Ann. Henri Poincaré 24 (2023), 2929–2948

1424-0637/23/092929-20
published online March 30, 2023
https://doi.org/10.1007/s00023-023-01280-7

Annales Henri Poincaré

Adiabatic Ground States in Non-smooth Spacetimes

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Abstract. Ground states are a well-known class of Hadamard states in smooth spacetimes. In this paper, we show that the ground state of the Klein–Gordon field in a non-smooth ultrastatic spacetime is an adiabatic state. The order of the state depends linearly on the regularity of the metric. We obtain the result by combining a propagation of singularities result for non-smooth pseudodifferential operators, properties of the causal propagator, and eigenvalue asymptotics for elliptic operators of low regularity.

1. Introduction

In the smooth setting, the analysis of the singular structure of physical quantum states goes back to Fulling, Narcowich and Wald [10]. Later on, Kay and Wald further developed the notion of Hadamard states in globally hyperbolic spacetimes possessing a one-parameter group of isometries with a bifurcate Killing horizon [19]. Then in 1996, Radzikowski introduced a characterisation of the singularity structure of these states in terms of the wavefront set [23]. This result was key to using microlocal tools for the construction of quantum states as done by Junker [16], Junker and one of the authors [15], and Gérard and Wrochna [11].

The analysis of quantum states in non-smooth spacetimes has two main motivations. First, there are several models of physical phenomena that require spacetime metrics with finite regularity. These include models of gravitational collapse [1], astrophysical objects [22] and general relativistic fluids [3]. Second, the well-posedness of Einstein's equations, viewed as a system of hyperbolic PDE requires spaces with finite regularity [20].

In this paper, we focus on scalar fields ϕ that satisfy the Klein–Gordon equation

$$(\Box_g + m^2)\phi := g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi = 0 \tag{1.1}$$

on a manifold $M = \mathbb{R} \times \Sigma$ where Σ is a compact Cauchy hypersurface, $g^{\mu\nu}$ is the inverse metric tensor of an ultrastatic metric, ∇_μ is the covariant derivative and m^2 is a positive real number.

In a non-smooth spacetime the quantisation requires in a first instance that the classical system be well-posed. Several results in this direction have been obtained for different degrees of regularity in the time and space variables [5]. Moreover, even when one has classical well-posedness, the quantisation procedure is a significant further challenge. However, some progress has been made for certain degrees of spacetime regularity. For example: Dereziński and Siemssen showed the existence of classical and nonclassical propagators under weak regularity assumptions [6,7]. Hörmann, Spreitzer, Vickers and one of the authors gave the construction of quantisation functors that satisfy the Haag-Kastler axioms in the $C^{1,1}$ case [14]. In this paper we prove that the ground state of the quantum linear scalar field is an adiabatic state and that the adiabatic order is a linear function with respect to the metric regularity (Theorem 4.17).

Outline of the paper. In Sect. 2, we show the algebraic quantisation of fields satisfying Eq.(1.1) in spacetimes of finite regularity. We give details about the construction of the algebra of observables and precise definitions of the states considered. In Sect. 3, we state the main definitions and theorems regarding non-smooth pseudodifferential operators. In Sect. 4, we focus on ultra-static spacetimes and show that the ground state is an adiabatic state.

2. Quantum Field Theory in Non-smooth Spacetimes

The quantisation of the linear scalar field is a procedure to change the mathematical structure of the theory. On the one hand in the classical theory, the states are represented by vectors in a symplectic space, (V, Ξ), and the classical observables are defined as smooth functionals on (V, Ξ). On the other hand, in the framework of algebraic quantisation, the quantum observables of the theory are represented as the elements of a unique up to $*$-isomorphism C^*-algebra which satisfies the canonical commutation relations (CCR) and the quantum states, ω, are given by certain positive linear functionals on the C^*-algebra [2,30]. Below we give details of the quantisation procedure.

2.1. Observables

For a classical system with equations of motion given by Eq. (1.1) in a globally hyperbolic spacetime (M, g) of regularity $C^{1,1}$, it was shown that the space (V, Ξ) is given by $V = H^1_{\mathrm{comp}}(M)/\ker G$ and $\Xi([f],[g]) = ([f], G[g])_{L^2_{\mathbb{R}}(M)}$ where $H^1_{\mathrm{comp}}(M)$ denotes compactly supported functions in the Sobolev space $H^1(M)$ and $\ker G$ is the kernel of the causal propagator [14]. In fact, this symplectic space is symplectically isomorphic to the classical phase space (Γ, σ)

given by the space $\Gamma := H^2_{\text{comp}}(\Sigma) \oplus H^1_{\text{comp}}(\Sigma)$ of real-valued initial data with compact support and the symplectic bilinear form

$$\sigma(F_1, F_2) = \int_\Sigma [q_1 p_2 - q_2 p_1] dv$$

with $F_i := (q_i, p_i) \in \Gamma, i = 1, 2$ and dv the induced volume form on Σ.

Moreover, to the symplectic space (V, Ξ) one can associate a C^*-algebra $\mathcal{A} = \mathcal{A}[V, \Xi]$ that satisfies the CCR, known as the Weyl algebra. It is generated by the elements $W([f])$, $[f] \in V$, that satisfy

$$W([f])^* = W([f])^{-1} = W([-f])$$
$$W([f_1])W([f_2]) = e^{-\frac{i}{2}\Xi([f_1],[f_2])} W([f_1 + f_2])$$

for all $[f], [f_1], [f_2] \in V$ (see e.g. [2,14]).

As (V, Ξ) and (Γ, σ) are isomorphic as symplectic spaces, one can construct a C^*-algebra, $\mathcal{B} = \mathcal{B}[\Gamma, \sigma]$, using the map $\beta : \mathcal{A} \to \mathcal{B}$ given by $\beta(W([f])) := W((\rho^t_0 Gf, \rho^t_1 Gf))$ where $\rho^t_0 \phi := \phi|_{\Sigma_t}$, $\rho^t_1 \phi := \frac{\partial \phi}{\partial t}|_{\Sigma_t}$. The algebra $\mathcal{B}$ is $*$-isomorphic to the Weyl algebra $\mathcal{A}$ described above. Each of these algebras represents the quantum observables of the theory.

Moreover, one can localise this construction to suitable subsets of M following the approach of local quantum physics. In fact, one can do these local constructions in a functorial way and the functors satisfy the Haag-Kastler axioms (see [14, Theorem 6.12]).

2.2. States

The quantum states as defined above need to be further restricted in order to be physically relevant. A candidate for physical quantum states, ω, is quasifree states that satisfy the microlocal spectrum condition.

To be precise, given a real scalar product $\mu : \Gamma \times \Gamma \to \mathbb{R}$ satisfying

$$|\sigma(F_1, F_2)|^2 \leq \mu(F_1, F_1)\mu(F_2, F_2) \tag{2.1}$$

for all $F_1, F_2 \in \Gamma$, there exists a quasifree state ω_μ acting on the algebra $\mathcal{B}$ associated with μ given by $\omega_\mu(W(F)) = e^{-\frac{1}{2}\mu(F,F)}$. Moreover, one can determine the ("symplectically smeared") two-point function of ω_μ by

$$\lambda(F_1, F_2) = \mu(F_1, F_2) + \frac{i}{2}\sigma(F_1, F_2) \tag{2.2}$$

for $F_1, F_2 \in \Gamma$. The Wightman two-point function $\omega^{(2)}_\mu$ associated with the state ω_μ, is given by

$$\omega^{(2)}_\mu(f_1, f_2) = \lambda\left(\begin{pmatrix}\rho_0 Gf_1 \\ \rho_1 Gf_1\end{pmatrix}, \begin{pmatrix}\rho_0 Gf_2 \\ \rho_1 Gf_2\end{pmatrix}\right) \tag{2.3}$$

for $f_1, f_2 \in H^1_{\text{comp}}(M)$. By restricting the two point function $\omega^{(2)}_\mu$ to $\mathcal{D}(M) \otimes \mathcal{D}(M)$ one obtains a bidistribution in $M \times M$.

To define the microlocal spectrum condition, we use the inverse of the spacetime metric $g^{-1} := (g^{\mu\nu})_{\mu,\nu=0}^n$ in order to introduce the sets

$$C = \left\{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in T^*(M\times M)\backslash 0; g^{\mu\nu}(\tilde{x})\tilde{\xi}_\mu\tilde{\xi}_\nu = g^{\mu\nu}(\tilde{y})\tilde{\eta}_\mu\tilde{\eta}_\nu = 0, (\tilde{x},\tilde{\xi}) \sim (\tilde{y},\tilde{\eta})\right\}$$
$$C^+ = \left\{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in C; \tilde{\xi}^0 \geq 0, \tilde{\eta}^0 \geq 0\right\}, \tag{2.4}$$

where $(\tilde{x},\tilde{\xi}) \sim (\tilde{y},\tilde{\eta})$ means that $\tilde{\xi},\tilde{\eta}$ are cotangent to the null geodesic γ at $\tilde{x}$ resp. $\tilde{y}$ and parallel transports of each other along γ.

Using the above sets one can define the microlocal spectrum condition which goes back to Radzikowski [23]:

Definition 2.1. A quasifree state ω_H on the algebra of observables satisfies the microlocal spectrum condition if its two point function $\omega_H^{(2)}$ is a distribution in $\mathcal{D}'(M \times M)$ and satisfies the following wavefront set condition

$$WF'(\omega_H^{(2)}) = C^+,$$

where $WF'(\omega_H^{(2)}) := \{(\tilde{x},\tilde{\eta};\tilde{y},-\tilde{\eta}) \in T^*(M \times M); (\tilde{x},\tilde{\eta};\tilde{y},\tilde{\eta}) \in WF(\omega_{2\,H})\}$.

These states are called Hadamard states and include ground states in smooth spacetimes [9–11,16,24].

A larger class of states called adiabatic states of order N characterised in terms of their Sobolev-wavefront set has been obtained by Junker and one of the authors [15]. These states are the natural generalisation of Hadamard states suitable for spacetimes with limited regularity.

Definition 2.2. A quasifree state ω_N on the algebra of observables is called an adiabatic state of order $N \in \mathbb{R}$ if its two-point function $\omega_N^{(2)}$ is a bidistribution that satisfies the following H^s-wavefront set condition for all $s \leq N + \frac{3}{2}$

$$WF'^s(\omega_N^{(2)}) \subset C^+,$$

where WF^s is a refinement of the notion of the wavefront set in terms of Sobolev spaces. To be precise, a distribution u is *microlocally in* H^s at $(x_0,\xi_0) \in T^*M\backslash 0$ if there exist a conic neighbourhood Γ of ξ_0 and a smooth function $\varphi \in \mathcal{D}(M)$ with $\varphi(x_0) \neq 0$ such that

$$\int_\Gamma \langle\xi\rangle^{2s}|\mathcal{F}(\varphi u)(\xi)|^2 d^n\xi < \infty.$$

Otherwise we say that (x_0,ξ_0) lies in the s-wave front set $WF^s(u)$.

If u is microlocally in H^s in an open conic subset $\Gamma \subset T^*M\backslash 0$ we write $u \in H^s_{mcl}(\Gamma)$.

3. Pseudodifferential Operators with Non-smooth Symbols

3.1. Symbol Classes

Let $\{\psi_j; j = 0,1,\ldots\}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, i.e. a partition of unity $1 = \sum_{j=0}^\infty \psi_j$, where $\psi_0 \equiv 1$ for $|\xi| \leq 1$ and $\psi_0 \equiv 0$ for $|\xi| \geq 2$

and $\psi_j(\xi)=\psi_0(2^{-j}\xi)-\psi_0(2^{1-j}\xi)$. The support of ψ_j, $j\geq 1$, then lies in an annulus around the origin of interior radius 2^{j-1} and exterior radius 2^{1+j}.

Definition 3.1. (a) For $\tau\in(0,\infty)$, the Hölder space $C^\tau(\mathbb{R}^n)$ is the set of all functions f with

$$\|f\|_{C^\tau}:=\sum_{|\alpha|\leq[\tau]}\|\partial_x^\alpha f\|_{L^\infty(\mathbb{R}^n)}+\sum_{|\alpha|=[\tau]}\sup_{x\neq y}\frac{|\partial_x^\alpha f(x)-\partial_x^\alpha f(y)|}{|x-y|^{\tau-[\tau]}}<\infty. \tag{3.1}$$

(b) For $\tau\in\mathbb{R}$ the Zygmund space $C_*^\tau(\mathbb{R}^n)$ consists of all functions f with

$$\|f\|_{C_*^\tau}=\sup_j 2^{j\tau}\|\psi_j(D)f\|_{L^\infty}<\infty. \tag{3.2}$$

Here $\psi_j(D)$ is the Fourier multiplier with symbol ψ_j, i.e. $\psi_j(D)u=\mathcal{F}^{-1}\psi_j\mathcal{F}u$, where $(\mathcal{F}u)(\xi)=(2\pi)^{-n/2}\int e^{-ix\xi}u(x)\,d^nx$ is the Fourier transform.

We have the following relations $C^\tau=C_*^\tau$ if $\tau\notin\mathbb{Z}$ and $C^\tau\subset C_*^\tau$ if $\tau\in\mathbb{N}$.

We next introduce symbol classes of finite Hölder or Zygmund regularity, following Taylor [27]. We use the notation $\langle\xi\rangle:=(1+|\xi|^2)^{\frac{1}{2}}$, $\xi\in\mathbb{R}^n$.

Definition 3.2. Let $p(x,\xi):\mathbb{R}^{2n}\to\mathbb{R}$ be a continuous function, smooth with respect to ξ.

(a) Let $0\leq\delta<1$. A symbol $p(x,\xi)$ belongs to $C_*^\tau S_{1,\delta}^m$ if for all α

$$\|D_\xi^\alpha p(x,\xi)\|_{C_*^\tau}\leq C_\alpha\langle\xi\rangle^{m-|\alpha|+\tau\delta}\text{ and }|D_\xi^\alpha p(x,\xi)|\leq C_\alpha\langle\xi\rangle^{m-|\alpha|}.$$

(b) We obtain the symbol class $C^\tau S_{1,\delta}^m$ for $\tau>0$ by requiring that for all α

$$\|D_\xi^\alpha p(x,\xi)\|_{C^s}\leq C_\alpha\langle\xi\rangle^{m-|\alpha|+s\delta},\quad 0\leq s\leq\tau,\text{ and }|D_\xi^\alpha p(x,\xi)|\leq C_\alpha\langle\xi\rangle^{m-|\alpha|}.$$

(c) A symbol $p(x,\xi)$ is in $C^\tau S_{cl}^m$ provided $p(x,\xi)\in C^\tau S_{1,0}^m$ and $p(x,\xi)$ has a classical expansion

$$p(x,\xi)\sim\sum_{j\geq 0}p_{m-j}(x,\xi)$$

in terms p_{m-j} homogeneous of degree $m-j$ in ξ for $|\xi|\geq 1$, in the sense that the difference between $p(x,\xi)$ and the sum over $0\leq j<N$ belongs to $C^\tau S_{1,0}^{m-N}$.

3.2. Characteristic Set and Pseudodifferential Operators

Let $p\in C^\tau S_{1,\delta}^m$, $\tau>0$, with $\delta<1$. Suppose that there is a conic neighbourhood Γ of (x_0,ξ_0) and constants $c,C>0$ such that $|p(x,\xi)|\geq c|\xi|^m$ for $(x,\xi)\in\Gamma$, $|\xi|\geq C$. Then (x_0,ξ_0) is called *non-characteristic*. If p is classical and has the homogeneous principal symbol p_m, the condition is equivalent to $p_m(x_0,\xi_0)\neq 0$. The complement of the set of non-characteristic points is the set of characteristic points denoted by $\mathrm{Char}(p)$.

The pseudodifferential operator $p(x,D_x)$ with the symbol $p(x,\xi)\in C^\tau S_{1,\delta}^m$ is given by

$$p(x,D_x)u=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot\xi}p(x,\xi)(\mathcal{F}u)(\xi)d^n\xi,\quad u\in\mathcal{S}(\mathbb{R}^n). \tag{3.3}$$

It extends to continuous maps

$$p(x, D_x) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad -\tau(1-\delta) < s < \tau. \tag{3.4}$$

3.3. Symbol Smoothing

Given $p(x,\xi) \in C^\tau S^m_{1,\delta}$ and $\gamma \in (\delta, 1)$ let

$$p^\#(x,\xi) = \sum_{j=0}^{\infty} J_{\epsilon_j} p(x,\xi)\psi_j(\xi). \tag{3.5}$$

Here J_ϵ is the smoothing operator given by $(J_\epsilon f)(x) = (\phi(\epsilon D)f)(x)$ with $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(\xi) = 1$ for $|\xi| \le 1$, and we take $\epsilon_j = 2^{-j\gamma}$.

Letting $p^b(x,\xi) = p(x,\xi) - p^\#(x,\xi)$ we obtain the decomposition

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi), \tag{3.6}$$

where $p^\#(x,\xi) \in S^m_{1,\gamma}$ and $p^b(x,\xi) \in C^\tau S^{m-\tau(\gamma-\delta)}_{1,\gamma}$.

If $p \in C^\tau S^m_{1,0}$, then we additionally have $p^b \in C^{\tau-t} S^{m-t\gamma}_{1,0}$ with $\tau - t > 0$ by [27, Proposition 1.3.B]. Furthermore, we have better estimates, see [27, Proposition 1.3.D]:

$$D_x^\beta p^\#(x,\xi) \in \begin{cases} S^m_{1,\gamma}, & |\beta| \le \tau \text{ and} \\ S^{m+\gamma(|\beta|-\tau)}_{1,\gamma}, & |\beta| > \tau \end{cases}. \tag{3.7}$$

4. Ground States in Ultrastatic Spacetimes

Let $M = \mathbb{R} \times \Sigma$, where Σ is a 3-dimensional compact manifold and the Lorentzian metric $g := (g_{\mu\nu})^3_{\mu,\nu=0}$ is of the form

$$ds^2 = dt^2 - h_{ij}(x)dx^i dx^j, \tag{4.1}$$

where $h_{ij}(x)$ are the components of a time independent Riemannian metric of Hölder regularity C^τ (when $\tau \in \mathbb{N}$ we will consider the Zygmund spaces C^τ_*, introduced in Definition 3.1). As usual, the tensor $(g^{\mu\nu})$ in (1.1) is the inverse to $(g_{\mu\nu})$.

Moreover, the vector field ∂_t induces a one-parameter group of isometries $\tau_t : M \to M, t \in \mathbb{R}$, such that $\tau_t(\Sigma_{t_o}) = \Sigma_{t_o+t}$.

This group induces a one-parameter group of automorphisms in the C^*-algebras as follows. Define $\mathcal{T}(t) : \Gamma_{t_o} \to \Gamma_{t_o+t}$ by

$$\mathcal{T}(t)\tilde{F}_{t_o} := \tilde{F}_{t_o+t},$$

where Γ_s is the initial data at Σ_s, $\tilde{F}_s := (\rho_0^s\phi, \rho_1^s\phi)$ and $\phi \in \ker(\Box_g + m^2)$.

Since the symplectic form σ is invariant under the action of $\mathcal{T}(t)$ and since $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s)$ $t,s \in \mathbb{R}$, $\mathcal{T}$ is a one-parameter group of symplectic transformations (also called Bogoliubov transformations), it gives rise to a group of automorphisms $\tilde{\alpha}(t), t \in \mathbb{R}$, (Bogoliubov automorphisms) on the algebra $\mathcal{B}$ via

$$\tilde{\alpha}(t)W(F) = W(\mathcal{T}(t)F).$$

In this case, there exists a preferred class of states on $\mathcal{A}$, namely those invariant under $\alpha(t) = \tilde{\alpha}(t) \circ \beta$ with β as defined in Section 2.1. A quasifree state ω_μ will be invariant under this symmetry if and only if

$$\mu(\mathcal{T}(t)F_1, \mathcal{T}(t)F_2) = \mu(F_1, F_2) \; \forall t \in \mathbb{R} \; \forall F_1, F_2 \in \Gamma.$$

The specification of μ is equivalent to the specification of a one-particle structure as established by the following theorem of Kay and Wald [19, Proposition 3.1]:

Theorem 4.1. *Let ω_μ be a quasifree state on $\mathcal{B}[\Gamma, \sigma]$. Then there exists a* **one-particle Hilbert space structure***, i.e. a Hilbert space $\mathcal{H}$ and a real-linear map $k : \Gamma \to \mathcal{H}$ such that*

(i) $k\Gamma + ik\Gamma$ *is dense in* $\mathcal{H}$,
(ii) $\mu(F_1, F_2) = \mathrm{Re}\langle kF_1, kF_2\rangle_{\mathcal{H}} \; \forall F_1, F_2 \in \Gamma$,
(iii) $\sigma(F_1, F_2) = 2\mathrm{Im}\langle kF_1, kF_2\rangle_{\mathcal{H}} \; \forall F_1, F_2 \in \Gamma$. *The pair $(k, \mathcal{H})$ is uniquely determined up to unitary equivalence. Moreover, ω_μ is pure if and only if $k(\Gamma)$ is dense.*

Remark 4.2. Notice that the specification of a Hilbert space $\mathcal{H}$ together with a real-linear map $k : \Gamma \to \mathcal{H}$ such that $k\Gamma + ik\Gamma$ is dense in $\mathcal{H}$ and $2\mathrm{Im}\langle kF_1, kF_2\rangle_{\mathcal{H}} = \sigma(F_1, F_2)$ gives rise via Eq.(2.2) to a real scalar product μ satisfying Eq.(2.1).

Moreover, the automorphism group $\tilde{\alpha}(t)$ can be unitarily implemented in the one-particle Hilbert space structure $(k, \mathcal{H})$ of an invariant state ω_μ, i.e. there exists a unitary group $U(t), t \in \mathbb{R}$, on $\mathcal{H}$ satisfying

$$\begin{aligned} U(t)k &= k\mathcal{T}(t) \\ U(t)U(s) &= U(t+s). \end{aligned} \tag{4.2}$$

If $U(t)$ is strongly continuous it takes the form $U(t) = e^{-iHt}$ for some self-adjoint operator H on $\mathcal{H}$.
We define now the notion of ground states following Kay [18]:

Definition 4.3. Let the phase space $(\Gamma, \sigma, \mathcal{T}(t))$ be given. A quasifree ***ground state*** is a quasifree state over $\mathcal{B}[\Gamma, \sigma]$ with one-particle Hilbert space structure $(k, \mathcal{H})$ and a strongly continuous unitary group $U(t) = e^{-iHt}$ (satisfying (4.2)) such that H is a positive operator (the "one-particle Hamiltonian").

In the ultrastatic case we define the ground state, ω_G by the one-particle Hilbert space structure $(k_G, \mathcal{H}_G)$

$$\begin{aligned} k_G : \Gamma &\to \mathcal{H}_G := L^2_{\mathbf{C}}(\Sigma_{t_0}) \\ F = (q, p) &\mapsto \frac{1}{\sqrt{2}} \left(A^{1/4} q - i A^{-1/4} p \right), \end{aligned} \tag{4.3}$$

where $A := -\Delta_h + m^2$ and $t_0 \in \mathbb{R}$ (invariance under time translations makes any choice of $t \in \mathbb{R}$ equivalent to any other) and the strongly continuous unitary group is given by $U(t) := e^{iA^{\frac{1}{2}}t}$.

The Wightman two-point function of ω_G is:

$$\omega_G^{(2)}(h_1, h_2) = \lambda_G \left(\begin{pmatrix} \rho_0^t G h_1 \\ \rho_1^t G h_1 \end{pmatrix}, \begin{pmatrix} \rho_0^t G h_2 \\ \rho_1^t G h_2 \end{pmatrix} \right), \tag{4.4}$$

for $h_1, h_2 \in \mathcal{D}(\mathcal{M})$.

Moreover using Eq. 2.3, Eq. (4.3) and Theorem 4.1 the "symplectically smeared two-point function" λ_G is given on the initial data $F_i = \binom{q_i}{p_i} \in \Gamma$ by Eq.(2.2),

$$\begin{aligned} \lambda_G(F_1, F_2) &= \langle k_G F_1, k_G F_2 \rangle_{L^2_{\mathbb{C}}(\Sigma)} \\ &= \frac{1}{2} \left\langle A^{1/4} q_1 - i A^{-1/4} p_1, A^{1/4} q_2 - i A^{-1/4} p_2 \right\rangle_{L^2_{\mathbb{C}}(\Sigma)} \\ &= \frac{1}{2} \left\langle (A^{1/2} q_1 - i p_1), A^{-1/2} \left(A^{1/2} q_2 - i p_2 \right) \right\rangle_{L^2_{\mathbb{C}}(\Sigma)}, \end{aligned} \tag{4.5}$$

since A is selfadjoint. Combining (4.4) and (4.5) we obtain

$$\omega_G^{(2)}(h_1, h_2) = \frac{1}{2} \left\langle \left(A^{1/2} \rho_0^t - i \rho_1^t \right) G h_1, A^{-1/2} \left(A^{1/2} \rho_o^t - i \rho_1^t \right) G h_2 \right\rangle_{L^2_{\mathbb{C}}(\Sigma_t)}. \tag{4.6}$$

The two-point function, $\omega_G^{(2)}$, of the ground state, ω_G, is the Schwartz kernel of the operator $e^{iA^{\frac{1}{2}}(t-s)} A^{-\frac{1}{2}}$.

Explicitly, for $u, v \in \mathcal{D}(M)$ we have

$$\omega_G^{(2)}(u, v) = \int_M \left(e^{iA^{\frac{1}{2}}(t-s)} A^{-\frac{1}{2}} u \right)(s, y) v(s, y) ds dy,$$

which gives the singular integral kernel representation

$$\omega_G^{(2)}(t, x; s, y) = \sum_j \lambda_j^{-1} e^{i\lambda_j (t-s)} \phi_j(x) \phi_j(y). \tag{4.7}$$

where $\{\phi_j, j = 1, 2, \ldots\}$ is an orthonormal basis of eigenfunctions of $L^2(\Sigma)$ associated with the eigenvalues λ_j^2 of the operator $m^2 - \Delta_h$.

The ground state in a smooth ultrastatic space-time is a Hadamard state [10,11,16,24]. In the following section we will show that, in the non-smooth case, the ground state is an adiabatic state.

4.1. Microlocal Analysis for Bisolutions of the Klein–Gordon Operator

We write local coordinates on $\mathbb{R} \times \Sigma$ in the form

$$\tilde{x} = (t, x), \ \tilde{y} = (s, y) \tag{4.8}$$

and the associated covariables as

$$\tilde{\xi} = (\xi_0, \xi), \ \tilde{\eta} = (\eta_0, \eta). \tag{4.9}$$

Using the notation above and in (4.1), we have:

Remark 4.4. The Klein–Gordon operator on M is given by

$$P\phi = (\Box_g + m^2)\phi = \partial_{tt}\phi - \Delta_h\phi + m^2\phi. \tag{4.10}$$

It has the symbol $P(\tilde{x}, \tilde{\xi}) = (-\xi_0^2 + h^{ij}\xi_i\xi_j) + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h})\xi_j + m^2$. For a metric of regularity C^τ, the symbol $P(\tilde{x}, \tilde{\xi})$ belongs to $C^{\tau-1}S^2_{cl}$ and

$$\mathrm{Char}(P) = \{(t, x, \xi_0, \xi) \in T^*M \setminus \{0\}; -\xi_0^2 + h^{ij}\xi_i\xi_j = 0\}.$$

In the sequel we shall apply the Klein–Gordon operator to functions and distributions on $M \times M$, say with variables $((t, x), (s, y))$ in the notation (4.8). In order to make clear whether P acts on the first set of variables (t, x) or on the second set (s, y) we will write $P_{(t,x)}$ and $P_{(s,y)}$, respectively. Using the coordinates in Eqs.(4.8) and (4.9), the associated symbols $P_{(t,x)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})$ and $P_{(s,y)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})$ formally depend on the full set of (co-)variables $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})$, however, only the (co-)variables associated with either (t, x) or (s, y) show up:

$$P_{(t,x)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) = \underbrace{(-{\xi^0}^2 + h^{ij}(x)\xi_i\xi_j)}_{p_2(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})} + \underbrace{i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j}_{p_1(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})} + \underbrace{m^2}_{p_0(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta})}.$$

$$P_{(s,y)}(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \quad = (-{\eta_0}^2 + h^{ij}(x)\eta_i\eta_j) + i\tfrac{1}{\sqrt{h}}\partial_{y^i}(h^{ij}\sqrt{h}(y))\eta_j + m^2.$$

In particular,

$$\begin{aligned}\mathrm{Char}(P_{(t,x)}) &= (\mathrm{Char}(P) \times T^*M) \cup \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in T^*(M \times M)\backslash\{0\}, \tilde{\xi} = 0\}\\ \mathrm{Char}(P_{(s,y)}) &= (T^*M \times \mathrm{Char}(P)) \cup \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in T^*(M \times M)\backslash\{0\}, \tilde{\eta} = 0\}.\end{aligned} \tag{4.11}$$

Now we will state a microelliptic estimate tailored for bisolutions of the Klein–Gordon operator

Theorem 4.5. *Let the metric g be of class C^τ, $\tau > 1$, $0 \leq \sigma < \tau - 1$ and $v \in H^{2+\sigma-\tau+\epsilon}_{loc}(M \times M)$ for some $\epsilon > 0$ with $P_{(t,x)}(\tilde{x}, \tilde{y}, D)v = P_{(s,y)}(\tilde{x}, \tilde{y}, D)v = 0$. Then*

$$WF^{\sigma+2}(v) \subset \mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)}).$$

The proof can be found in [26, Theorem 3.4].

Remark 4.6. To obtain Theorem 4.5 we smooth each of the non-smooth symbols (the principal symbol and the sub-leading term) separately to obtain the remainder $p_2^b + p_1^b$ for $p_2^b \in C^\tau S^{2-\tau\delta}_{1,\delta}$ and $p_1^b \in C^{\tau-1}S^{1-(\tau-1)\delta}_{1,\delta}$. Applying the symbol smoothing directly to $P_{(t,x)} \in C^{\tau-1}S^2_{1,0}$ would leave us with $P^b_{(t,x)} \in C^{\tau-1}S^{2-(\tau-1)\delta}_{1,\delta}$.

Furthermore, the main results on the microlocal propagation of singularities in the non-smooth setting that we will apply can be found in [27, Proposition 6.1.D] or [28, Proposition 11.4]. In particular, the theorem below holds for spacetime metrics belonging to the space C^2_* [28, p.215].

Theorem 4.7. *Let $u \in \mathcal{D}'(M \times M)$ solve $P_{(t,x)}u = f$. Let γ be an integral curve of the Hamiltonian vector field H_{p_2} with p_2 the principal symbol of $P_{(t,x)}$. If for some $s \in \mathbb{R}$, $f \in H^s_{mcl}(\Gamma)$ and $P^b_{(t,x)}u \in H^s_{mcl}(\Gamma)$ where $\gamma \subset \Gamma$ with Γ a conical neighbourhood and $u \in H^{s+1}_{mcl}(\gamma(0))$ then $u \in H^{s+1}_{mcl}(\gamma)$.*

Remark 4.8. If $u \in H^{2+s-\tau\delta}_{loc}(M \times M)$, then $P^b_{(t,x)}u \in H^s_{loc}(M \times M)$ for $-(1-\delta)(\tau-1) \le s \le \tau - 1$, see Remark 4.6.

4.2. The Microlocal Spectrum Condition

Now we will show that the Wightman two-point function of the ground state described above satisfies Definition 2.2. We will assume throughout this section that the metric is of regularity C^τ with $\tau > 2$.

Let $\{\phi_j \otimes \phi_k; j,k = 1,2,\ldots\}$ be an orthonormal basis of $L^2(\Sigma) \otimes L^2(\Sigma)$ associated with the eigenfunctions $\{\phi_j\}$ and the eigenvalues $\{\lambda_j^2\}$ of the operator $m^2 - \Delta_h$. Then, for $u \in L^2(M \times M)$ we have the representation

$$u(t,s,x,y) = \sum_{j,k} u_{jk}(t,s)\phi_j(x)\phi_k(y) \quad \text{with } u_{jk} = \langle u, \phi_j \otimes \phi_k\rangle \in L^2(\mathbb{R}^2). \tag{4.12}$$

Moreover, we have the following generalisation for $u \in H^{2\theta}(M \times M)$ shown in [26, Proposition 4.1, Corollary 4.4]

Theorem 4.9. *For $-1 \le \theta \le 1$*

$$H^{2\theta}(\mathbb{R}^2 \times \Sigma^2) = \Bigg\{ u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2); \sum_{j,k} \int_{\mathbb{R}^2} (|\xi_0|^2 + |\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^{2\theta} \times |\mathcal{F}u_{jk}(\xi_0,\eta_0)|^2 d\xi_0 d\eta_0 < \infty \Bigg\},$$

with $u_{jk} = \langle u, \phi_j \otimes \phi_k\rangle \in \mathcal{S}'(\mathbb{R}^2)$.

The previous theorem allows us to establish the local Sobolev regularity of the two-point function.

Theorem 4.10. $\omega_G^{(2)} \in H^{-\frac12-\epsilon}_{loc}(M \times M)$ *for every $\epsilon > 0$*

Proof. Let $\psi \in \mathcal{D}(M \times M)$. Without loss of generality assume that $\psi = \psi(s,t)$. We will show that $\psi\omega_G^{(2)} \in H^{-\frac12-\epsilon}(M \times M)$.

According to Eq. (4.7) and Theorem 4.9

$$\|\psi\omega_G^{(2)}\|^2_{H^{-\frac12-\epsilon}(M\times M)} \tag{4.13}$$

$$= \sum_{j=k} \int_{\mathbb{R}^2} (|\xi_0|^2 + |\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^{-\frac12-\epsilon}$$

$$\left|\mathcal{F}_{(t,s)\to(\xi_0,\eta_0)}\Big(\frac{\psi(t,s)}{\lambda_j} e^{i\lambda_j(t-s)}\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0. \tag{4.14}$$

We have by direct computation that

$$\left|\mathcal{F}_{(t,s)\to(\xi_0,\eta_0)}\Big(\frac{\psi(t,s)}{\lambda_j}e^{i\lambda_j(t-s)}\Big)(\xi_0,\eta_0)\right|^2 \\ = \frac{1}{\lambda_j^2}\left|\mathcal{F}(\psi)(\xi_0-\lambda_j,\eta_0+\lambda_j)\right|^2. \tag{4.15}$$

Taking into account that $\|\psi\|_{L^2(\mathbb{R}^2)} = \|\mathcal{F}(\psi)\|_{L^2(\mathbb{R}^2)} < \infty$ we have (with constants possibly changing from line to line)

$$\begin{aligned}
&\|\psi\omega_G^{(2)}\|_{H^{-\frac{1}{2}-\epsilon}(M\times M)} \\
&= \sum_{j=k}\int_{\mathbb{R}^2}(|\xi_0|^2+|\eta_0|^2+\lambda_j^2+\lambda_k^2)^{-\frac{1}{2}-\epsilon} \\
&\quad \left|\mathcal{F}_{(t,s)\to(\xi_0,\eta_0)}\Big(\frac{\psi(t,s)}{\lambda_j}e^{i\lambda_j(t-s)}\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0 \\
&\leq \sum_{j=1}^{\infty}\int_{\mathbb{R}^2}(\lambda_j^2+\lambda_j^2)^{-\frac{1}{2}-\epsilon}\frac{1}{\lambda_j^2}\left|\mathcal{F}(\psi)(\xi_0-\lambda_j,\eta_0+\lambda_j)\right|^2 d\xi_0 d\eta_0 \\
&\leq C\sum_{j=1}^{\infty}\frac{(\lambda_j^2+\lambda_j^2)^{-\frac{1}{2}-\epsilon}}{\lambda_j^2} \\
&\leq C\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{3+2\epsilon}}.
\end{aligned}$$

From Weyl's law for non-smooth metrics [31, Theorem 1.1] we obtain the estimate $j^{\frac{2}{3}} \leq C\lambda_j^2$ for a suitable constant C which gives

$$\|\psi\omega_G^{(2)}\|^2_{H^{-\frac{1}{2}-\epsilon}(M\times M)} \leq \sum_{j=1}^{\infty}\frac{C}{\lambda_j^{3+2\epsilon}} \leq \sum_{j=1}^{\infty}\frac{C'}{j^{1+\frac{2\epsilon}{3}}} < \infty$$

for a suitable constant C'. □

It will be useful to consider the following bidistribution:

Corollary 4.11. *Let $\omega_A \in \mathcal{D}'(M\times M)$ be the bidistribution given by*

$$\omega_A(u\otimes v) := -\int_{M\times M}\sum_j \lambda_j^{-2}e^{i\lambda_j(t-s)}\phi_j(x)\phi_j(y)\sqrt{h(y)}\sqrt{h(x)} \\ \times u(t,x)v(s,y)dsdydtdx$$

Then,

$$\omega_A \in H_{loc}^{\frac{1}{2}-\epsilon}(M\times M) \text{ for every } \epsilon > 0. \tag{4.16}$$

Proof. Direct computation shows that for ψ as in the previous proof and $s \geq 0$ we have

$$\begin{aligned}
&\|\psi\omega_A\|_{H^s(M\times M)}\\
&=\sum_{j=1}^{\infty}\int_{\mathbb{R}^2}(|\xi_0|^2+|\eta_0|^2+\lambda_j^2+\lambda_j^2)^s\\
&\quad\times\left|\mathcal{F}_{(t,s)\to(\xi_0,\eta_0)}\Big(\frac{\psi(t,s)}{\lambda_j^2}e^{-i\lambda_j(t-s)}\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0\\
&=\sum_{j=1}^{\infty}\int_{\mathbb{R}^2}(|\xi_0|^2+|\eta_0|^2+\lambda_j^2+\lambda_j^2)^s\left|\mathcal{F}\Big(\frac{\psi}{\lambda_j^2}\Big)(\xi_0-\lambda_j,\eta_0+\lambda_j)\right|^2 d\xi_0 d\eta_0\\
&=\sum_{j=1}^{\infty}\int_{\mathbb{R}^2}(|\xi_0-\lambda_j|^2+|\eta_0+\lambda_j|^2+\lambda_j^2+\lambda_j^2)^s\left|\mathcal{F}\Big(\frac{\psi}{\lambda_j^2}\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0\\
&\le\sum_{j=1}^{\infty}\int_{\mathbb{R}^2}(2|\xi_0|^2+2|\eta_0|^2+3\lambda_j^2+3\lambda_j^2)^s\left|\mathcal{F}\Big(\frac{\psi}{\lambda_j^2}\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0\\
&\le C\sum_{j=1}^{\infty}\frac{(1+\lambda_j^2)^s}{\lambda_j^4}\int_{\mathbb{R}^2}(1+|\xi_0|^2+|\eta_0|^2)^s\left|\mathcal{F}\Big(\psi\Big)(\xi_0,\eta_0)\right|^2 d\xi_0 d\eta_0\\
&\le C\sum_{j=1}^{\infty}\frac{(1+\lambda_j^2)^s}{\lambda_j^4}|\psi|^2 H^s(\mathbb{R}^2)\\
&\le C\sum_{j=1}^{j_0}\frac{(1+\lambda_j^2)^s}{\lambda_j^4}+C\sum_{j=j_0}^{\infty}\frac{(\lambda_j^2)^s}{\lambda_j^4},
\end{aligned}$$

where we have chosen j_0 large enough such that $\lambda_{j_0}>1$.

According to Weyl's law for non-smooth metrics [31, Theorem 1.1] we have the estimate $j^{\frac{2}{3}}\le C\lambda_j^2$ for a suitable constant C. This gives for $s=\frac{1}{2}-\epsilon$

$$\|\psi\omega_A\|^2_{H^{\frac{1}{2}-\epsilon}(M\times M)}\le C+\sum_j\frac{C}{\lambda_j^{3+2\epsilon}}\le C+\sum_j\frac{C}{j^{1+\frac{2\epsilon}{3}}}<\infty \tag{4.17}$$

for a suitable constant C. □

Remark 4.12. Notice that $i\partial_t\omega_A=\omega_G^{(2)}$.

Lemma 4.13. *For any* $\tilde\epsilon>0$

$$WF^{-\frac{1}{2}-\tilde\epsilon+\tau}(\omega_G^{(2)})\subset \mathrm{Char}(P)\times\mathrm{Char}(P). \tag{4.18}$$

Proof. Since $\omega_G^{(2)}$ satisfies $(\partial_t+\partial_s)\omega_G^{(2)}=0$ we conclude that for all $l\in\mathbb{R}$

$$\begin{aligned}WF^l(\omega_G^{(2)})\subset WF(\omega_G^{(2)})&\subset\mathrm{Char}(\partial_t+\partial_s)\\
&=\{(\tilde x,\xi_0,\xi,\tilde y,\eta_0,\eta)\in T^*(M\times M)\backslash\{0\};\xi_0+\eta_0=0\},\end{aligned} \tag{4.19}$$

where the second inclusion follows from the standard theory of pseudodifferential operators.

Now we have $P_{(t,x)}(\tilde{x},\tilde{y},D)\omega_A = P_{(s,y)}(\tilde{x},\tilde{y},D)\omega_A = 0$. Choose $\epsilon < \tilde{\epsilon}/2$. Since $\omega_A \in H^{\frac{1}{2}-\frac{\epsilon}{2}}_{loc}(M\times M) = H^{(\frac{1}{2}-\epsilon)+\frac{\epsilon}{2}}_{loc}(M\times M)$, an application of Theorem 4.5 with $\sigma = -\frac{3}{2}+\tau-\epsilon < \tau - 1$ shows that

$$WF^{\frac{1}{2}+\tau-\tilde{\epsilon}}(\omega_A) \subset \mathrm{Char}(P_{(t,x)}) \cap \mathrm{Char}(P_{(s,y)});$$

here we assume without loss of generality that ϵ is so small that $-\frac{3}{2}+\tau-\epsilon \geq 0$. Equation (4.11) implies that

$$\begin{aligned} WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_A) \subset &(\mathrm{Char}(P)\times \mathrm{Char}(P)) \\ &\cup (\{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in T^*(M\times M)\backslash\{0\}; \tilde{\xi}=0, (\tilde{y},\tilde{\eta})\in \mathrm{Char}(P)\} \\ &\cup \{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta}) \in T^*(M\times M)\backslash\{0\}; (\tilde{x},\tilde{\xi})\in \mathrm{Char}(P), \tilde{\eta}=0\}. \end{aligned}$$

If $\tilde{\eta}=0$, then $\eta_0=0$, and $\xi_0=0$ by Eq. (4.19). Since $\mathrm{Char}\, P = \{(\tilde{x},\tilde{\xi}); (\xi_0)^2 = h^{ij}(x)\xi_i\xi_j\}$ we then have $\tilde{\xi}=0$. Together with the corresponding argument for the case $\tilde{\xi}=0$ this shows that

$$WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_A) \subset \mathrm{Char}(P)\times \mathrm{Char}(P); \tag{4.20}$$

otherwise $0\in T^*(M\times M)$ would be in $WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_A)$.

Since $WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(i\partial_t\omega_A) \subset WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_A)$ by [15, Proposition B.3], we have

$$\begin{aligned} WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) = WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(i\partial_t\omega_A) \subset WF^{\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_A) \\ \subset (\mathrm{Char}(P)\times \mathrm{Char}(P)). \end{aligned}$$

□

Theorem 4.14. *For all $s\in\mathbb{R}$, $WF^s(\omega_G^{(2)}) \subset \{(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta})\in T^*(M\times M); \tilde{\xi}^0 > 0\}$.*

Proof. We define $F:\mathbb{R}+i\,]0,\delta[\,\subset\mathbb{C}\to\mathcal{D}'(\Sigma\times M)$ for $\delta>0$ by

$$\begin{aligned} \langle F(z), \psi_1(s)\psi_2(x)\psi_3(y)\rangle = \int_M \sum_j \lambda_j^{-1} e^{iz\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right) \\ \times \phi_j(y)\psi_1(s)\psi_3(y)dyds. \end{aligned} \tag{4.21}$$

Notice that $\partial_z F:\mathbb{R}+i]0,\delta[\subset\mathbb{C}\to\mathcal{D}'(\Sigma\times M)$ is given by

$$\begin{aligned} \langle \partial_z F(z), \psi_1(s)\psi_2(x)\psi_3(y)\rangle = i\int_M \sum_j e^{iz\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right) \\ \phi_j(y)\psi_1(s)\psi_3(y)dyds \end{aligned} \tag{4.22}$$

and therefore F is a holomorphic function with values in $\mathcal{D}'(\Sigma \times M)$, see [17, Theorem 10.11]. Moreover, for $\varphi(t) \in \mathcal{D}(\mathbb{R})$ we have

$$|\langle\langle F(t+i\epsilon), \psi_1(s)\psi_2(x)\psi_3(y)\rangle, \varphi(t)\rangle| \tag{4.23}$$

$$= \left|\int_{\mathbb{R}}\int_M \sum_j \lambda_j^{-1} e^{i(t+i\epsilon)\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right)\phi_j(y)\psi_1(s)\psi_3(y)dyds\varphi(t)dt\right| \tag{4.24}$$

$$= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_j \underbrace{\lambda_j^{-1} e^{i(t+i\epsilon)\lambda_j} e^{-is\lambda_j} (\psi_2, \phi_j)_{L^2(\Sigma)}(\psi_3, \phi_j)_{L^2(\Sigma)}\psi_1(s)\varphi(t)}_{h_\epsilon(s,t,j)} dsdt\right|. \tag{4.25}$$

Now let $g(s,t,j) := |(\psi_2, \phi_j)_{L^2(\Sigma)}(\psi_3, \phi_j)_{L^2(\Sigma)}| \cdot |\psi_1(s)\varphi(t)|$, then

$$|h_\epsilon(s,t,j)| \le g(s,t,j). \tag{4.26}$$

Moreover,

$$\sum_j (\psi_2, \phi_j)_{L^2(\Sigma)}(\psi_3, \phi_j)_{L^2(\Sigma)} = \int_\Sigma \psi_2(w)\psi_3(w)\sqrt{h(w)}dw. \tag{4.27}$$

This implies the sequence is unconditionally convergent and therefore absolutely convergent.

Hence, $g(s,t,j) \in L^1(dt \times ds \times \mu)$, where μ is the counting measure on $\mathbb{N}$.

Using dominated convergence in Eq. (4.30) and Parseval's Identity in Eq. (4.31) we obtain for $\epsilon_n \to 0^+$

$$\lim_{\epsilon\to 0^+} \langle\langle F(t+i\epsilon), \psi_1(s)\psi_2(x)\psi_3(y)\rangle, \varphi(t)\rangle \tag{4.28}$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}}\int_M \sum_j \lambda_j^{-1} e^{i(t+i\epsilon_n)\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right) \times \phi_j(y)\psi_1(s)\psi_3(y)dyds\varphi(t)dt \tag{4.29}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_j \lim_{n\to\infty} \lambda_j^{-1} e^{i(t+i\epsilon_n)\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_3(y)\phi_j(y)dy\right)\left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right) \times \psi_1(s)ds\varphi(t)dt \tag{4.30}$$

$$= \int_{\mathbb{R}}\int_M \sum_j \lambda_j^{-1} e^{it\lambda_j} e^{-is\lambda_j} \left(\int_\Sigma \psi_2(x)\phi_j(x)dx\right)\phi_j(y)\psi_1(s)\psi_3(y)dyds\varphi(t)dt \tag{4.31}$$

$$= \omega_G^{(2)}(\varphi\psi_1\psi_2\psi_3). \tag{4.32}$$

Therefore $\lim_{\epsilon\to 0^+}\langle F(t+i\epsilon), \cdot\rangle = \omega_G^{(2)} \in \mathcal{D}'(M\times M)$. Applying [12, Proposition 7.5] we obtain

$$WF^s(\omega_G^{(2)}) \subset WF(\omega_G^{(2)}) \subset \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in T^*(M \times M); \xi_0 > 0\} \tag{4.33}$$

which gives $\sum_{\mu=0}^{3} g^{0\mu}\xi_\mu = \xi^0 > 0$. □

Lemma 4.15. *Let $(\tilde{x}, \tilde{y}) \in M \times M$ be such that $\tilde{x}$ and $\tilde{y}$ are not causally related, i.e. $\tilde{x} \notin J(\tilde{y})$. Then $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \notin WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)})$ for every $\epsilon > 0$.*

Proof. From Eq. (4.19), Lemma 4.13 and Theorem 4.14 we conclude that

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset N_+ \times N_-, \tag{4.34}$$

where $N_\pm := \{(t, x, \xi_0, \xi) \in \mathrm{Char}(P); \pm\xi_0 > 0\}$

Now consider the restriction $\omega_G^{(2)}|_{\mathcal{Q}} := \omega_G^{(2)} : \mathcal{D}(M \times M)|_{\mathcal{Q}} \to \mathbb{C}$, where the set $\mathcal{Q}$ is defined as the set of pairs of causally separated points $(\tilde{x}, \tilde{y}) \in M \times M$.

Notice that $\omega_G^{(2)} = \omega^+ + iK_G$, where ω^+ is the Schwartz kernel of $A^{-\frac{1}{2}}$ $\cos(A^{\frac{1}{2}}(t-s))$ and K_G is the causal propagator, which is the Schwartz kernel of $A^{-\frac{1}{2}}\sin(A^{\frac{1}{2}}(t-s))$. Since $K_G|_{\mathcal{Q}} = 0$ by [26, Lemma 5.1] we have $\omega_G^{(2)}|_{\mathcal{Q}} = \omega^+|_{\mathcal{Q}}$.

Also, the “flip” map $\rho(\tilde{x}, \tilde{y}) = (\tilde{y}, \tilde{x})$ is a diffeomorphism of $\mathcal{Q}$ and we have $\rho^*\omega^+ = \omega^+$. Moreover, using the covariance of the Sobolev wavefront set under diffeomorphisms (see Appendix 5.1), we have

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega^+|_{\mathcal{Q}}) = WF^{-\frac{1}{2}-\epsilon+\tau}(\rho^*\omega^+|_{\mathcal{Q}}) = \rho^* WF^{-\frac{1}{2}-\epsilon+\tau}(\omega^+|_{\mathcal{Q}}). \tag{4.35}$$

Moreover, $\rho^*(N_+ \times N_-) = N_- \times N_+$ which implies

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) \subset (N_+ \times N_-) \cap (N_- \times N_+) = \emptyset. \tag{4.36}$$

□

Lemma 4.16. *If $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)})$ for some $\tilde{\epsilon} > 0$, then $\tilde{\eta} = -\tilde{\xi}$.*

Proof. Note that $WF^{-\frac{3}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) \subset WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ for $0 < \epsilon < \tilde{\epsilon}$, so that we may possibly decrease $\tilde{\epsilon}$. Suppose $\tilde{\eta}$ and $\tilde{\xi}$ are linearly independent, i.e. $\tilde{\eta} \neq \lambda\tilde{\xi}$ for $\lambda \in \mathbb{R}$. By Lemma 4.13, $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in \mathrm{Char}(P) \times \mathrm{Char}(P)$.

Now we choose a Cauchy hypersurface $\Sigma_{t_0} = \{t_0\} \times \Sigma$ such that the null geodesic with initial data $(\tilde{x}, \tilde{\xi})$ and the null geodesic with initial data $(\tilde{x}, \tilde{\eta})$ intersect it. These points of intersections are unique by global hyperbolicity (see [4,21,25] for low regularity definitions). Moreover, using the condition $\tilde{\eta} \neq \lambda\tilde{\xi}$, we can choose Σ_{t_0} such that these points are distinct. We denote these points by $(t_0, x_0), (t_0, y_0)$. Clearly, these points are not causally related.

Notice that ω_A satisfies $P_{(t,x)}\omega_A = 0$. Given any $\epsilon > 0$ we can achieve $P^b_{(t,x)}\omega_A \in H^{-\frac{3}{2}-\epsilon+\tau}$ by fixing δ in Remark 4.6 close to 1. Taking ϵ small, this allows us to choose $s = -\frac{3}{2} - \epsilon + \tau$ with $0 < s < \tau - 1$ in Theorem 4.7. This propagation of singularities result applied to the distribution ω_A and the operator $P_{(t,x)}$ guarantees that if $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset$

$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)$ then the full null bicharacteristic is contained in the wavefront set i.e. $(\gamma(\tilde{x},\tilde{\xi}),(\tilde{x},\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)$, where $\gamma(\tilde{x},\tilde{\xi})$ is the null bicharacteristic with initial data $(\tilde{x},\tilde{\xi})$. Similarly, using the operator $P_{(s,y)}$, we obtain $(\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{x},\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)$, where $\gamma(\tilde{x},\tilde{\eta})$ is the null bicharacteristic with initial data $(\tilde{x},\tilde{\eta})$.

Now we show that $(\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{x},\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)})$.

By Theorem 6.1.1' from [8] we have

$$\begin{aligned} WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)\backslash WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}) &= WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)\backslash WF^{-\frac{1}{2}-\epsilon+\tau}(i\partial_t\omega_A) \\ &\subset \mathrm{Char}(i\partial_t). \end{aligned} \tag{4.37}$$

However, using Eq.(4.20) and that $(\partial_t+\partial_s)\omega_A = 0$, we have the inclusion

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A) \subset WF^{\frac{1}{2}-\epsilon+\tau}(\omega_A) \subset (\mathrm{Char}(P)\times\mathrm{Char}(P)) \cap \mathrm{Char}(\partial_t+\partial_s). \tag{4.38}$$

Since $(\mathrm{Char}(P)\times\mathrm{Char}(P)) \cap \mathrm{Char}(\partial_t+\partial_s) \cap \mathrm{Char}(i\partial_t) = \emptyset$, taking the intersection between Eq.(4.37) and Eq.(4.38), we obtain that the left hand side of Eq.(4.37) must be empty. Therefore,

$$WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A) \subset WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}). \tag{4.39}$$

Hence, $(\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{x},\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)})$. In particular $(t_0,x_0,\tilde{\xi},t_0,y_0,\tilde{\eta}) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}))$. However, this is a contradiction to Lemma 4.15. Therefore, $\tilde{\eta} = \lambda\tilde{\xi}$ for some $\lambda \in \mathbb{R}$. Using Eq.(4.19) we have $\xi_0 = -\eta_0$ which gives $\lambda = -1$, i.e. $\tilde{\eta} = -\tilde{\xi}$. □

We have used the distribution ω_A, because a direct application of Theorem 4.5 for $\omega_G^{(2)}$ is not possible, since for δ close to 1, σ cannot take the value $-\frac{1}{2}$.

Now we state the main result

Theorem 4.17. *$WF'^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C^+$ for every $\epsilon > 0$ and C^+ as in Eq.(2.4).*

Proof. Let $(\tilde{x},\tilde{\xi},\tilde{y},-\tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)$, where the inclusion follows from [15, Proposition B.3] since $\omega_G^{(2)} = i\partial_t\omega_A$. The propagation of singularities result (Theorem 4.7) implies that $(\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{y},-\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_A)$, where $\gamma(\tilde{x},\tilde{\xi})$ is the null bicharacteristic with initial data $(\tilde{x},\tilde{\xi})$ and $\gamma(\tilde{y},-\tilde{\eta})$ is the null bicharacteristic with initial data $(\tilde{y},-\tilde{\eta})$. Hence, by Eq.(4.39), we have $(\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{y},-\tilde{\eta})) \in WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)})$.

Now we choose a Cauchy surface $\Sigma_{t_1} = \{t_1\}\times\Sigma$ and suppose that $(t_1,x_1,\tilde{\xi}_1,t_1,x_2,\tilde{\xi}_2) \in (\gamma(\tilde{x},\tilde{\xi}),\gamma(\tilde{y},-\tilde{\eta})) \cap (\Sigma_{t_1}^2)$. By Lemmas 4.15 and 4.16, $(t_1,x_1,\tilde{\xi}_1),(t_1,x_2,\tilde{\xi}_2) \in \mathrm{Char}(P)$, $x_1 = x_2$, and $\tilde{\xi}_2 = -\tilde{\xi}_1$.

Next we define a curve $\tilde{\gamma} : (-\infty,\infty) \to M$ as follows. First, we shift the parametrisation λ in the definition of the null bicharacteristics so that

$$\gamma(\tilde{x},\tilde{\xi})(t_1) = (t_1,x_1,\tilde{\xi}_1) \quad \gamma(\tilde{y},-\tilde{\eta})(t_1) = (t_1,x_1,-\tilde{\xi}_1).$$

Then, we denote by $\Pi : T^*M \to M$ the canonical projection and define two curves in M by

$$\gamma_1(\lambda) := \Pi(\gamma(\tilde{x}, \tilde{\xi})(\lambda)) \quad \gamma_2(\lambda) := \Pi(\gamma(\tilde{y}, -\tilde{\eta})(\lambda)).$$

Notice that we have $\gamma_1(t_1) = (t_1, x_1), \dot{\gamma_1}(t_1) = g^{-1}(\tilde{\xi}_1, \cdot)$ and $\gamma_2(t_1) = (t_1, x_1), \dot{\gamma_2}(t_1) = g^{-1}(-\tilde{\xi}_1, \cdot)$. Moreover, we can assume that $\tilde{x} = \gamma_1(a)$ and $\tilde{y} = \gamma_2(b)$ for suitable $a, b \in \mathbb{R}$ with $a < t_1 < b$.

Finally, let

$$\tilde{\gamma}(\lambda) = \begin{cases} \gamma_1(\lambda) & \lambda \in (-\infty, t_1] \\ -\gamma_2(\lambda) & \lambda \in (t_1, \infty) \end{cases} \tag{4.40}$$

where $-\gamma_2$ denotes the curve with opposite orientation.

Then $\tilde{\gamma}(a) = \tilde{x}$, $\tilde{\gamma}(b) = \tilde{y}$; moreover $g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{x}}M} = \tilde{\xi}$, $g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{y}}M} = \tilde{\eta}$ and therefore, $\tilde{\gamma}$ is a null geodesic between $\tilde{x}$ and $\tilde{y}$ with cotangent vectors $\tilde{\xi}$ at $\tilde{x}$ and $\tilde{\eta}$ at $\tilde{y}$, i.e. $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in C' := \{(\underline{\tilde{x}}, \underline{\tilde{\xi}}, \underline{\tilde{y}}, -\underline{\tilde{\eta}}); (\underline{\tilde{x}}, \underline{\xi}; \underline{\tilde{y}}, \underline{\tilde{\eta}}) \in C\}$. This shows

$$WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C' \tag{4.41}$$

Using the definition of $WF^{l'}(u) := \{(\tilde{x}, \tilde{\eta}; \tilde{y}, -\tilde{\eta}) \in T^*(M \times M); (\tilde{x}, \tilde{\xi}; \tilde{y}, \tilde{\eta}) \in WF^l(u)\}$ and Theorem 4.14 gives the result. □

Remark 4.18. For a $C^{1,1}$ metric the same arguments as used in [26, Theorem 7.1] apply and therefore in that scenario we have for every $\epsilon > 0$

$$WF^{\frac{1}{2}-\epsilon}(\omega_G^{(2)}) \subset C'^{+}.$$

Acknowledgements

We are grateful to Chris Fewster, Bernard Kay and James Vickers for helpful discussions. We also thank the anonymous reviewers for their comments.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflicts of interest We have no conflicts of interest to disclose.

5. Appendix

5.1. Covariance of the Sobolev Wavefront Set under Diffeomorphisms

The following is a variant of Theorem [13, Theorem 8.2.4] adapted to the H^s-wave front set.

Lemma 5.1. *Let $\varphi : M \to M$ be a C^∞ diffeomorphism and $u \in \mathcal{D}'(M)$. Then*

$$WF^s(\varphi^* u) = \varphi^* WF^s(u), \quad s \in \mathbb{R}. \tag{5.1}$$

Proof. Let $(x,\xi) \notin \varphi^* WF^s(u)$ which by definition implies $(\varphi(x),{}^t\,\partial\varphi(x)^{-1}\xi) \notin WF^s(u)$. By [8, p. 202], we can write $u = u_1 + u_2$ where $u_1 \in H^s_{loc}$ and $(\varphi(x),{}^t\,\partial\varphi(x)^{-1}\xi) \notin WF(u_2)$. By the covariance of the Sobolev spaces in compact sets [29, Chapter 4, Section 2] we have $\varphi^* u_1 \in H^s_{loc}$ and by the covariance under diffeomorphism of the wavefront set $(x,\xi) \notin WF(\varphi^* u_2) = \varphi^* WF(u_2)$. Putting this together gives $(x,\xi) \notin WF^s(\varphi^* u)$, i.e. $WF^s(\varphi^* u) \subset \varphi^* WF^s(u)$. Applying this relation to φ^{-1} we conversely see that

$$\varphi^* WF^s(u) = \varphi^* WF^s(\varphi^{-1*}\varphi^* u) \subset \varphi^*\varphi^{-1*} WF^s(\varphi^* u) = WF^s(\varphi^* u).$$

This completes the argument. □

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Communicated by Karl-Henning Rehren.

Received: July 19, 2022.

Accepted: February 2, 2023.