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$U_q[OSp(3|2)]$ quantum chains with quantum group invariant boundaries

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Abstract

Based on the finite-size analysis of the spectrum of the quantum group invariant deformation of the $OSp(3|2)$ superspin chain we identify the operator content of the conformal field theory describing the model in its scaling limit. We find that the macroscopic degeneracy of the conformal weights observed in the thermodynamic limit of the isotropic superspin chain is lifted by the deformation.

1. Introduction

During the recent past years there has been an increasing activity on the study of families of massless one-dimensional integrable models with intricate critical behaviour similar to that expected for conformal field theories (CFTs) based on non-compact symmetries. A common property among of these lattice models is the presence of towers of lower energy states leading to the same scaling dimension in the thermodynamic limit whose degeneracies are typically lifted by logarithmic corrections on the finite interval. It has been argued that examples of such systems are not restricted to specific models but instead range from staggered two-dimensional vertex

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models to families of spin chains based on twisted Lie algebras and as well as on supergroup symmetries [1–7]. Probably best understood among these models is the staggered six-vertex model [2,8–12]: here the spectrum of conformal weights has been shown to have continuous components arising from the non-compact target space of the underlying CFT. Based on finite size studies probing properties of the algebra of extended conformal symmetry the low energy effective theory of this model has been argued to be a Lorentzian black hole nonlinear sigma model [12,13].

For conventional conformal field theories it is well known that boundary conditions are able to select some of the conformal dimensions predicted for the operator content of the underlying conformal field theory from symmetry considerations. Therefore, a natural question to ask is whether or not the presence of towers of low energy states and the presence of continuous components in the finite-size spectrum is robust against changing the system boundary conditions. Indeed, this kind of question has recently been investigated in the context of the staggered six-vertex model [14–17]: in this case fine-tuning of the boundary conditions is necessary for the low energy degrees of freedom to be described by a non-compact boundary CFT. Moreover, based on numerical work a boundary RG flow to a fixed point with discrete conformal spectrum has been identified.

The purpose of this work is to begin an investigation about the effect of boundary conditions on the operator content of the q-deformed $OSp(3|2)$ superspin chain. In the isotropic $q \to 1$ limit this model is a lattice realization of the Goldstone phase of intersecting loops and it has been pointed out that, for both periodic and free boundary conditions, there exist several towers of states with integer conformal dimensions subject to strong subleading logarithmic corrections to scaling in the finite system [18–21]. On the other hand, it has been observed that some of the towers are eliminated in the integrable anisotropic deformation of the periodic model [7]. At the same time the corresponding corrections to scaling vanish as power laws depending on the deformation parameter q rather than logarithmic in the thermodynamic limit. This behaviour has been linked to the observation that part of the energies of the q-deformed $OSp(3|2)$ superspin chain coincides with the eigenvalues of the integrable XXZ spin $S = 1$ chain for a special value of the deformation parameter q.

Here we will study the critical properties of such q-deformed superspin chain with boundary conditions restoring the full $U_q[OSp(3|2)]$ quantum group invariance. This is the simplest boundary condition retaining part of the spectral degeneracies of the isotropic model and at the same time recovers the free boundary conditions in the isotropic limit $q \to 1$. In addition, the $U_q[OSp(3|2)]$ spin chain is solvable by the Bethe ansatz which allows to study its spectral properties for very large systems. For our analysis of the critical properties we exploit the relationship between the eigenenergies $E_n(L)$ of the finite system and the central charge c and the conformal weights h_n (appearing as surface exponents describing the asymptotic behaviour of boundary correlation functions) of the corresponding operators in the boundary CFT describing the model in the scaling limit [22,23],

$$E_n(L) = L\varepsilon_\infty + f_\infty - \frac{\pi v_F}{24L}\, c + \frac{\pi v_F}{L}\, h_n + o\left(\frac{1}{L}\right) , \tag{1.1}$$

where ε_∞ is the energy per site of the ground state, f_∞ is the surface energy resulting from the open boundary conditions, and v_F is the Fermi velocity of the massless low-lying excitations. A peculiar feature of $U_q[OSp(3|2)]$-invariant model is that the ground state energy has no finite-size corrections beyond the surface energy f_∞ implying a vanishing central charge c for generic values of the deformation parameter q in the massless regime. This has to be contrasted with other integrable quantum group invariant spin chains such as those based on the spin-s representation of the $U_q[SU(2)]$ quantum algebra in which the ground state finite-size corrections only vanish for a specific choice of q [23,24].

Given the quantum group symmetry the eigenenergies of the superspin chain appear in the multiplets of $OSp(3|2)$. Following Ref. [25] they may be labelled by two indices $(\mathrm{p};\mathrm{q})$. Except for the trivial representation $(0;0)$ the index p takes nonnegative integer values while $\mathrm{q} \geq \frac{1}{2}$ is an integer or half integer – as in the isotropic case. The presence of these levels in the spectrum of the superspin chain of length L requires the selection rule $L - \mathrm{p} - 2\mathrm{q} \in 2\mathbb{N}$ to be satisfied. An exception are the 'atypical' representations with $\mathrm{p} - 2\mathrm{q} + 1 = 0$ such as $(1;1)$. For $\mathrm{p} \geq 1$ these appear as part of reducible but indecomposable representations in the Hilbert space of chains of any length.

Combining exact diagonalization for small systems with the numerical solution of the Bethe equations we have investigated a significant number of low energy states in the sectors with different values of the two $U(1)$ charges of the $U_q[OSp(3|2)]$ superspin chain. Using (1.1) with data for sufficiently large systems we provide strong evidence that the conformal weights in terms of the quantum numbers $(\mathrm{p};\mathrm{q})$ are given as

$$h_{(\mathrm{p};\mathrm{q})} = \frac{\pi - \gamma}{2\pi}\,\mathrm{p}(\mathrm{p}+1) + \frac{\gamma}{2\pi}\,2\mathrm{q}(2\mathrm{q}-1)\,. \tag{1.2}$$

The anisotropy parameter γ is related to the quantum group deformation by $q = e^{i\gamma/2}$. We expect that the massless regime of the $U_q[OSp(3|2)]$ chain is in the interval $\gamma \in [0,\pi]$, as in the periodic model [7], and that (1.2) holds throughout this interval. Note, however, that most of our numerical analysis is on the region $\gamma \leq \pi/2$. A significant difference to the periodic case studied in [7] is the absence of towers of states in the eigenspectrum for the quantum group boundary condition. We anticipate however that some of the low lying states have strong power law corrections to finite-size scaling which required the study of large system sizes. In any case this result further elucidates our earlier findings for the isotropic model [20,21]: (i) as $\gamma \to 0$ the second term of Eq. (1.2) vanishes and we are left with the conformal weights identified in the finite-size analysis of the periodic chain. (ii) For the isotropic model with free boundary conditions (1.2) results in a tower of states with integer conformal dimensions labelled on the finite lattice by the index p. Since the number of available values for q grows with the lattice size this leads to a macroscopic degeneracy of the low energy states which is lifted by corrections to scaling vanishing as $1/\log L$ in the thermodynamic limit.

This paper is organized as follows: in the following section we formulate the integrable $U_q[OSp(3|2)]$-invariant superspin chain based on particular solutions of the boundary Yang-Baxter algebra [26,27]. Then we present the solution to this spectral problem of the model in terms of Bethe equations. Being based on a superalgebra there exist two such solutions corresponding to different orderings of the fermionic and bosonic states in the local basis. In Section 4 we uncover an exact correspondence between the spectra of the q-deformed $OSp(3|2)$ superspin chain and the XXZ spin-1 spin chain, both with quantum-group invariant boundary conditions, for a particular value of the deformation parameter q. Together with our finite-size analysis of the $U_q[OSp(3|2)]$-symmetric super spin chain this correspondence supports our proposal (1.2) for the operator content of the boundary CFT describing the critical point of the model. We close with a discussion putting the present results in the context of earlier work on the $OSp(3|2)$ superspin chain and its deformations as well as on the influence of boundary conditions in other integrable spin chains with a continuous component to the conformal spectrum.

The $U_q[OSp(3|2)]$-invariant Hamiltonian in terms of the $OSp(3|2)$ superalgebra is presented in Appendix A.

2. Formulation of the model

In this section we describe the integrable q-deformed $OSp(3|2)$ spin chain with quantum group invariance. We recall that the main tools for dealing with open boundary conditions within the quantum inverse scattering method have been introduced by Cherednik and Sklyanin [26,27]. This pioneering work was further elaborated and in particular it has been shown how integrable quantum group invariant models can be constructed for a variety of distinct affine Lie algebras in [28,29]. Later on similar constructions have been pursued to include integrable models with underlying graded affine superalgebras [30,31]. Specifically, the right and left boundary conditions are encoded in reflection matrices $K^{\pm}(\lambda)$ satisfying the so-called boundary Yang-Baxter equations, e.g.

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu)=K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu) \tag{2.1}$$

for the left boundary matrix $K^-(\lambda)$. Here $K_j^-(\lambda)$ is a copy of the reflection matrix acting non-trivially on the space V_j and $R_{ij}(\lambda)\in\mathrm{End}\left(V_i\otimes V_j\right)$ is the R-matrix of the q-deformed $OSp(3|2)$ vertex model in the fundamental representation for which V_j is a five-dimensional $\mathbb{Z}_2$-graded vector space [32]. The R-matrix satisfies the commutation property $[\check{R}_{12}(\lambda),\check{R}_{12}(\mu)]=0$ where $\check{R}_{12}(\lambda)=P_{12}\,R_{12}(\lambda)$ with the graded permutation operator $P_{12}\,a\otimes b=(-1)^{p_a p_b}b\otimes a$ on $V\otimes V$.[1] Recall here, that p_a stands for the grading (or parity), i.e. $p_a=0$ if a is an even (bosonic) state and $p_a=1$ if a is odd (fermionic). As a consequence it is easy to see that (2.1) is satisfied by the trivial solution $K^-(\lambda)\equiv\mathbf{1}$. Given this solution of (2.1) $K^+(\lambda)$ is easily obtained using the crossing properties of the R-matrix [27]. Since the explicit form of the R-matrix depends on the choice of the $\mathbb{Z}_2$-grading the same is true for the corresponding $K^+(\lambda)$. For instance, ordering the elements of the five dimensional basis in the $fbbbf$ grading we obtain,

$$K^+(\lambda)=\begin{pmatrix} q^2 & 0 & 0 & 0 & 0\\ 0 & q^2 & 0 & 0 & 0\\ 0 & 0 & q & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}. \tag{2.2}$$

With these objects the double row transfer matrix generating commuting integrals of motion acting on the Hilbert space $\otimes_{j=1}^{L}V_j$ of a chain of length L then reads

$$T(\lambda)=Str_0[K_0^+(\lambda)R_{0L}(\lambda)\cdots R_{01}(\lambda)K_0^-(\lambda)R_{01}^{-1}(-\lambda)\cdots R_{0L}^{-1}(-\lambda)] \tag{2.3}$$

where the symbol Str_0 denotes the supertrace taken over an auxiliary five-dimensional $OSp(3|2)$ superspace. The respective Hamiltonian with open boundary is obtained by expanding the double-row transfer matrix (2.3) up to the first order in the spectral parameter λ. For the choice of reflection matrices introduced above the Hamiltonian can be written as

[1] This is because $\check{R}$ belongs to the algebra generated by a braid operator and can therefore be obtained by means of the Baxterization method [33].

Table 1
The five-dimensional representation of the six bosonic and six fermionic generators of the $OSp(3|2)$ superalgebra in terms of the Weyl matrices ordered in the $fbbbf$ grading.

Generator		Weyl Matrix	$\mathbb{Z}_2$ parity
our notation	notation from [25]		
τ^z	s^z	$e_{22}-e_{44}$	even
τ^+	s^+	$\sqrt{2}(e_{23}-e_{34})$	even
τ^-	s^-	$\sqrt{2}(e_{32}-e_{43})$	even
σ^z	t^z	$(e_{11}-e_{55})/2$	even
σ^+	t^+	$-e_{15}$	even
σ^-	t^-	$-e_{51}$	even
c^+	$R_{1,\frac{1}{2}}$	$e_{25}-e_{14}$	odd
c^-	$R_{-1,-\frac{1}{2}}$	$-e_{52}-e_{41}$	odd
d^+	$R_{0,-\frac{1}{2}}$	$e_{31}+e_{53}$	odd
d^-	$R_{0,\frac{1}{2}}$	$e_{35}-e_{13}$	odd
f^+	$R_{-1,\frac{1}{2}}$	$e_{12}-e_{45}$	odd
f^-	$R_{1,-\frac{1}{2}}$	$e_{21}+e_{54}$	odd

$$H=\sum_{j=1}^{L-1}\frac{\partial}{\partial\lambda}\check{R}_{j,j+1}(\lambda)\Big|_{\lambda=0} \tag{2.4}$$

up to an additive constant. Recall that the R-matrix is constructed by means of Baxterization from its braid limit and the $\check{R}$-matrix can be written as a sum over projectors appearing in the tensor product of two copies of the five-dimensional vector representations. As has been argued before this property ensures that the Hamiltonian (2.4) commutes with the generators of $U_q[OSp(3|2)]$ (see also our discussion of the isotropic model below) [30].

In principle it is also possible to express the Hamiltonian (2.4) in terms of the generators of the $OSp(3|2)$ superalgebra. To this end we recall that representations of this superalgebra have been previously investigated by Van der Jeugt [25]. The even part of the $OSp(3|2)$ superalgebra is isomorphic to $SO(3)\oplus Sp(2)$ and we shall denote the corresponding generators by the operators $\{\tau^z,\tau^\pm\}$ and $\{\sigma^z,\sigma^\pm\}$, respectively. The odd subspace of the $OSp(3|2)$ is constituted by another six fermionic generators which here are going to be represented by the operators $\{c^\pm,d^\pm,f^\pm\}$. In terms of the 5×5 Weyl matrices e_{ij}, whose entries are 1 on the i-th row and the j-th column and zero elsewhere, the twelve generators of the $OSp(3|2)$ superalgebra can be given explicitly in the $fbbbf$ grading. For the reader's convenience we have listed them together with the original notation used in Ref. [25] in Table 1. The Hamiltonian of the open spin chain (2.4) commutes with $U(1)$ charges of the Cartan subalgebra of $OSp(3|2)$ which are directly related to the azimuthal bosonic generators of the $SO(3)$ and $Sp(2)$ subalgebras:

$$[H,\sum_{j=1}^{L}\tau_j^z]=[H,\sum_{j=1}^{L}\sigma_j^z]=0\,. \tag{2.5}$$

We further remark that the quadratic Casimir $C_{j,j+1}$ of $OSp(3|2)$ acting on pair of sites $(j,j+1)$ in terms of these generators is

$$C_{j,j+1} = \frac{\left(\tau_j^+\tau_{j+1}^- + \tau_j^-\tau_{j+1}^+\right)}{2} + \tau_j^z\tau_{j+1}^z - 2\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^+\sigma_{j+1}^-\right) - 4\sigma_j^z\sigma_{j+1}^z + \left(c_j^+c_{j+1}^- - c_j^-c_{j+1}^+\right) + \left(d_j^+d_{j+1}^- - d_j^-d_{j+1}^+\right) + \left(f_j^+f_{j+1}^- - f_j^-f_{j+1}^+\right) \tag{2.6}$$

where the tensor products among the fermionic degrees of freedom have to be understood in the graded sense. The degeneracies of the eigenvalues of this Casimir operator are compatible with expected Clebsch-Gordon decomposition $(0;\frac{1}{2})\otimes(0;\frac{1}{2}) = (0;0)\oplus(0;1)\oplus(1;\frac{1}{2})$, corresponding to the identity, the 12-dimensional adjoint and a non-fundamental representation [25]. At this point we have the basic ingredients to represent the quantum group Hamiltonian (2.4) in terms of the generators of the superalgebra $OSp(3|2)$. The final expression for the Hamiltonian is quite cumbersome and is given in Appendix A.

In the limit $\gamma \to 0$ the Hamiltonian (2.4) can be written in terms of the nearest-neighbour Casimir operator,

$$H = -\sum_{j=1}^{L-1}\left(C_{j,j+1} - 3C_{j,j+1}^2\right) - 4(L-1)\,, \tag{2.7}$$

making explicit the $OSp(3|2)$ invariance at the isotropic point.

3. The Bethe ansatz solution

The Hamiltonian of the q-deformed $OSp(3|2)$ superspin chain with *periodic* boundary condition has been diagonalized using the algebraic Bethe ansatz in Ref. [32]. Within the framework of the analytical Bethe ansatz it has been found that the Bethe equations for the corresponding *open* spin chain with quantum algebra invariance can be obtained from those for the periodic model based on the so-called 'doubling postulate' [34–37]. We have checked that the postulate applies for the $U_q[OSp(3|2)]$ model by comparing the eigenenergies obtained by exact diagonalization of the Hamiltonian (2.4) with those obtained solving the doubled Bethe equations given below for several low-lying states up to $L = 8$.

With these Bethe equations the spectrum of the model is built starting from suitable highest weight reference states. For models based on superalgebras it is well known that the explicit form of the Bethe equations depends on the choice of the grading, see. e.g. [38]. For the $U_q[OSp(3|2)]$ superspin chain this amounts to two different formulations of the Bethe ansatz, i.e. in the grading $fbbbf$ starting from a reference state in the $(L-1;\frac{1}{2})$-multiplet and in the grading $bfbfb$ starting from the $(0;L/2)$-multiplet [20]. In what follows we shall discuss the form of the Bethe equations for these two possible gradings.

3.1. The $fbbbf$ grading

Applying the doubling procedure to the $fbbbf$ Bethe equations of the periodic $U_q[OSp(3|2)]$ model [7,32] we find that the eigenstates of the open $U_q[OSp(3|2)]$ invariant superspin chain (2.4) are parametrized by solutions to the following set of Bethe equations,

$$\left[f_{1/2}\left(\lambda_j^{(1)}\right)\right]^{2L} = \prod_{k=1}^{L-n_1-n_2} f_{1/2}\left(\lambda_j^{(1)} - \lambda_k^{(2)}\right) f_{1/2}\left(\lambda_j^{(1)} + \lambda_k^{(2)}\right)\,, \quad j = 1,\cdots,L-n_1,$$

$$\prod_{k=1}^{L-n_1} f_{1/2}\left(\lambda_j^{(2)}-\lambda_k^{(1)}\right) f_{1/2}\left(\lambda_j^{(2)}+\lambda_k^{(1)}\right)= \\ =\prod_{\substack{k=1\\k\neq j}}^{L-n_1-n_2} f_{1/2}\left(\lambda_j^{(2)}-\lambda_k^{(2)}\right) f_{1/2}\left(\lambda_j^{(2)}+\lambda_k^{(2)}\right), \quad j=1,\cdots,L-n_1-n_2, \tag{3.1}$$

where the function $f_s(\lambda)$ is defined as

$$f_s(\lambda)=\frac{\sinh(\lambda+is\gamma)}{\sinh(\lambda-is\gamma)}. \tag{3.2}$$

In (3.1) n_1, n_2 are the eigenvalues of the $U(1)$ charges (2.5) for a highest weight state in the multiplet $(\mathrm{p};\mathrm{q})=(n_1-1;\frac{1}{2}(n_2+1))$. Its energy is given in terms of the Bethe roots from the first level as,

$$E(\{\lambda_j^{(a)}\},L)=\sum_{j=1}^{L-n_1}\frac{2\sin\gamma}{\cos\gamma-\cosh(2\lambda_j^{(1)})}. \tag{3.3}$$

In order to study the thermodynamic limit properties we first diagonalized the Hamiltonian (2.4) for lattice sizes $L\leq 8$. We next solve numerically the Bethe equations (3.1) for some $U(1)$-sectors (n_1,n_2) and compare the eigenenergies (3.3) with the spectrum obtained by exact diagonalization of Hamiltonian (2.4). For low-lying energy states, similar as in the periodic case [7], we find that the Bethe root configurations on both levels are dominated by two-strings with $\mathrm{Re}\left(\lambda_j^{(1,2)}\right)\geq 0$ and $\mathrm{Im}\left(\lambda_j^{(1,2)}\right)\simeq\pm\gamma/4$. As the system size grows the difference among the root configurations of the two levels becomes exponentially close. Therefore, for $L\to\infty$ the respective string hypothesis may be formulated as

$$\lambda_j^{(1)}=\xi_j\pm i\frac{\gamma}{4},\quad \lambda_j^{(2)}=\xi_j\pm i\frac{\gamma}{4},\quad \xi_j\in\mathbb{R}^+. \tag{3.4}$$

In the thermodynamic limit these strings fill the positive part of the real axis and the root configuration can be described in terms of their density $\sigma_L(\xi)$ within the root density approach [39]. Symmetrizing the density around the origin we obtain the following linear integral equation for $\sigma_L(\xi)$:

$$2\pi\sigma_L(\xi)+\int_{-\infty}^{+\infty}d\xi'\left[2\Psi\left(\xi-\xi',\frac{\gamma}{2}\right)+\Psi\left(\xi-\xi',\gamma\right)\right]\sigma_L(\xi') \\ =2\left[\Psi\left(\xi,\frac{3\gamma}{4}\right)+\Psi\left(\xi,\frac{\gamma}{4}\right)\right]+\frac{1}{L}\left[2\Psi\left(\xi,\frac{\gamma}{2}\right)+\Psi(\xi,\gamma)-2\Psi(2\xi,\gamma)\right], \tag{3.5}$$

where $\Psi(\xi,\gamma)=\frac{2\sin(2\gamma)}{\cosh(2x)-\cos(2\gamma)}$. This equation can be solved by Fourier transformation order by order in L. To leading order one finds

$$\sigma_\infty(x)=\frac{2}{\gamma\cosh(2\pi x/\gamma)}, \tag{3.6}$$

from which we reproduce the ground state energy density

$$\varepsilon_\infty=-2\cot\frac{\gamma}{2}. \tag{3.7}$$

The low energy excitations above the ground state are gapless with a linear dispersion relation $\epsilon(p) \simeq v_F|p|$ where the Fermi velocity is $v_F = 2\pi/\gamma$. These quantities are already known from the q-deformed model with periodic boundary conditions [7]. Similarly, one obtains the surface energy f_∞ from the $\mathcal{O}(L^{-1})$ contribution to the solution of (3.5). After some simplifications we find

$$f_\infty^{(OSp)} = 2\cot\frac{\gamma}{2}\,, \tag{3.8}$$

which is just the value for the ground state energy per site with the opposite sign.

3.2. The grading $bfbfb$

Alternatively we can use the doubling procedure in the Bethe ansatz solution of the model with periodic boundaries for the grading $bfbfb$ [7]. In this case the spectrum of the open $U_q[OSp(3|2)]$ invariant superspin chain is parametrized by solutions of a different set of Bethe equations,

$$\begin{aligned}
&\left[f_{1/2}\left(\lambda_j^{(1)}\right)\right]^{2L} = \prod_{k=1}^{L-m_1-m_2} f_{1/2}\left(\lambda_j^{(1)}-\lambda_k^{(2)}\right) f_{1/2}\left(\lambda_j^{(1)}+\lambda_k^{(2)}\right), \quad j=1,\cdots,L-m_2,\\
&\prod_{k=1}^{L-m_2} f_{1/2}\left(\lambda_j^{(2)}-\lambda_k^{(1)}\right) f_{1/2}\left(\lambda_j^{(2)}+\lambda_k^{(1)}\right) =\\
&\quad= \prod_{\substack{k=1\\k\neq j}}^{L-m_1-m_2} f_{-1/2}\left(\lambda_j^{(2)}-\lambda_k^{(2)}\right) f_{-1/2}\left(\lambda_j^{(2)}+\lambda_k^{(2)}\right) f_1\left(\lambda_j^{(2)}-\lambda_k^{(2)}\right) f_1\left(\lambda_j^{(2)}+\lambda_k^{(2)}\right),\\
&\qquad j=1,\cdots,L-m_1-m_2\,,
\end{aligned} \tag{3.9}$$

and the energy of the Hamiltonian (2.4) corresponding to a particular root configuration is

$$E\left(\{\lambda_j^{(a)}\},L\right) = -\sum_{j=1}^{L-m_2}\frac{2\sin\gamma}{\cos\gamma-\cosh(2\lambda_j^{(1)})} - 2(L-1)\cot\gamma\,. \tag{3.10}$$

As mentioned above, the Bethe states for $fbbbf$ and $bfbfb$ are constructed starting from different reference states. Solutions to (3.9) parametrize the state with $U(1)$-charges $(n_1,n_2) = (m_1+1, m_2-1)$ in a $(\mathrm{p};\mathrm{q}) = (m_1; m_2/2)$ multiplet.

It turns out that the thermodynamic limit in the $bfbfb$ grading simplifies studies in the subsector with quantum number $m_1 = 0$ because the number of Bethe roots in both levels are the same. Numerical solution of the Bethe equations (3.9) for small systems tells us that the Bethe root configurations are dominated by pairs of complex rapidities with positive real parts and $\mathrm{Im}\left(\lambda_j^{(1)}\right) \simeq \pm 3\gamma/4$, $\mathrm{Im}\left(\lambda_j^{(2)}\right) \simeq \pm\gamma/4$. Once again the difference of their real parts becomes exponentially small for large L. Similar to what has been found for the periodic case [7], the string hypothesis for the $bfbfb$ grading is given by,

$$\lambda_{j,\pm}^{(1)} = \xi_j \pm i\frac{3\gamma}{4}\,, \quad \lambda_{j,\pm}^{(2)} = \xi_j \pm i\frac{\gamma}{4}\,, \quad \xi_j \in \mathbb{R}^+\,. \tag{3.11}$$

The root density approach based on this string assumption can be used to reproduce the bulk energy density, Fermi velocity of low energy excitations and surface energy obtained in the grading $fbbbf$.

4. The $U_q[SU(2)]$ spin $S = 1$ XXZ model

The Hamiltonian of the integrable $S = 1$ XXZ model with quantum algebra symmetry is [40,41]

$$H = \sum_{j=1}^{L-1} H_{j,j+1} + i(S_L^z - S_1^z) \tag{4.1}$$

where the nearest-neighbour bulk terms $H_{j,j+1}$ are those of the integrable spin-1 model introduced by Zamolodchikov and Fateev [42],

$$\begin{aligned} H_{j,j+1} =& \frac{1}{\sin(\gamma)}\left[\mathbf{S}_j \cdot \mathbf{S}_{j+1} - (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2\right] \\ & - \tan(\frac{\gamma}{2})\left[S_j^z S_{j+1}^z + (S_j^z)^2 + (S_{j+1}^z)^2 - (S_j^z S_{j+1}^z)^2\right] \\ & + \frac{4\sin^2(\frac{\gamma}{4})}{\sin(\gamma)}\left[(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)S_j^z S_{j+1}^z + S_j^z S_{j+1}^z (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)\right] \\ & + \left(1 + 2\tan(\frac{\gamma}{2})\right)\mathbf{1}_j \mathbf{1}_{j+1}, \end{aligned} \tag{4.2}$$

where the last constant term is a convenient normalization for the spectral relationship to the $U_q[OSp(3|2)]$ superspin chain. The $\mathbf{S} = (S^x, S^y, S^z)$ are the spin-1 generators of $SU(2)$

$$S^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.3}$$

and $\mathbf{1}$ is the 3×3 identity. The Hamiltonian (4.1) has been diagonalized using the algebraic Bethe ansatz [43]. The eigenstates of this model are parameterized by the $L - n$ complex roots λ_k of the Bethe equations,

$$\left[f_{1/2}(\lambda_k)\right]^{2L} = \prod_{\substack{\ell \neq k \\ \ell=1}}^{L-n} f_{1/2}(\lambda_k - \lambda_\ell)\, f_{1/2}(\lambda_k + \lambda_\ell), \quad k = 1, \cdots, L-n, \tag{4.4}$$

where n is the $U(1)$-charge $\sum_{j=1}^L S_j^z$ for the highest weight state in a $U_q[SU(2)]$ spin-j multiplet, i.e. $n = j$. In terms of these roots the corresponding energies of the Hamiltonian (4.1) are

$$E(\{\lambda_k\}, L) = \sum_{k=1}^{L-n} \frac{2\sin\gamma}{\cos\gamma - \cosh(2\lambda_k)}. \tag{4.5}$$

For large L, the Bethe roots configurations for the low-lying energies are essentially dominated by pairs of complex conjugate roots

$$\lambda_{k,\pm} \simeq \xi_k \pm i\frac{\gamma}{4}, \quad \xi_k \in \mathbb{R}^+. \tag{4.6}$$

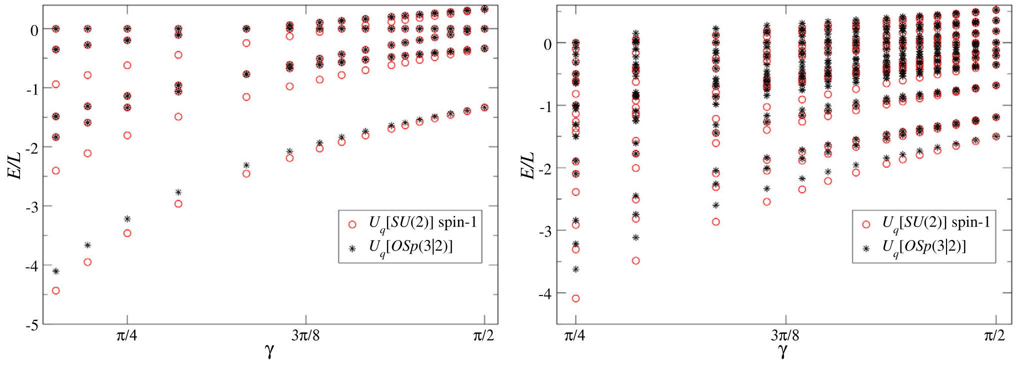

Fig. 1. Spectra of the $U_q[OSp(3|2)]$ superspin chain and the quantum group invariant XXZ spin-1 model as a function of the anisotropy γ for $L = 3$ (left panel) and $L = 4$ (right panel). Note that for $\gamma = \pi/2$ the spectra coincide up to degeneracies.

In the thermodynamic limit the density of these 2-strings is given by the integral equation

$$2\pi\sigma_L(\xi) + \int_{-\infty}^{+\infty} d\xi' \left[2\Psi\left(\xi-\xi',\frac{\gamma}{2}\right) + \Psi(\xi-\xi',\gamma)\right]\sigma_L(\xi') \\ = 2\left[\Psi\left(\xi,\frac{3\gamma}{4}\right)+\Psi\left(\xi,\frac{\gamma}{4}\right)\right] + \frac{1}{L}\left[2\Psi\left(\xi,\frac{\gamma}{2}\right)+\Psi(\xi,\gamma)+2\Psi(2\xi,\gamma)\right] \tag{4.7}$$

obtained within the root density approach.

Comparing Eq. (4.7) with the corresponding result for the $U_q[OSp(3|2)]$ superspin chain given by Eq. (3.5) we note that they only differ in the sign of the driving term $2\Psi(2\xi,\gamma)/L$. Therefore both the bulk energy density ε_∞ and the Fermi velocity v_F of the models (2.4) and (4.1) coincide in the critical region and only the corresponding surface energies f_∞ differ. Considering the thermodynamic limit of the relation (4.7) we find

$$f_\infty^{(XXZ)} = 2\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\cosh[(\pi-\gamma)x]\tanh(\gamma x)}{\sinh(\pi x)} \tag{4.8}$$

for the integrable quantum group invariant $U_q[SU(2)]$ spin-1 chain.

Note, however, that $\Psi(2\xi,\gamma)$ vanishes for $\gamma = \pi/2$ and therefore both the bulk and surface energies of the two models coincide. It has already been observed that the XXZ spin-1 chain and the q-deformed $OSp(3|2)$ model with periodic boundary conditions have some common eigenvalues for this value of the anisotropy [7]. Motivated by this we have compared the eigenspectra of the quantum group invariant Hamiltonians (2.4) and (4.1) for small lattice sizes using exact diagonalizations. Remarkably, we find perfect matching of the spectra of these Hamiltonians for $\gamma = \frac{\pi}{2}$ apart from degeneracies[2] due to the different sizes of the Hilbert space, see Fig. 1. Within

[2] Note that there appear additional degeneracies in the quantum group invariant models at $\gamma = \pi/2$: for this value of the anisotropy the lowest excitation of the $U_q[SU(2)]$-invariant model is eight-fold degenerate and decomposes into a triplet with total spin $j = 1$ and a quintet with total spin $j = 2$. For L even the corresponding level in the $U_q[OSp(3|2)]$-

our numerical precision this feature has also been checked for the complete spectrum of both models up to the size $L = 8$. In addition we have observed this correspondence of the spectra continues to hold from the point of view of solutions of the corresponding Bethe ansatz equations: for small L we find that the level-1 Bethe roots for the state with $(\mathrm{p};\mathrm{q}) = (0; 1)$ in the $fbbbf$ grading approach those for the $n = 1$ state in the $U_q[SU(2)]$ chain as $\gamma \to \pi/2$. As a consequence of (3.3) and (4.5) their energies coincide in this limit.

The critical behaviour. The low energy spectrum of the integrable spin-1 XXZ spin chain in the critical regime $0 \le \gamma \le \pi$ subjected to various boundary conditions has been studied extensively [24,44–48]. The finite-size spectrum of the quantum group invariant model has first been investigated for the lowest states with $U_q[SU(2)]$-spin $j = 0$ and 1 in Ref. [24] for even length chains giving the conformal anomaly and the conformal weight $h_{j=1}$ as a function of the anisotropy

$$c_{\mathrm{qg}} = \frac{3}{2}\left(1 - \frac{2\gamma^2}{\pi(\pi-\gamma)}\right), \quad h_{j=1} = 1 - \frac{\gamma}{\pi}. \tag{4.9}$$

Later on it has been proposed that the full operator content of the $U_q[SU(2)]$-invariant spin chain can be obtained by combining a twisted free boson field with compactification radius depending on the anisotropy and an Ising field with free boundaries. More precisely the conformal weights, in our notation, are [48]

$$h_j = \frac{j}{2}\left(j - \frac{\gamma}{\pi}(j+1)\right) + h_I, \tag{4.10}$$

where $h_I = 0$ or $\frac{1}{2}$ depending on whether j is even or odd is related to the Ising degree of freedom [49]. Note that c_{qg} vanishes for $\gamma = \frac{\pi}{2}$ together with the subleading corrections to scaling to the ground state. This is in accordance with the mentioned correspondence with the $U_q[OSp(3|2)]$ symmetric superspin chain.

As a consequence of the antiferromagnetic character of the quantum group invariant spin chain the ground state is typically frustrated which may lead to an excited state with distinct exponent when the thermodynamic limit is taken for L odd. Based on our numerical results shown in Fig. 2 we are led to complement the proposal of Ref. [48] for the operator content of the conformal field theory describing the continuum limit of the $U_q[SU(2)]$ spin-1 chain to be given by (4.10) but with

$$h_I = \begin{cases} 0 & \text{for } L+j \text{ even} \\ \frac{1}{2} & \text{for } L+j \text{ odd} \end{cases}. \tag{4.11}$$

In addition we have identified the Bethe root configuration for the first excitation in the $j = 0$ sector for chains with even L. This is a descendent of the identity with conformal weight $h = 2$ independent of the anisotropy which we identify with the stress tensor. The results are exhibited Fig. 3.

We emphasize that this spectrum of conformal weights is discrete for all values of γ. At rational values of γ/π the conformal weights can be rearranged in terms of an extended algebra

invariant model consists of a $(\mathrm{p};\mathrm{q}) = (0; 1)$ multiplet and a $(1; 1/2)$ multiplet. For L odd it is an indecomposable consisting of two $(0; \frac{1}{2})$ quintets and one $(1; 1)$ multiplet. The total degeneracies are $12 + 12 = 24$ for L even and $5 + 5 + 30 = 40$ for L odd.

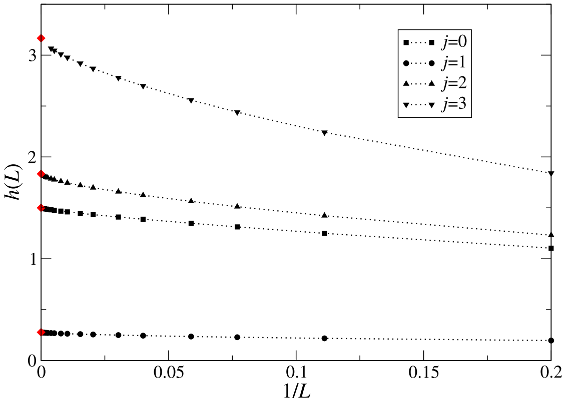

Fig. 2. Finite-size estimates $h_j(L) = (L/\pi v_F)\left(E_j(L) - E_0(L)\right)$ of the conformal weights corresponding to the ground states of the quantum group invariant spin-1 chain (4.1) in the sectors with total spin $j = 0, 1, 2, 3$ for $\gamma = 2\pi/9$ and odd length L. Red diamonds indicate the values proposed in (4.10) with (4.11).

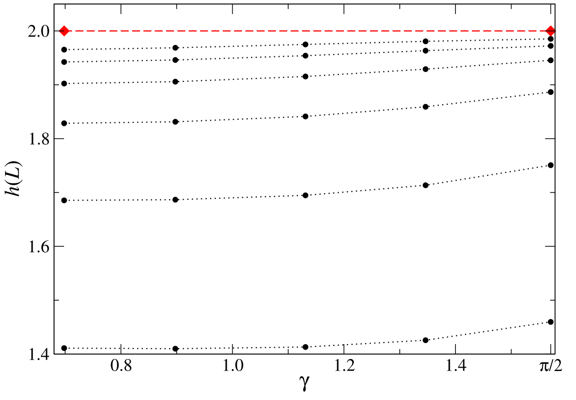

Fig. 3. Finite-size estimates $h(L) = (L/\pi v_F)\left(E_{0,x}(L) - E_0(L)\right)$ of the conformal weight corresponding to the first excitation in the total spin $j = 0$ sector of the quantum group invariant spin-1 chain (4.1) for various γ. The dotted lines connecting data for even lengths $L = 2^k$, $k = 2, \ldots, 7$ are a guide to the eye only. The data converge to $h \equiv 2$ independent of γ as $L \to \infty$ (red dashed line).

with a finite number of primaries. Specifically, for $\gamma = \pi/2$ (4.10) with (4.11) take only integer and half-odd integer values.

Given the spectral correspondence with the $U_q[OSp(3|2)]$ symmetric superspin chain the critical properties of the quantum group invariant spin-1 chain will provide additional input to elaborate on the conformal content of the $U_q[OSp(3|2)]$ symmetric superspin chain below.

5. Finite-size spectrum of the $U_q[OSp(3|2)]$ chain

Based on the exact diagonalization of small systems we find that the ground state of the $U_q[OSp(3|2)]$ superspin chain with quantum-group invariant boundary conditions is a (0; 0)-singlet for L even ((0; 1/2)-quintet for L odd) with energy

$$E_0 \equiv L\epsilon_\infty + f_\infty^{(OSp)} = -2(L-1)\cot(\gamma/2)\,, \tag{5.1}$$

i.e. without any finite-size corrections implying that the effective central charge of the field theory is $c_{\rm eff} = 0$.[3] For odd L Eq. (5.1) can be verified by solving the $bfbfb$ Bethe equations (3.9) for the quintet where we find root configurations containing $(L-1)/2$ complex conjugate pairs of roots on each level which are arranged in groups similar to (3.11). The root configurations for even L contain degenerate roots.

For the finite-size spectrum of the lowest states in the sectors ($\mathfrak{p} = 0$; $\mathfrak{q} \geq 1$) we have solved the Bethe equations in $bfbfb$ grading (3.9). For $\mathfrak{q} = 1$ we find that the numerical estimates for the conformal weights

$$h_{\rm eff}(L) = \frac{L}{\pi v_F}\left(E(L) - L\,\epsilon_\infty - f_\infty^{(OSp)}\right), \tag{5.2}$$

converge to the $h_{(0;1)} = \gamma/\pi$ (plus positive integers for the descendent fields) for $L \to \infty$, in accordance with our proposal Eq. (1.2). For all states we observe subleading corrections to scaling in the conformal weights vanishing as $h_{\rm eff}(L) - h_{(0;1)} \propto L^{-\alpha}$ with an exponent $\alpha = \gamma/(\pi - \gamma)$. Such power law corrections have also been observed in the periodic $U_q[OSp(3|2)]$ model [7]. Typically they originate from the presence of irrelevant operators with scaling dimensions larger than two [50]. Note that this perturbation becomes marginal ($\alpha \to 0$) in the isotropic limit where it becomes the source of the logarithmic fine structure observed in the conformal spectrum of the $OSp(3|2)$ chain with free boundaries [21]. We note that the finite size estimates of the conformal weights for some of the descendent fields are complex. We find, however, that the imaginary parts are again subleading corrections to scaling which vanish as $L^{-\alpha}$ in the thermodynamic limit, see Fig. 4.

Proceeding in the same way for the lowest states in the sectors (0; $\mathfrak{q}$) with $\mathfrak{q} = \frac{3}{2}$ and 2 we find that the finite size data converge to the proposed values $h_{(0;\frac{3}{2})} = 3\gamma/\pi$ and $h_{(0;2)} = 6\gamma/\pi$ again with subleading power law corrections $\propto L^{-\alpha}$, $\alpha = \gamma/(\pi - \gamma)$, see Fig. 5. Again, the energies of the level-2 and -3 descendents of the (0; $\frac{3}{2}$) primary are complex for finite chains with their imaginary parts vanishing as $L^{-\alpha}$ the thermodynamic limit.

Note that for $\gamma \to 0$ the conformal weights of the (0; $\mathfrak{q}$)-primaries for $\mathfrak{q} = 1, \frac{3}{2}, 2, \ldots$ degenerate with those of the (0; 0)- and (0; $\frac{1}{2}$)-vacua of the even and odd length chain, i.e. $h = 0$, in the thermodynamic limit. The apparent degeneracy is lifted by the subleading corrections which become logarithmic in the limit $\gamma \to 0$. This is consistent with what has been found for the periodic model [18–20] and the observation in our previous study of the $OSp(3|2)$-symmetric superspin chain subject to free boundary conditions [21] where we found

$$h_{(0;\mathfrak{q})}\Big|_{\gamma=0} \simeq \frac{2\mathfrak{q}(2\mathfrak{q}-1)}{\log L}. \tag{5.3}$$

A similar behaviour can be observed for the conformal weights of the (1; $\mathfrak{q}$)-primaries (the $fbbbf$ Bethe equations (3.1) have been solved to study the lowest states in the sectors ($\mathfrak{p} \geq 1$; $\mathfrak{q}$)): for $\mathfrak{q} = \frac{1}{2}$ and $\frac{3}{2}$ these are realized in the finite size spectrum of even L chains. The corresponding finite size data for the primaries and some descendents are shown in Fig. 6. In the thermodynamic limit the effective conformal weights of the primaries in these sectors converge to $h_{(1,\frac{1}{2})} = 1 - \gamma/\pi$ and $h_{(1,\frac{3}{2})} = 1 + 2\gamma/\pi$, respectively, in agreement with our proposal (1.2). Again one observes power law corrections to scaling vanishing as $L^{-\alpha}$.

[3] This has also been found for the isotropic $OSp(3|2)$ superspin chain with both periodic and free boundary conditions [20,21]. Note that (5.1) is *not* the lowest energy in the q-deformed $OSp(3|2)$ model subject to periodic boundary conditions [7].

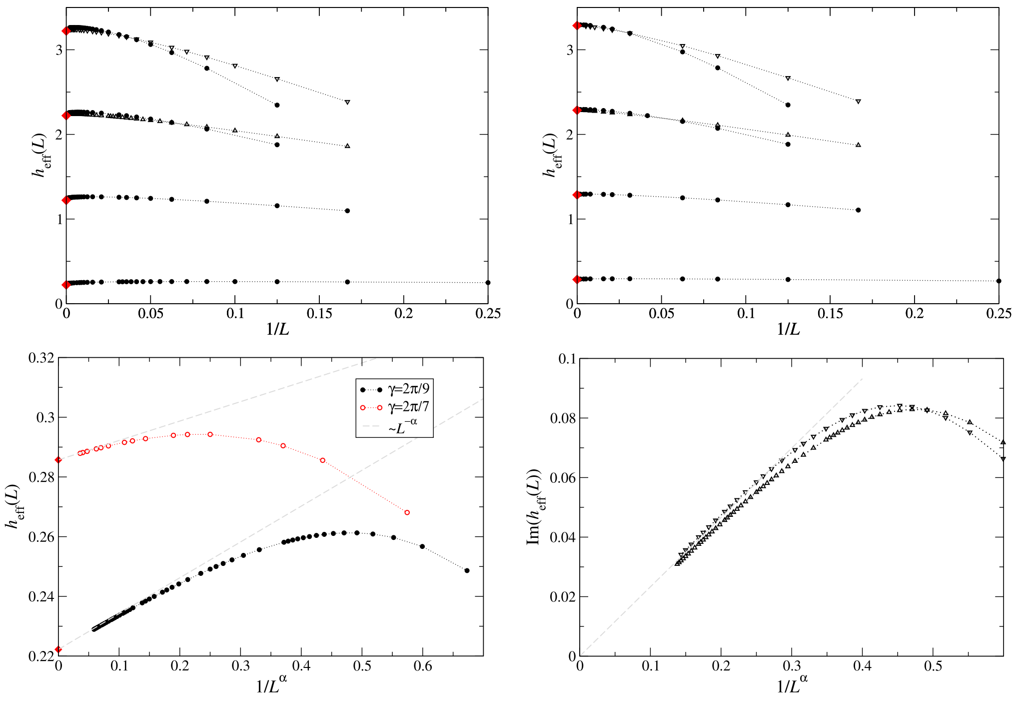

Fig. 4. Effective conformal weights $h_{\mathrm{eff}}(L)$ of the lowest states in the sector $(\mathfrak{p};\mathfrak{q}) = (0; 1)$: in the top panels the finite size data are displayed for $\gamma = 2\pi/9$ (left panel) and $\gamma = 2\pi/7$ (right panel). Red diamonds indicate the conformal weights $h_{(0;1)} = \gamma/\pi + k$, $k = 0, 1, 2, 3$. Note that two of the descendent fields correspond to complex energies in the finite size spectrum. For these the real part of the conformal weight is shown (marked by $\triangle$ and ∇). The lower left panel shows finite size data of the lowest states which extrapolate to $h_{(0;1)} = \gamma/\pi$ in the thermodynamic limit (indicated by red $\diamond$). The dashed lines indicate the conjectured corrections to scaling $\propto L^{-\alpha}$ with $\alpha = \gamma/(\pi - \gamma)$. Similarly, in the lower right panel the vanishing of the imaginary parts of the complex energies for $\gamma = 2\pi/9$ with the same subleading power law $\propto L^{-\alpha}$ is shown.

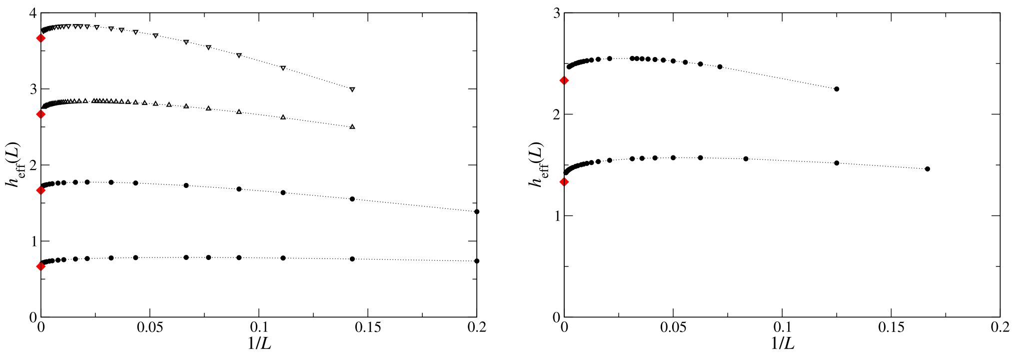

Fig. 5. Finite size data for the primary and some descendents in the $(\mathfrak{p};\mathfrak{q}) = (0; \frac{3}{2})$- (left panel) and (0; 2)-sector (right panel) for $\gamma = 2\pi/9$. Red diamonds indicate the conformal weights $h_{(0;\frac{3}{2})} = 3\gamma/\pi + k$, $k = 0, 1, 2, 3$ and $h_{(0;2)} = 6\gamma/\pi + k$, $k = 0, 1$. Note that the level-2 and -3 descendents of the $(0; \frac{3}{2})$ appear as complex energies in the finite size spectrum (triangular symbols). Similar as for the complex levels in the (0; 1)-sector shown in Fig. 4 the imaginary part of the effective conformal weights vanishes with the subleading power law in the thermodynamic limit.

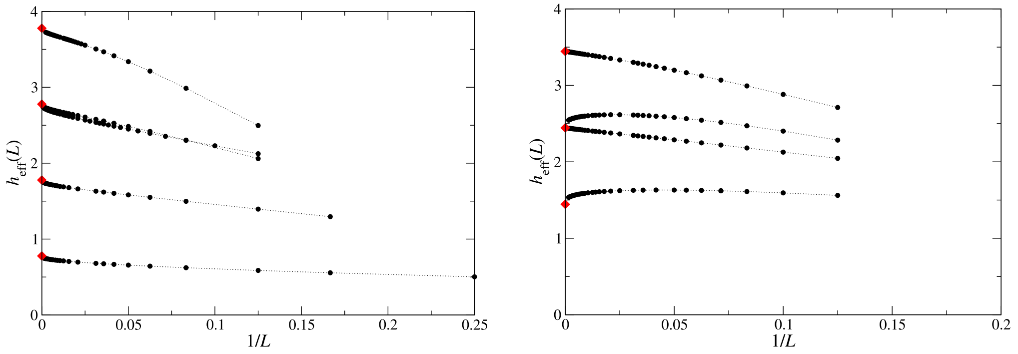

Fig. 6. Finite size data for the primary and some descendents in the $(\mathrm{p};\mathrm{q}) = (1;\frac{1}{2})$- (left panel) and $(1;\frac{3}{2})$-sector (right panel) for $\gamma = 2\pi/9$. Red diamonds indicate the conformal weights $h_{(1;\frac{1}{2})} = 1 - \gamma/\pi + k$, $k = 0, 1, 2, 3$ and $h_{(1;\frac{3}{2})} = 1 + 2\gamma/\pi + k$, $k = 0, 1, 2$.

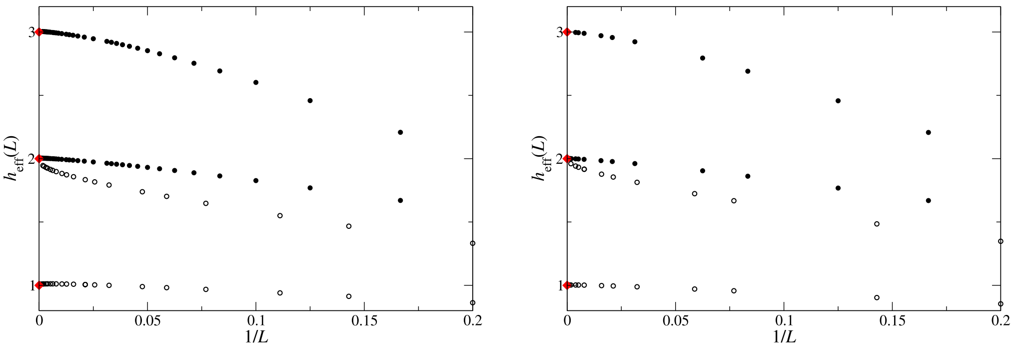

Fig. 7. Finite size data for the primary and some descendents in the $(\mathrm{p};\mathrm{q}) = (1;1)$-sector for $\gamma = 2\pi/9$ (left panel) and $\gamma = 2\pi/7$ (right panel). Red diamonds indicate the conformal weights $h_{(1;1)} = 1 + k$, $k = 0, 1, 2$. Open (filled) symbols denote data for odd (even) lattice length.

As mentioned in the introduction the $(1;1)$ representation is atypical and appears chains of length with either parity as part of indecomposables. According to (1.2) the conformal weights of operators for this representation are integers, independent of the anisotropy γ. This is supported by the finite size estimates of the lowest two conformal weights for both odd and even length presented in Fig. 7. For L odd the ground state in this sector extrapolates to $h_{(1;1)} = 1$, independent of the anisotropy. The lowest $(1;1)$-level observed for even length corresponds to a field with conformal weight $h = 2$ which we may identify with the stress tensor.

As in the case of the $\mathrm{p} = 0$ the $(\mathrm{p};\mathrm{q}) = (1,\mathrm{q})$-primaries degenerate to give the integer conformal weight $h_{(1;\mathrm{q})}\big|_{\gamma=0} = 1$ in the thermodynamic limit. The subleading corrections become logarithmic consistent with the critical behaviour observed for the isotropic $OSp(3|2)$ model [20].

Finally, we have studied the finite size spectrum of levels in the $(\mathrm{p};\mathrm{q}) = (2;\mathrm{q})$-representations, see Fig. 8. For $\mathrm{q} = \frac{1}{2}, 1$ the effective conformal weights of primaries and first descendents extrapolate to the proposal (1.2) for $h_{(2;\mathrm{q})}$ and $h_{(2;\mathrm{q})} + 1$. As for the sectors with $\mathrm{p} = 0, 1$ they degenerate in the isotropic limit $\gamma \to 0$, consistent with what has been found previously [19,20].

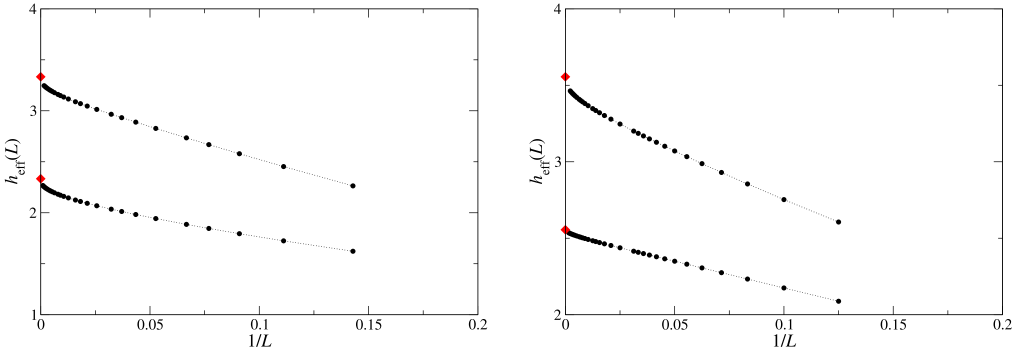

Fig. 8. Finite size data for the primary and first descendents in the $(\mathrm{p};\mathrm{q}) = (2;\frac{1}{2})$- (left panel) and (2; 1)-sector (right panel) for $\gamma = 2\pi/9$. Red diamonds indicate the conformal weights $h_{(2;\frac{1}{2})} = 3 - 3\gamma/\pi + k$ and $h_{(2;1)} = 3 - 2\gamma/\pi + k$, both for $k = 0, 1$.

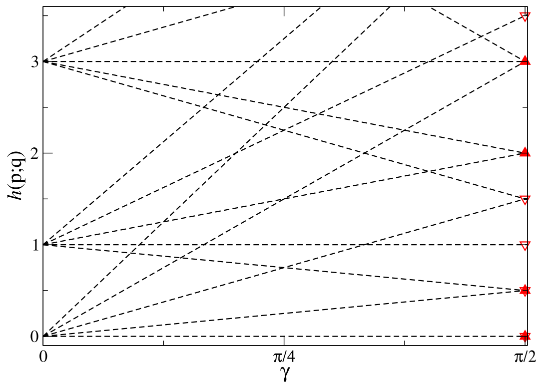

Fig. 9. Conformal weights (1.2) of primaries $(\mathrm{p};\mathrm{q})$ with $\mathrm{p} = 0, 1, 2, 3$, $\mathrm{q} \le \frac{5}{2}$ in the $U_q[OSp(3|2)]$-invariant superspin as function of the anisotropy: the degeneracies of weights with different q in the Goldstone phase of the isotropic model are lifted for any $\gamma > 0$. For $\gamma = \pi/2$ the possible weights coincide with those of the quantum group invariant XXZ spin-1 chain (4.10) and (4.11) (red triangles, filled (open) for even (odd) chain lengths).

6. Summary and conclusion

Based on our analysis of the finite-size spectrum of the $U_q[OSp(3|2)]$-invariant superspin chain in the previous section together with the spectral correspondence to the quantum group invariant XXZ spin-1 spin chain we arrive at the proposal (1.2) for the operator content of the former. We emphasize that this proposal is consistent with previous results for the isotropic superspin chain: for $\gamma \to 0$ the conformal weights $h_{(p;q)} = \mathrm{p}(\mathrm{p}+1)/2$ coincide with those identified in the periodic model [20]. Moreover, the amplitudes of the subleading logarithmic corrections in the $(0;\mathrm{q})$-multiplets of the isotropic model with free boundaries [21] have the same q-dependence as (1.2).

For the models with $\gamma > 0$ a similar link to the q-deformed superspin chain with periodic boundary conditions can not be established. In view of what is known for the periodic model the most striking property of the quantum group invariant one studied in the present paper is its purely discrete spectrum of conformal weights for $0 < \gamma < \pi$, see Fig. 9, while there have been indications for the presence of continuous components in the periodic model [7]. This resembles the behaviour observed in the open staggered six-vertex model: depending on the choice of boundary conditions this system flows to different fixed points with the non-compact one being unstable, see Refs. [14–17].

At this point we remark that solutions of the boundary Yang-Baxter equation for the q-deformed $OSp(3|2)$ spin chain beyond the quantum group invariant one have been investigated in the literature: in [51] the author claims that there exist two additional diagonal solutions (the most amenable ones for a Bethe ansatz solution). Just as the constant boundary matrices $K^{\pm}$ considered here, none of these does contain any free parameter though. It may well be that non-compact degrees of freedom are present in the scaling limit of the superspin chain with one of these additional boundary conditions (or a combination of two of the three known ones). In the absence of a free parameter such integrable models would however be isolated points in the space of boundary parameters and the system sizes needed for tests of whether or not the corresponding critical properties are robust against perturbations appear to be out of reach for numerical approaches.

Another obstacle is the lack of knowledge on the conformal spectrum of the periodic superspin chain where the connection to the spectrum of the periodic Zamolodchikov-Fateev spin-1 chain for $\gamma = \pi/2$ is limited to certain charge-sectors. While this has been taken as a hint that the fields appearing in the effective low energy theory for the periodic $U_q[OSp(3|2)]$ model may be composites of two Gaussian fields and an Ising operator the preliminary proposal for the scaling dimensions of the periodic superspin chain is far from a complete description of the full operator content of the critical model. In particular, the dependence of the scaling dimensions on the two possible twist angles related to the conserved $U(1)$ charges has not been addressed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The Bethe roots and finite size spectral data used in Figs. 2–8 are available from the Research Data Repository at Leibniz Universität Hannover [52].

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Appendix A. The quantum group invariant Hamiltonian

Here we write the quantum group invariant Hamiltonian (2.4) in the following form,

$$H = \sum_{j=1}^{L-1} H^{b}_{j,j+1} + H^{s}_{L,1} \tag{A.1}$$

where $H^b_{j,j+1}$ represents the two-body bulk term and $H^s_{L,1}$ encodes the surface interactions.

The surface term can be represented in the terms of azimuthal bosonic operators as follows

$$H^s_{L,1} = -i(\tau^z_L - \tau^z_1) - 2i(\sigma^z_L - \sigma^z_1) \tag{A.2}$$

The bulk term encodes both the bosonic and the fermionic generators and it is given by,

$$\begin{aligned}
H^b_{j,j+1} = &-\frac{\left(C_{j,j+1} - 3C^2_{j,j+1}\right)}{\sin(\gamma)} + \delta_1 \tau^z_j \tau^z_{j+1} + \delta_2 \sigma^z_j \sigma^z_{j+1} + \sin(\gamma)\left(\tau^z_j\tau^z_{j+1}\right)^2 + \delta_3\left(\sigma^z_j\sigma^z_{j+1}\right)^2 \\
&+ \frac{\tan(\frac{\gamma}{2})}{2}\left[(\tau^+_j\tau^-_{j+1} + \tau^-_j\tau^+_{j+1})^2 + 2(\tau^z_j)^2 + 2(\tau^z_{j+1})^2 + 24(\sigma^z_j)^2 + 24(\sigma^z_{j+1})^2\right] \\
&- i\left[2+\cos(\gamma)\right]\tau^z_j\tau^z_{j+1}(\tau^z_{j+1} - \tau^z_j) + 8i\cos(\gamma)\sigma^z_j\sigma^z_{j+1}(\sigma^z_{j+1} - \sigma^z_j) \\
&- t^-_1\left(c^+_j c^-_{j+1} c^-_j c^+_{j+1} + f^-_j f^+_{j+1} f^+_j f^-_{j+1} + \tau^z_j\tau^z_{j+1} c^+_j c^-_{j+1} - 4\sigma^z_j\sigma^z_{j+1} c^-_j c^+_{j+1}\right) \\
&- t^+_1\left(c^-_j c^+_{j+1} c^+_j c^-_{j+1} + f^+_j f^-_{j+1} f^-_j f^+_{j+1} - \tau^z_j\tau^z_{j+1} c^-_j c^+_{j+1} + 4\sigma^z_j\sigma^z_{j+1} c^+_j c^-_{j+1}\right) \\
&- \frac{t^+_2}{2}\left[(\tau^+_j\tau^-_{j+1} + \tau^-_j\tau^+_{j+1})\tau^z_j\tau^z_{j+1} - (\tau^+_j\tau^-_{j+1} + \tau^-_j\tau^+_{j+1}) f^-_j f^+_{j+1} - 8\sigma^z_j\sigma^z_{j+1} d^-_j d^+_{j+1}\right] \\
&- \frac{t^-_2}{2}\left[\tau^z_j\tau^z_{j+1}(\tau^+_j\tau^-_{j+1} + \tau^-_j\tau^+_{j+1}) + (\tau^+_j\tau^-_{j+1} + \tau^-_j\tau^+_{j+1}) f^+_j f^-_{j+1} + 8\sigma^z_j\sigma^z_{j+1} d^+_j d^-_{j+1}\right] \\
&+ t^+_3\left(\tau^z_j\tau^z_{j+1} f^-_j f^+_{j+1} - 4\sigma^z_j\sigma^z_{j+1} f^+_j f^-_{j+1}\right) - t^-_3\left(\tau^z_j\tau^z_{j+1} f^+_j f^-_{j+1} - 4\sigma^z_j\sigma^z_{j+1} f^-_j f^+_{j+1}\right) \\
&+ 4i\cos(\frac{\gamma}{2})\left(f^-_j f^+_{j+1} d^-_j d^+_{j+1} - d^-_j d^+_{j+1} f^-_j f^+_{j+1}\right) - 2\left(\tan(\frac{\gamma}{2}) + \frac{2}{\sin(\gamma)}\right)\mathrm{I_j I_{j+1}}
\end{aligned} \tag{A.3}$$

where I_j denotes the 5×5 identity acting on the jth lattice site. The dependence of the coupling parameters on the anisotropy γ is,

$$\begin{aligned}
&\delta_1 = 2\sin^2(\frac{\gamma}{2})\tan(\frac{\gamma}{2}), \quad \delta_2 = 4\left[2+\cos(\gamma)\right]\tan(\frac{\gamma}{2}), \quad \delta_3 = -16\left[6+\cos(\gamma)\right]\tan(\frac{\gamma}{2}) \\
&t^\pm_1 = \pm i\frac{\exp(\pm i\frac{\gamma}{2})}{\cos(\frac{\gamma}{2})}, \quad t^\pm_2 = \pm i\frac{\exp(\pm i\frac{\gamma}{2})}{\cos(\frac{\gamma}{2})}\left(1 + \exp(\pm i\frac{\gamma}{4})\frac{\cos(\frac{\gamma}{2})}{\cos(\frac{\gamma}{4})}\right) \\
&t^\pm_3 = \pm i\frac{\exp(\pm i\frac{\gamma}{2})}{\cos(\frac{\gamma}{2})}\left(1 + 2\exp(\pm i\frac{\gamma}{2})\cos(\frac{\gamma}{2})\right)
\end{aligned} \tag{A.4}$$

We finally remark that one may use the commutation relations among the bosonic and fermionic generators of the $OSp(3|2)$ superalgebra to write alternative expressions to the bulk Hamiltonian.

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