}$$

Therefore, setting $\vartheta^{\pm}(t) := (\pm 1 - \mathbb{A}(f(t)^*)^{-1}[\kappa(f(t))] \in H^{\bar{r}-1}(\mathbb{R})$, $t \in (0, T]$, we infer from Equations (3.21) and (3.22) that there exists a constant $C > 0$ such that for all $t \in (0, T]$ we have

$$\begin{aligned}
\|\vartheta^{\pm}(t)\|_{H^{r-2}} &= \sup_{\|\psi\|_{H^{2-r}}=1} |\langle \vartheta^{\pm}(t) | \psi \rangle_2| = \sup_{\|\psi\|_{H^{2-r}}=1} |\langle (\pm 1 - \mathbb{A}(f(t)^*)^{-1}[\kappa(f(t))] | \psi \rangle_2| \\
&= \sup_{\|\psi\|_{H^{2-r}}=1} |\langle \kappa(f(t)) | (\pm 1 - \mathbb{A}(f(t))^{-1}[\psi] \rangle_2| \\
&\leq \sup_{\|\psi\|_{H^{2-r}}=1} \|\kappa(f(t))\|_{H^{r-2}} \|(\pm 1 - \mathbb{A}(f(t))^{-1}[\psi]\|_{H^{2-r}} \\
&\leq C.
\end{aligned}$$

Above $\langle\cdot|\cdot\rangle_2$ is the L_2-scalar product. Since $\Phi(f(t))[f(t)] = (\mathbb{B}(f(t))^*[\vartheta^-(t) - \vartheta^+(t)])'$ for $t \in (0, T]$, see Equation (3.2), it follows now from Lemma 2.1 (iv) and Equation (2.8) there exists a constant $C > 0$ such that for all $t \in (0, T]$

$$\|\Phi(f(t))[f(t)]\|_{H^{r-3}} \leq \|\mathbb{B}(f(t))^*[\vartheta^-(t) - \vartheta^+(t)]\|_{H^{r-2}} \leq C(1 + \|(f(t))'\vartheta^{\pm}(t)\|_{H^{r-2}} \leq C.$$

To derive the last inequality, we have use the continuity of the multiplication operator

$$[(g_1, g_2) \mapsto g_1 g_2] \,:\, H^{r-1}(\mathbb{R}) \times H^{2-r}(\mathbb{R}) \to H^{2-r}(\mathbb{R}),$$

see [29, Equation (1.8)]. To summarize, we have shown that

$$\sup_{t\in(0,T]} \left\| \frac{df}{dt}(t) \right\|_{H^{r-3}} \leq C.$$

Since $f \in \mathrm{C}([0,T], H^r(\mathbb{R}))$, the latter estimate together with the mean value theorem and the observation that for $\zeta := (r-\bar{r})/3$ it holds that $[H^{r-3}(\mathbb{R}), H^r(\mathbb{R})]_{1-\zeta} = H^{\bar{r}}(\mathbb{R})$, see Equation (3.16), yields

$$\|f(t_2) - f(t_1)\|_{H^{\bar{r}}} \leq C\|f(t_2) - f(t_1)\|^{\zeta}_{H^{\bar{r}-3}} \leq C|t_2 - t_1|^{\zeta} \qquad \text{for all } 0 \leq t_1 \leq t_2 \leq T,$$

which proves Equation (3.20). Recalling Proposition 2.4, we have established the existence and uniqueness of maximal classical solutions to Equation (1.1). Finally, the parabolic smoothing property (1.2) may be shown by using a parameter trick employed also in other settings, see [7, 18, 32, 40]. Since the arguments are more or less identical to those used in [32, Theorem 1.3], we refrain to present them here. □

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APPENDIX A: SOME PROPERTIES OF THE SINGULAR INTEGRAL OPERATORS $B^0_{n,m}(f)$

We recall some recent results that are available for the singular integrals operators $B^0_{n,m}(f)$ introduced in Equation (2.7) and which are used in the analysis in Section 3. We begin with a commutator type estimate.

Lemma A.1. *Let $n, m \in \mathbb{N}$, $r \in (3/2, 2)$, $f \in H^r(\mathbb{R})$, and $\varphi \in \mathrm{BUC}^1(\mathbb{R})$ be given. Then, there exists a constant K that depends only on n, m, $\|\varphi'\|_\infty$, and $\|f\|_{H^r}$ such that for all $\alpha \in L_2(\mathbb{R})$ we have*

$$\|\varphi B^0_{n,m}(f)[\alpha] - B^0_{n,m}(f)[\varphi\alpha]\|_{H^1} \leq K\|\alpha\|_2$$

Proof. See [1, Lemma 12]. □

The next results describe how to localize the singular integrals operators $B^0_{n,m}(f)$. They may be viewed as generalizations of the method of freezing the coefficients of elliptic differential operators.

Lemma A.2. *Let $n, m \in \mathbb{N}$, $r \in (3/2, 2)$, $r' \in (3/2, r)$, and $\nu \in (0, 1)$ be given. Let further $f \in H^r(\mathbb{R})$ and $a \in \{1\} \cup H^{r-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0, 1)$, there exists a constant K that depends only on ε, n, m, $\|f\|_{H^r}$, and $\|a\|_{H^{r-1}}$ (if*

$a \neq 1$) such that for all $|j| \leq N-1$ and $\alpha \in H^{r-1}(\mathbb{R})$ we have

$$\left\|\pi_j^\varepsilon B_{n,m}^0(f)[a\alpha] - \frac{a(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m} H[\pi_j^\varepsilon \alpha]\right\|_{H^{r-1}} \leq \nu \|\pi_j^\varepsilon \alpha\|_{H^{r-1}} + K\|\alpha\|_{H^{r'-1}}.$$

Proof. See [1, Lemma 13] if $a = 1$, respectively [33, Lemma D.5] if $a \in H^{r-1}(\mathbb{R})$. □

Lemma A.3 describes how to localize the operators $B_{n,m}^0(f)$ "at infinity."

Lemma A.3. *Let $n, m \in \mathbb{N}$, $r \in (3/2, 2)$, $r' \in (3/2, r)$, and $\nu \in (0,1)$ be given. Let further $f \in H^r(\mathbb{R})$ and $a \in \{1\} \cup H^{r-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there exists a constant K that depends only on ε, n, m, $\|f\|_{H^r}$, and $\|a\|_{H^{r-1}}$ (if $a \neq 1$) such that for all $\alpha \in H^{r-1}(\mathbb{R})$*

$$\|\pi_N^\varepsilon B_{n,m}^0(f)[a\alpha]\|_{H^{r-1}} \leq \nu \|\pi_N^\varepsilon \alpha\|_{H^{r-1}} + K\|\alpha\|_{H^{r'-1}} \qquad \text{if } n \geq 1 \text{ or } a \in H^{r-1}(\mathbb{R}),$$

and

$$\|\pi_N^\varepsilon B_{0,m}^0(f)[\alpha] - H[\pi_N^\varepsilon \alpha]\|_{H^{r-1}} \leq \nu \|\pi_N^\varepsilon \alpha\|_{H^{r-1}} + K\|\alpha\|_{H^{r'-1}}$$

Proof. See [1, Lemma 15] if $a = 1$, respectively [33, Lemma D.6] if $a \in H^{r-1}(\mathbb{R})$. □