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Prescribed mean curvature flow of non-compact space-like Cauchy hypersurfaces

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Abstract

In this paper we consider the prescribed mean curvature flow of a non-compact space-like Cauchy hypersurface of bounded geometry in a generalized Robertson–Walker space-time. We prove that the flow preserves the space-likeness condition and exists for infinite time. We also prove convergence in the setting of manifolds with boundary. Our discussion generalizes previous work by Ecker, Huisken, Gerhardt and others with respect to a crucial aspects: we consider any non-compact Cauchy hypersurface under the assumption of bounded geometry. Moreover, we specialize the aforementioned works by considering globally hyperbolic Lorentzian space-times equipped with a specific class of warped product metrics.

Keywords Mean curvature flow · Non-compactness · Generalized Robertson–Walker space-times · Prescribed mean curvature

Mathematics Subject Classification 53E10 · 58J35 · 83C05

1 Introduction and statement of the main result

We are interested in maximal space-like Cauchy hypersurfaces, where maximality refers to vanishing mean curvature, and more generally in space-like Cauchy hypersurfaces with prescribed mean curvature in globally hyperbolic Lorentzian space-times. These play an important role in gravitational physics, such as in the first proof of the positive mass theorem by Schoen and Yau [27, 28] and the analysis of the Cauchy problem for asymptotically flat space-times by Choquet-Bruhat and York [7] and Lichnerowicz [22]. We also refer the reader, for example, to Bartnik [3] and references therein for an overview.

Construction of such Cauchy hypersurfaces using the prescribed mean curvature flow has been pioneered by Ecker and Huisken [9]. The prescribed mean curvature flow in a semi-

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Riemannian manifold $(N, \overline{g})$ is a family of embeddings $F(t) \equiv F(\cdot, t) : M \to N$ of a smooth manifold M with parameter $t \in [0, T]$ in some interval, satisfying an initial value problem

$$\partial_t F(t) = -(H - \mathcal{H})\mu \quad F(t=0) = F_0, \tag{1.1}$$

where $\mathcal{H} : M \to \mathbb{R}$ is the prescribing function, H is the mean curvature of $F(t)M \subset N$ and F_0 is some initial embedding. Under the mean curvature flow, for every point p in M the normal velocity at which $F(p, t)$ moves is given by the mean curvature of $F(t)(M)$ at $F(t, p)$ minus $\mathcal{H}$. If $\mathcal{H} \equiv 0$, the flow is referred to as the (usual) mean curvature flow.

Mean curvature flows have been extensively studied in various scenarios. Though we are rather interested in the Lorentzian setting, let us mention some results in case of M being a compact hypersurface of a Riemannian manifold N. Mean curvature flows in this setting have been studied, for example, by Huisken [16, 17], Ecker [10], Colding and Minicozzi [8], White [31], Mantegazza [23]) and Smoczyk [30], to cite just a few. The list is far from complete.

Prescribed mean curvature flows in a globally hyperbolic Lorentzian space-time N have been studied for compact hypersurfaces M by Ecker and Huisken [9] and Gerhardt [14]. Without spatial compactness, Ecker [11] proved long time existence and convergence for (1.1) when N is the Minkowski space-time $\mathbb{R} \times \mathbb{R}^m$. Recently, in [18] the authors proved convergence of (1.1) with $\mathcal{H} = 0$ under the assumption $N = \mathbb{R} \times M$ with Lorentzian product metric $\overline{g} = -dx_0^2 + \widetilde{g}$ and $(M, \widetilde{g})$ being asymptotically flat.

1.1 Setting and notation

Up to isometry, a globally hyperbolic Lorentzian space-time $(N, \overline{g})$ is given by a product $\mathbb{R} \times M$ with Lorentzian metric

$$\overline{g} = e^{\phi}\left(-\left(dx^0\right)^2 + \widetilde{g}_{x^0}\right),$$

where M is a smooth space-like Cauchy hypersurface, the natural projection $x^0 : \mathbb{R} \times M \to \mathbb{R}$ is a time-function, $\phi \in C^\infty(\mathbb{R} \times M)$ is a smooth function and $\widetilde{g}_{x^0}$ restricts to a Riemannian metric on $\{x^0\} \times M$. This statement is due to Bernal and Sanchez [5, Theorem 1.1].

In the preceeding work [12], the first named author has studied the prescribed mean curvature flow (1.1) in the special case where the metric $\overline{g}$ is given by a warped product

$$\overline{g} = -\left(dx^0\right)^2 + f(x^0)^2 \widetilde{g} \tag{1.2}$$

for some positive smooth function $f : \mathbb{R} \to \mathbb{R}^+$, bounded away from zero. Assume that for each $t \in [0, T]$ the embedding $F(t)(M) \subset N$ is a space-like Cauchy hypersurface given by the graph of a function $u(t) : M \to \mathbb{R}$. Then the flow (1.1) can be written as an evolution equation for $u(t)$. As asserted by [12, Proposition 3.1], the evolution is explicitly given by

$$\begin{aligned} &\partial_t u + \Delta u = \frac{f'(u)}{f(u)}\left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}, \\ &u(0, \cdot) = u_0, \end{aligned} \tag{1.3}$$

where $m = \dim M$, $\widetilde{\nabla}$ is the gradient on M defined by $\widetilde{g}$ and Δ is the (positive) Laplace–Beltrami operator induced by the s-dependent metric $g = F(t)^*\overline{g}$, which is Riemannian by the space-likeness assumption.

In this paper we will always stay in the following setting:

Setting 1.1 *Consider the following setting.*

(1) *Assume* $(M, \widetilde{g})$*, to be a stochastically complete Riemannian manifold of bounded geometry. Assume furthermore that its embedding* $F_0(M) \subset N$ *is a space-like Cauchy hypersurface given by the graph of a function* $u_0 : M \to \mathbb{R}$.
(2) *Let* $f : \mathbb{R} \to \mathbb{R}^+$ *be smooth, uniformly bounded away from zero, with uniformly bounded first and second derivatives. We consider a warped product Lorentzian metric (usually referred to as generalized Robertson–Walker metric) on* $N = \mathbb{R} \times M$

$$\overline{g} = -\left(dx^0\right)^2 + f(x^0)^2 \widetilde{g} \tag{1.4}$$

(3) *The solution* $u = u(\cdot, t)$ *to* (1.3)*, if it exists, defines a family of embeddings*

$$F = F(\cdot, t) : M \times [0, T] \to N, \quad F(p, t) := (p, u(p, t)), p \in M.$$

The induced family of metrics on M *is defined by* $g = F^* \overline{g}$.

We want to point out that bounded geometry of $(M, \widetilde{g})$, see Definition 3.4, is required in order to apply parabolic Schauder and Krylov–Safonov estimates; and it is required in [18] as well. Stochastic completeness, see Sect. 4, allows for applications of the Omori-Yau maximum principle.

Remark 1.2 We will prove long-time existence and convergence of (1.3) under the assumption that f as well as its derivatives are uniformly bounded and $f \geqslant \varepsilon > 0$ on $\mathbb{R}$. However, once we deduce existence of a uniformly bounded u, a posteriori uniform bounds on f and its derivatives are not necessary, since f appears in (1.3) only as $f(u)$; thus, the only relevant values of f are over the bounded range of u.

We will consistently use the following notation and conventions:

Notation 1.3 (1) *Sometimes we drop the* t*-dependence notationally. When referring to the evolution in* t*, we will refer to the parameter* t *as* time *as well.*
(2) *The upper script* $\sim$*, as for* $\widetilde{\nabla}, \widetilde{\Delta}$*, stands for differential operators defined in terms of the metric* $\widetilde{g}$ *on* M*. We omit any upper script, as for* ∇*,* Δ*, to denote the* t*-dependent operators defined with respect to the induced metric* $g = g(t)$ *on* M*. The upper script* $^-$*, as for* $\overline{\nabla}$ *will be used for operators on* $(N, \overline{g})$.
(3) *We use summation convention on repeated indices. Latin indices will run in* $\{1, \ldots, m\}$ *while the Greek ones are ranging in* $\{0, \ldots, m\}$*. Finally, we will write* $f(u)$ *instead of* $f \circ u$*; we will write* ∂_i *instead of* $\partial / \partial x^i$ *and, as a convention, we will use* ∂_0 *for* ∂_{x^0} *in* N.
(4) *We will consider* Δ *to be the positive Laplace–Beltrami operator, that is*

$$\Delta u = -\operatorname{div}(\nabla u). \tag{1.5}$$

1.2 Statement of the main result

In this paper we present three main results, one on short-time existence of the flow, the second on long-time existence, and the third one on convergence. These results will require varying sets of analytic assumptions, which we now list.

Assumptions 1.4 Consider the classical Hölder spaces $C^{k,\alpha}(M)$ with integer $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, defined with respect to the Riemannian metric $\widetilde{g}$. We impose

(1) **initial regularity:** $u_0 \in C^{3,\alpha}(M)$ and $\mathcal{H} \in C^{\ell,\alpha}(M)$ with $\ell \geqslant 2$.
(2) **upper barrier:** $H(t=0) - \mathcal{H} \geqslant \delta > 0$ for some positive δ.
(3) **Time-like convergence:** $\mathrm{Ric}^N(X, X) > 0$ for any time-like $X \in TN$.

While the initial regularity assumption is natural, the other two assumptions are rather restrictive. Still, they already appear in [9], cf. also page 606 therein for the time-like convergence assumption. Gerhardt [14] studies mean curvature flow without time-like convergence assumption; however, the authors did not succeed in extending his arguments to the non-compact setting.
Our first main result is on the short-time existence and it is proved in Sect. 3.4.

Theorem 1.5 *Impose Assumptions 1.4 (1). Then the solution* u *to the mean curvature flow* (1.3) *exists with* $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{\ell+2,\alpha}(M \times [\sigma, T])$ *for* $T > 0$ *sufficiently small and for every* $0 < \sigma < T$. *The embeddings* $F(t)M$ *are space-like Cauchy hypersurfaces in* N.

Remark 1.6 Note that in what follows we will be denoting the space $C^{\ell+2,\alpha}(M \times [\sigma, T])$ with $0 < \sigma < T$ simply by $C^{\ell+2,\alpha}(M \times (0, T])$.

Our next result concerns the long-time existence.

Theorem 1.7 *Consider the Setting 1.1. Then the mean curvature flow* (1.3).

(i) admits a global solution $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{\ell+2,\alpha}(M \times (0,\infty))$ *in* $(0,\infty)$ *locally uniformly bounded Hölder norm, if Assumptions 1.4 (1) and (2) hold;*
(ii) admits a global solution $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{\ell+2,\alpha}(M \times (0,\infty))$ *with uniformly bounded Hölder norm, if Assumptions 1.4 (1)–(3) hold. Moreover,* $\|\partial_t u\|_\infty$ *is exponentially decreasing.*

Our final main result is about convergence of the flow and we state it as follows.

Theorem 1.8 *Consider the Setting 1.1 and impose Assumptions 1.4 (1)–(3). Assume that* M *is the open interior of a compact manifold* $\overline{M}$ *with boundary* ∂M. *Then the prescribed mean curvature flow* (1.3)*, starting at* u_0 *exists for all times and converges to* $u^* \in L^\infty(M)$ *as* $t \to \infty$. *Moreover,* $u^* \in C^{\ell+2}$ *in the open interior* M *with well-defined mean curvature* $H^* \equiv \mathcal{H}$.

Remark 1.9 The strict positivity $\mathrm{Ric}^N(X, X) > 0$ for any time-like $X \in TN$ in the time-like convergence assumption can be relaxed. Alternatively, one may only assume $\mathrm{Ric}^N(X, X) \geqslant 0$ for any time-like $X \in TN$, and require additionally $\mathcal{H} \geqslant \delta > 0$. Then the results of Theorem 1.8 still hold.

Both Theorems 1.7 and 1.8 will be proved in Sect. 11.

1.3 Distinct arguments due to non-compactness

We should emphasize here that the arguments in our basic references [9] and [14] in fact do not simply carry over to the non-compact setting. Therefore, it might be beneficial for the reader to list those points where the arguments had to be adapted to the non-compact setting.

(i) As is usual in the analysis of geometric flows, a priori estimates are a consequence of the maximum principle. In the non-compact setting, we apply the Omori-Yau maximum principle on stochastically complete manifolds. In particular, we prove that the (graphical) mean curvature flow stays stochastically complete.

(ii) The a priori C^0 estimates, as derived, for example, in [13], use barrier functions. This approach is not easily adapted to the non-compact setting, compare the very intricate barrier function construction [18]. That barrier function argument does not carry over to a general bounded geometry setting in any obvious way.
(iii) The a priori C^2 estimates, as derived, for example, in [14], require certain local coordinates around some maximum point. In the non-compact setting we cannot expect the maximum to be attained. Instead, one works with the supremum of a solution, which may lie "at infinity" of the manifold. Thus, a different argument, cf. [9], without using special coordinates is necessary.
(iv) Convergence of the flow in the compact setting is usually a consequence of a compact embedding of Hölder spaces. On manifolds with bounded geometry the embedding $C^{k,\alpha}(M) \subset C^{k,\beta}(M)$ with $\beta < \alpha$, is not necessarily compact. We overcome this difficulty by specializing to the case of manifolds with boundary, where a similar compact embedding holds in the setting of weighted Hölder spaces.

1.4 Outline of the paper

We begin in Sect. 2 with the geometry of generalized Robertson–Walker space-times and their space-like hypersurfaces. In Sect. 3 we discuss parabolic Schauder and Krylov Safonov estimates on manifolds of bounded geometry. These estimates are applied twice: first in order to establish short-time existence of the flow, and later to turn a priori estimates into Hölder regularity, concluding long-time existence. In Sect. 4 we discuss the Omori-Yau maximum principle on stochastically complete manifolds.

In Sect. 5 we derive the evolution equation of the main object of our analysis, the gradient function. The proof of the aforementioned evolution equation will be divided in two steps in Sects. 5.1 and 5.2. This is due to a lack of literature about (prescribed) mean curvature flows in warped product-type Lorentzian manifolds. Thus, all the "classical" evolution equations, e.g. in [9], had to be re-derived. Evolution equations for the mean curvature and for the scalar second fundamental form are derived in Sects. 6 and 7.

Uniform a priori bounds are derived in the subsequent three sections, Sects. 8, 9 and 10. The upper bound in the C^0-estimates follows a classical argument, while the lower bound uses a trick to overcome absence of a lower barrier. For the C^1-estimates in Sect. 9, we follow Gerhardt's argument, cf. [14], to conclude that a space-like prescribed graphical mean curvature flow stays uniformly space-like. From here we deduce long-time existence and convergence in Sect. 11.

2 Geometry of generalized Robertson–Walker space-times

In this work we are interested in generalized Robertson–Walker space-times, abbreviated as (GRWST), whose definition we now state explicitly once again, before continuing in studying its intrinsic geometry.

Definition 2.1 Let $(M, \widetilde{g})$ be an m-dimensional Riemannian manifold. A generalized Robertson–Walker space-time (GRWST) is an $(m+1)$-dimensional Lorentzian manifold $(N, \overline{g})$ satisfying the following:
there exist a diffeomorphism $\Phi : \mathbb{R} \times M \to N$ and a function $f \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that $\overline{g}$ is a warped product, i.e.

$$\Phi^*\overline{g} = -dx_0^2 + f(x_0)^2\widetilde{g}, \tag{2.1}$$

where x_o denotes the coordinate on $\mathbb{R}$. Below we will always identify $(N, \overline{g})$ with $(\mathbb{R} \times M, \Phi^*\overline{g})$. GRWSTs are automatically time-oriented, i.e. admit a nowhere vanishing time-like vector field T. Here, we can obviously take $T = \partial_0$.

We continue in the setting of Definition 2.1 and consider a family of embeddings $F(\cdot, t) : M \to N$ arising as graphs of a family of functions $u(\cdot, t) : M \to \mathbb{R}$ with $t \in [0, T]$ so that

$$F(p, t) = (u(p, t), p). \tag{2.2}$$

The induced metric on M is given by $g = F^*\overline{g}$ and is explicitly determined in terms of u and $\widetilde{g}$, as asserted by the next lemma, cf. [12, Proposition 2.2] for the proof.

Lemma 2.2 *The induced metric* $g = F^*\overline{g}$ *is given in local coordinates by*

$$g_{ij} = -u_i u_j + f(u)^2\widetilde{g}_{ij}. \tag{2.3}$$

The inverse of the metric tensor g *can be locally expressed as*

$$g^{ij} = \frac{1}{f(u)^2}\widetilde{g}^{ij} + \frac{1}{f(u)^2}\frac{\widetilde{g}^{jl}u_l\widetilde{g}^{im}u_m}{f(u)^2 - |\widetilde{\nabla}u|^2}. \tag{2.4}$$

The prescribed mean curvature flow is a family of metrics on M, embedded into N as a space-like graphs, satisfying some mean curvature flow evolution equation. In order to be precise, we need to gather some geometric quantities and present some useful facts about graphical space-like hypersurfaces.

2.1 Space-like graphs

A space-like hypersurface of a Lorentzian manifold $(N, \overline{g})$ is a codimension 1 submanifold so that the induced metric is Riemannian. Equivalently, a hypersurface is space-like if its unit normal μ is time-like. We choose, as a convention, that the unit normal μ is future oriented, i.e.

$$-\overline{g}(T, \mu) \equiv -\overline{g}(\partial_0, \mu) > 0. \tag{2.5}$$

We codify this expression as the *gradient function* in the next definition. Our analysis will revolve around that gradient function, as, for example, in [9].

Definition 2.3 Let A be a space-like hypersurface of a time orientable Lorentzian manifold $(N, \overline{g})$ with a nowhere vanishing time-like vector field T. The gradient function ν is then defined by

$$\nu := -\overline{g}(T, \mu). \tag{2.6}$$

We now provide an explicit expression for the gradient function of the hypersurface $F(M, t) \subset N$ in the GRWST $(N, \overline{g})$. In local coordinates, induced from M, the (future oriented) unit normal μ of $F(M, t) \subset N$ is given by (see [12] for more details and computations)

$$\mu = \frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla}u|^2}}\Big(\partial_0 + \frac{1}{f(u)^2}\widetilde{g}^{ij}u_j\partial_i\Big). \tag{2.7}$$

From there, using $T = \partial_0$, we obtain by Definition 2.3

$$v = \frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2}}. \tag{2.8}$$

Remark 2.4 The graph of a function $u : M \to \mathbb{R}$ immersed in the GRWST $(N, \overline{g})$ is spacelike if and only if

$$|\widetilde{\nabla} u|^2_{\widetilde{g}} < f(u)^2 \mathfrak{I} \tag{2.9}$$

See Sect. 2 in [12] for more details.

We emphasize that the geometry induced on M by the embeddings $F \equiv F(\cdot, t)$ is different from the geometry arising from the metric $\widetilde{g}$. Therefore, we distinguish the geometric quantities associated to $g = F^* \overline{g}$ from those associated to $\widetilde{g}$: those associated to the latter are indicated by an upper script $\sim$. For instance, ∇ and $\widetilde{\nabla}$ denote the gradient (or covariant derivative) on M with respect to g and $\widetilde{g}$ respectively. We compute

$$\nabla u = \frac{\widetilde{\nabla} u}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}, \quad |\nabla u|^2_g = \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}, \tag{2.10}$$

where $|\cdot|_g$ denotes the pointwise norm with respect to g, while $|\cdot|_{\widetilde{g}}$ refers to the pointwise norm with respect to $\widetilde{g}$. From (2.8) to (2.10) we conclude the following list of properties for the gradient function.

Proposition 2.5 *The gradient function* v *in* (2.8) *satisfies the following properties.*

(i) *the gradient function* v *and the gradient of* u *are related by*

$$|\nabla u|^2_g = \frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} - 1 = v^2 - 1. \tag{2.11}$$

(ii) *The gradient function* v *satisfies* $v \geqslant 1$.

(iii) *The pointwise* g*-norm of* ∇u *is bounded from above by*

$$|\nabla u|^2_g \leqslant v^2. \tag{2.12}$$

(iv) *The following equality holds*

$$v^2 |\widetilde{\nabla} u|^2_{\widetilde{g}} = f(u)^2 |\nabla u|^2_g. \tag{2.13}$$

2.2 Intrinsic geometry

Consider local coordinates $(x^1, \ldots, x^m)$ on M, with the corresponding local frame $(\partial_1, \ldots, \partial_m)$ on TM. Identifying N with $\mathbb{R} \times M$, local coordinates on N are given by $(x^0, x^1, \ldots, x^m)$ and a local frame for TN is

$$(\partial_0, \partial_1, \ldots, \partial_m).$$

The intrinsic geometry of a GRWST $(N, \overline{g})$ is described completely by the Christoffel symbols, which are given explicitly by the following formulae.

Lemma 2.6 *The Christoffel symbols* $\overline{\Gamma}^{\alpha}_{\beta\eta}$ *of the metric tensor* $\overline{g}$ *over* N *are given by*

$$\begin{aligned} &\overline{\Gamma}^{\alpha}_{00} = 0,\ \overline{\Gamma}^{0}_{0i} = 0,\ \overline{\Gamma}^{k}_{ij} = \widetilde{\Gamma}^{k}_{ij}, \\ &\overline{\Gamma}^{0}_{ij} = f(x^0)f'(x^0)\widetilde{g}_{ij},\ \overline{\Gamma}^{k}_{0i} = \frac{f(x^0)f'(x^0)}{f(x^0)^2}\delta^k_i. \end{aligned} \tag{2.14}$$

By making use of the Christoffel symbols listed above, we can compute the local expression for the Riemannian curvature tensor on $(N, \overline{g})$. Recall, for $\overline{\nabla}$ being the Levi-Civita connection of TN, its Riemann curvature R^N is a $(1,3)$-tensor

$$R^N : \Gamma(TN) \times \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN).$$

In terms of any coordinate frame $(\partial_\alpha)_\alpha$ for TN, the components

$$R^N_{\alpha\beta\gamma}{}^{\delta}\partial_\delta = R^N(\partial_\alpha, \partial_\beta)\partial_\gamma. \tag{2.15}$$

can be expressed in terms of Christoffel symbols by

$$R^N_{\alpha\beta\gamma}{}^{\delta} = \overline{\Gamma}^{\delta}_{\beta\gamma,\alpha} - \overline{\Gamma}^{\delta}_{\alpha\gamma,\beta} + \sum_\eta \overline{\Gamma}^{\eta}_{\beta\gamma}\overline{\Gamma}^{\delta}_{\alpha\eta} - \overline{\Gamma}^{\eta}_{\alpha\gamma}\overline{\Gamma}^{\delta}_{\beta\eta}. \tag{2.16}$$

By making use of the metric tensor we can contract indices gaining a $(0,4)$-tensor. Such a $(0,4)$-tensor will be denoted by R^N as well and is given by

$$\begin{aligned} &R^N : \Gamma(TN) \times \Gamma(TN) \times \Gamma(TN) \times \Gamma(TN) \to C^\infty(N), \\ &R^N(X, Y, Z, W) := \overline{g}(R^N(X, Y)Z, W), \\ &R^N_{\alpha\beta\gamma\delta} := R^N(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta) = \overline{g}_{\delta\zeta}R^N_{\alpha\beta\gamma}{}^{\zeta}. \end{aligned} \tag{2.17}$$

Using local coordinates $(x^0, x^1, \ldots, x^m)$, we obtain from (2.14), (2.16) and (2.17) the following list of formulas for R^N.

Lemma 2.7 *For every* $i, j, k \in \{1, \ldots, m\}$ *we have for the Riemann curvature tensor*

$$\begin{aligned} &R^N(\partial_0, \partial_i, \partial_j, \partial_k) = R^N_{0ijk} = 0, \\ &R^N(\partial_0, \partial_i, \partial_j, \partial_0) = R^N_{0ij0} = -f(x^0)f''(x^0)\widetilde{g}_{ij}. \end{aligned} \tag{2.18}$$

We can now compute the values of the Ricci tensor Ric^N, defined in terms of any coordinate frame $(\partial_\alpha)_\alpha$ for TN by (we sum over $j, k = 1, \ldots, m$)

$$\begin{aligned} Ric^N(\partial_\alpha, \partial_\beta) &= \overline{g}^{\delta\gamma}R^N(\partial_\delta, \partial_\alpha, \partial_\beta, \partial_\gamma) \\ &= \overline{g}^{jk}R^N(\partial_j, \partial_\alpha, \partial_\beta, \partial_k) - R^N(\partial_0, \partial_\alpha, \partial_\beta, \partial_0). \end{aligned} \tag{2.19}$$

Corollary 2.8 *For every* $i, j = 1, \ldots, m$ *we have for the Ricci curvature tensor*

$$\begin{aligned} &Ric^N(\partial_0, \partial_0) = -m\frac{f''(x^0)}{f(x^0)}, \quad Ric^N(\partial_0, \partial_i) = 0, \\ &Ric^N(\partial_i, \partial_j) = \widetilde{Ric}(\partial_i, \partial_j) + f(x^0)f''(x^0)\widetilde{g}_{ij} + (m-1)(f'(x^0))^2\widetilde{g}_{ij}. \end{aligned} \tag{2.20}$$

2.3 Extrinsic geometry

Consider the graphical embedding $F \equiv F(\cdot, t) : M \to N$. Its image $F(M)$ is an hypersurface in N, i.e. a codimension one submanifold of $N = \mathbb{R} \times M$. Consider the pull-back bundle F^*TN over M. The Riemannian metric $\overline{g}$ induces an inner product on F^*TN by setting for any $p \in M$ and any $W_1 = (p, w_1), W_2 = (p, w_2) \in F_p^*TN$ with $w_1, w_2 \in T_{F(p)}N$

$$\overline{g}_p(W_1, W_2) := \overline{g}_{F(p)}(w_1, w_2). \tag{2.21}$$

We will denote the pull-back connection on F^*TN (of the Levi-Civita connection on TN) by ∇^{F^*TN}. It is easy to see, e.g. by computations in local coordinates, that the pull-back connection satisfies the following metric property.

Lemma 2.9 *For every* $X \in \Gamma(TM)$ *and for every* $Y, Z \in \Gamma(F^*TN)$*, one has*

$$X(\overline{g}(Y, Z)) = \overline{g}\left(\nabla_X^{F^*TN} Y, Z\right) + \overline{g}\left(Y, \nabla_X^{F^*TN} Z\right). \tag{2.22}$$

Remark 2.10 Notice that the metric property above holds in general whenever considering the pull-back connection of the Levi-Civita connection; i.e. it does not depend on the topology of N nor in the codimension of M.

Note that the total differential $DF(p)$ maps T_pM to F_p^*TN. If μ is the (time-like) unit normal of $F(M)$ and $(\partial_1, \ldots, \partial_m)$ denotes the coordinate frame for TM then

$$(\mu, DF(\partial_1), \ldots, DF(\partial_m)),$$

is a local frame for F^*TN. The main object needed for describing the extrinsic geometry of a submanifold is the second fundamental form. We recall the usual definitions here briefly and then specify the results to our GRWST setting.

Definition 2.11 Let $F : M \to (N, \overline{g})$ be an immersion and $g = F^*\overline{g}$ the metric induced on M by pulling-back $\overline{g}$. Denote by ∇ the Levi-Civita connection on TM associated with the induced metric g. The second fundamental form is defined for every $X, Y \in \Gamma(TM)$ by

$$\mathrm{II}(X, Y) := \nabla_X^{F^*TN}(DF(Y)) - DF(\nabla_X Y). \tag{2.23}$$

The second fundamental form is normal. That is, for every $X, Y, Z \in \Gamma(TM)$

$$\overline{g}(\mathrm{II}(X, Y), DF(Z)) = 0. \tag{2.24}$$

In particular, $\mathrm{II}(X, Y)$ lies in the $C^\infty(M)-$span of μ. We can define the scalar second fundamental form as follows.

Definition 2.12 The scalar second fundamental form of $F(M) \subset N$ is a map $h : \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$ so that for every vector fields X and Y over M

$$h(X, Y) = \overline{g}(\mathrm{II}(X, Y), \mu). \tag{2.25}$$

Proposition 2.13 *Let* $F(M) \subset N$ *be a space-like hypersurface. Then, for any* $X, Y \in \Gamma(TM)$ *we have the following relations between* II *and* h *and between the operator* $\nabla^{F^*TN}\mu$: $\Gamma(TM) \to \Gamma(F^*TN)$ *and* h

$$\mathrm{II}(X, Y) = -h(X, Y)\mu\mathcal{J} \tag{2.26}$$

$$-\overline{g}\left(\nabla_X^{F^*TN}\mu, DF(Y)\right) = \overline{g}(\mu, \mathrm{II}(X, Y)) = h(X, Y). \tag{2.27}$$

Proof Since $(\mu, DF(\partial_1), \dots, DF(\partial_m))$ is a local frame for F^*TN, with μ orthogonal to $DF(\partial_i)$ for every i and time-like,

$$\mathrm{II}(X,Y) = -\overline{g}(\mathrm{II}(X,Y),\mu)\mu + g^{ij}\overline{g}(\mathrm{II}(X,Y), DF(\partial_i))DF(\partial_j)$$

where g^{ij} denotes the inverse of the induced metric on M. Equation (2.26) now follows from (2.24).
Equation (2.27) follows from the metric property of ∇^{F^*TN}, (2.23), (2.26) and the fact that μ is time-like. □

Definition 2.14 For an immersion $F : M \to (N, \overline{g})$, we define

(1) the mean curvature vector $\mathbf{H} := \operatorname{trace} \mathrm{II}$,
(2) the mean curvature $H := \operatorname{trace} h$.

From Proposition 2.13 we conclude

Corollary 2.15 *Let* $F : M \to (N, \overline{g})$ *be a space-like hypersurface. Then*

$$\mathbf{H} = -H\,\mu.$$

Finally, let us state a formula that will be useful later. From the local expression of the Christoffel symbols Γ^k_{ij} and of the scalar second fundamental form h_{ij}, e.g. equations (2.6) and (2.15) in [12] we infer

$$\nu\, h_{ij} = -(u_{ij} - \Gamma^k_{ij}u_k) - f(u)f'(u)\widetilde{g}_{ij}. \tag{2.28}$$

Remark 2.16 We want to point out that Eq. (2.28) is exactly the same as equation (1.16) in [14] once substituting the appropriate values of the Christoffel symbols of $(N, \overline{g})$ expressed in (2.14).

3 Parabolic Schauder and Krylov–Safonov estimates

In this section we review parabolic Schauder and Krylov–Safonov estimates on manifolds with bounded geometry.

3.1 Classical Hölder spaces

Consider a Riemannian manifold $(M, \widetilde{g})$.

Definition 3.1 The Hölder space $C^\alpha \equiv C^\alpha(M \times [0,T])$, for $\alpha \in (0,1)$, is defined as the space of continuous functions $u \in C^0(M \times [0,T])$ which satisfy

$$[u]_\alpha := \sup_{M_T^2} \left\{ \frac{|u(p,t) - u(p',t')|}{d(p,p')^\alpha + |t-t'|^{\alpha/2}} \right\} < \infty, \tag{3.1}$$

where the supremum is over M_T^2 with $M_T := M \times [0,T]$; the distance d is induced by the metric $\widetilde{g}$. The Hölder norm of any $u \in C^\alpha(M \times [0,T])$ is defined by

$$\|u\|_\alpha := \|u\|_\infty + [u]_\alpha. \tag{3.2}$$

The resulting normed vector space $C^\alpha(M \times [0,T])$ is a Banach space. As asserted in the next result, cf. [6, Lemma 2] for a similar statement and its proof, an equivalent Hölder norm is obtained with spatial and time differences taken only within bounded local regions.

Lemma 3.2 *The following defines an equivalent norm on* $C^\alpha(M \times [0,T])$ *(we will not distinguish equivalent norms notationally)*

$$\|u\|_\alpha := \|u\|_\infty + [u]'_\alpha, \quad [u]'_\alpha := \sup_{M^2_{T,\delta}} \left\{ \frac{|u(p,t) - u(p',t')|}{d(p,p')^\alpha + |t-t'|^{\alpha/2}} \right\}, \tag{3.3}$$

where $M^2_{T,\delta} := \{(p,t),(p',t') \in M_T \mid d(p,p')^\alpha + |t-t'|^{\alpha/2} \leqslant \delta\}$.

We will only use the Hölder norm $\|u\|_\alpha$ as in (3.3). We also define the higher order Hölder spaces for any given $k \in \mathbb{N}$ in terms of the gradient $\widetilde{\nabla}$ and pointwise norms $|\cdot|_{\widetilde{g}}$ induced by $\widetilde{g}$ by setting

$$C^{k,\alpha} \equiv C^{k,\alpha}(M \times [0,T]) := \left\{ u \in C^k \;\middle|\; |\widetilde{\nabla}^{\ell_1} \partial_t^{\ell_2} u|_{\widetilde{g}} \in C^\alpha, \quad \ell_1 + 2\ell_2 \leqslant k \right\}$$

which is a Banach space with the norm

$$\|u\|_{k,\alpha} := \sum_{l_1 + 2l_2 \leqslant k} \left\| |\widetilde{\nabla}^{\ell_1} \partial_s^{\ell_2} u|_{\widetilde{g}} \right\|_\alpha . \tag{3.4}$$

We will also use Hölder spaces for functions depending either only on spatial variables or only on the s-time variables. We denote the former by $C^{k,\alpha}(M)$ and the latter by $C^{k,\alpha}([0,T])$.

We conclude this subsection with the following observation.

Lemma 3.3 *Let* $u \in C^\alpha(M \times [0,T])$. *Then the functions* $u_{\sup}(t) := \sup_{p \in M} u(p,t)$ *and* $u_{\inf}(t) := \inf_{p \in M} u(p,t)$ *are continuous in* $[0,T]$.

Proof We will prove the statement only for $u_{\sup}(t)$ since a similar argument holds also for $u_{\inf}(t)$. Let $\varepsilon > 0$ and $t_0 \in [0,T]$ be given. By definition of the supremum we know that, for every $(p,t) \in M \times [0,T]$,

$$u(p,t) \leqslant u_{\sup}(t) + \varepsilon/2.$$

Consider now $t \in [0,T]$ so that $\|u\|_\alpha |t - t_0|^{\alpha/2} < \varepsilon/2$. Since $u \in C^\alpha(M \times [0,T])$ one has

$$|u(p,t) - u(p,t_0)| \leqslant \|u\|_\alpha |t - t_0|^{\alpha/2}.$$

From the above one concludes the following chain of inequalities:

$$u(p,t) \leqslant u(p,t_0) + \|u\|_\alpha |t-t_0|^{\alpha/2} \leqslant u_{\sup}(0) + \varepsilon/2 + \|u\|_\alpha |t-t_0|^{\alpha/2} = u_{\sup}(t_0) + \varepsilon,$$

implying $u_{\sup}(t) \leqslant u_{\sup}(t_0) + \varepsilon$. With similar arguments one shows that the other inequality holds as well, thus providing

$$|u_{\sup}(t) - u_{\sup}(t_0)| \leqslant \varepsilon$$

for every $t \in B_\delta(t_0)$ with δ depending on ε and $\|u\|_\alpha$. Since $t_0 \in [0,T]$ was arbitrary, the statement follows. □

3.2 Manifolds of bounded geometry

Definition 3.4 We say that a Riemannian manifold $(M, \widetilde{g})$ has bounded geometry if its injectivity radius is bounded away from 0 and its Ricci curvature is uniformly bounded, i.e. if for any vector field X on M we have $|\widetilde{\mathrm{Ric}}(X,X)| \leqslant c\widetilde{g}(X,X)$ for some uniform constant $c > 0$.

The hypothesis of bounded geometry implies, in particular, that for some $\delta > 0$, all (open) balls $B_\delta(x)$ of radius δ, centred at $x \in M$, are uniformly quasi-isometric to the Euclidean ball $B_\delta(0) \in \mathbb{R}^m$. This means that for each $B_\delta(x)$ there exists a diffeomorphism

$$\Psi_x : B_\delta(0) \to B_\delta(x),$$

which changes the distances at most by a constant factor that can be chosen independently of x. Using these quasi-isometries Ψ_x, we can define in view of Lemma 3.2 an equivalent norm on $C^{k,\alpha}(M \times [0,T])$ as follows. Ψ_x defines a diffeomorphism

$$\Psi_x : B_\delta(0) \times [0,T] \to B_\delta(x) \times [0,T].$$

We denote the Hölder norm on $C^{k,\alpha}(B_\delta(0) \times [0,T])$, defined as in Definition 3.1, by $\|\cdot\|_{k,\alpha,B_\delta(0)\times[0,T]}$ and obtain an equivalent norm on $C^{k,\alpha}(M \times [0,T])$ given by

$$\|u\|_{k,\alpha} = \sup_{x \in M} \left\| \Psi^* u|_{B_\delta(x)} \right\|_{k,\alpha,B_\delta(0)\times[0,T]}. \tag{3.5}$$

3.3 Parabolic Schauder and Krylov–Safonov estimates

The classical Krylov–Safonov estimates, see [20], as well as the classical parabolic Schauder estimates, see [19], can be obtained for manifolds of bounded geometry in the sense of Sect. 3.2. We refer the reader to a nice exposition about Krylov–Safonovand parabolic Schauder estimates in [25]. We sum up over repeated indices and consider a uniformly elliptic symmetric differential operator L acting on $C_0^\infty(M)$. Here by uniform ellipticity we mean that in local coordinates

$$\begin{aligned} &\Psi^* \circ L \circ (\Psi^*)^{-1} = -a^{ij}(s,x)\partial_{x_i}\partial_{x_j} + b^j(s,x)\partial_{x_j} + c(s,x), \\ &\text{where} \quad \Lambda^{-1}\|\xi\|^2 \leqslant a^{ij}(s,x)\xi_i\xi_j \leqslant \Lambda\|\xi\|^2, \\ &\text{and} \quad \|b(s,x)\| \leqslant \Lambda^{-1},\ 0 \leqslant c(s,x) \leqslant \Lambda^{-1}, \end{aligned} \tag{3.6}$$

for some uniform $\Lambda > 0$.

Proposition 3.5 *Consider a uniformly elliptic symmetric differential operator* L*, as in* (3.6)*, acting on* $C_0^\infty(M)$*. Let* $\varphi : M \times [0,T] \to \mathbb{R}$ *be uniformly bounded and consider a uniformly bounded solution* ω *to*

$$(\partial_t + L)\,\omega = \varphi. \tag{3.7}$$

(1) *Then there exists a constant* $C > 0$ *depending only on* m *and* Λ *such that*

$$\|\omega\|_\alpha \leqslant C\Big(\|\omega\|_\infty + \|\varphi\|_\infty\Big),$$

(2) *If additionally,* $a^{ij}, b^j, c, \varphi \in C^{k,\alpha}(M \times [0,T])$*, then there exists a constant* $C > 0$ *depending only on* m, Λ *and the Hölder norms of the local coefficients of* L*, such that*

$$\|\omega\|_{k+2,\alpha} \leqslant C\Big(\|\omega\|_\infty + \|\varphi\|_{k,\alpha}\Big).$$

Proof The solution ω satisfies for each $x \in M$

$$\left(\partial_t + \Psi^* \circ L \circ (\Psi^*)^{-1}\right) \Psi^* \omega|_{B_\delta(x)} = \Psi^* \varphi|_{B_\delta(x)} .$$

Let us set $Q_\delta := B_\delta(0) \times [0, \delta^2]$. By the Krylov–Safonov estimate, see [20, Theorem 4.2] and cf. [25, Theorem 12], we find for some uniform constant $C > 0$, depending only on $\delta, m = \dim M$ and the ellipticity constant $\Lambda > 0$ from (3.6)

$$\begin{aligned} \| \Psi^* \omega|_{B_\delta(x)} \|_{\alpha, Q_{\delta/2}} &\leqslant C\Big(\| \Psi^* \omega|_{B_\delta(x)} \|_{\infty, Q_\delta} + \| \Psi^* \varphi|_{B_\delta(x)} \|_{\infty, Q_\delta}\Big) \\ &\leqslant C\Big(\|\omega\|_\infty + \|\varphi\|_\infty\Big). \end{aligned}$$

Thus, using the Hölder norm in (3.5), we find

$$\|\omega\|_\alpha \leqslant C\Big(\|\omega\|_\infty + \|\varphi\|_\infty\Big).$$

By Lemma 3.2 we conclude $\omega \in C^\alpha(M \times [0, (\delta/2)^2])$. We extend the regularity statement to the whole time interval $[0, T]$ (with constants independent of T) iteratively. By setting $t = (\delta/2)^2 + t'$, from the argument above we obtain $\omega \in C^\alpha(M \times [(\delta/2)^2, 2(\delta/2)^2])$. The first statement now follows by repeating the iteration, till we reach T.

For the second statement, standard parabolic Schauder estimates, see [19, Theorem 8.12.1] and cf. [25, Theorem 6], assert that for some uniform constant $C > 0$, depending only on δ, m, Λ and the Hölder norms of the coefficients

$$\begin{aligned} \| \Psi^* \omega|_{B_\delta(x)} \|_{k+2,\alpha, Q_{\delta/2}} &\leqslant C\Big(\| \Psi^* \omega|_{B_\delta(x)} \|_{\infty, Q_\delta} + \| \Psi^* f|_{B_\delta(x)} \|_{k,\alpha, Q_\delta}\Big) \\ &\leqslant C\Big(\|\omega\|_\infty + \|f\|_{k,\alpha}\Big). \end{aligned}$$

By Lemma 3.2 we conclude $\omega \in C^{k,\alpha}(M \times [0, (\delta/2)^2])$. Extension to $C^{k,\alpha}(M \times [0, T])$ goes exactly as before. □

We conclude the subsection by presenting some mapping properties for the parametrix to the inhomogeneous heat Eq. (3.7). These can be deduced from Proposition 3.5 exactly as in [6, Proposition 10.1], cf. [19, Theorem 8.10.1] for the first claim in (3.8) below.

Proposition 3.6 *Consider an* s*-independent uniformly elliptic symmetric differential operator* L *acting on* $C_0^\infty(M)$ *as above. The inhomogeneous heat equation* $(\partial_t + L)\omega = \varphi$*, with* $\omega(t = 0) = 0$ *and* $\varphi \in C^{k,\alpha}(M \times [0, T])$*, has a parametrix* Q *acting as a bounded linear map*

$$\begin{aligned} Q &: C^{k,\alpha}(M \times [0, T]) \to C^{k+2,\alpha}(M \times [0, T]), \\ Q &: C^{k+2,\alpha}(M \times [0, T]) \to s\, C^{k+2,\alpha}(M \times [0, T]). \end{aligned} \tag{3.8}$$

3.4 Application: short-time existence of the flow

We present a weaker analogue of the main result by the first named author [12], proving short-time existence of the prescribed mean curvature flow with space-like Cauchy hypersurfaces of bounded geometry. In contrast, [12] asserts that starting with a Φ-manifold, the Cauchy hypersurfaces remain generalized Φ-manifolds for short time. The following proves Theorem 1.5.

Proof of Theorem 1.5 For convenience of the reader, we shall repeat briefly the argument, that is worked out in detail in [12]. We need to linearize the evolution Eq. (1.3). The most complicated term is the linearization of Δu. Here, Δ is the Laplace–Beltrami operator on M with respect to the metric g on the graph of u, given explicitly by

$$g_{ij} = -u_i u_j + f(u)^2 \tilde{g}_{ij}.$$

From here one computes explicitly

$$\begin{aligned}\Delta h = &\frac{1}{f(u)^2}\tilde{\Delta} h + \frac{1}{f(u)^2}\hat{\Delta} h \\ &+ \tilde{g}(\tilde{\nabla} u, \tilde{\nabla} h)\frac{1}{f(u)^2(f(u)^2 - |\tilde{\nabla} u|^2_{\tilde{g}})}\tilde{\Delta} u \\ &+ \tilde{g}(\tilde{\nabla} u, \tilde{\nabla} h)\frac{1}{f(u)^2(f(u)^2 - |\tilde{\nabla} u|^2_{\tilde{g}})}\hat{\Delta} u \\ &- (m-1)\tilde{g}(\tilde{\nabla} u, \tilde{\nabla} h)\frac{f'(u)}{f(u)(f(u)^2 - |\tilde{\nabla} u|^2_{\tilde{g}})} \\ &+ \tilde{g}(\tilde{\nabla} u, \tilde{\nabla} h)\frac{f(u)f'(u)}{(f(u)^2 - |\tilde{\nabla} u|^2_{\tilde{g}})^2}\end{aligned} \tag{3.9}$$

where in the above $\hat{\Delta}$ is an operator acting on functions over M defined by

$$\hat{\Delta} h = -\frac{v^2}{f(u)^2}\tilde{\nabla}^2 h\left(\tilde{\nabla} u, \tilde{\nabla} u\right) = -\frac{\tilde{g}^{jl} u_l \tilde{g}^{im} u_m}{f(u)^2 - |\tilde{\nabla} u|^2_{\tilde{g}}}\left(h_{ij} - \tilde{\Gamma}^k_{ij} h_k\right), \tag{3.10}$$

where $\tilde{\nabla}^2$ denotes the Hessian and the second expression is an expression in local coordinates. Plugging in $u = u_0 + \omega$ we obtain (writing $\hat{\Delta}_0$ for (3.10) with u_0 instead of u, and writing $\Delta_{g(0)}$ for the Laplace–Beltrami operator of $g(0)$)

$$\begin{aligned}\Delta u = &\Delta_{g(0)} u_0 + \Delta_{g(0)}\omega + \frac{|\tilde{\nabla} u_0|^2_{\tilde{g}}}{f(u_0)^2(f(u_0)^2 - |\tilde{\nabla} u_0|^2_{\tilde{g}})}\left(\tilde{\Delta}\omega + \hat{\Delta}_0\omega\right) \\ &+ F_1'(\omega, \tilde{\nabla}\omega) + F_2'(\omega, \tilde{\nabla}\omega, \tilde{\nabla}^2\omega),\end{aligned} \tag{3.11}$$

where $F_1'(\omega, \tilde{\nabla}\omega)$ denotes an expression depending at most linearly on the entries in brackets, with coefficients given in terms of u_0, $\tilde{\nabla} u_0$ and $\tilde{\nabla}^2 u_0$. The summand $F_2'(\omega, \tilde{\nabla}\omega, \tilde{\nabla}^2\omega)$ denotes an expression depending at least quadratically on the entries in brackets, with coefficients given in terms of u_0, $\tilde{\nabla} u_0$ and $\tilde{\nabla}^2 u_0$. On the other hand, (3.9) implies

$$\Delta_{g(0)}\omega = \frac{1}{f(u_0)^2}\left(\tilde{\Delta}\omega + \hat{\Delta}_0\omega\right) + F_1''(\omega, \tilde{\nabla}\omega), \tag{3.12}$$

where $F_1''(\omega, \tilde{\nabla}\omega)$, similar to $F_1'(\omega, \tilde{\nabla}\omega)$, denotes an expression depending at most linearly on the entries in brackets, with coefficients given in terms of u_0, $\tilde{\nabla} u_0$ and $\tilde{\nabla}^2 u_0$. Combining (3.11) and (3.12) we obtain

$$\begin{aligned}\Delta u = &\Delta_{g(0)} u_0 + \frac{f(u_0)^2}{(f(u_0)^2 - |\tilde{\nabla} u_0|^2_{\tilde{g}})}\Delta_{g(0)}\omega \\ &+ F_1'(\omega, \tilde{\nabla}\omega) - f(u_0)^2 F_1''(\omega, \tilde{\nabla}\omega) + F_2'(\omega, \tilde{\nabla}\omega, \tilde{\nabla}^2\omega),\end{aligned} \tag{3.13}$$

Linearizing similarly the remaining terms of (1.3) we obtain

$$\begin{aligned}
&\left(\partial_t + \frac{f(u_0)^2}{(f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}})}\Delta_{g(0)}\right)\omega \\
&\quad = -\Delta_{g(0)} u_0 + \frac{f'(u_0)}{f(u_0)}\left(m + \frac{|\widetilde{\nabla} u_0|^2_{\widetilde{g}}}{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u_0)}{\sqrt{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}}} \\
&\qquad + F_1(\omega, \widetilde{\nabla}\,\omega) + F_2(\omega, \widetilde{\nabla}\,\omega, \widetilde{\nabla}^2\,\omega),
\end{aligned} \tag{3.14}$$

where $F_1(\omega, \widetilde{\nabla}\,\omega)$, similar to F_1' and F_1'', denotes an expression depending at most linearly on the entries in brackets, with coefficients given in terms of $\mathcal{H}, u_0, \widetilde{\nabla}\,u_0$ and $\widetilde{\nabla}^2\,u_0$. Similarly, $F_2(\omega, \widetilde{\nabla}\,\omega, \widetilde{\nabla}^2\,\omega)$ denotes an expression depending at least quadratically on the entries in brackets, with coefficients given in terms of $\mathcal{H}, u_0, \widetilde{\nabla}\,u_0$ and $\widetilde{\nabla}^2\,u_0$.

Assuming $u_0 \in C^{2,\alpha}(M)$, we find in view of (3.9) and space-likeness condition (2.9) that

$$L := \frac{f(u_0)^2}{(f(u_0)^2 - |\widetilde{\nabla}\,u_0|^2_{\widetilde{g}})}\Delta_{g(0)},$$

is uniformly elliptic in the sense of Proposition 3.6. Provided $\mathcal{H} \in C^{\alpha}(M \times [0, T])$, the solution $\omega \in C^{2,\alpha}(M \times [0, T])$ to (3.15) is obtained as a fixed point of the bounded map

$$\begin{aligned}
&\Phi : C^{2,\alpha}(M \times [0, T]) \to C^{2,\alpha}(M \times [0, T]), \\
&\omega \mapsto Q(F_1(\omega, \widetilde{\nabla}\,\omega) + F_2(\omega, \widetilde{\nabla}\,\omega, \widetilde{\nabla}^2\,\omega)).
\end{aligned} \tag{3.15}$$

The higher regularity assumption $u_0 \in C^{3,\alpha}(M)$ and $\mathcal{H} \in C^{\ell,\alpha}(M)$ with $\ell \geqslant 1$ implies, exactly as in [12, Theorem 6.14], that the mapping above is a contraction on a closed subset of $C^{2,\alpha}(M \times [0, T])$, if $T > 0$ is sufficiently small. Thus, we have proved existence of a solution $u \in C^{2,\alpha}(M \times [0, T])$ for $T > 0$ sufficiently small.

Let us now prove that $u \in C^{3,\alpha}$, where we abbreviate $C^{k,\alpha} \equiv C^{k,\alpha}(M \times [0, T])$. That gain in regularity is not a consequence of a fixed point argument, but rather of the Krylov–Safonov estimates in Proposition 3.5 (ii). More precisely, $u \in C^{2,\alpha}$ implies in view of (3.9) that Δu is a uniformly elliptic operator with coefficients being $C^{1,\alpha}$. Moreover, $u \in C^{2,\alpha}$ and $\mathcal{H} \in C^{\ell,\alpha}(M)$ with $\ell \geqslant 1$ imply that the right-hand side of (1.3) is $C^{1,\alpha}$. Thus, applying Proposition 3.5 (ii) directly to the evolution Eq. (1.3) implies that $u \in C^{3,\alpha}$.

Repeating the argument of the last paragraph allows for bootstrapping: Even if the initial data is only $u_0 \in C^{3,\alpha}(M)$, we find that, provided $\mathcal{H} \in C^{\ell,\alpha}(M)$ for any $\ell \in \mathbb{N}_0$, u admits the following Hölder regularity

$$u \in C^{3,\alpha}(M \times [0, T]) \cap C^{\ell+2,\alpha}(M \times (0, T]).$$

□

Remark 3.7 Note that the regularity $C^{3,\alpha}(M \times [0, T]) \cap C^{\ell+2,\alpha}(M \times (0, T])$ is due to the initial condition u_0 being merely $C^{3,\alpha}(M)$; thus, one has that $u(\_, 0) = u_0 \in C^{3,\alpha}(M)$. But as it is usually the case for parabolic equations, the flow is instantaneously "smoothing" meaning that, even if we start the flow with lower regularity, the solutions gains regularity right away. It is also worth pointing out that if $u_0 \in C^{4,\alpha}(M)$ and $\mathcal{H} \in C^{2,\alpha}(M)$ then the solution $u \in C^{4,\alpha}(M \times [0, T])$.

4 Omori-Yau parabolic maximum principle

In order to study the behaviour of solutions of parabolic PDEs one usually proceeds by gaining a priori estimates. One of the tools employed to obtain such estimates is the parabolic maximum principle. We will therefore formulate a parabolic maximum principle for manifolds satisfying the Omori-Yau maximum principle.

4.1 The Omori-Yau maximum principles

We denote by (X, g_X) a Riemannian manifold non-necessarily compact, non-necessarily complete. One says that the Riemannian manifold (X, g_X) satisfies the Omori-Yau maximum principle for the Laplacian if for any function $u \in C^2(X)$ with bounded supremum there is a sequence $\{p_k\}_k \subset X$ satisfying

$$u(p_k) > \sup_X u - \frac{1}{k} \quad \text{and} \quad -\Delta_X u(p_k) < \frac{1}{k}. \tag{4.1}$$

Similarly, provided u has bounded infimum, there exists a sequence $\{p'_k\}_k \subset M$ such that

$$u(p'_k) < \inf_X u + \frac{1}{k} \quad \text{and} \quad -\Delta_X u(p'_k) > \frac{1}{k}. \tag{4.2}$$

The above definition can be find, for example, in [1, Definition 2.3].

Remark 4.1 We want to point out a difference with [1] in the different sign convention for the Laplace–Beltrami operator.

The notion of a manifold satisfying the Omori-Yau maximum principle can be a bit abstract. So next we point out some examples which are known to satisfy the required condition. In [24, Theorem A'] Omori showed that the Omori-Yau maximum principle holds for complete Riemannian manifolds with sectional curvature bounded from below. Subsequently [32, Theorem 1] Yau proved that the Omori-Yau maximum principle for the Laplacian holds for complete Riemannian manifolds with Ricci curvature bounded from below. A detailed description of the above can be also found in [1].

The previous examples seem to suggest a strict relation between the geometry of a manifold and the Omori-Yau maximum principle. But it turns out that the Omori-Yau maximum principle condition is actually more analytic than geometric. In particular, Pigola, Rigoli and Setti showed in [26, Theorem 1.1] that a stochastically complete Riemannian manifold satisfies the Omori-Yau maximum principle.

For completeness we recall here the notion of stochastically complete manifolds.

Definition 4.2 A Riemannian manifold (X, g_X) is stochastically complete if the heat kernel H of the (positive) Laplace–Beltrami operator Δ_X, associated to g_X, satisfies

$$\int_X H(t, p, \widetilde{p})\, d\mathrm{vol}_{g_X}(\widetilde{p}) = 1. \tag{4.3}$$

4.2 An enveloping theorem and applications

Based on the Omori-Yau maximum principle above, the second named author proved in [6, Proposition 3.1] jointly with Caldeira and Hartmann the following enveloping theorem, that is formulated for Φ-manifolds but holds on all stochastically complete spaces with exactly the same proof.

Proposition 4.3 *Let* (X, g_X) *be a Riemannian manifold satisfying the Omori-Yau maximum principle. Consider any* $u \in C^{2,\alpha}(X \times [0,T])$. *Then*

$$u_{\sup}(t) := \sup_X u(\cdot, t), \quad u_{\inf}(t) := \inf_X u(\cdot, t)$$

are locally Lipschitz and differentiable almost everywhere in $(0,T)$. *Moreover, at those differentiable times* $t \in (0,T)$ *we find, in the notation of* (4.1) *and* (4.2),

$$\begin{aligned} \frac{\partial}{\partial t} u_{\sup}(t) &\leqslant \lim_{\epsilon \to 0^+} \left(\limsup_{k \to \infty} \frac{\partial u}{\partial t} (p_k(t+\epsilon), t+\epsilon) \right), \\ \frac{\partial}{\partial t} u_{\inf}(t) &\geqslant \lim_{\epsilon \to 0^+} \left(\liminf_{k \to \infty} \frac{\partial u}{\partial t} (p'_k(t+\epsilon), t+\epsilon) \right). \end{aligned} \tag{4.4}$$

Finally, we are in the position to prove the parabolic maximum principle for stochastically complete manifolds.

Theorem 4.4 *Let* $(X, g_X(t))$ *be a family of Riemannian manifolds satisfying the Omori-Yau maximum principle as above with* $t \in [0,T]$. *Denote the corresponding family of Laplace–Beltrami operators by* Δ_s. *Consider solutions* $u^\pm \in C^{1,\alpha}(M \times [0,T]) \cap C^{2,\alpha}(M \times (0,T])$, *solving the differential inequalities*

$$\left(\frac{\partial}{\partial t} + \Delta_t \right) u^+ \leqslant 0, \quad \left(\frac{\partial}{\partial t} + \Delta_t \right) u^- \geqslant 0. \tag{4.5}$$

Then $u^+_{\sup}(t) \leqslant u^+_{\sup}(0)$ *and* $u^-_{\inf}(t) \geqslant u^-_{\inf}(0)$ *for every* $t \in [0,T]$.

Proof Note first by (4.1) and (4.2)

$$\frac{\partial}{\partial t} u^+(p_k(t), t) \leqslant \frac{1}{k}, \quad \frac{\partial}{\partial t} u^-(p'_k(t), t) \geqslant -\frac{1}{k}.$$

Then in view of Proposition 4.3 we find almost everywhere

$$\frac{\partial}{\partial t} u^+_{\sup}(t) \leqslant 0, \quad \frac{\partial}{\partial t} u^-_{\inf}(t) \geqslant 0.$$

The above shows that the functions $u^+_{\sup}$ and $u^-_{\inf}$ are decreasing and increasing, respectively, in $(0,T)$. The claim now follows by recalling that the functions $u^+_{\sup}$ and $u^-_{\inf}$ are continuous in $[0,T]$ from Lemma 3.3. □

4.3 Omori-Yau maximum principle along the flow

Our aim is to make use of the Omori-Yau maximum principle along the flow. This means we have to make sure that a mean curvature flow (M, g_t) satisfies the Omori-Yau maximum principle for every $t \in [0,T]$. It is worth noticing that the statements in Sect. 4.2 hold for a (family of) Riemannian manifolds satisfying the Omori-Yau maximum principle. Therefore, if we show that a mean curvature flow preserves the Omori-Yau maximum principle for all of its time existence, then we can employ the Omori-Yau maximum principle to deduce a priori estimates.

Due to the geometry involved in our problem, a control on the Ricci or sectional curvature is not really at our disposal. Nonetheless, as mentioned in Sect. 4, stochastically complete manifolds do satisfy the Omori-Yau maximum principle. Therefore, if we show that a mean curvature flow stays stochastically complete then it will automatically satisfy the Omori-Yau maximum principle. Stochastic completeness looks like a "non-easy" to handle definition as well, but it can be equivalently characterized by a volume growth condition, due to Grigor'yan [15], cf. also Theorem 2.11 in [1].

Theorem 4.5 *Let* (X, g_X) *be a complete Riemannian manifold. Consider for some reference point* $p \in X$ *the geodesic ball* $B_R(p)$ *of radius* R *around* p. *If the function*

$$\frac{R}{\log\left(\mathrm{Vol}\left(B_R(p)\right)\right)} \notin L^1(1, \infty) \tag{4.6}$$

then (X, g_X) *is stochastically complete.*

Thus, due to Grigor'yan's result we have a geometric/analytic condition to ensure that a complete Riemannian manifold is stochastically complete. This makes our work way easier, since in order to make sure that the Omori-Yau maximum principle is satisfied at every time, it is enough to prove that the assumptions in Theorem 4.5 are satisfied. Let us begin with an easy observation.

Lemma 4.6 *Let* (X, g_X) *be a Riemannian manifold and consider a* $(0, 2)$*-tensor* A *over* X. *Its norm* $|A|_{g_X}$ *with respect to* g_X *is given in local coordinates by*

$$|A|^2_{g_X} = g_X^{il} g_X^{jq} A_{ij} A_{lq}. \tag{4.7}$$

Then for any two vector fields Y *and* Z, *one has*

$$|A(Y, Z)| \leqslant |A|_{g_X} |Y|_{g_X} |Z|_{g_X}. \tag{4.8}$$

Corollary 4.7 *Let* $(M, g = F^*\overline{g})$ *be the prescribed graphical mean curvature flow* (1.1), *arising as a family of graphs of functions* u *over the Riemannian manifold* $(M, \widetilde{g})$. *Let* v *denote the associated family of gradient functions, as defined in Definition* 2.3. *Then, as long as the flow exists and* v *is finite, there exist positive* $c, C > 0$ *(depending on* v*) such that for any* $p, q \in M$

$$c\, d_{\widetilde{g}}(p, q) \leqslant d_g(p, q) \leqslant C\, d_{\widetilde{g}}(p, q). \tag{4.9}$$

In the above $d_{\widetilde{g}}$ *and* d_g *denote the distance on* M *with respect to* $\widetilde{g}$ *and the induced metric* $g = F^*\overline{g}$, *respectively.*

Proof Let p and q be any two fixed points on M and consider a connecting differentiable curve $\gamma := \gamma(\tau) : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Recall the curve length with respect to $\widetilde{g}$ is explicitly given by

$$|\gamma|_{\widetilde{g}} = \int_0^1 \sqrt{\widetilde{g}\left(\gamma'(\tau), \gamma'(\tau)\right)}\, d\tau.$$

Equation (4.8) and the fact that the distance is by definition the infimum of the lengths of paths joining p and q give

$$d_g(p, q) \leqslant \int_0^1 \sqrt{g\left(\gamma'(\tau), \gamma'(\tau)\right)}\, d\tau \leqslant \sqrt{|g|_{\widetilde{g}}} \int_0^1 \sqrt{\widetilde{g}\left(\gamma'(\tau), \gamma'(\tau)\right)}\, d\tau.$$

Taking infimum over all such paths γ, we find

$$d_g(p, q) \leqslant \sqrt{|g|_{\widetilde{g}}}\, d_{\widetilde{g}}(p, q).$$

The same holds with the roles of g and $\widetilde{g}$ reversed. This leads to

$$\frac{1}{\sqrt{|\widetilde{g}|_g}}\, d_{\widetilde{g}}(p,q) \leqslant d_g(p,q) \leqslant \sqrt{|g|_{\widetilde{g}}}\, d_{\widetilde{g}}(p,q).$$

The only thing left to do is to estimate $|g|_{\widetilde{g}}$ and $|\widetilde{g}|_g$. In view of (4.7), and keeping in mind that the metric tensor g can be expressed as in (2.3), we compute in local coordinates

$$\begin{aligned}
|g|^2_{\widetilde{g}} &= \widetilde{g}^{il}\widetilde{g}^{jm}\left(-u_iu_j + f(u)^2\widetilde{g}_{ij}\right)\left(-u_lu_m + f(u)^2\widetilde{g}_{lm}\right)\\
&= |\widetilde{\nabla} u|^4_{\widetilde{g}} - 2f(u)^2|\widetilde{\nabla} u|^2_{\widetilde{g}} + f(u)^4 m\\
&= (f(u)^2 - |\widetilde{\nabla} u|_{\widetilde{g}})^2 + f(u)^4(m-1) = \left(\frac{f(u)^2}{v^2}\right)^2 + f(u)^4(m-1)\\
&\leqslant f(u)^4 + f(u)^4(m-1) = f(u)^4 m \leqslant c_1.
\end{aligned}$$

In the above the last inequality follows from $v \geqslant 1$ and assuming that f is uniformly bounded. With similar arguments we compute, recalling that the inverse of the metric tensor g is expressed as in (2.4)

$$\begin{aligned}
|\widetilde{g}|^2_g &= \frac{1}{f(u)^4}\left(\widetilde{g}^{il} + \frac{\widetilde{g}^{ia}u_a\widetilde{g}^{lb}u_b}{f(u)^2 - |\widetilde{\nabla} u|_{\widetilde{g}}}\right)\left(\widetilde{g}^{jm} + \frac{\widetilde{g}^{jc}u_c\widetilde{g}^{mk}u_k}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\right)\widetilde{g}_{ij}\widetilde{g}_{lm}\\
&= \frac{1}{f(u)^4}\left(m + 2\frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} + \frac{|\widetilde{\nabla} u|^4_{\widetilde{g}}}{\left(f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}\right)^2}\right)\\
&= \frac{1}{f(u)^4}\left(m - 1 + \left(1 + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2}\right)^2\right)\\
&= \frac{1}{f(u)^4}(m - 1 + v^4) \leqslant c_2 v^4.
\end{aligned}$$

From the above expression we can further notice that $|\widetilde{g}|_g$ is nonzero, since $v \geqslant 1$, thus proving the claim. □

Proposition 4.8 *Let* $(M, g = g(t))$ *be as above in Corollary* 4.7. *Then*

$$\mathrm{dvol}_g = \frac{f(u)^m}{v}\,\mathrm{dvol}_{\widetilde{g}} \tag{4.10}$$

Proof There exists some $\lambda \in C^\infty(M)$ so that $\mathrm{dvol}_g = \lambda\,\mathrm{dvol}_{\widetilde{g}}$. By the local expression of the volume form we conclude $\lambda = \sqrt{\det(g\widetilde{g}^{-1})}$. By expressing the induced metric tensor g in coordinates, cf. (2.4), one has

$$(g\widetilde{g}^{-1})_{ij} = -\widetilde{g}^{jk}u_iu_k + f(u)^2\delta^j_i = f(u)^2\left(\delta^j_i - \frac{1}{f(u)^2}\widetilde{g}^{kj}u_ku_i\right).$$

Let us set $Du^T := -1/f(u)^2(u_1, \ldots, u_m)$, where the lower indices denote partial derivatives with respect to the coordinate frame $(\partial_1, \ldots, \partial_m)$. This implies

$$\det\left(g\widetilde{g}^{-1}\right) = f(u)^{2m}\det\left(\mathrm{id} + \widetilde{\nabla} u \cdot Du^T\right) = f(u)^{2m}\left(1 + \widetilde{\nabla} u^T \cdot Du\right)$$
$$= f(u)^{2m}\left(1 - \frac{|\widetilde{\nabla} u|_{\widetilde{g}}}{f(u)^2}\right) = \frac{f(u)^{2m}}{v^2}.$$

□

Proposition 4.9 *Let $(M, g = g(t))$ be a prescribed graphical mean curvature flow as above in Corollary 4.7. Assuming that $(M, \widetilde{g})$ is stochastically complete, the flow $(M, g(t))$ stays stochastically complete for each fixed t, as long as $(M, g(t))$ are space-like, i.e. as long as the gradient function $v(t)$ is finite. In particular, the Omori-Yau maximum principle holds for every time.*

Proof This is a straightforward consequence of Corollary 4.7 and Proposition 4.8. Together they imply that volume of R-balls with respect to $g = g(t)$ and with respect to $\widetilde{g}$ are comparable up to constants depending on $v(t)$. Thus, by Theorem 4.5, $(M, g = g(t))$ is also stochastically complete as long as $v(t)$ is bounded, i.e. as long as $(M, g(t))$ are space-like. This proves the claim □

5 Evolution equation for the gradient function

Our central aim is to prove that a space-like prescribed graphical mean curvature flow stays uniformly space-like along the flow. To this end we will prove that the gradient function v, defined in 2.3, satisfies a partial differential inequality of the form (4.5). Such an inequality will follow from the next theorem.

Theorem 5.1 *Let $u(t)$ be a solution to the prescribed graphical mean curvature flow (1.3) of an m-dimensional space-like Cauchy hypersurface. Then the gradient function $v \equiv v(t)$ for the graph of $u(t), t \in [0, T]$ satisfies the following evolution equation*

$$\begin{aligned}(\partial_t + \Delta)v = &-\| h \|^2 v - \mathrm{Ric}^N(\mu,\mu)v - 2\frac{f'(u)}{f(u)}H + \frac{f'(u)}{f(u)}\mathcal{H} - V(\mathcal{H})\\ &- \frac{f'(u)}{f(u)}\mathcal{H}v^2 + 2\frac{f'(u)}{f(u)}g(\nabla u, \nabla v) + m\frac{f''(u)}{f(u)}v\\ &- \left(\frac{f'(u)}{f(u)}\right)^2\|\nabla u\|^2 v - \frac{f''(u)}{f(u)}\|\nabla u\|^2 v - m\left(\frac{f'(u)}{f(u)}\right)^2 v.\end{aligned} \tag{5.1}$$

In the above V is a vector field over M so that[1]

$$DF(V) = \overline{g}^{ij}\overline{g}(\partial_0, DF(\partial_i))DF(\partial_j).$$

The above theorem is a direct consequence of the following propositions.

Proposition 5.2 *The gradient function v evolves as*

$$\partial_t v = V(H - \mathcal{H}) - (H - \mathcal{H})\frac{f'(u)}{f(u)} + (H - \mathcal{H})\frac{f'(u)}{f(u)}v^2. \tag{5.2}$$

[1] Recall that $\overline{g}$ defines an inner product on F^*TN by (2.21).

Proposition 5.3 *The Laplacian of the gradient function v can be expressed as*

$$\begin{aligned}\Delta v = &-\frac{f'(u)}{f(u)}H - \frac{f'(u)}{f(u)}Hv^2 - V(H) + 2\frac{f'(u)}{f(u)}g(\nabla u, \nabla v) \\ &- \| h \|^2 v - \mathrm{Ric}^N(\mu,\mu)v + m\frac{f''(u)}{f(u)}v + \left(\frac{f'(u)}{f(u)}\right)\|\nabla u\|^2 v \\ &- \frac{f''(u)}{f(u)}\|\nabla u\|^2 v - 2\left(\frac{f'(u)}{f(u)}\right)\|\nabla u\|^2 v - m\left(\frac{f'(u)}{f(u)}\right)^2 v.\end{aligned} \tag{5.3}$$

We will prove Propositions 5.2 and 5.3 in Sects. 5.1 and 5.2, respectively.

5.1 Time derivative of the gradient function

It is important to notice that the unit normal $\mu = \mu(t)$ is a section of a t-dependent vector bundle over M, namely $F^*TN \equiv F(t)^*TN$ with $F(t) : M \to N$ being the graphical embedding given by $F(p,t) = (u(p,t), p)$ for any $p \in M$. The pull-back connection on F^*TN is denoted by ∇^{F^*TN}, as introduced in Sect. 2.3.

Our next aim is to treat the partial derivative in t, ∂_t, as a vector field; this will be achieved by following [30, Sect. 3.2] as well as [2, Sect. 2.3]. In Sect. 2.3 we have introduced the pull-back bundle and the pull-back connection arising from an embedding $F : M \to N$. Moreover, as we have also done in the above, we have made the identification $F = F(t)$. To avoid confusion, in order to introduce a connection on a time-dependent vector bundle, we denote by $\mathcal{F}$ a time-dependent embedding, that is $\mathcal{F} : M \times [0,T] \to N$. In particular, the above means that for every fixed $t \in [0,T]$ we have $\mathcal{F}(\_, t) = F(t) = F$ is an embedding.

Now we can proceed by pulling-back the vector bundle TN to $M \times [0,T]$ by means of the time-dependent embedding $\mathcal{F}$. Such a bundle will be denoted, as in Sect. 2.3, by $\mathcal{F}^*TN$ which is now a bundle over $M \times [0,T]$. In a similar fashion we pull-back the connection $\overline{\nabla}$ to $\mathcal{F}^*TN$ and we denote it by $\nabla^{\mathcal{F}^*TN}$. This works exactly as it has already been done in Sect. 2.3. Now, as in [2] we denote by $\mathcal{S}$ the vector bundle over $M \times [0,T]$ given by $\mathcal{S} = \{v \in T(M \times [0,T]) \,|\, Dt(v) = 0\}$. By means of the time-dependent embedding $\mathcal{F}$, the vector bundle $\mathcal{S}$ gives rise to a sub-bundle $D\mathcal{F}(\mathcal{S})$ of $\mathcal{F}^*TN$ of rank m. Furthermore, since $\mathcal{F}^*TN$ is trivially equipped with a metric, then one finds the orthogonal complement to $D\mathcal{F}(\mathcal{S})$ which is denoted by $\mathcal{N}$ and it is of rank 1. Notice that for a fixed time t one has the identifications $D\mathcal{F}(\mathcal{S}) = DF(TM)$ and $\mathcal{N} = DF(TM)^\perp = \mathrm{span}(\mu)$. From the above one deduced the following.

Remark 5.4 For every $i = 1, \ldots, m = \dim M$, at a point $(p,t) \in M \times [0,T]$, we obtain for the differential of $\mathcal{F}$

$$\begin{aligned} D\mathcal{F}(\partial_t) &= -(H - \mathcal{H})\mu, \\ D\mathcal{F}(\partial_i) &= DF(\partial_i). \end{aligned} \tag{5.4}$$

Also one can treat a partial t-derivative as a covariant derivative in the direction of the vector field ∂_t. At a point $(p,t) \in M \times [0,T]$ for every $\sigma \in \Gamma(\mathcal{F}^*TN)$, given in local coordinates by $\sigma = \sigma^\alpha \partial_\alpha$ (recall, $\overline{\nabla}$ is the covariant derivative of $(N, \overline{g})$), with $\sigma^\alpha \in C^\infty(M \times [0,T])$

$$\begin{aligned} \partial_t \sigma := \nabla^{\mathcal{F}^*TN}_{\partial_t} \sigma &= \frac{\partial}{\partial t}\sigma^\alpha \cdot \partial_\alpha + \sigma^\alpha \cdot \overline{\nabla}_{D\mathcal{F}(\partial_t)}\partial_\alpha, \\ \nabla^{\mathcal{F}^*TN}_{\partial_i}\sigma &= \nabla^{F^*TN}_{\partial_i}\sigma. \end{aligned} \tag{5.5}$$

Finally, due to the symmetry of the second fundamental form associated to $\mathcal{F}$ (cf. [2, Proposition 3]) one has

$$\nabla^{\mathcal{F}^*TN}_{\partial_t} D\mathcal{F}(\partial_i) = \nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_t) = -\partial_i(H-\mathcal{H})\cdot\mu - (H-\mathcal{H})\cdot\nabla^{F^*TN}_{\partial_i}\mu. \quad (5.6)$$

With the above machinery one can easily compute the t-evolution of the unit normal μ.

Proposition 5.5 *For u being a solution of (1.3), the unit normal μ evolves as*

$$\partial_t\mu := \nabla^{\mathcal{F}^*TN}_{\partial_t}\mu = -DF(\nabla(H-\mathcal{H})). \quad (5.7)$$

Proof Viewing μ as a section of the pull-back bundle $\mathcal{F}^*TN$, $\partial_t\mu$ lies in $\Gamma(\mathcal{F}^*TN)$ as well. Thus, taking $(\partial_1,\dots,\partial_m)$ as a local coordinate frame on TM, we get a local frame for $\mathcal{F}^*(TN)$ given by $(D\mathcal{F}(\partial_t), D\mathcal{F}(\partial_1),\dots,D\mathcal{F}(\partial_m))$. We can therefore express $\partial_t\mu \equiv \nabla^{\mathcal{F}^*TN}_{\partial_t}\mu$ with respect to that frame (recall that $\overline{g}$ defines an inner product on F^*TN by (2.21))

$$\begin{aligned}\partial_t\mu = |D\mathcal{F}(\partial_t)|^{-1}_{\overline{g}}\cdot\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t}\mu, D\mathcal{F}(\partial_t)\right)D\mathcal{F}(\partial_t)\\ + g^{ij}\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t}\mu, D\mathcal{F}(\partial_i)\right)D\mathcal{F}(\partial_j).\end{aligned}$$

From (5.4) the first term reads

$$\begin{aligned}\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t}\mu, D\mathcal{F}(\partial_t)\right)D\mathcal{F}(\partial_s) &= (H-\mathcal{H})^2\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t}\mu,\mu\right)\mu\\ &= \frac{1}{2}(H-\mathcal{H})^2\cdot\partial_t\overline{g}(\mu,\mu)\cdot\mu = 0,\end{aligned} \quad (5.8)$$

where the second equality follows by the metric property of the pull-back connection, and the last equality follows by μ being of unit length. Note that $\overline{g}(\mu, DF(\partial_i)) = 0$, since μ is normal. We conclude again by the metric property of the pull-back connection ∇^{F^*TN}, and using (5.6) in the second equality

$$\begin{aligned}\partial_t\mu &= -g^{ij}\overline{g}\left(\mu,\nabla^{F^*TN}_{\partial_t}DF(\partial_i)\right)DF(\partial_j)\\ &= -g^{ij}\left(-\partial_i(H-\mathcal{H})\right)\overline{g}(\mu,\mu)DF(\partial_j) + g^{ij}(H-\mathcal{H})\overline{g}\left(\mu,\nabla^{F^*TN}_{\partial_i}\mu\right)DF(\partial_j)\\ &= -g^{ij}\partial_i(H-\mathcal{H})DF(\partial_j) + \frac{1}{2}g^{ij}(H-\mathcal{H})\partial_i\overline{g}(\mu,\mu)DF(\partial_j),\end{aligned}$$

where we used $\overline{g}(\mu,\mu) = -1$ in the last equation. The second summand vanishes by unitarity of μ, which is a similar argument as in (5.8), and thus the statement follows. □

We are now in the position to prove Proposition 5.2.

Proof of Proposition 5.2 Recall that, by definition $\nu = -\overline{g}(\mu,\partial_t)$ hence

$$\partial_t\nu = -\overline{g}(\partial_t\mu,\partial_0) - \overline{g}\left(\mu,\nabla^{F^*TN}_{\partial_t}\partial_0\right). \quad (5.9)$$

Formula (5.7) implies that $\partial_t\mu$ lies in $\Gamma(F^*TN)$ and is tangential to the graph of u; that is $\overline{g}(\partial_t\mu,\mu) = 0$. Now $(\mu, DF(\partial_1),\dots,DF(\partial_m))$ is a local frame for F^*TN, with μ orthogonal to the other frame elements and time-like. Thus, we can write

$$\partial_0 = -\overline{g}(\partial_0,\mu)\mu + \partial_0^\top = \nu\mu + \partial_0^\top,\quad \partial_0^\top := g^{ij}\overline{g}(\partial_0, DF(\partial_i))DF(\partial_j). \quad (5.10)$$

Defining a local vector field $V\in\Gamma(TM)$ by $V = g^{ij}\overline{g}(\partial_0, DF(\partial_i))\partial_j$, so that $DF(V) = \partial_0^\top$, we conclude from Proposition 5.5 (recall $g = F^*\overline{g}$)

$$\begin{aligned}\overline{g}\left(\partial_t\mu,\partial_0\right) &= -\overline{g}\left(DF\left(\nabla(H-\mathcal{H})\right),DF(V)\right)\\ &= -g\left(\nabla(H-\mathcal{H}),V\right) = -V(H-\mathcal{H}).\end{aligned}$$

For the second term in (5.9) let us express μ in the local frame $(\partial_0, \partial_1, \dots, \partial_m)$

$$\mu = -\overline{g}(\mu,\partial_0)\partial_0 + \overline{g}^{ij}\overline{g}(\mu,\partial_i)\partial_j = \nu\partial_0 + \nu b^j\partial_j \tag{5.11}$$

where $b^j : M \to \mathbb{R}$, $b^j := \tilde{g}^{ij}u_i/f(u)^2$, using (2.7) in the last equality. In particular, one writes

$$\nabla^{\mathcal{F}^*TN}_{\partial_t}\partial_0 = -(H-\mathcal{H})\nu\overline{\nabla}_{\partial_0}\partial_0 - (H-\mathcal{H})\nu b^j\overline{\nabla}_{\partial_j}\partial_0.$$

From equation (2.14) and applying (5.11) one concludes

$$\nabla^{\mathcal{F}^*TN}_{\partial_t}\partial_0 = -\frac{f'(u)}{f(u)}(H-\mathcal{H})\mu + \nu\frac{f'(u)}{f(u)}(H-\mathcal{H})\partial_0.$$

The result now follows by substituting the above in (5.9). □

5.2 Laplacian of the gradient function

In order to prove Theorem 5.1 it remains to compute $\Delta\nu$ (recall Δ is the Laplacian with respect to $g = F^*\overline{g}$) at a fixed time $t \in [0,T]$; for simplicity we will suppress the parameter t. All the upcoming computations will be performed at a fixed point $(p,t) \in M \times [0,T]$. In particular, t will be fixed and we choose an arbitrary point $p \in M$. Let us consider a local parallel orthonormal frame at p (with respect to g), that is an orthonormal frame $(e_i)_i$ of TM over an open neighbourhood U, such that $\nabla_{e_i}e_j(p) = 0$ for every i, j at the fixed point $p \in M$ (recall that ∇ here denotes the Levi-Civita connection of $F(t)(M)$; also the existence of such a frame is a consequence of the existence of normal coordinates). Then we can write for $\Delta\nu$ at p

$$\Delta\nu = e_ie_i\overline{g}(\mu,\partial_0) = e_i(\overline{g}(\nabla^{F^*TN}_{e_i}\mu,\partial_0)) + e_i(\overline{g}(\mu,\nabla^{F^*TN}_{e_i}\partial_0)). \tag{5.12}$$

The second summand in (5.12) will be computed using the next proposition (in 5.16). The first summand is computed below in Lemma 5.8.

Let u be a solution of (1.3) and $(e_i)_i$ a local parallel orthonormal frame at $p \in M$ as above. With respect to a local coordinate frame $(\partial_k)_k$ we can write $e_i = e_i^k\partial_k$ for some smooth coefficients $e_i^k : U \to \mathbb{R}$. Note also that $e_i(u) = -\overline{g}\left(DF(e_i),\partial_0\right)$. We then obtain the following useful formulae at $p \in M$ (recall the definition in (2.23) and the fact that we assumed $\nabla_{e_i}e_j(p) = 0$)

$$-h(e_i,e_j)\mu = \mathrm{II}(e_i,e_j) = \nabla^{F^*TN}_{e_i}DF(e_j), \tag{5.13}$$

$$DF(e_i) = -\overline{g}\left(DF(e_i),\partial_0\right)\partial_0 + e_i^k\partial_k = e_i(u)\partial_0 + e_i^k\partial_k. \tag{5.14}$$

Proposition 5.6 *Let u be a solution of (1.3) and F the corresponding family of graphical embeddings. For fixed $t \in [0,T]$ and $p \in M$, consider $(e_i)_i$ to be a local parallel orthonormal frame at p as above, that is $\nabla_{e_i}e_j(p) = 0$ for every i, j. Then at p we have*

(1) *the covariant derivative of ∂_0, as a section of F^*TN, can be expressed as*

$$\begin{aligned}\nabla^{F^*TN}_{e_i}\partial_0 &= \frac{f'(u)}{f(u)}DF(e_i) + \frac{f'(u)}{f(u)}\overline{g}(DF(e_i),\partial_0)\partial_0\\ &= \frac{f'(u)}{f(u)}DF(e_i) - \frac{f'(u)}{f(u)}e_i(u)\partial_0.\end{aligned} \tag{5.15}$$

(2) *For* μ *being the unit normal to the graph of* u *one has*

$$\overline{g}\left(\mu, \nabla^{F^*TN}_{e_i}\partial_0\right) = \nu\frac{f'(u)}{f(u)}e_i(u). \tag{5.16}$$

(3) *For every* i *and* j *ranging between* 1 *and* $m = \dim M$, *one finds*

$$e_i\left(\overline{g}\left(\partial_0, DF(e_j)\right)\right) = \frac{f'(u)}{f(u)}\delta_{ij} + \frac{f'(u)}{f(u)}e_i(u)e_j(u) + \nu\, h(e_i, e_j)\mathcal{I} \tag{5.17}$$

where δ_{ij} *denotes the Kronecker delta. In particular, for* $i = j$, *with the obvious summation convention over repeated indices, one concludes*

$$e_i\left(\overline{g}\left(\partial_0, DF(e_i)\right)\right) = \nu\, H + \frac{f'(u)}{f(u)}\left(m + |\nabla u|^2_g\right). \tag{5.18}$$

Proof (1) From Eq. (5.14) we see that

$$\nabla^{F^*TN}_{e_i}\partial_0 = e_i(u)\overline{\nabla}_{\partial_0}\partial_0 + e_i^k\overline{\nabla}_{\partial_k}\partial_0.$$

Equation (5.15) now follows by substituting the appropriate values of the covariant derivatives on the right-hand side, described by the Christoffel symbols of $(N, \overline{g})$ in (2.14), and using (5.14) once more.

(2) Equation (5.16) is a direct consequence of (5.15) and the fact that μ is normal to the graph of u, that is $\overline{g}\left(\mu, DF(e_i)\right) = 0$ for every $i = 1, \ldots, m$.

(3) The metric property of the pull-back connection ∇^{F^*TN} gives

$$e_i\left(\overline{g}\left(\partial_0, DF(e_j)\right)\right) = \overline{g}\left(\nabla^{F^*TN}_{e_i}\partial_0, DF(e_i)\right) + \overline{g}\left(\partial_0, \nabla^{F^*TN}_{e_i}DF(e_j)\right).$$

From Eqs. (5.15) and (5.13) we deduce

$$\begin{aligned} e_i\left(\overline{g}\left(\partial_0, DF(e_j)\right)\right) &= \frac{f'(u)}{f(u)}\overline{g}\left(DF(e_i), DF(e_j)\right) \\ &\quad - \frac{f'(u)}{f(u)}e_i(u)\overline{g}\left(\partial_t, DF(e_j)\right) - h(e_i, e_j)\overline{g}\left(\partial_0, \mu\right). \end{aligned}$$

By assumption, $(e_i)_i$ is a local orthonormal frame, with respect to the metric $g = F^*\overline{g}$; thus, $\overline{g}\left(DF(e_i), DF(e_j)\right) = \delta_{ij}$. Moreover, from Eq. (5.14) we compute $-\overline{g}\left(\partial_0, DF(e_j)\right) = e_j(u)$. Finally, the result follows by recalling the definition of the gradient function (cf. Definition 2.3).

□

Remark 5.7 Notice that Eq. (5.18) is nothing but (at $p \in M$)

$$\Delta u = -e_i\left(e_i(u)\right) = \nu\, H + \frac{f'(u)}{f(u)}\left(m + |\nabla u|^2_g\right) = \nu\, H + \frac{f'(u)}{f(u)}(m + \nu^2 - 1)$$

where we have used (i) in Proposition 2.5. This is exactly the same result as Proposition 2.12 in [12].

Lemma 5.8 *In the notation of Proposition* 5.6 *we have at* p

$$\begin{aligned} e_i(\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_0)) &= -e_i(\overline{g}(\partial_0, DF(e_j))\, h(e_i, e_j) \\ &\quad - \overline{g}(\partial_0, DF(e_j))e_i(h(e_i, e_j)). \end{aligned} \tag{5.19}$$

Proof The result follows by the Leibniz rule once we prove that

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_0) = -\overline{g}(\partial_0, DF(e_j))\, h(e_i, e_j). \tag{5.20}$$

To this end, notice that ∂_0, as a section of F^*TN, decomposes with respect to the orthonormal frame $(\mu, DF(e_1), \ldots, DF(e_m))$ as

$$\partial_0 = -\overline{g}(\partial_0, \mu)\mu + \overline{g}(\partial_0, DF(e_j))DF(e_j).$$

Substituting this into the left-hand side of (5.20), one finds

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_0) = -\overline{g}\left(\partial_0, \mu\right) \overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, \mu\right) + \overline{g}\left(\partial_0, DF(e_j)\right) \overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, DF(e_j)\right).$$

The first summand now vanishes, since μ is of unit length. Using (5.13) we now conclude at $p \in M$

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_0) = -\overline{g}\left(\partial_0, DF(e_j)\right) h(e_i, e_j).$$

□

Lemma 5.9 *In the notation of Proposition 5.6 we have at* p [2]

$$e_i(h(e_i, e_j)) = e_j(H) - \mathrm{Ric}^N(DF(e_j), \mu)\mathcal{J} \tag{5.21}$$

with the obvious summation convention on repeated indices.

Proof From Eq. (2.27) in Proposition 2.13 we compute, by making use of the metric property of the pull-back connection ∇^{F^*TN},

$$\begin{aligned} e_i(h(e_i, e_j)) &= -\,\overline{g}(\nabla^{F^*TN}_{e_i} DF(e_i), \nabla^{F^*TN}_{e_j}\mu) - \overline{g}(DF(e_i), \nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu) \\ &= -\,\overline{g}(DF(e_i), \nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu), \end{aligned}$$

where the second equality follows from the fact that $\overline{g}(\nabla^{F^*TN}_{e_i} DF(e_i), \nabla^{F^*TN}_{e_j}\mu) = 0$ due to (5.13) and the fact that μ is of unit length, i.e. $\overline{g}\left(\mu, \mu\right) = -1$.

Recall that, at p, the curvature form of the pull-back connection is the pull-back of the curvature of the connection, that is

$$\nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu - \nabla^{F^*TN}_{e_j}\nabla^{F^*TN}_{e_i}\mu - \nabla^{F^*TN}_{[e_i,e_j]}\mu = R^N\left(DF(e_i), DF(e_j)\right)\mu.$$

Thus, using $\nabla^{F^*TN}_{[e_i,e_j]}\mu = 0$ due to naturality of the pull-back and the computations being performed at p, we obtain (summing over double indices i)

$$\begin{aligned} e_i\left(h(e_i, e_j)\right) &= -\overline{g}\left(\nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu, DF(e_i)\right) \\ &= -R^N\left(DF(e_i), DF(e_j), \mu, DF(e_i)\right) - \overline{g}\left(\nabla^{F^*TN}_{e_j}\nabla^{F^*TN}_{e_i}\mu, DF(e_i)\right) \\ &= -\,\mathrm{Ric}^N\left(DF(e_j), \mu\right) - e_j\left(\overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, DF(e_i)\right)\right) \\ &\quad + \overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, \nabla^{F^*TN}_{e_j}\mu\right) \\ &= -\,\mathrm{Ric}^N\left(DF(e_j), \mu\right) + e_j(H). \end{aligned}$$

[2] Ric^N applies to $DF(e_j) \in F^*TN$ similar to(2.21).

In the above, the second equality is obtained by making use of (2.17). The first term in the third equality is a consequence of $(\mu, DF(e_1), \ldots, DF(e_m))$ being an orthonormal basis of $T_{F(p)}N$ with μ time-like. The second term is instead a mere application of the metric property of the connection ∇^{F^*TN}. Finally the fourth identity is the result of the formula (2.27), Definition 2.14 and of μ being of unit length. □

We now conclude with the following expression for (5.12).

Proposition 5.10 *Let* u *be a solution of* (1.3). *Then the Laplacian of the gradient function* ν *can be expressed in terms of a local vector field* $V = g^{ij}\overline{g}(\partial_0, DF(\partial_i))\,\partial_j \in \Gamma(TM)$, *such that* $DF(V) = \partial_0^{\top}$, *as follows:*

$$\begin{aligned}\Delta\nu = &-\frac{f'(u)}{f(u)}H - \frac{f'(u)}{f(u)}h(\nabla u, \nabla u) - \nu\|h\|^2 - V(H) + \mathrm{Ric}^N(DF(V), \mu)\\ &+ \frac{f'(u)}{f(u)}g(\nabla u, \nabla\nu) + \frac{f''(u)}{f(u)}\|\nabla u\|^2\nu - 2\left(\frac{f'(u)}{f(u)}\right)^2\|\nabla u\|^2\nu\\ &- \frac{f'(u)}{f(u)}H\nu^2 - m\left(\frac{f'(u)}{f(u)}\right)^2\nu.\end{aligned} \tag{5.22}$$

Proof Plugging (5.16), Lemmas 5.8 and 5.9 into (5.12) yields the following intermediate expression that holds at $p \in M$

$$\begin{aligned}\Delta\nu = &-e_i(\overline{g}(\partial_0, DF(e_j))\,h(e_i, e_j)\\ &- \overline{g}(\partial_0, DF(e_j))\Big(e_j(H) - \mathrm{Ric}^N(DF(e_j), \mu)\Big) + e_i\Big(\nu\frac{f'(u)}{f(u)}e_i(u)\Big).\end{aligned} \tag{5.23}$$

Noticing that $V = \overline{g}(\partial_0, DF(e_k))\,e_k$ with summation over k, we conclude from formula (5.17), Lemmas 5.8 and 5.9

$$\begin{aligned}\Delta\nu = &-\Big(\frac{f'(u)}{f(u)}\delta_{ij} + \frac{f'(u)}{f(u)}e_i(u)e_j(u) + \nu h(e_i, e_j)\Big)h(e_i, e_j)\\ &- V(H) + \mathrm{Ric}^N(DF(V), \mu) + e_i\Big(\nu\frac{f'(u)}{f(u)}e_i(u)\Big)\\ = &-\frac{f'(u)}{f(u)}H - \frac{f'(u)}{f(u)}h(\nabla u, \nabla u) - \nu\|h\|^2\\ &- V(H) + \mathrm{Ric}^N(DF(V), \mu) + e_i\Big(\nu\frac{f'(u)}{f(u)}e_i(u)\Big).\end{aligned} \tag{5.24}$$

In order to conclude the statement, it remains to study the last term in (5.24). We compute using Remark 5.7, arriving at an expression that holds globally

$$\begin{aligned}e_i\Big(\nu\frac{f'(u)}{f(u)}e_i(u)\Big) &= e_i(\nu)\frac{f'(u)}{f(u)}e_i(u) + \nu e_i\left(\frac{f'(u)}{f(u)}\right)e_i(u) - \nu\frac{f'(u)}{f(u)}\Delta u\\ &= e_i(\nu)e_i(u)\frac{f'(u)}{f(u)} + \nu\frac{f''(u)}{f(u)}e_i(u)e_i(u) - \nu\left(\frac{f'(u)}{f(u)}\right)^2 e_i(u)e_i(u)\\ &\quad - \nu^2\frac{f'(u)}{f(u)}H - \nu\left(\frac{f'(u)}{f(u)}\right)^2 m - \nu\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\\ &= g(\nabla u, \nabla\nu)\frac{f'(u)}{f(u)} + \nu\frac{f''(u)}{f(u)}|\nabla u|_g^2 - 2\nu\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\end{aligned}$$

$$- v^2 \frac{f'(u)}{f(u)} H - mv \left(\frac{f'(u)}{f(u)}\right)^2 .$$

□

Notice that Proposition 5.10 is not yet a proof for Proposition 5.3. In particular, the terms $h(\nabla u, \nabla u)$ and $\mathrm{Ric}^N(DF(V), \mu)$ appearing in (5.22) need to be simplified.

Proposition 5.11 *Let* u *be a solution of* (1.3). *Then*

$$h(\nabla u, \nabla u) = -g(\nabla u, \nabla v) - \frac{f'(u)}{f(u)} |\nabla u|^2 v. \tag{5.25}$$

Proof Notice that the statement is a direct consequence of a local identity

$$v_i = -g^{jk} u_j h_{ki} - \frac{f'(u)}{f(u)} v u_i. \tag{5.26}$$

Indeed, assuming (5.26) to hold locally, we find

$$\begin{aligned} g(\nabla u, \nabla v) &= g^{im} u_m v_i = -g^{im} u_m g^{jk} u_j h_{ik} - g^{im} u_m \frac{f'(u)}{f(u)} v u_i \\ &= -h(\nabla u, \nabla u) - \frac{f'(u)}{f(u)} |\nabla u|_g^2 v. \end{aligned}$$

This is precisely the statement after rearrangement. Let us therefore prove (5.26). By making use of (2.11) one has

$$v v_i = \frac{1}{2} \partial_i v^2 = \frac{1}{2} \partial_i (1 + |\nabla u|^2) = g(\nabla_{\partial_i} \nabla u, \nabla u).$$

Furthermore, $\nabla_{\partial_i} \nabla u$ can be expressed locally as

$$\nabla_{\partial_i} \nabla u = \partial_i (g^{jk} u_k) \partial_j + g^{jk} u_k \Gamma^l_{ij} \partial_l.$$

By keeping in mind that $\partial_i g^{jk} = -g^{jl} \Gamma^k_{il} - \Gamma^j_{il} g^{lk}$, one finds

$$g(\nabla_{\partial_i} \nabla u, \nabla u) = g^{jk} u_j (u_{ki} - \Gamma^l_{ik} u_l).$$

The result now follows by substituting (2.28) and by noticing that

$$g^{jk} u_j \widetilde{g}_{ki} = \frac{v^2}{f(u)^2} u_i.$$

□

The only thing left to prove Proposition 5.3 is a formula for $\mathrm{Ric}^N(DF(V), \mu)$.

Proposition 5.12 *Let* u *be a solution for* (1.3) *and* F *the corresponding family of graphical embeddings. Then we have the following formula for the vector field* $V = g^{ij} \overline{g}(\partial_0, DF(\partial_i)) \partial_j \in \Gamma(TM)$, *with* $DF(V) = \partial_0^\top$

$$\mathrm{Ric}^N(DF(V), \mu) = -\frac{f''(u)}{f(u)} mv - v \,\mathrm{Ric}^N(\mu, \mu). \tag{5.27}$$

Proof By definition of V we have $\partial_0^\top = DF(V) = \partial_0 + \overline{g}(\partial_0, \mu)\mu$. Thus,

$$\begin{aligned}\mathrm{Ric}^N(DF(V), \mu) &= \mathrm{Ric}^N(\partial_0, \mu) - \nu\,\mathrm{Ric}^N(\mu, \mu)\\ &= \nu\,\mathrm{Ric}^N(\partial_0, \partial_0) + \nu b^i\,\mathrm{Ric}^N(\partial_0, \partial_i) - \nu\,\mathrm{Ric}^N(\mu, \mu)\\ &= -\nu m\frac{f''(u)}{f(u)} + \nu\,\mathrm{Ric}^N(\mu, \mu).\end{aligned}$$

The second identity is obtained by considering the orthogonal decomposition of the unit normal μ with respect to the local frame $(\partial_0, \partial_1, \ldots, \partial_m)$ of F^*TN (cf. formula 5.11); in particular, $b^i = \widetilde{g}^{ij}u_j/f(u)^2$. The last identity is a consequence of the values for the Ricci tensor described in Corollary 2.8. □

6 Evolution equation for the mean curvature

As before, let $u(\cdot, t)$ be a solution to the prescribed graphical mean curvature flow (1.3) of an m-dimensional space-like Cauchy hypersurface. The solution induces a family of embeddings $F(t) : M \to N$ with $F(p, t) = (u(p, t), p)$ for any $p \in M$. The induced metric g is defined by the pull-back $g = F(t)^*\overline{g}$. We begin with the following basic evolution equations for the metric tensor.

Proposition 6.1 *The metric tensor* (g_{ij}) *and its inverse* (g^{ij})*, written in local coordinates, satisfy the following evolution equations along* (1.3)

$$\partial_t g_{ij} = 2(H - \mathcal{H})\, h_{ij}, \tag{6.1}$$

$$\partial_t g^{ij} = -2(H - \mathcal{H}) g^{ik}\, h_{kl}\, g^{lj}. \tag{6.2}$$

Proof Notice that (6.2) is a direct consequence of (6.1). So we only need to prove (6.1). Using the same notation as in Sect. 5.1 we compute (see right below for the explanation of the individual steps)

$$\begin{aligned}\partial_t g_{ij} &= \partial_t \overline{g}\left(DF(\partial_i), DF(\partial_j)\right)\\ &= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t} DF(\partial_i), DF(\partial_j)\right) + \overline{g}\left(DF(\partial_i), \nabla^{\mathcal{F}^*TN}_{\partial_t} DF(\partial_j)\right)\\ &= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_t), DF(\partial_j)\right) + \overline{g}\left(DF(\partial_i), \nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_t)\right)\\ &= -(H - \mathcal{H})\left(\overline{g}(\nabla^{F^*TN}_{\partial_i}\mu, DF(\partial_j)) + \overline{g}(DF(\partial_i), \nabla^{F^*TN}_{\partial_j}\mu)\right) = 2\,(H - \mathcal{H})\, h_{ij}\,.\end{aligned}$$

In the above the first identity follows by definition of the induced metric tensor $g = F^*\overline{g}$. The second is just a consequence of the metric property of the pull-back derivative $\nabla^{\mathcal{F}^*TN}$. The third line comes from the commutativity $[\partial_t, \partial_i] = 0$ of local coordinate fields. Finally the last equality is a consequence of (5.6) and the fact that μ is normal, i.e. $\overline{g}\,(DF(X), \mu) = 0$ for any vector field X over M. □

Next we study the evolution of the scalar second fundamental form.

Proposition 6.2 *The tensor* (h_{ij}) *of the scalar second fundamental form satisfies the following evolution equation (summing over double indices as usual) along* (1.3)

$$\begin{aligned} \partial_t h_{ij} &= (H-\mathcal{H})\left(g^{kl} h_{ik} h_{jl} - R^N(\mu, DF(\partial_i), DF(\partial_j), \mu)\right) \\ &\quad + (H-\mathcal{H})_{ij} - \Gamma_{ij}^k \partial_k (H-\mathcal{H}) \\ &= (H-\mathcal{H})\left(g^{kl} h_{ik} h_{jl} - R^N(\mu, DF(\partial_i), DF(\partial_j), \mu)\right) + \nabla^2_{ij}(H-\mathcal{H}). \end{aligned} \tag{6.3}$$

Proof We compute with respect to a local coordinate frame (∂_i)

$$\begin{aligned} \partial_t h_{ij} &= \partial_t h(\partial_i, \partial_j) = \partial_t \overline{g}\left(\nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) \\ &= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) + \overline{g}\left(\nabla^{F^*TN}_{\partial_i} DF(\partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_t} \mu\right) \\ &= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) - h_{ij} \overline{g}\left(\mu, \nabla^{\mathcal{F}^*TN}_{\partial_t} \mu\right) \\ &\quad + \overline{g}\left(DF(\nabla_{\partial_i} \partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_t} \mu\right) \\ &= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) + \overline{g}\left(DF(\nabla_{\partial_i} \partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_t} \mu\right). \end{aligned}$$

In the first line we just used (2.27). The second line is obtained by making use of the metric property of the pull-back connection $\nabla^{\mathcal{F}^*TN}$. The third is a consequence of (2.23). The last equality follows from the fact that μ is of unit length, which implies vanishing of the second term in the third line.

We shall now compute these two terms above. By Proposition 5.5

$$\begin{aligned} \overline{g}\left(DF(\nabla_{\partial_i} \partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_t} \mu\right) &\equiv \overline{g}\left(DF(\nabla_{\partial_i} \partial_j), \partial_t \mu\right) \\ &= -g\left(\nabla_{\partial_i} \partial_j, \nabla(H-\mathcal{H})\right) = -\Gamma^k_{ij} \partial_k (H-\mathcal{H}). \end{aligned}$$

This computes the last term. For the first term we proceed as follows. Noting that that $[\partial_i, \partial_t] = 0$, we obtain from the definition of the Riemann curvature tensor

$$\nabla^{\mathcal{F}^*TN}_{\partial_t} \nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_j) - \nabla^{\mathcal{F}^*TN}_{\partial_i} \nabla^{\mathcal{F}^*TN}_{\partial_t} D\mathcal{F}(\partial_j) = R^N\left(D\mathcal{F}(\partial_t), D\mathcal{F}(\partial_i)\right) D\mathcal{F}(\partial_j).$$

This implies directly

$$\begin{aligned} &\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_t} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) \\ &= R^N\left(D\mathcal{F}(\partial_t), D\mathcal{F}(\partial_i), D\mathcal{F}(\partial_j), \mu\right) + \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_i} \nabla^{\mathcal{F}^*TN}_{\partial_t} D\mathcal{F}(\partial_j), \mu\right) \\ &= -(H-\mathcal{H}) R^N\left(\mu, D\mathcal{F}(\partial_i), D\mathcal{F}(\partial_j), \mu\right) + \partial_i\left(\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_t), \mu\right)\right) \\ &\quad - \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_t), \nabla^{\mathcal{F}^*TN}_{\partial_i} \mu\right), \end{aligned} \tag{6.4}$$

where the second equality is a consequence of the (5.4), (5.6) and the metric property of the pull-back connection. Let us now describe the second term on the right-hand side of the second equation in (6.4). From (5.6) we write

$$\begin{aligned}&\partial_i\left(\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j}D\mathcal{F}(\partial_t),\mu\right)\right)\\&=-\partial_i\left(\partial_j\left(H-\mathcal{H}\right)\overline{g}(\mu,\mu)\right)-\partial_i\left(\left(H-\mathcal{H}\right)\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j}\mu,\mu\right)\right)\\&=\partial_i\partial_j\left(H-\mathcal{H}\right),\end{aligned}\tag{6.5}$$

where in the second equation we used the fact that $\overline{g}\,(\mu,\mu)=-1$.

To conclude the computation of (6.4), we need $\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j}D\mathcal{F}(\partial_t),\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu\right)$. To express this we will use Eq. (5.6) once more. Before presenting the expression let us notice the following. In view of (5.4), $\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu=\nabla^{F^*TN}_{\partial_i}\mu$ is a section of the pull-back bundle F^*TN. Hence it can be linearly decomposed in terms of the local frame $(\mu, DF(\partial_1),\dots,DF(\partial_m))$. In particular, by keeping in mind that μ is a unit length time-like vector we conclude

$$\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu=g^{jk}\overline{g}\left(\nabla^{F^*TN}_{\partial_i}\mu,DF(\partial_j)\right)DF(\partial_k)=-g^{jk}\,h_{ij}\,DF(\partial_k)$$

with the obvious summation over the indices j and k. Thus, we find by (5.4)

$$\begin{aligned}&\overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j}D\mathcal{F}(\partial_s),\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu\right)\\&=-\partial_j\left(H-\mathcal{H}\right)\cdot\overline{g}\left(\mu,\nabla^{F^*TN}_{\partial_i}\mu\right)-(H-\mathcal{H})\overline{g}\left(\nabla^{F^*TN}_{\partial_j}\mu,\nabla^{F^*TN}_{\partial_i}\mu\right)\\&=-(H-\mathcal{H})g^{kl}\,h_{ik}\,h_{jl},\end{aligned}\tag{6.6}$$

where we used $\overline{g}\left(\mu,\nabla^{F^*TN}_{\partial_i}\mu\right)=0$ by the metric property of the pull-back connection and the fact that μ is of unit length. Equation (6.7) now follows by substituting (6.6) and (6.5) in (6.4). □

Corollary 6.3 *The mean curvature evolves along* (1.3) *by*

$$\begin{aligned}(\partial_t+\Delta)(H-\mathcal{H})&=-(H-\mathcal{H})\left(\|h\|^2+\mathrm{Ric}^N(\mu,\mu)\right),\\(\partial_t+\Delta)(H-\mathcal{H})^2&=-2\,(H-\mathcal{H})^2\left(\|h\|^2+\mathrm{Ric}^N(\mu,\mu)\right)\\&\quad-2|\nabla\left(H-\mathcal{H}\right)|^2.\end{aligned}\tag{6.7}$$

Proof The second evolution equation is a direct consequence of the first one. For the first equation we compute by Propositions 6.1 and 6.2

$$\begin{aligned}\partial_t H=&\partial_t\left(g^{ij}\,h_{ij}\right)=\partial_t g^{ij}\cdot h_{ij}+g^{ij}\cdot\partial_t h_{ij}\\=&-2\left(H-\mathcal{H}\right)\|h\|^2+g^{ij}\partial_t h_{ij}\\=&-2\left(H-\mathcal{H}\right)\|h\|^2+\left(H-\mathcal{H}\right)\left(\|h\|^2-\mathrm{Ric}^N(\mu,\mu)\right)-\Delta\left(H-\mathcal{H}\right).\end{aligned}$$

□

Remark 6.4 We want to point out a difference between the first equation in (6.7) and the same evolution equation in the proof of [9, Proposition 4.6]. In the latter one sees an extra term $\overline{g}\left(\overline{\nabla}\mathcal{H},\mu\right)$. Its presence is due to the function $\mathcal{H}$ being defined in [9] on the ambient Lorentzian manifold $(N,\overline{g})$ while in our case $\mathcal{H}$ is defined on $(M,\widetilde{g})$. In particular, in our case $\partial_t\mathcal{H}$ is just vanishing.

7 Evolution of the scalar second fundamental form

In this section we derive an evolution equation for the norm (with respect to g) of the scalar second fundamental form. This will play an essential role for the uniform C^0 and C^2-estimates of u. We begin by recalling some useful formulae, to be consistent with other references we will also write them in abstract index notation.
First we recall the Codazzi-Mainardi equation, cf. [21, Theorem 8.9].

Proposition 7.1 *For every* $X, Y, Z \in \Gamma(TM)$ *one has*

$$\nabla_X \mathrm{II}(Y, Z) - \nabla_Y \mathrm{II}(X, Z) = R^N(DF(X), DF(Y))DF(Z) - DF(R(X, Y)Z). \tag{7.1}$$

Corollary 7.2 *For every* $X, Y, Z \in \Gamma(TM)$ *one has*

$$\nabla h(X, Y, Z) - \nabla h(Y, X, Z) = R^N(DF(X), DF(Y), DF(Z), \mu). \tag{7.2}$$

Proof The result follows by taking the inner product with the unit normal on both sides of (7.1) and using the formula for the covariant derivative of tensors. □

Next we recall Gauß' Theorema Egregium.

Theorem 7.3 *For every* $X, Y, Z, W \in \Gamma(TM)$ *one has*

$$\begin{aligned} &R^N(DF(X), DF(Y), DF(Z), DF(W)) \\ &= R^M(X, Y, Z, W) + h(Y, Z)\, h(X, W) - h(X, Z)\, h(Y, W). \end{aligned} \tag{7.3}$$

Proof For a proof of this we refer to [21, Theorem 8.5], where we used (2.26) and (2.25).

□

Let A now be a $(0, 2)$-tensor over (M, g). Setting for every $X, Y, Z, W \in \Gamma(TM)$

$$\nabla^2 A(X, Y, Z, W) = \nabla(\nabla A)(X, Y, Z, W),$$

one has by direct computation of $\nabla^2 A(X, Y, Z, W)$ and $\nabla^2 A(Y, X, Z, W)$

$$\begin{aligned} \nabla^2 A(X, Y, Z, W) - \nabla A(Y, X, Z, W) &= A\,(R(Y, X)Z, W) + A\,(Z, R(Y, X)W) \\ &= -A\,(R(X, Y)Z, W) - A\,(Z, R(X, Y)W)\,. \end{aligned} \tag{7.4}$$

Next we give an expression for the well-known Simons identities (cf. [29, Theorem 4.2.1]) We begin by presenting how to interchange second-order covariant derivatives of the (scalar) second fundamental form.

$$\begin{aligned} \nabla_k \nabla_l h_{ij} = \nabla_i \nabla_j h_{kl} &+ g^{pq} R^N_{iklp}\, h_{qj} + g^{pq} R^N_{ikjp}\, h_{ql} - h_{kl}\, R^N_{0ij0} \\ &- h_{kp}\, g^{pq} R^N_{lijq} - h_{ij}\, R^N_{k0l0} - h_{ip}\, g^{pq} R^N_{kjlq} + \nabla_k R^N_{lij0} + \nabla_i R^N_{kjl0} \\ &- g^{pq}\, h_{kl}\, h_{ip}\, h_{qj} + g^{pq}\, h_{il}\, h_{kp}\, h_{qj} - g^{pq}\, h_{kj}\, h_{ip}\, h_{ql} + g^{pq}\, h_{ij}\, h_{kp}\, h_{ql}\,. \end{aligned} \tag{7.5}$$

We want to point out a difference in signs with the classical result cited above due to a different sign convention for the scalar second fundamental form h. By taking the trace of (7.5) we find an expression for the Laplacian of the (scalar) second fundamental form which is as follows:

$$\begin{aligned} \Delta h_{ij} = -\nabla_i \nabla_j H - h_{ij} &\left(\| h \|^2 + \mathrm{Ric}^N(\mu, \mu) \right) + H\, h_{ik}\, h_{kj} \\ &+ 2\, h_{kl}\, R^N_{kijl} - h_{pj}\, R^N_{ikkp} + h_{ip}\, R^N_{kjkp} \\ &+ H\, R^N_{0ij0} - \nabla_k R^N_{kij0} - \nabla_i R^N_{kjk0}. \end{aligned} \tag{7.6}$$

By summing the above with (6.3) we find

$$\begin{aligned}(\partial_t+\Delta)\,h_{ij} &= -\nabla_i\nabla_j\mathcal{H}-\mathcal{H}\left(h_{ik}\,h_{kj}-R^N_{0ij0}\right)\\ &\quad+2\,H\,h_{ik}\,h_{kj}-h_{ij}\left(\|\,h\,\|^2+\mathrm{Ric}^N(\mu,\mu)\right)\\ &\quad+2\,h_{kl}\,R^N_{kijl}-h_{jl}\,R^N_{ikkl}+h_{il}\,R^N_{kjkl}-\nabla_k R^N_{kij0}-\nabla_i R^N_{kjk0}\end{aligned} \tag{7.7}$$

Remark 7.4 Due to different sign conventions, Eq. (7.7) has slight differences in signs with the one in [9, Proposition 3.2 (i)].

Although the slight change in signs between Eq. (7.7) and the corresponding one in [9] we can conclude by straightforward estimates the same inequality as [9, Proposition 3.2 (iii)], which is the assertion of the final result in this section.

Proposition 7.5

$$\begin{aligned}(\partial_t+\Delta)\,\|\,h\,\|^2 &\leqslant -2\|\nabla\,h\,\|^2-\|\,h\,\|^4+c_0\cdot\left(1+\|\,h\,\|^2+\|\nabla^2\mathcal{H}\|^2\right)\\ &\leqslant -2\|\nabla\,h\,\|^2-\|\,h\,\|^4+c_1\cdot\left(1+\|\,h\,\|+\|\,h\,\|^2\right)\end{aligned} \tag{7.8}$$

where the constants

$$\begin{aligned}c_0 &= c_0\left(m,\nu,\|R^N\|,\|\nabla R^N\|,\|\mathcal{H}\|_\infty\right),\\ c_1 &= c_1\left(m,\nu,\|R^N\|,\|\nabla R^N\|,\|\mathcal{H}\|_\infty,\|\mathcal{H}\|_{C^2}\right),\end{aligned}$$

depend on the entries in the brackets.

Proof We only indicate the proof idea. We conclude first from (7.6)

$$\begin{aligned}\Delta\|\,h\,\|^2 &= -2\|\nabla\,h\,\|^2-2\,h_{ij}\,\nabla_i\nabla_j\,H-2\|\,h\,\|^2\left(\|\,h\,\|^2+\mathrm{Ric}^N(\mu,\mu)\right)\\ &\quad+2\,H\,h_{ij}\,h_{jk}\,h_{ki}+4\,h_{ij}\,h_{kl}\,R^N_{kijl}-2\,h_{ij}\,h_{lj}\,RN_{ikkl}+2\,h_{ij}\,h_{il}\,R^N_{kjkl}\\ &\quad+2\,H\,h_{ij}\,R^N_{0ij0}-2\,h_{ij}\,\nabla_k R^N_{kij0}-2\,h_{ij}\,\nabla_i R^N_{kjk0}.\end{aligned} \tag{7.9}$$

With similar arguments by (6.3) we infer

$$\partial_t\|\,h\,\|^2=-2\,(H-\mathcal{H})\left(h_{ij}\,h_{jk}\,h_{ki}+R^N_{0ij0}\,h_i\,j\right)+2\,h_{ij}\,\nabla_i\nabla_j\,(H-\mathcal{H})\,. \tag{7.10}$$

Summing up (7.9) and (7.10) we find the following evolution equation for the g-norm of the scalar second fundamental form.

$$\begin{aligned}(\partial_s+\Delta)\,\|\,h\,\|^2 &= -2\|\nabla\,h\,\|^2-2\|\,h\,\|^2\left(\|\,h\,\|^2+\mathrm{Ric}^N(\mu,\mu)\right)-2\,h_{ij}\,\nabla_i\nabla_j\mathcal{H}\\ &\quad+2\mathcal{H}\left(h_{ij}\,h_{jk}\,h_{ki}+R^N_{0ij0}\,h_{ij}\right)+4\,h_{ij}\,h_{kl}\,R^N_{kijl}-2\,h_{ij}\,h_{jl}\,R^N_{ikkl}\\ &\quad+2\,h_{ij}\,h_{il}\,R^N_{kjkl}-2\,h_{ij}\,\nabla_k R^N_{kij0}-2\,h_{ij}\,\nabla R^N_{kjk0}.\end{aligned} \tag{7.11}$$

From here the first inequality follows by bounded geometry.

For the second inequality, the problem is controlling $\|\nabla^2\mathcal{H}\|^2$. We may want to use the second displayed equation in the proof of [9, Proposition 4.7]; however, in their setting $\mathcal{H}$ is a function on N. Instead, note that

$$\|\nabla^2\mathcal{H}\|^2=g^{ik}g^{jl}\nabla^2\mathcal{H}(\partial_i,\partial_j)\nabla^2\mathcal{H}(\partial_k,\partial_l),$$

$$\nabla^2\mathcal{H}(\partial_i,\partial_j)=\partial_i\partial_j\mathcal{H}-\nabla_{\nabla_{\partial_i}\partial_j}\mathcal{H}=\mathcal{H}_{ij}-\Gamma^k_{ij}\mathcal{H}_k.$$

From [12, (2.6), (2.15)] we can conclude

$$\Gamma^k_{ij}\mathcal{H}_k=\widetilde{\Gamma}^k_{ij}\mathcal{H}_k+\frac{f'(u)}{f(u)}\left(u_i\mathcal{H}_j+u_j\mathcal{H}_i\right)+\frac{\nu}{f(u)^2}\widetilde{g}\left(\widetilde{\nabla}u,\widetilde{\nabla}\mathcal{H}\right)h_{ij}\,.$$

Thus, we find for some uniform constant $c>0$, using (2.3) and uniform bounds on f and its derivatives

$$\|\nabla^2\mathcal{H}\|\leqslant c\nu^2\left(\|\widetilde{\nabla}^2\mathcal{H}\|_{\widetilde{g}}+\nu^2\|\,h\,\|\|\widetilde{\nabla}\mathcal{H}\|_{\widetilde{g}}\right).$$

This yields the second inequality and proves the statement. □

8 C^0-estimates: uniform bounds on the solution u

Consider now a solution $u\in C^{3,\alpha}(M\times[0,T])\cap C^{4,\alpha}(M\times(0,T])$ of (1.3), which exists for $T>0$ sufficiently small by Theorem 1.5, provided $\mathcal{H}\in C^{2,\alpha}(M)$. We first prove a uniform upper bound.

Proposition 8.1 *Consider Setting 1.1 and impose Assumptions 1.4 (1), (2). Then u is bounded uniformly from above by $u_{\sup}(0)$.*

Proof Notice that the prescribed mean curvature flow (1.3) can be written as

$$\partial_t u=-\left(H-\mathcal{H}\right)\nu. \tag{8.1}$$

The statement will follow once we prove that $(H-\mathcal{H})\geqslant 0$. Indeed, due to $\nu\geqslant 1$, $\partial_t u\leqslant 0$ and thus u is non-increasing with upper bound $u_{\sup}(0)$.

Since $(H-\mathcal{H})(t=0)>0$, there exists some maximal interval $[0,\varepsilon)\subseteq[0,T]$ such that $(H-\mathcal{H})(t)>0$ for $t\in[0,\varepsilon)$. If $\varepsilon=T$, then the right-hand side in (8.1) is negative and the statement follows. Let us now assume that $\varepsilon<T$. From (6.7) we see that, by differentiating in $(0,\varepsilon]$,

$$(\partial_t+\Delta)(H-\mathcal{H})\geqslant -c(H-\mathcal{H}),$$

for some positive constant c, depending on bounded geometry and $\|h(t)\|$ for $t\in(0,\varepsilon]$. Since $u\in C^{3,\alpha}(M\times[0,T])\cap C^{4,\alpha}(M\times(0,T])$, we note that $\|h(t)\|$ is uniformly bounded for $t\in[0,\varepsilon]$. These bounds need not be uniform in T (we have not proved this yet), but this is not necessary for the argument here.

Using now the Omori-Yau maximum principle in the form (4.2), we conclude from the enveloping theorem in Proposition 4.3 that

$$\partial_t(H-\mathcal{H})_{\inf}\geqslant -c(H-\mathcal{H})_{\inf}.$$

Integrating this differential inequality and the fact that $(H-\mathcal{H})_{\inf}$ is locally Lipschitz yields for $t\in(0,\varepsilon]$

$$(H-\mathcal{H})_{\inf}(t)\geqslant e^{-c(t-\sigma)}(H-\mathcal{H})_{\inf}(\sigma),$$

for $0<\sigma<t\leqslant\varepsilon$. Since $\sigma\in(0,\varepsilon)$, $(H-\mathcal{H})(\sigma)>0$. Thus, $(H-\mathcal{H})(t=\varepsilon)>0$ and hence by maximality of the interval $[0,\varepsilon)$, we conclude that $\varepsilon=T$, that is $(H-\mathcal{H})>0$ on $M\times[0,T]$. The statement now follows from (8.1). □

For the uniform lower bounds the following lemma is useful.

Lemma 8.2 *Let* $\theta \in C^{1,\alpha}(M \times [0,T]) \cap C^{2,\alpha}(M \times (0,T])$. *If* θ *satisfies the differential inequality*

$$(\partial_t + \Delta)\,\theta \leqslant -a^2\theta^2 + b, \tag{8.2}$$

with $a > 0$ *and* b *constants, then* θ *is uniformly bounded from above.*

Proof We begin by noticing in the inequality above we can always replace b by some positive nonzero $b^2 > 0$, which we henceforth do.

Furthermore, if $\sup_M \theta(t) \leqslant b/a$ there is nothing to prove. Hence let us assume there exists some $t_0 \in [0,T]$ so that $\sup_M \theta(t_0) > b/a$.

Since $\theta \in c^{1,\alpha}(M \times [0,T]) \cap C^{2,\alpha}(M \times (0,T])$, from Proposition 4.3, $\theta_{\sup}(t)$ is a locally Lipschitz function and hence positive in a neighbourhood $(t_1,t_2) \subset [0,T]$ containing t_0. Let us then consider the minimal such $t_1 \geqslant 0$. Now, by Lemma 3.3 we have that the function $\theta_{\sup}$ is continuous, thus either $t_1 = 0$ or $\theta_{\sup}(t_1) = b/a$.

Let $t \in (t_1,t_2)$ and $(p_k(t))_k \subset M$ a sequence satisfying the estimates (4.2) for the Omori-Yau maximum principle. For $k \in \mathbb{N}$ large enough $\theta\,(p_k(t),t)) > b/a$. In particular, at these points, θ satisfies the differential inequality

$$(\partial_t + \Delta)\theta\,(p_k(t),t) \leqslant 0.$$

In conclusion, in view of Theorem 4.4,

$$\theta(\cdot,t) \leqslant \theta_{\sup}(t) \leqslant \theta_{\sup}(t_1) = \frac{b}{a}\mathcal{J}$$

thus providing the required uniform upper bound. □

Now we establish a lower bound on u for any finite T.

Proposition 8.3 *Consider Setting* 1.1 *and impose Assumptions* 1.4 *(1), (2). Then* $\|\,h\,\|$ *and* H *are uniformly bounded. Moreover,* u *is bounded uniformly for finite times.*

Proof In Theorem 9.3 in the next section we will prove that, as a consequence of the upper bound in Proposition 8.1, ν is uniformly bounded.
By playing with binomial formulae we find from inequality (7.8)

$$(\partial_t + \Delta)\,\|\,h\,\|^2 \leqslant -a^2\|\,h\,\|^4 + b^2, \tag{8.3}$$

for some uniform $a, b > 0$. Note that a, b depend on ν and thus uniform bounds on ν from Theorem 9.3 below are crucial. By Lemma 8.2, we conclude that $\|\,h\,\|$ is uniformly bounded and hence H is bounded uniformly as well. Thus, the right-hand side of (8.1) is uniformly bounded and thus u is bounded uniformly for finite times. □

Deriving a uniform time-independent lower bound for u is harder and is usually done by a barrier argument. In the non-compact setting the barrier argument is somewhat intricate and we present here a different approach without using barriers.

Proposition 8.4 *Consider Setting* 1.1 *and impose Assumptions* 1.4 *(1)–(3). Then* $\|\partial_t u\|_\infty$ *is exponentially decreasing. In particular,* u *is bounded uniformly.*

Proof As explained in Proposition 8.3, $\| h \|$ is uniformly bounded. Anticipating uniform space-likeness as asserted in Theorem 9.3, by the time-like convergence assumption we may take $\delta' > 0$ small enough such that $\mathrm{Ric}^N(\mu, \mu) \geqslant \delta' > 0$. By (6.7) we conclude

$$(\partial_t + \Delta)(H - \mathcal{H})^2 \leqslant -\delta'(H - \mathcal{H})^2. \tag{8.4}$$

By the Omori-Yau estimates (4.1) and Proposition. 4.3, we find

$$\partial_t (H - \mathcal{H})^2_{\sup} \leqslant -\delta'(H - \mathcal{H})^2_{\sup}. \tag{8.5}$$

This differential inequality can be integrated and yields

$$0 \leqslant (H - \mathcal{H})^2_{\sup}(t) \leqslant e^{-\delta'(t-\sigma)}(H - \mathcal{H})^2_{\sup}(\sigma), \tag{8.6}$$

for every $0 < \sigma < t$. Since $(H - \mathcal{H})^2 \in C^{1,\alpha}(M \times [0, T]) \cap C^{2,\alpha}(M \times (0, T])$, Lemma 3.3 implies that $(H - \mathcal{H})^2_{\sup}$ is continuous. Therefore, we can take the limit for σ going to 0 in (8.7) resulting in

$$0 \leqslant (H - \mathcal{H})^2_{\sup}(t) \leqslant e^{-t\delta'}(H - \mathcal{H})^2_{\sup}(0), \tag{8.7}$$

for every $t \in [0, T]$. As already noted in the previous proposition, Theorem 9.3 asserts that as a consequence of the upper bound in Proposition 8.1, ν is uniformly bounded. Thus, by (8.1) there exists a uniform constant $c > 0$ such that

$$\|\partial_t u\|_\infty \leqslant c e^{-t\delta'}. \tag{8.8}$$

This proves the statement. □

9 C^1-estimates: preserving the space-like property

In this section we will prove the first main result of this paper, namely that a prescribed mean curvature flow stays uniformly space-like for as long as the flow exists, if u is uniformly bounded from above. The argument presented here follows in spirit the work of Gerhardt in [14] and is concluded by an application of the parabolic maximum principle. We begin by noticing the following.

Proposition 9.1 *If the gradient function ν is uniformly bounded along the flow (1.3), then the prescribed mean curvature flow (1.3) stays space-like.*

Proof Assume there exists some $K > 1$ so that $\nu = \nu(p, t) \leqslant K$ for every $(p, t) \in M \times [0, T]$. Note that the requirement $K > 1$ follows from Proposition 2.5 (ii). Equation (2.8) implies

$$f(u) \leqslant K\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}},$$

where $\widetilde{\nabla} u$ is as before the gradient of u with respect to $\widetilde{g}$. We conclude

$$|\widetilde{\nabla} u|^2_{\widetilde{g}} \leqslant \left(1 - \frac{1}{K^2}\right) f(u)^2 < f(u)^2.$$

Notice that the above is precisely the condition required for a graph to be space-like as pointed out in [12, Remark 2.6]. □

Remark 9.2 Note that a solution $u \in C^{2,\alpha}(M \times [0, T])$ is guaranteed to exist if $\mathcal{H} \in C^{1,\alpha}(M)$ and the initial condition is merely $u_0 \in C^{2,\alpha}(M)$.

In order to prove that the flow stays space-like, it is therefore enough to prove that the gradient function ν is uniformly bounded along the flow. This is precisely the main conclusion of this section, which we now put as a separate theorem

Theorem 9.3 *Consider the flow* (1.3) *with* $\mathcal{H} \in C^{1,\alpha}(M)$ *and solution* $u \in C^{2,\alpha}(M \times [0, T])$. *Assume that* u *is uniformly bounded from above. Then the gradient function* ν *is uniformly bounded along the flow, with the bound depending only on the upper bound of* u. *In particular, the prescribed mean curvature flow stays space-like as long as the flow exists.*

It is worth noticing that Theorem 9.3 is a statement about properties preserved along the flow. Usually one deals with these problems by finding appropriate parabolic differential inequalities and employing the parabolic maximum principle. Having a parabolic maximum principle at our disposal (cf. Theorem 4.4), following along the same lines of the proof of [14, Proposition 3.7], the reminder of this section will be devoted to gaining the claimed parabolic differential inequalities. In order to do so, some preparation is needed.

9.1 Preliminaries

First we recall the evolution Eq. (1.3) for u. For $\mathcal{H} : M \to \mathbb{R}$ being a fixed prescribing function, one can rewrite (1.3) in terms of ν that is,

$$(\partial_t + \Delta)u = \mathcal{H}\nu + \frac{f'(u)}{f(u)}(m + \nu^2 - 1). \tag{9.1}$$

We will now proceed by presenting some estimates which will be useful for the proof of Theorem 9.3, and hold for any given graphical embedding (not necessarily along the (1.3) flow).

Proposition 9.4 *Consider Setting* 1.1. *Assume the embeddings* $F(M) \equiv F(t)(M)$ *are space-like for* $t \in [0, T]$. *Recall,* $g = F^*\overline{g}$ *denotes the induced metric on* M, h *the scalar second fundamental and* ν *the gradient function. Then there exists a constant* $c > 0$ *independent of* u, *such that*

$$|g(\nabla u, \nabla \nu)| \leqslant \| h \| |\nabla u|_g^2 + c|\nabla u|_g^2 \nu. \tag{9.2}$$

Proof In local coordinates one has

$$g(\nabla u, \nabla \nu) = g^{ik} u_k \nu_i.$$

From Eq. (5.26) we infer

$$\begin{aligned} g(\nabla u, \nabla \nu) &= -g^{ik} u_k g^{jm} u_m \, h_{ij} - \frac{f'(u)}{f(u)} g^{im} u_m u_i \nu \\ &= -g^{ik} u_k g^{jm} u_m \, h_{ij} - \frac{f'(u)}{f(u)} |\nabla u|_g^2. \end{aligned}$$

Recall, by Setting 1.1 there exists some constant $c > 0$ so that $\|f'/f\|_\infty \leqslant c$. Thus, we conclude

$$|g(\nabla u, \nabla \nu)| \leqslant |g^{ik} u_k g^{jm} u_m \, h_{ij}| + \left| \frac{f'(u)}{f(u)} \right| |\nabla u|_g^2 \nu \leqslant \| h \| |\nabla u|_g^2 + c|\nabla u|_g^2 \nu,$$

where $\| h \|$ denotes the norm of the scalar second fundamental form h with respect to the metric g. □

Next we present an estimate for $\mathrm{Ric}^N(\mu, \mu)$.

Proposition 9.5 *We continue as in Proposition 9.4. Then there exists a constant* $c > 0$ *independent of* u*, such that*

$$|\mathrm{Ric}^N(\mu, \mu)| \leqslant c\nu^2. \tag{9.3}$$

Proof From the local expression of the unit normal μ in (2.7) we find

$$\mathrm{Ric}^N(\mu, \mu) = \nu^2 \mathrm{Ric}^N(\partial_0, \partial_0) + \frac{2\nu^2}{f(u)^2} \mathrm{Ric}^N(\partial_0, \widetilde{\nabla} u) + \frac{\nu^2}{f(u)^4} \mathrm{Ric}^N(\widetilde{\nabla} u, \widetilde{\nabla} u).$$

Proposition 2.8 gives

$$\mathrm{Ric}^N(\mu, \mu) = -m\nu^2 \frac{f''(u)}{f(u)} + \frac{2\nu^2}{f(u)^2} \mathrm{Ric}^N(\partial_0, \widetilde{\nabla} u) + \frac{\nu^2}{f(u)^4} \mathrm{Ric}^N(\widetilde{\nabla} u, \widetilde{\nabla} u).$$

The second term vanishes due to Proposition 2.8. Again, from Proposition 2.8 we infer for the third term

$$\frac{\nu^2}{f(u)^4} \mathrm{Ric}^N(\widetilde{\nabla} u, \widetilde{\nabla} u) = \frac{\nu^2}{f(u)^4} \widetilde{\mathrm{Ric}}(\widetilde{\nabla} u, \widetilde{\nabla} u) + \frac{f''(u)}{f(u)^3} \nu^2 |\widetilde{\nabla} u|^2_{\widetilde{g}} + (m-1) \frac{f'(u)^2}{f(u)^4} \nu^2 |\widetilde{\nabla} u|^2_{\widetilde{g}}.$$

We plug this back into the expression for $\mathrm{Ric}^N(\mu, \mu)$ and conclude from (iv) in Proposition 2.5

$$\begin{aligned} \mathrm{Ric}^N(\mu, \mu) = m\nu^2 \frac{f''(u)}{f(u)} + \frac{\nu^2}{f(u)^4} \widetilde{\mathrm{Ric}}(\widetilde{\nabla} u, \widetilde{\nabla} u) \\ + \frac{f''(u)}{f(u)} |\nabla u|^2_g + (m-1) \left(\frac{f'(u)}{f(u)} \right)^2 |\nabla u|^2_g. \end{aligned}$$

By Setting 1.1 there exist some constants $c_1, c_2, c_3 > 0$ so that

$$|f(x^0)| \geqslant c_1 \mathfrak{I} \quad \left| \frac{f'(x^0)}{f(x^0)} \right| \leqslant c_2 \mathfrak{I} \quad \left| \frac{f''(x^0)}{f(x^0)} \right| \leqslant c_3 \mathfrak{I} \quad \forall x^0 \in \mathbb{R}, \tag{9.4}$$

By taking the absolute value and keeping in mind that $(M, \widetilde{g})$ is of bounded geometry (i.e. in particular $\widetilde{\mathrm{Ric}}(X, X) \leqslant c_4 \widetilde{g}(X, X)$ for any vector field X and some uniform constant $c_4 > 0$), we obtain the following estimate

$$|\mathrm{Ric}^N(\mu, \mu)| \leqslant m c_3 \nu^2 + \frac{c_4}{f(u)^4} \nu^2 + c_3 |\nabla u|^2_g + (m-1) c_2^2 |\nabla u|^2_g.$$

The statement now follows by noticing that $|\nabla u|^2_g \leqslant \nu^2$ by Proposition 2.5 and since $|f(x^0)| \geqslant c_1 > 0$ is bounded uniformly from below away from zero. ☐

Next we prove that hypersurfaces of $(N, \overline{g})$ arising as graphs of some Hölder regular functions satisfy the mean curvature structure condition, cf. [4, chapter 3].

Proposition 9.6 *We continue as in Proposition 9.4. Recall,* H *denotes the scalar mean curvature and* h *the scalar second fundamental form. Then for any* $\varepsilon > 0$ *and some uniform constant* $c > 0$ *(independent of* u*) we have*

$$|H + h(\nabla u, \nabla u)| \leqslant \varepsilon \nu \|h\| + c\varepsilon^{-1} \nu^3. \tag{9.5}$$

Proof At any fixed $(p,t) \in M \times [0,T]$ there exists an orthonormal (with respect to g) basis $\{e_i\}$ of h-eigenvectors, i.e. for the Kronecker delta δ_{ij}

$$h(e_i,e_j) = h_i\delta_{ij}, \quad g(e_i,e_j) = \delta_{ij}.$$

With respect to that basis we compute at (p,t) (writing $(\nabla u)_i := g(\nabla u, e_i)$)

$$\begin{aligned} |H + h(\nabla u, \nabla u)| &= \left|\sum_{i=1}^m h_i + \sum_{i=1}^m h_i(\nabla u)_i^2\right| \leqslant \sum_{i=1}^m \Big(\frac{1+(\nabla u)_i^2}{\sqrt{\varepsilon\nu}}\Big)\sqrt{\varepsilon\nu}|h_i| \\ &\leqslant \sum_{i=1}^m (\nu\varepsilon)^{-1}\Big(1+(\nabla u)_i^2\Big)^2 + \varepsilon\nu\sum_{i=1}^m h_i^2 \\ &\leqslant (\nu\varepsilon)^{-1}\Big(m + 2|\nabla u|_g^2 + |\nabla u|_g^4\Big) + \varepsilon\nu\|h\|^2. \end{aligned}$$

By (2.12) we conclude for some $c > 0$ (independent of u and (p,t))

$$H + h(\nabla u, \nabla u) \leqslant c\varepsilon^{-1}\nu^3 + \varepsilon\nu\|h\|^2.$$

□

We will need one last estimate.

Proposition 9.7 *We continue as in Proposition 9.4. Consider as above the (local) vector field V on M, so that $DF(V) = \partial_0^\top$. Then for every function $\mathcal{H} \in C^{1,\alpha}(M)$ there exists some uniform constant $c > 0$ (independent of u) such that*

$$|V(\mathcal{H})| \leqslant c\|\nabla u\|_g\|\mathcal{H}\|_{1,\alpha}. \tag{9.6}$$

Proof It is easy to see that the condition $DF(V) = \partial_0^\top$ gives $V = -\frac{\widetilde{\nabla} u}{f(u)^2}$. Therefore, in local coordinates, we obtain using (2.10) in the last estimate

$$|V(\mathcal{H})| = \left|\frac{1}{f(u)^2}\widetilde{g}^{ij}u_i\mathcal{H}_j\right| \leqslant c|\widetilde{\nabla} u|_{\widetilde{g}}\|\mathcal{H}\|_{1,\alpha} \leqslant c|\nabla u|_g\|\mathcal{H}\|_{1,\alpha},$$

where we used the fact that $f > 0$ is uniformly bounded away from zero. □

We are now ready to prove Theorem 9.3.

9.2 Proof of Theorem 9.3

We will use the ideas of the argument of [14] with some adaptations due to non-compact geometry. In the upcoming computations we will systematically suppress the point $(p,t) \in M \times [0,T]$ from notation. We consider some constants $\lambda, \rho > 0$, which we will specify later.

Let $\varphi = e^{\rho e^{\lambda u}}$. Assume, without loss of generality that $u > 1$, if it is not the case we can consider $u + C$ for some constant $C > 0$ large enough. An easy computation gives

$$(\partial_t + \Delta)\varphi = -\rho\lambda^2 e^{\lambda u}(1 + \rho e^{\lambda u})\varphi|\nabla u|_g^2 + \rho\lambda e^{\lambda u}\varphi(\partial_t + \Delta)u. \tag{9.7}$$

Let us now set $w = \varphi\nu$. Therefore, we find (recall μ is defined in 2.7)

$$\begin{aligned} (\partial_t + \Delta)w &= \nu(\partial_t + \Delta)\varphi + \varphi(\partial_t + \Delta)\nu - 2g(\nabla\varphi, \nabla\nu) \\ &= \nu(\partial_t + \Delta)\varphi + \varphi(\partial_t + \Delta)\nu - 2\rho\lambda e^{\lambda u}\varphi g(\nabla u, \nabla\nu). \end{aligned}$$

Substituting (9.7) and (5.1) in the above, we obtain

$$(\partial_t + \Delta)w = I_1 + I_2.$$

where I_1 and I_2 are explicitly given as follows (recall μ is defined in (2.7))

$$\begin{aligned} I_1 :=& -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2\varphi v - \| h \|^2 \varphi v - V(\mathcal{H})\varphi - 2\frac{f'(u)}{f(u)} H\varphi \\ & - 2\left(\rho\lambda e^{\lambda u} - \frac{f'(u)}{f(u)}\right) g(\nabla u, \nabla v)\varphi - \left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2 \varphi v, \\ I_2 :=& \rho\lambda e^{\lambda u}\varphi v(\partial_t + \Delta)u - \mathrm{Ric}^N(\mu,\mu)\varphi + \frac{f'(u)}{f(u)}\mathcal{H}\varphi - \frac{f'(u)}{f(u)}\mathcal{H}\varphi v^2 \\ & + m\frac{f''(u)}{f(u)}\varphi v - \frac{f''(u)}{f(u)}|\nabla u|_g^2\varphi v - m\left(\frac{f'(u)}{f(u)}\right)^2 \varphi v \end{aligned}$$

First, we estimate I_2 from above. By Setting 1.1 there exist some constants $c_1, c_2 > 0$ such that the warping function $f : \mathbb{R} \to \mathbb{R}^+$ satisfies for any $x^0 \in \mathbb{R}$

$$|f(x^0)| \geqslant c_1, \quad \left|\frac{f'(x^0)}{f(x^0)}\right| \leqslant c_2, \quad \left|\frac{f''(x^0)}{f(x^0)}\right| \leqslant c_2.$$

From Eq. (9.1) we now deduce for some $c_3 > 0$ depending on $\|\mathcal{H}\|_\infty$

$$(\partial_t + \Delta)u \leqslant c_3 v^2.$$

Since $|\nabla u|_g \leqslant v$ by (iii) in Proposition 2.5, we arrive by Propositions 9.5 and 9.7 at the following estimate of I_2 (we write $c > 0$ for any, positive, uniform constant)

$$I_2 \leqslant c\rho\lambda e^{\lambda u}\varphi v^2 + c|\nabla u|_g^2\varphi v + c|\nabla u|_g\varphi \leqslant c\rho\lambda e^{\lambda u}\varphi v^3.$$

The estimate of I_1 is slightly more involved. Using the formula from Proposition (5.11)

$$h(\nabla u, \nabla u) = -g(\nabla u, \nabla v) - \frac{f'(u)}{f(u)}|\nabla u|^2 v,$$

we can rewrite I_1 as follows

$$\begin{aligned} I_1 =& -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2\varphi v - \| h \|^2\varphi v - 2\frac{f'(u)}{f(u)}\Big(H + h(\nabla u, \nabla u)\Big)\varphi \\ & - 3\left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2\varphi v - m\left(\frac{f'(u)}{f(u)}\right)^2 \varphi v - 2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi. \end{aligned}$$

By Proposition 9.6 we find for some uniform constant $c > 0$ (in fact we will not differentiate between all the, positive, uniform constants and denote them all by c)

$$\begin{aligned} I_1 \leqslant& -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2\varphi v \\ & - \left(1 - 2\left|\frac{f'(u)}{f(u)}\right|\varepsilon\right)\| h \|^2\varphi v - 3\left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2\varphi v \\ & + 2c\left|\frac{f'(u)}{f(u)}\right|\varepsilon^{-1}\varphi v^3 - 2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi. \end{aligned} \tag{9.8}$$

We now want to estimate the last term above. By Proposition 9.4 we have for some uniform constant $c > 0$

$$-2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi \leqslant 2\rho\lambda e^{\lambda u}|g(\nabla u, \nabla v)|\varphi$$

$$\leqslant 2\rho\lambda e^{\lambda u}\Big(\|h\||\nabla u|_g^2 + c|\nabla u|_g^2 v\Big)\varphi.$$

We estimate this further for any $\varepsilon' > 0$ and using (2.12) in the last step

$$\begin{aligned} -2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi &\leqslant \frac{2\rho\lambda e^{\lambda u}|\nabla u|_g^2}{\sqrt{2(1-\varepsilon')v}}\sqrt{2(1-\varepsilon')v}\| h \|\varphi + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2 \varphi v \\ &\leqslant \frac{\rho^2\lambda^2 e^{2\lambda u}|\nabla u|_g^4}{(1-\varepsilon')v}\varphi + (1-\varepsilon')\| h \|^2\varphi v + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2\varphi v \\ &\leqslant \frac{\rho^2\lambda^2 e^{2\lambda u}}{(1-\varepsilon')}|\nabla u|_g^2\varphi v + (1-\varepsilon')\| h \|^2\varphi v + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2\varphi v. \end{aligned}$$

Choosing, for any given $\varepsilon' \in (0,1)$, an $\varepsilon > 0$ sufficiently small such that $\varepsilon' > 2\left|\frac{f'(u)}{f(u)}\right|\varepsilon$ and plugging the last estimate into (9.8), we arrive at

$$\begin{aligned} I_1 \leqslant\ & -\rho\lambda e^{\lambda u}\Big(\lambda - 2c\Big)|\nabla u|_g^2\varphi v - 3\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\varphi v \\ & + 2c\left|\frac{f'(u)}{f(u)}\right|\varepsilon^{-1}\varphi v^3 + \frac{\varepsilon'}{(1-\varepsilon')}\rho^2\lambda^2 e^{2\lambda u}|\nabla u|_g^2\varphi v. \end{aligned} \tag{9.9}$$

Set $\varepsilon' = e^{-\lambda u}$ and $\rho = 1/2$. Choose $\overline{\lambda} > 0$ so that for every $\lambda > \overline{\lambda}$

$$\frac{\rho}{1-e^{-\lambda u}} \leqslant \frac{3}{4}.$$

Then we can estimate I_1 even further by (recall $|\nabla u|_g \leqslant v$ by Proposition 2.5)

$$I_1 \leqslant -\frac{1}{8}\lambda e^{\lambda u}\Big(\lambda - c\Big)|\nabla u|_g^2\varphi v + c\lambda e^{\lambda u}\varphi v^3.$$

We want to point out that the above estimates follows by considering $\overline{\lambda}$ to be large enough so that the second and third term in (9.9) can be estimated by the second term in the equation above.

Summarizing, we arrive at the following intermediate estimate

$$(\partial_t + \Delta)w \leqslant -\frac{1}{8}\lambda e^{\lambda u}\Big(\Big(\lambda - c\Big)|\nabla u|_g^2 - cv^2\Big)w. \tag{9.10}$$

We want to turn this into a differential inequality for the supremum

$$v_{\sup}(t) = \sup_{p\in M} v(p,t).$$

Let us assume that there exists some $t_0 \in [0,T]$ such that $v_{\sup}(t_0) > 2$, otherwise the statement is trivial. Since by Proposition 4.3, $v_{\sup}(t)$ is locally Lipschitz, $v_{\sup}(t) > 2$ in an open interval $I = (a,b) \subset [0,T]$ containing t_0. We take the minimal possible such $a \geqslant 0$, such that by continuity of $v_{\sup}(t)$ we have either $a = 0$ or $v_{\sup}(a) = 2$.

Consider $t \in (a,b)$ and a sequence $(p_k(t)) \subset M$ satisfying (4.1). Then for $k \in \mathbb{N}$ sufficiently large, $v(p_k(t),t) > 2$ and we establish a differential evolution inequality for v at those points as follows. We consider v and w evaluated at $(p_k(t),t)$ without making it notationally explicit. Since $v \geqslant 2$, we have $-4 \geqslant -v^2$ and from (i) in Proposition 2.5 we find

$$|\nabla u|_g^2 = v^2 - 1 \geqslant v^2 - \frac{v^2}{4} = \frac{3}{4}v^2. \tag{9.11}$$

Choosing $\lambda > \overline{\lambda}$ sufficiently large (note that these choices do not depend on u) the right-hand side of (9.10), evaluated at $(p_k(t))$ for $k \in \mathbb{N}$ sufficiently large, turns negative and we conclude

$$(\partial_t + \Delta)w(p_k(t), t) \leqslant 0.$$

This implies by Proposition 4.3 for any $t \in (a, b)$ in the limit $k \to \infty$

$$\partial_t w_{\sup}(t) \leqslant 0.$$

Thus, for any $t \in (a, b)$ we conclude $w(\cdot, t) \leqslant w_{\sup}(t) \leqslant w_{\sup}(a)$. In particular, we find for any $(p, t) \in M \times (a, b)$ and some constant $c > 0$, depending only on $\mathcal{H}$, $u(t = a)$ and the ambient geometry, that (note that $e^{\rho e^{\lambda u}} > 1$)

$$\nu(p, t) \leqslant \exp\left(\rho e^{\lambda u_{\sup}(a)}\right)\nu_{\sup}(a) < c\nu_{\sup}(a), \tag{9.12}$$

where the second estimate holds, provided u is bounded uniformly from above. Now, since we have either $a = 0$ or $\nu_{\sup}(a) = 2$, we conclude that ν is uniformly bounded.

Corollary 9.8 *We continue in the Setting 1.1. Assume that u is uniformly bounded from above. Then ν is uniformly bounded and hence, provided f is uniformly bounded, as assumed in Setting 1.1, $|\widetilde{\nabla} u|_{\widetilde{g}}$ is uniformly bounded as well.*

10 C^2-estimates: bounds of the second fundamental form

Uniform boundedness of $\|h\|$ and hence also of the mean curvature H has been established already in Proposition 8.3. Now, as computed in the preceeding work by the first named author [12, (2.15)]

$$h_{ij} = -\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}\left(u_{ij} - \widetilde{\Gamma}^k_{ij}u_k - 2\frac{f(u)f'(u)}{f(u)^2}u_iu_j + f(u)f'(u)\widetilde{g}_{ij}\right). \tag{10.1}$$

From here it is clear in view of uniform bounds of f and its derivatives, as well as Corollary 9.8 that each u_{ij} is uniformly bounded. We thus arrive at the C^2 estimates

Proposition 10.1 *We continue in the Setting 1.1. Assume that u is uniformly bounded from above. Then $\|h\|$ and hence also of the mean curvature H are uniformly bounded and hence $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ is uniformly bounded as well.*

Taken altogether, results on the last three sections yield the following

Theorem 10.2 *Consider Setting 1.1 and a solution $u \in C^{4,\alpha}(M \times (0, T])$ to (1.3).*

(1) *Impose Assumptions 1.4 (1) and (2). Then $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded uniformly for finite $T > 0$, with bounds possibly depending on T.*
(2) *Impose Assumptions 1.4 (1)–(3). Then $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded uniformly independent of $T > 0$. Moreover, $\|\partial_t u\|_\infty$ is exponentially decreasing.*

11 Long-time existence and convergence

As before, we continue in the Setting 1.1 and consider a local solution u to (1.3) extended to a maximal time interval $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{4,\alpha}(M \times (0,T_{max}))$. Let us assume without loss of generality that $T_{max} > 0$ is finite. The Hölder norm is bounded for each compact interval in $[0, T_{max})$, but may a priori blow up the closer we get to T_{max}. The main point of this section is show that a posteriori this does not happen.

We first use uniform estimates from Theorem 10.2 to establish uniform ellipticity in the sense of (3.6) for the Laplacian Δ of $g = g(t) = F(t)^*\widetilde{g}$. Recall also the constant $\Lambda > 0$ in the definition of uniform ellipticity in (3.6).

Proposition 11.1 *Consider a solution $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{4,\alpha}(M \times (0,T_{max}))$ to (1.3).*

(1) *If $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded uniformly for any finite $T_{max} > 0$, then Δ is uniformly elliptic for each $t \in [0, T_{max})$ with $\Lambda > 0$ bounded for any finite maximal time T_{max}.*
(2) *If $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded uniformly independent of $T_{max} > 0$, then Δ is uniformly elliptic for each $t \in [0, T_{max})$ where $\Lambda > 0$ can be chosen independent of T_{max}.*

Proof From (3.9) we obtain after cancellations

$$\begin{aligned} \Delta u &= \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \left(\widetilde{\Delta} + \widehat{\Delta}\right) u \\ &\quad + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{(f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}})} \left(\frac{f(u)f'(u)}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} - (m-1)\frac{f'(u)}{f(u)} \right). \end{aligned} \tag{11.1}$$

Thus, in view of uniform bounds, it suffices to prove uniform ellipticity for $(\widetilde{\Delta} + \widehat{\Delta})$. We compute from (3.10) in local coordinates

$$\widetilde{\Delta} + \widehat{\Delta} = \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \left(-\widetilde{g}^{ij} - \frac{\widetilde{g}^{iq} u_q \widetilde{g}^{jm} u_m}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \right) \left(u_{ij} - \widetilde{\Gamma}^k_{ij} u_k \right)$$

From here we obtain for the symbol of $(\widetilde{\Delta} + \widehat{\Delta})$ in local coordinates

$$\begin{aligned} \sigma(\widetilde{\Delta} + \widehat{\Delta})(p, \xi) &= \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \left(\widetilde{g}^{ij} \xi_i \xi_j + \frac{\widetilde{g}^{iq} u_q \xi_i \widetilde{g}^{jm} u_m \xi_j}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \right) \\ &= \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \left(\|\xi\|^2_{\widetilde{g}} + \frac{\widetilde{g}(du, \xi)^2}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \right) \end{aligned} \tag{11.2}$$

This implies uniform ellipticity as asserted. □

We can now establish Hölder regularity of the gradient function.

Proposition 11.2 *Consider a solution $u \in C^{3,\alpha}(M \times [0,T]) \cap C^{4,\alpha}(M \times (0,T_{max}))$ to (1.3) and assume that $\mathcal{H}$ is bounded.*

(1) *If $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded uniformly for finite $T_{max} > 0$, then*

$$u, \nu \in C^{\alpha}(M \times [0, T_{max}]),$$

with the Hölder norm bounded for finite $T_{max} > 0$.

(2) *If* $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ *and* $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ *are bounded uniformly independent of* $T_{max} > 0$, *then*

$$u, v \in C^{\alpha}(M \times [0, T_{max}]),$$

with a T_{max}*-independent bound for the Hölder norm.*

Proof Consider the evolution equation for the gradient function v, as derived in Theorem 5.1. Since the right-hand side of (5.1) is bounded uniformly (with bounds possibly depending on T depending on whether $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ and $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ are bounded independent of T or not), Hölder regularity follows by uniform ellipticity of Δ in Proposition 11.1 and the Krylov–Safonov estimates in the first statement of Proposition 3.5. The statement for u follows in exactly the same way from the evolution equation (1.3). □

We can now bootstrap to improve upon regularity of u.

Proposition 11.3 *Consider a solution* $u \in C^{3,\alpha}(M \times [0, T]) \cap C^{4,\alpha}(M \times (0, T_{max}))$ *to* (1.3). *Assume that* $\mathcal{H} \in C^{\ell,\alpha}(M)$ *for some* $\ell \in \mathbb{N}_0$. *Then the following is true.*

(1) *If* $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ *and* $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ *are bounded uniformly for finite* $T_{max} > 0$, *then*

$$u \in C^{3,\alpha}(M \times [0, T]) \cap C^{2+\ell,\alpha}(M \times (0, T_{max}]),$$

with the Hölder norm bounded for finite $T_{max} > 0$.

(2) *If* $u, |\widetilde{\nabla} u|_{\widetilde{g}}$ *and* $|\widetilde{\nabla}^2 u|_{\widetilde{g}}$ *are bounded uniformly independent of* $T_{max} > 0$, *then*

$$u \in C^{3,\alpha}(M \times [0, T]) \cap C^{2+\ell,\alpha}(M \times (0, T_{max}]),$$

with a T_{max}*-independent bound for the Hölder norm.*

Proof Consider the evolution Eq. (1.3) for the solution u. By Proposition 11.2, the right-hand side of (1.3) as well as the coefficients of Δ (cf. 11.1) lie in $C^{\alpha}(M \times [0, T_{max}])$. Thus, by Proposition 3.5 (ii), we conclude

$$u \in C^{2,\alpha}(M \times [0, T_{max}]).$$

Now we can bootstrap exactly as at the end of the proof of Theorem 1.5. □

Therefore, assuming that $\mathcal{H} \in C^{2,\alpha}(M)$, we have $u \in C^{3,\alpha}(M \times [0, T]) \cap C^{4,\alpha}(M \times (0, T_{max}])$ and hence by Theorem 1.5 we can restart the flow with $u(T_{max}) \in C^{4,\alpha}(M)$ as a new initial condition. Therefore, proving the long-time existence statement in Theorem 1.7.

It remains to discuss convergence under the conditions of Theorem 1.7 (ii). First we note that exponential decay of $\|\partial_t u\|_\infty$ implies that $u(t)$ admits a well-defined limit $u^* \in L^\infty(M)$ as $t \to \infty$. We need to conclude at least that $u^* \in C^2(M)$ in order for u^* to admit a well-defined mean curvature H^*, which can then be shown to equal $\mathcal{H}$. We can therefore prove our final main result Theorem 1.8.

Proof of Theorem 1.8 As mentioned above, convergence to $u^* \in L^\infty(M)$ follows from the exponential decay of $\|\partial_t u\|_\infty$; therefore, it remains to prove that the limit is twice differentiable in M. Let $x : \overline{M} \to \mathbb{R}^+$ be a defining function of ∂M. Then, cf. [6, Proposition 11.2], for any $\varepsilon > 0$ and $\alpha' < \alpha$ the inclusion of weighted Hölder spaces

$$\iota : C^{\ell+2,\alpha}(M) \hookrightarrow x^{-\varepsilon} C^{\ell+2,\alpha'}(M),$$

is compact. Consider the global solution $u \in C^{3,\alpha}(M \times [0, T]) \cap C^{\ell+2,\alpha}(M \times (0, \infty))$, whose existence follows by the previous Theorem 1.7. Since the sequence $(u(t))_{t>0} \subset$

$C^{\ell+2,\alpha}(M)$ is uniformly bounded, by compactness of ι, there exists a convergent subsequence $(u(t_n))_n \subset x^{-\varepsilon}C^{\ell+2,\alpha'}(M)$. Consequently the pointwise limit u^* lies in $x^{-\varepsilon}C^{\ell+2,\alpha'}(M)$. In particular, it admits a well-defined mean curvature H^*. By (8.7), $H(t_n) - \mathcal{H}$ converges to zero and hence indeed $H^* = \mathcal{H}$. □

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