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Letter

Equation governing the probability density evolution of multi-dimensional linear fractional differential systems subject to Gaussian white noise

Yi Luo[a,b], Meng-Ze Lyu[c,*], Jian-Bing Chen[d], Pol D. Spanos[a]

[a] George R. Brown School of Engineering, Rice University, Houston 77005, TX, USA
[b] Institute for Risk & Reliability, Leibniz University Hannover, Hannover 30167, Germany
[c] College of Civil Engineering, Tongji University, Shanghai 200092, China
[d] State Key Laboratory of Disaster Reduction in Civil Engineering, Tongji University, Shanghai 200092, China

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A B S T R A C T

Stochastic fractional differential systems are important and useful in the mathematics, physics, and engineering fields. However, the determination of their probabilistic responses is difficult due to their non-Markovian property. The recently developed globally-evolving-based generalized density evolution equation (GE-GDEE), which is a unified partial differential equation (PDE) governing the transient probability density function (PDF) of a generic path-continuous process, including non-Markovian ones, provides a feasible tool to solve this problem. In the paper, the GE-GDEE for multi-dimensional linear fractional differential systems subject to Gaussian white noise is established. In particular, it is proved that in the GE-GDEE corresponding to the state-quantities of interest, the intrinsic drift coefficient is a time-varying linear function, and can be analytically determined. In this sense, an alternative low-dimensional equivalent linear integer-order differential system with exact closed-form coefficients for the original high-dimensional linear fractional differential system can be constructed such that their transient PDFs are identical. Specifically, for a multi-dimensional linear fractional differential system, if only one or two quantities are of interest, GE-GDEE is only in one or two dimensions, and the surrogate system would be a one- or two-dimensional linear integer-order system. Several examples are studied to assess the merit of the proposed method. Though presently the closed-form intrinsic drift coefficient is only available for linear stochastic fractional differential systems, the findings in the present paper provide a remarkable demonstration on the existence and eligibility of GE-GDEE for the case that the original high-dimensional system itself is non-Markovian, and provide insights for the physical-mechanism-informed determination of intrinsic drift and diffusion coefficients of GE-GDEE of more generic complex nonlinear systems.

Fractional derivatives have a long mathematical history. In recent decades, the fractional derivative has received increasing attention due to its applicability in the fields of physics and engineering [1–3], such as hydraulics [4], optimal control [5,6], dynamic stability [7], and thermodynamics [8], etc. Stochastic dynamic systems endowed with fractional derivative element enforced by random excitations have been investigated widely [9,10]. Analytical solution of one-dimensional linear stochastic fractional differential systems was given firstly in a Duhamel integral form by Agrawal [11], and then was investigated for more cases [12–14], while effective numerical solutions were developed for multi-dimensional linear cases [15,16]. The difficulty of random vibration of systems endowed with fractional derivative element is that the Markovian property is no longer applicable. Thus, there are many approximate techniques developed for nonlinear stochastic fractional differential systems, such as the equivalent linearization [17–20], Wiener path integral [21], the harmonic wavelets [22–25], the stochastic averaging [26], and the stochastic perturbation [27], etc. Recently, the globally-evolving-based generalized density evolution equation (GE-GDEE) was proposed as a unified partial differential equation (PDE) governing the transient probability density function (PDF) of a generic path-continuous non-Markov process [28]. In particular, even the problem is multi-dimensional, nonlin-

* Corresponding author.
E-mail address: lyumz@tongji.edu.cn (M.-Z. Lyu).

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ear, and non-Markovian, the transient PDF of an arbitrary component of the system satisfies the GE-GDEE, a PDE in only one or two dimensions, exactly. It was shown that the GE-GDEE exactly holds for multi-dimensional nonlinear stochastic fractional differential systems subject to Gaussian white noise [29]. Generally, the intrinsic drift coefficient in the GE-GDEE must usually be determined by some numerically techniques [30–32], though analytical expressions are available in a few special cases when the underlying physics can be fully made use of, such as a class of energy-equipartition autonomous systems [33]. The intrinsic drift coefficient of GE-GDEE, as a physically driving factor of the evolution of the PDF, is a conditional expectation function of the drift coefficient of the original multi-dimensional system. In this paper, a closed-form expression of the intrinsic drift coefficient of GE-GDEE for multi-dimensional linear fractional differential systems subject to Gaussian white noise is derived based on the linearity of fractional derivative element preserving Gaussian property of the systems. This implies that an exactly equivalent linear stochastic integer-order differential systems with analytical time-variant coefficients can be constructed. The investigation will provide a remarkable family of theoretically proved illustrations on the existence and eligibility of GE-GDEE for the case that the original high-dimensional system is itself essentially non-Markovian, and a demonstration of the physical-mechanism-informed determination of the intrinsic drift coefficient of fractional differential systems, and provides insight for more complex nonlinear systems.

Consider a stochastic process $Y(t)$ governed by the one-dimensional linear stochastic fractional differential equation

$$\mathrm{d}Y(t) = -\left[kY(t) + c\mathrm{D}^{\alpha}Y(t)\right]\mathrm{d}t + \sigma_{\mathrm{W}}\mathrm{d}W(t), \tag{1}$$

where k and c are constants; σ_{W} is the intensity. Further, $W(t)$ is a standard Wiener process defined by the properties [34]:

(1) $W(t) \sim \mathcal{N}(0,t)$, i.e., $W(t)$ is Gaussian and satisfies
  (a) $W(0) = 0$ with probability one;
  (b) $\mathrm{E}[W(t)] = 0$, for $t \geq 0$;
  (c) $\mathrm{E}\left[W^2(t)\right] = t$, for $t \geq 0$;
(2) $W(t) - W(\tau) \sim \mathcal{N}(0, t-\tau)$, for $0 < \tau < t$.

Herein, $\mathrm{E}(\cdot)$ denotes the expectation operator, and $\mathrm{D}^{\alpha}Y(t)$ is the α -order fractional derivative of $Y(t)$, $0 < \alpha < 1$. Next, adopt the Caputo fractional derivative definition [35],

$$\mathrm{D}^{\alpha}Y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{Y}(\tau)\mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. The initial condition of $Y(t)$ can be denoted as

$$Y(0) = y_0. \tag{3}$$

Clearly, $Y(t)$ governed by Eq. (1) is a path-continuous process of with probability one which satisfied the Dynkin-Kinney condition[1] [36,37], but it is non-Markovian. In general, it is difficult to directly expand $Y(t)$ into a finite-dimensional Markov system exactly, because the fractional derivative of $Y(t)$ is dependent on all the information before the time instant as seen in Eq. (2). However, using the GE-GDEE for generic path-continuous non-Markovian process [28], the transient PDF of $Y(t)$, denoted as $p_Y(y,t)$, satisfies a one-dimensional PDE exactly, and the coefficients in the PDE can be achieved analytically. Specifically, following the GE-GDEE [28], the GE-GDEE for $p_Y(y,t)$ is

$$\frac{\partial p_Y(y,t)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{eff})}(y,t)p_Y(y,t)\right]}{\partial y} + \frac{\partial^2\left[b^{(\mathrm{eff})}(y,t)p_Y(y,t)\right]}{2\partial y^2}, \tag{4}$$

where the intrinsic drift and diffusion coefficients are defined as [28]

$$\begin{cases} a^{(\mathrm{eff})}(y,t) = \lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathrm{E}[\Delta Y(t)|\, Y(t)=y], \\ b^{(\mathrm{eff})}(y,t) = \lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathrm{E}\left\{[\Delta Y(t)]^2|\, Y(t)=y\right\}, \end{cases} \tag{5}$$

with $\Delta Y(t) = Y(t+\Delta t) - Y(t)$ denoting the increment during time interval $[t, t+\Delta t]$. The intrinsic drift and diffusion coefficients can also be called as effective drift and diffusion coefficients in the senses that they can be constructed analytically or numerically. By substituting Eq. (1) into Eq. (5), the second equation becomes

$$b^{(\mathrm{eff})}(y,t) = \sigma_{\mathrm{W}}^2, \tag{6}$$

while the first equation yields

$$\begin{aligned} a^{(\mathrm{eff})}(y,t) &= -\mathrm{E}\left[kY(t) + c\mathrm{D}^{\alpha}Y(t)|\, Y(t)=y\right] \\ &= -ky - c\mathrm{E}\left[\mathrm{D}^{\alpha}Y(t)|\, Y(t)=y\right] \\ &= -ky - \frac{c}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\mathrm{E}\left[\dot{Y}(\tau)|\, Y(t)=y\right]\mathrm{d}\tau \\ &= -ky - \frac{c}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{E}[Y(\tau)|\, Y(t)=y]\mathrm{d}\tau. \end{aligned} \tag{7}$$

The analytical determination of the intrinsic drift coefficient in the GE-GDEE is a daunting task for generic nonlinear stochastic fractional differential systems [29]. However, for linear stochastic fractional differential systems, the closed-form expression of Eq. (7) can be investigated. To determine the intrinsic drift coefficient in Eq. (7), the Gaussian property and auto-covariance of $Y(t)$ are reviewed firstly. Note that the solution of Eq. (1) with the initial condition Eq. (3) can be expressed as [11,12,41]

$$Y(t) = \sigma_{\mathrm{W}}\int_0^t g(t-\tau)\mathrm{d}W(\tau) + [g(t) + cg_1(t)]y_0, \tag{8}$$

where $g(\cdot)$ and $g_1(\cdot)$ are fractional Green functions. They are given by Miller and Ross [42], West et al. [43]

$$g(t) = \sum_{j=0}^{\infty}\frac{(-kt)^j}{j!}\epsilon^{(j)}_{1-\alpha,1+j\alpha}\left(-ct^{1-\alpha}\right), \tag{9}$$

and

$$g_1(t) = \sum_{j=0}^{\infty}\frac{(-kt)^j t^{1-\alpha}}{j!}\epsilon^{(j)}_{1-\alpha,2+(j-1)\alpha}\left(-ct^{1-\alpha}\right), \tag{10}$$

respectively. In these equations, $\epsilon^{(j)}_{p,q}(\cdot)$ is the th-order derivative of the generalized Mittag-Leffler function, namely,

$$\epsilon^{(j)}_{p,q}(x) = \sum_{l=0}^{\infty}\frac{(j+l)!x^l}{l!\,\Gamma[(j+l)p+q]}. \tag{11}$$

Equation (8) shows that the response solution of a linear fractional system is similar to that of a linear integer order system expressed via a Duhamel integral [11]. That is, the fractional derivative element does not break the linear superposition property of the system. This means that the response of a linear fractional system excited by Gaussian white noise is still Gaussian, and can be fully characterized by its first two order moments. According to Eq. (8), the expectation and variance of $Y(t)$ can be given as [11,12,41]

$$\mu_Y(t) = \mathrm{E}[Y(t)] = [g(t) + cg_1(t)]y_0, \tag{12}$$

[1] The path-continuous condition was proved for Markov process by Dynkin [36] in 1952 and Kinney [37] in 1953 independently. Further, it was generalized for separable process by Dobrushin [38]. Hence, it was referred to as Dynkin-Kinney condition [39]. However, in some literatures it is also called as Lindeberg's condition [40].

and

$$\sigma_Y^2(t) = \mathrm{E}\left\{[Y(t) - \mu_Y(t)]^2\right\} = \sigma_W^2 \int_0^t g^2(\tau)\mathrm{d}\tau, \tag{13}$$

respectively. Further, the auto-covariance function of $Y(t)$ can be written as

$$\begin{aligned}\varsigma_Y(t_1, t_2) &= \mathrm{E}\{[Y(t_1) - \mu_Y(t_1)][Y(t_2) - \mu_Y(t_2)]\} \\ &= \sigma_W^2 \int_0^{t_1} g(\tilde{\tau})g(t_2 - t_1 + \tilde{\tau})\mathrm{d}\tilde{\tau}, \text{ for } t_1 \le t_2.\end{aligned} \tag{14}$$

For two different instants τ and t, $Y(\tau)$ and $Y(t)$ are two dependent Gaussian variables. Thus,

$$\begin{aligned}\mathrm{E}[Y(\tau)|Y(t) = y] &= \mu_Y(\tau) + \frac{[y - \mu_Y(t)]\varsigma_Y(\tau, t)}{\sigma_Y^2(t)} \\ &= \mu_Y(\tau) + \frac{\sigma_W^2[y - \mu_Y(t)]}{\sigma_Y^2(t)} \int_0^{\tau} g(\tilde{\tau})g(t - \tau + \tilde{\tau})\mathrm{d}\tilde{\tau}.\end{aligned} \tag{15}$$

The derivation of Eq. (15) with respect to τ leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{E}[Y(\tau)|Y(t) = y] \\ &= \dot{\mu}_Y(\tau) + \frac{\sigma_W^2[y - \mu_Y(t)]}{\sigma_Y^2(t)}\left[g(\tau)g(t) - \int_0^{\tau} g(\tilde{\tau})\dot{g}(t - \tau + \tilde{\tau})\mathrm{d}\tilde{\tau}\right].\end{aligned} \tag{16}$$

Substituting Eq. (16) into Eq. (7) yields

$$a^{(\mathrm{eff})}(y, t) = -k^{(\mathrm{eff})}(t)y + f^{(\mathrm{eff})}(t), \tag{17}$$

in which

$$k^{(\mathrm{eff})}(t) = k + \frac{cr(t)}{\Gamma(1 - \alpha)\sigma_Y^2(t)}, \tag{18}$$

and

$$f^{(\mathrm{eff})}(t) = \frac{c}{\Gamma(1 - \alpha)}\left[\frac{r(t)}{\sigma_Y^2(t)}\mu_Y(t) - m(t)\right]. \tag{19}$$

In Eqs. (18) and (19), the functions $m(t)$ and $r(t)$ are given by

$$m(t) = y_0 \int_0^t (t - \tau)^{-\alpha}[\dot{g}(\tau) + c\dot{g}_1(\tau)]\mathrm{d}\tau, \tag{20}$$

and

$$r(t) = \sigma_W^2 \int_0^t (t - \tau)^{-\alpha}\left[g(t)g(\tau) - \int_0^{\tau} g(\tilde{\tau})\dot{g}(t - \tau + \tilde{\tau})\mathrm{d}\tilde{\tau}\right]\mathrm{d}\tau, \tag{21}$$

respectively; $\mu_Y(t)$ and $\sigma_Y^2(t)$ are the expectation and variance of $Y(t)$ given by Eqs. (12) and (13), respectively; $\dot{g}(\cdot)$ and $\dot{g}_1(\cdot)$ are the derivative of $g(\cdot)$ and $g_1(\cdot)$ given by Eqs. (9) and (10), respectively, i.e.,

$$\begin{aligned}\dot{g}(t) = -\sum_{j=0}^{\infty} \frac{(-kt)^j}{j!}\Big[&k\epsilon_{1-\alpha,1+(j+1)\alpha}^{(j+1)}\left(-ct^{1-\alpha}\right) \\ &+(1 - \alpha)ct^{-\alpha}\epsilon_{1-\alpha,1+j\alpha}^{(j+1)}\left(-ct^{1-\alpha}\right)\Big],\end{aligned} \tag{22}$$

and

$$\begin{aligned}\dot{g}_1(t) = \sum_{j=0}^{\infty} \frac{(-kt)^j t^{-\alpha}}{j!}\Big[&(j + 1 - \alpha)\epsilon_{1-\alpha,2+(j-1)\alpha}^{(j)}\left(-ct^{1-\alpha}\right) \\ &-(1 - \alpha)ct^{1-\alpha}\epsilon_{1-\alpha,2+(j-1)\alpha}^{(j+1)}\left(-ct^{1-\alpha}\right)\Big].\end{aligned} \tag{23}$$

Note that once the analytical form of the intrinsic drift coefficient of GE-GDEE is obtained, a one-dimensional Markov diffusion process $\tilde{Y}(t)$ can be constructed to satisfy the following Itô stochastic differential equation (SDE):

$$\mathrm{d}\tilde{Y}(t) = a^{(\mathrm{eff})}\left[\tilde{Y}(t), t\right]\mathrm{d}t + \sigma_W \mathrm{d}W(t) \tag{24}$$

with the same initial condition $\tilde{Y}(0) = y_0$ as that of $Y(t)$, where $a^{(\mathrm{eff})}(\cdot)$ is given by Eq. (17) as a linear function with respect to $\tilde{Y}$. Clearly, since the GE-GDEEs corresponding to $Y(t)$ and $\tilde{Y}(t)$ are identical, the transient PDFs of the two processes are identical. Further, the transition probability density (TPD) of $\tilde{Y}(t)$ during two different instants $t' < t$, denoted as $p_{\tilde{Y}}\left(y, t\middle|y', t'\right)$, satisfies the following Fokker-Planck equation

$$\frac{\partial p_{\tilde{Y}}\left(y, t|y', t'\right)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{eff})}(y, t)p_{\tilde{Y}}\left(y, t|y', t'\right)\right]}{\partial y} + \frac{\sigma_W^2}{2}\frac{\partial^2 p_{\tilde{Y}}\left(y, t|y', t'\right)}{\partial y^2}, \tag{25}$$

which is in the same form of PDE as the GE-GDEE (4). The difference is that Eq. (4) is applicable to one-dimensional transient PDF of both the non-Markov process $Y(t)$ and Markov process $\tilde{Y}(t)$; whereas Eq. (25) is only applicable to the TPD between any two instants of the Markov diffusion process $\tilde{Y}(t)$. However, the one-dimensional transient PDF of $\tilde{Y}(t)$, which can be considered as the TPD under the given initial condition, can also be determined via Eq. (25).

Consider a single-degree-of-freedom (SDOF) linear oscillator endowed with fractional derivative elements subject to Gaussian white noise. Its equation of motion reads

$$\ddot{X}(t) + c\mathrm{D}^{\alpha}X(t) + kX(t) = \xi(t), \tag{26}$$

where $X(t)$ and $\ddot{X}(t)$ are the displacement and acceleration, respectively; k and c are constants. Further, $\mathrm{D}^{\alpha}X(t)$ is the -order fractional derivative of defined by Eq. (2), $0 < \alpha < 1$; $\xi(t)$ is a Gaussian white noise process with intensity σ_W, i.e.,

$$\begin{cases}\mathrm{E}[\xi(t)] = 0, \\ \mathrm{E}[\xi(t)\xi(t + \tau)] = \sigma_W^2\delta(\tau),\end{cases} \tag{27}$$

in which $\delta(\cdot)$ is Dirac delta function. The initial condition of Eq. (26) is prescribed by

$$\begin{cases}X(0) &= x_0, \\ \dot{X}(0) &= v_0.\end{cases} \tag{28}$$

By denoting $V(t) = \dot{X}(t)$, Eq. (26) can be rewritten as the two-dimensional linear stochastic fractional differential equations

$$\begin{cases}\mathrm{d}X(t) = V(t)\mathrm{d}t, \\ \mathrm{d}V(t) = -\left[c\mathrm{D}^{\alpha}X(t) + kX(t)\right]\mathrm{d}t + \sigma_W\mathrm{d}W(t).\end{cases} \tag{29}$$

Clearly, $X(t)$ and $V(t)$ governed by Eq. (29) are path-continuous processes but non-Markovian. Hence, in accordance with the GE-GDEE for generic path-continuous non-Markovian process [28], the transient joint PDF of $(X(t), V(t))^{\mathrm{T}}$, denoted as $p_{XV}(x, v, t)$, satisfies a two-dimensional PDE exactly, and the coefficients in the PDE can be achieved analytically. To this end, using the unified formalism of the GE-GDEE [28], the GE-GDEE for $p_{XV}(x, v, t)$ is

$$\begin{aligned}\frac{\partial p_{XV}(x, v, t)}{\partial t} = &-\frac{\partial\left[a_X^{(\mathrm{eff})}(x, v, t)p_{XV}(x, v, t)\right]}{\partial x} \\ &-\frac{\partial\left[a_V^{(\mathrm{eff})}(x, v, t)p_{XV}(x, v, t)\right]}{\partial v} + \frac{1}{2}\frac{\partial^2\left[b_{XX}^{(\mathrm{eff})}(x, v, t)p_{XV}(x, v, t)\right]}{\partial x^2} \\ &+\frac{\partial^2\left[b_{XV}^{(\mathrm{eff})}(x, v, t)p_{XV}(x, v, t)\right]}{\partial x\partial v} \\ &+\frac{1}{2}\frac{\partial^2\left[b_{VV}^{(\mathrm{eff})}(x, v, t)p_{XV}(x, v, t)\right]}{\partial v^2},\end{aligned} \tag{30}$$

where the intrinsic drift and diffusion coefficients are defined as [28]

$$\begin{cases} a_X^{(\text{eff})}(x,v,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathrm{E}[\Delta X(t)|\ X(t)=x;\ V(t)=v], \\ a_V^{(\text{eff})}(x,v,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathrm{E}[\Delta V(t)|\ X(t)=x;\ V(t)=v], \\ b_{XX}^{(\text{eff})}(x,v,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathrm{E}\{[\Delta X(t)]^2|\ X(t)=x;\ V(t)=v\}, \\ b_{XV}^{(\text{eff})}(x,v,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathrm{E}[\Delta X(t)\Delta V(t)|\ X(t)=x;\ V(t)=v], \\ b_{VV}^{(\text{eff})}(x,v,t) = \lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}\mathrm{E}\{[\Delta V(t)]^2|\ X(t)=x;\ V(t)=v\}, \end{cases} \tag{31}$$

with $\Delta X(t) = X(t+\Delta t) - X(t)$ and $\Delta V(t) = V(t+\Delta t) - V(t)$ being the increments during time interval $[t, t+\Delta t]$. By substituting Eq. (29) into Eq. (31), the equations, except the second one, become

$$a_X^{(\text{eff})}(x,v,t) = v,\ \ b_{XX}^{(\text{eff})}(x,v,t) = 0, \\ b_{XV}^{(\text{eff})}(x,v,t) = 0,\ \ b_{VV}^{(\text{eff})}(x,v,t) = \sigma_{\mathrm{W}}^2, \tag{32}$$

while the second equation yields

$$\begin{aligned} a_V^{(\text{eff})}(x,v,t) &= -\mathrm{E}\left[kX(t) + c\mathrm{D}^\alpha X(t)|X(t)=x;\ V(t)=v\right] \\ &= -kx - c\mathrm{E}\left[\mathrm{D}^\alpha X(t)|X(t)=x;\ V(t)=v\right] \\ &= -kx - \frac{c}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\mathrm{E}[V(\tau)|X(t)=x;\ V(t)=v]\mathrm{d}\tau. \end{aligned} \tag{33}$$

The closed-form expression of Eq. (33) will be investigated later. Note that the solutions of Eq. (26) (or Eq. (29)) with the initial condition (28) can be expressed as [11,12,41]

$$X(t) = \sigma_{\mathrm{W}}\int_0^t \hat{g}(t-\tau)\mathrm{d}W(\tau) + x_0\hat{g}_1(t) + v_0\hat{g}(t), \tag{34}$$

and

$$V(t) = \sigma_{\mathrm{W}}\int_0^t \dot{\hat{g}}(t-\tau)\mathrm{d}W(\tau) + x_0\dot{\hat{g}}_1(t) + v_0\dot{\hat{g}}(t), \tag{35}$$

where $\hat{g}(\cdot)$ and $\hat{g}_1(\cdot)$ are fractional Green functions expressed as [42,43]

$$\hat{g}(t) = \sum_{j=0}^{\infty}\frac{(-k)^j t^{2j+1}}{j!}\epsilon^{(j)}_{2-\alpha,2+j\alpha}\left(-ct^{2-\alpha}\right), \tag{36}$$

and

$$\hat{g}_1(t) = \sum_{j=0}^{\infty}\frac{(-k)^j t^{2j}}{j!}\left[\epsilon^{(j)}_{2-\alpha,1+j\alpha}\left(-ct^{2-\alpha}\right) + ct^{2-\alpha}\epsilon^{(j)}_{2-\alpha,3+(j-1)\alpha}\left(-ct^{2-\alpha}\right)\right], \tag{37}$$

respectively; $\epsilon^{(j)}_{p,q}(\cdot)$ is the jth-order derivative of the generalized Mittag-Leffler function given by Eq. (11). Clearly, the responses $X(t)$ and $V(t)$ are Gaussian, and the expectation vector and variance matrix of $(X(t), V(t))^{\mathrm{T}}$ can be given according to Eqs. (34) and (35) as

$$\boldsymbol{\mu}(t) = \begin{pmatrix}\mu_X(t)\\ \mu_V(t)\end{pmatrix} = \mathrm{E}\left[\begin{pmatrix}X(t)\\ V(t)\end{pmatrix}\right] = \begin{pmatrix}x_0\hat{g}_1(t) + v_0\hat{g}(t)\\ x_0\dot{\hat{g}}_1(t) + v_0\dot{\hat{g}}(t)\end{pmatrix}, \tag{38}$$

and

$$\begin{aligned}\boldsymbol{\Sigma}(t) &= \begin{pmatrix}\sigma_X^2(t) & \varsigma_{XV}(t)\\ \varsigma_{XV}(t) & \sigma_V^2(t)\end{pmatrix} = \mathrm{E}\left[\begin{pmatrix}X(t)-\mu_X(t)\\ V(t)-\mu_V(t)\end{pmatrix}\begin{pmatrix}X(t)-\mu_X(t)\\ V(t)-\mu_V(t)\end{pmatrix}^{\mathrm{T}}\right] \\ &= \begin{pmatrix}\sigma_{\mathrm{W}}^2\int_0^t \hat{g}^2(\tau)\mathrm{d}\tau & \sigma_{\mathrm{W}}^2\int_0^t \hat{g}(\tau)\dot{\hat{g}}(\tau)\mathrm{d}\tau\\ \sigma_{\mathrm{W}}^2\int_0^t \hat{g}(\tau)\dot{\hat{g}}(\tau)\mathrm{d}\tau & \sigma_{\mathrm{W}}^2\int_0^t \dot{\hat{g}}^2(\tau)\mathrm{d}\tau\end{pmatrix},\end{aligned} \tag{39}$$

respectively. Further, the auto-/cross-covariance function of $X(t)$ and $V(t)$ can be written as

$$\begin{cases} \tilde{\varsigma}_{XX}(t_1,t_2) = \mathrm{E}\{[X(t_1)-\mu_X(t_1)][X(t_2)-\mu_X(t_2)]\} \\ \qquad = \sigma_{\mathrm{W}}^2\int_0^{t_1}\hat{g}(\tilde{\tau})\hat{g}(t_2-t_1+\tilde{\tau})\mathrm{d}\tilde{\tau}, \\ \tilde{\varsigma}_{XV}(t_1,t_2) = \mathrm{E}\{[X(t_1)-\mu_X(t_1)][V(t_2)-\mu_V(t_2)]\} \\ \qquad = \sigma_{\mathrm{W}}^2\int_0^{t_1}\hat{g}(\tilde{\tau})\dot{\hat{g}}(t_2-t_1+\tilde{\tau})\mathrm{d}\tilde{\tau}, \\ \tilde{\varsigma}_{VV}(t_1,t_2) = \mathrm{E}\{[V(t_1)-\mu_V(t_1)][V(t_2)-\mu_V(t_2)]\} \\ \qquad = \sigma_{\mathrm{W}}^2\int_0^{t_1}\dot{\hat{g}}(\tilde{\tau})\dot{\hat{g}}(t_2-t_1+\tilde{\tau})\mathrm{d}\tilde{\tau}, \qquad \text{for } t_1 \le t_2. \end{cases} \tag{40}$$

For different time instants τ and t, $X(t)$, $V(\tau)$ and $V(t)$ are three dependent Gaussian variables. Thus, there is

$$\mathrm{E}[V(\tau)|\ X(t)=x;\ V(t)=v] = \\ \mu_V(\tau) + \begin{pmatrix}\tilde{\varsigma}_{VX}(\tau,t)\\ \tilde{\varsigma}_{VV}(\tau,t)\end{pmatrix}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(t)\begin{pmatrix}x-\mu_X(t)\\ v-\mu_V(t)\end{pmatrix}. \tag{41}$$

Substituting Eq. (41) into Eq. (33) yields

$$a_V^{(\text{eff})}(x,v,t) = -\left[k + k^{(\text{eff})}(t)\right]x - c^{(\text{eff})}(t)v - f^{(\text{eff})}(t), \tag{42}$$

in which

$$\begin{cases}\left(k^{(\text{eff})}(t), c^{(\text{eff})}(t)\right) = \dfrac{c}{\Gamma(1-\alpha)}\boldsymbol{r}^{\mathrm{T}}(t)\boldsymbol{\Sigma}^{-1}(t), \\ f^{(\text{eff})}(t) = \dfrac{c}{\Gamma(1-\alpha)}\left[m(t) - \boldsymbol{r}^{\mathrm{T}}(t)\boldsymbol{\Sigma}^{-1}(t)\boldsymbol{\mu}(t)\right].\end{cases} \tag{43}$$

In Eq. (43), $\boldsymbol{\mu}(t)$ and $\boldsymbol{\Sigma}(t)$ are the expectation vector, and the covariance matrix of $(X(t), V(t))^{\mathrm{T}}$ given by Eqs. (38) and (39), respectively;

$$m(t) = \int_0^t (t-\tau)^{-\alpha}[x_0\dot{g}_1(\tau) + v_0\dot{g}(\tau)]\mathrm{d}\tau, \tag{44}$$

and

$$\boldsymbol{r}(t) = \begin{pmatrix}r_1(t)\\ r_2(t)\end{pmatrix} = \begin{pmatrix}\sigma_{\mathrm{W}}^2\int_0^t (t-\tau)^{-\alpha}\int_0^\tau \dot{g}(\tilde{\tau})g(t-\tau+\tilde{\tau})\mathrm{d}\tilde{\tau}\mathrm{d}\tau\\ \sigma_{\mathrm{W}}^2\int_0^t (t-\tau)^{-\alpha}\int_0^\tau \dot{g}(\tilde{\tau})\dot{g}(t-\tau+\tilde{\tau})\mathrm{d}\tilde{\tau}\mathrm{d}\tau\end{pmatrix}, \tag{45}$$

where $\dot{g}(\cdot)$ and $\dot{g}_1(\cdot)$ are the derivative of $g(\cdot)$ and $g_1(\cdot)$ given by Eqs. (36) and (37), respectively. Finally, one can rewrite the GE-GDEE (30) as

$$\frac{\partial p_{XV}(x,v,t)}{\partial t} = -v\frac{\partial p_{XV}(x,v,t)}{\partial x} - \frac{\partial\left[a_V^{(\text{eff})}(x,v,t)p_{XV}(x,v,t)\right]}{\partial v} \\ + \frac{\sigma_{\mathrm{W}}^2}{2}\frac{\partial^2 p_{XV}(x,v,t)}{\partial v^2}, \tag{46}$$

where the closed-form intrinsic drift coefficient $a_V^{(\text{eff})}(x,v,t)$ is given by Eq. (42). Note that an equivalent SDOF linear integer-order oscillator can be constructed. It is governed by the following equation of motion

$$\ddot{\tilde{X}}(t) + c^{(\text{eff})}(t)\dot{\tilde{X}}(t) + \left[k + k^{(\text{eff})}(t)\right]\tilde{X}(t) = \xi(t) - f^{(\text{eff})}(t), \tag{47}$$

with the same initial condition $\tilde{X}(0) = x_0$ and $\dot{\tilde{X}}(0) = v_0$ as Eq. (28), where $\tilde{X}(t)$, $\dot{\tilde{X}}(t)$, and $\ddot{\tilde{X}}(t)$ are the displacement, velocity, and acceleration responses, respectively; $k^{(\text{eff})}(t)$, $c^{(\text{eff})}(t)$, and $f^{(\text{eff})}(t)$ are given by Eq. (43) analytically. Clearly, transient joint PDFs of $\left(X(t), \dot{X}(t)\right)^{\mathrm{T}}$ and $\left(\tilde{X}(t), \dot{\tilde{X}}(t)\right)^{\mathrm{T}}$ are identical. The joint TPD of $\left(\tilde{X}(t), \dot{\tilde{X}}(t)\right)^{\mathrm{T}}$ during two different instants $t' < t$, denoted as

$p_{\tilde{X}\dot{\tilde{X}}}\left(x, v, t \middle| x', v', t'\right)$, satisfies the Fokker-Planck equation

$$\frac{\partial p_{\tilde{X}\tilde{V}}\left(x, v, t|x', v', t'\right)}{\partial t} = -v\frac{\partial p_{\tilde{X}\tilde{V}}\left(x, v, t|x', v', t'\right)}{\partial x} - \frac{\partial\left[a_V^{(\text{eff})}(x, v, t)p_{\tilde{X}\tilde{V}}\left(x, v, t|x', v', t'\right)\right]}{\partial v} + \frac{\sigma_W^2}{2}\frac{\partial^2 p_{\tilde{X}\tilde{V}}\left(x, v, t|x', v', t'\right)}{\partial v^2}, \tag{48}$$

which has the same form of PDE as GE-GDEE (46).

The preceding approach can be extended to multi-dimensional cases. Specifically, without loss of generality, consider an n -dimensional process vector $\boldsymbol{Y}(t) = (Y_1(t), \cdots, Y_n(t))^{\mathrm{T}}$ governed by the n -dimensional linear stochastic fractional differential equations

$$\mathrm{d}\boldsymbol{Y}(t) = -\left[\mathbf{C}\mathrm{D}^{\alpha}\boldsymbol{Y}(t) + \mathbf{K}\boldsymbol{Y}(t)\right]\mathrm{d}t + \mathbf{L}\mathrm{d}\boldsymbol{W}(t), \tag{49}$$

where $\mathbf{C} = \left[c_{ij}\right]_{n\times n}$ and $\mathbf{K} = \left[k_{ij}\right]_{n\times n}$ are $n\times n$ constant matrices; $\mathbf{L} = \left[l_{ij}\right]_{n\times r}$ is an $n\times r$ constant matrix, $r \le n$; $\mathrm{D}^{\alpha}\boldsymbol{Y}(t) = \left(\mathrm{D}^{\alpha}Y_1(t), \cdots, \mathrm{D}^{\alpha}Y_n(t)\right)^{\mathrm{T}}$ is the fractional derivative of $\boldsymbol{Y}(t)$ of order α, $0 < \alpha < 1$; $\boldsymbol{W}(t) = (W_1(t), \cdots, W_r(t))^{\mathrm{T}}$ are an r -dimensional standard Wiener process vector, i.e., $\boldsymbol{W}(t) \sim \mathcal{N}(\mathbf{0}_r, \mathbf{I}_r t)$, where $\mathbf{0}_r$ denotes an $r\times r$ matrix containing all elements as zeros and $\mathbf{I}_r$ denotes $r\times r$ identity matrix, and

$$\mathrm{E}\left[\boldsymbol{W}(t)\boldsymbol{W}^{\mathrm{T}}(t+\tau)\right] = \mathbf{I}_r t, \text{ for } \tau \ge 0. \tag{50}$$

The initial condition of $\boldsymbol{Y}(t)$ is prescribed as

$$\boldsymbol{Y}(0) = \boldsymbol{y}_0. \tag{51}$$

In general, the order of fractional derivative can be different for different dimensions of $\boldsymbol{Y}(t)$. However, to elucidate the basic concept clearly, an identical value of is employed herein for all components of $\boldsymbol{Y}(t)$. Note that even though in Eq. (49) α is considered within the range of $(0, 1)$, systems with the order of fractional derivative larger than 1, and even larger than the greatest order of the integer-order derivative involved, can be transferred into the form of Eq. (49) via dimension augmentation. For example, consider a one-dimensional system given by

$$\mathrm{D}^{\beta}X(t) + c\dot{X}(t) + kX(t) = \xi(t), \tag{52}$$

where $1 < \beta < 2$; and are constant coefficients; $\xi(t)$ denotes a stationary Gaussian white noise with $\mathrm{E}[\xi(t)\xi(t+\tau)] = \sigma_W^2\delta(\tau)$. Set $\boldsymbol{Y}(t) = [Y_1(t), Y_2(t)]^{\mathrm{T}} = \left[X(t), \mathrm{D}^{\beta-1}X(t)\right]^{\mathrm{T}}$, and $\alpha = 2-\beta$. Then, the governing differential equation of the two-dimensional stochastic process $\boldsymbol{Y}(t)$ can be cast as

$$\begin{bmatrix}\mathrm{d}Y_1(t)\\ \mathrm{d}Y_2(t)\end{bmatrix} = \begin{bmatrix}\mathrm{D}^{\alpha}Y_2(t)\\ -c\mathrm{D}^{\alpha}Y_2(t) - kY_1(t)\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0\\ \sigma_W\end{bmatrix}\mathrm{d}W(t). \tag{53}$$

If the transient PDF of one component of $\boldsymbol{Y}(t)$, denoted as $Y_l(t)$, $1 \le l \le n$, is of interest, then according to the unified formalism of the GE-GDEE [28], the GE-GDEE for $p_{Y_l}(y, t)$ is

$$\frac{\partial p_{Y_l}(y, t)}{\partial t} = -\frac{\partial\left[a^{(\text{eff})}(y, t)p_{Y_l}(y, t)\right]}{\partial y} + \frac{1}{2}\frac{\partial^2\left[b^{(\text{eff})}(y, t)p_{Y_l}(y, t)\right]}{\partial y^2}, \tag{54}$$

where the intrinsic drift and diffusion coefficients are defined as [28]

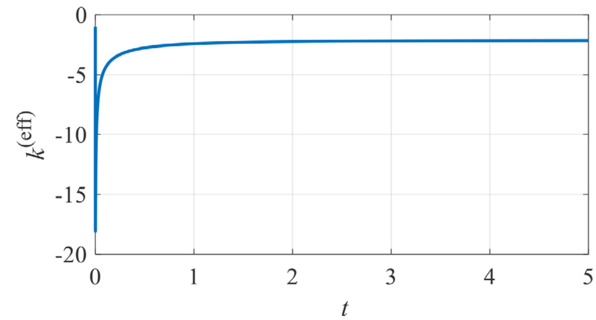


Fig. 1. Linear factor of the intrinsic drift coefficient in Example #1.

$$\begin{cases} a^{(\text{eff})}(y, t) = \lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathrm{E}[\Delta Y_l(t)|\ Y_l(t) = y],\\ b^{(\text{eff})}(y, t) = \lim\limits_{\Delta t\to 0}\frac{1}{\Delta t}\mathrm{E}\left\{[\Delta Y_l(t)]^2|\ Y_l(t) = y\right\},\end{cases} \tag{55}$$

where $\Delta Y_l(t) = Y_l(t+\Delta t) - Y_l(t)$ is the increment during time interval $[t, t+\Delta t]$. By substituting the lth component of Eq. (49), i.e.,

$$\mathrm{d}Y_l(t) = -\left[\boldsymbol{c}_{(l,\cdot)}\mathrm{D}^{\alpha}\boldsymbol{Y}(t) + \boldsymbol{k}_{(l,\cdot)}\boldsymbol{Y}(t)\right]\mathrm{d}t + \boldsymbol{l}_{(l,\cdot)}\mathrm{d}\boldsymbol{W}(t), \tag{56}$$

into Eq. (55), the second equation becomes

$$b^{(\text{eff})}(y, t) = \boldsymbol{l}_{(l,\cdot)}\boldsymbol{l}_{(l,\cdot)}^{\mathrm{T}} = b_{ll}, \tag{57}$$

while the first equation yields

$$a^{(\text{eff})}(y, t) = -\mathrm{E}\left[\boldsymbol{c}_{(l,\cdot)}\mathrm{D}^{\alpha}\boldsymbol{Y}(t) + \boldsymbol{k}_{(l,\cdot)}\boldsymbol{Y}(t)|\ Y_l(t) = y\right] = -\sum_{j=1}^{n}\left\{c_{lj}\mathrm{E}\left[\mathrm{D}^{\alpha}Y_j(t)|\ Y_l(t) = y\right] + k_{lj}\mathrm{E}\left[Y_j(t)|\ Y_l(t) = y\right]\right\}. \tag{58}$$

where $\boldsymbol{c}_{(l,\cdot)}$, $\boldsymbol{k}_{(l,\cdot)}$, and $\boldsymbol{l}_{(l,\cdot)}$ are the lth row vectors of $\mathbf{C}$, $\mathbf{K}$, and $\mathbf{L}$, respectively. To obtain the closed-form expression of Eq. (58), the Gaussian property and cross-covariance of $\boldsymbol{Y}(t)$ can be advocated. According to the property that the fractional derivative does not break the linear property of the system [11], $\boldsymbol{Y}(t)$ governed by Eq. (49) is still Gaussian. Denote the expectation vector and covariance matrix of $\boldsymbol{Y}(t)$ by

$$\boldsymbol{\mu}(t) = (\mu_1(t), \cdots, \mu_n(t))^{\mathrm{T}} = \mathrm{E}[\boldsymbol{Y}(t)], \tag{59}$$

and

$$\boldsymbol{\Sigma}(t) = \left[\varsigma_{ij}(t)\right]_{n\times n} = \mathrm{E}\left\{[\boldsymbol{Y}(t) - \boldsymbol{\mu}(t)][\boldsymbol{Y}(t) - \boldsymbol{\mu}(t)]^{\mathrm{T}}\right\}, \tag{60}$$

and denote the cross-covariance matrix of $\boldsymbol{Y}(t)$ by

$$\tilde{\boldsymbol{\Sigma}}(t_1, t_2) = \left[\tilde{\varsigma}_{ij}(t_1, t_2)\right]_{n\times n} = \mathrm{E}\left\{[\boldsymbol{Y}(t_1) - \boldsymbol{\mu}(t_1)][\boldsymbol{Y}(t_2) - \boldsymbol{\mu}(t_2)]^{\mathrm{T}}\right\}. \tag{61}$$

Then the conditional expectation of multivariate Gaussian distributions is

$$\mathrm{E}\left[Y_j(t)|\ Y_l(t) = y\right] = \mu_j(t) + \frac{[y - \mu_l(t)]\varsigma_{lj}(t)}{\sigma_l^2(t)}, \tag{62}$$

and

$$\begin{aligned}\mathrm{E}\left[\mathrm{D}^{\alpha}Y_j(t)|\ Y_l(t) = y\right] &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\mathrm{E}\left[\dot{Y}_j(\tau)|\ Y_l(t) = y\right]\mathrm{d}\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{E}\left[Y_j(\tau)|\ Y_l(t) = y\right]\mathrm{d}\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\mu_j(\tau) + \frac{\tilde{\varsigma}_{jl}(\tau, t)[y - \mu_l(t)]}{\sigma_l^2(t)}\right\}\mathrm{d}\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{\mu}_j(\tau)\mathrm{d}\tau + \frac{y - \mu_l(t)}{\sigma_l^2(t)\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial\tilde{\varsigma}_{jl}(\tau, t)}{\partial\tau}\mathrm{d}\tau\\ &= \mathrm{D}^{\alpha}\mu_j(t) + \frac{y - \mu_l(t)}{\sigma_l^2(t)}\mathrm{D}_1^{\alpha}\tilde{\varsigma}_{jl}(t, t),\end{aligned} \tag{63}$$

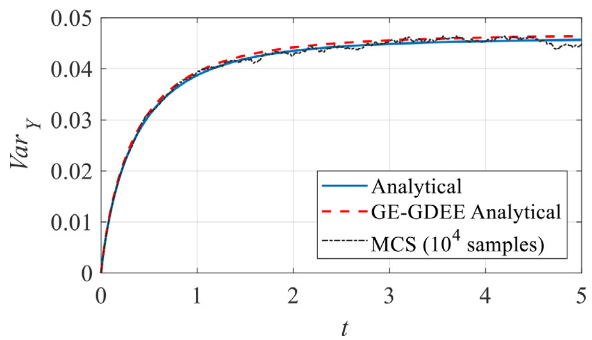


Fig. 2. Time histories of variance of $Y(t)$ in Example #1. (**a**) $Y(t)$; (**b**) $Y(t)$ (log-coordinate).

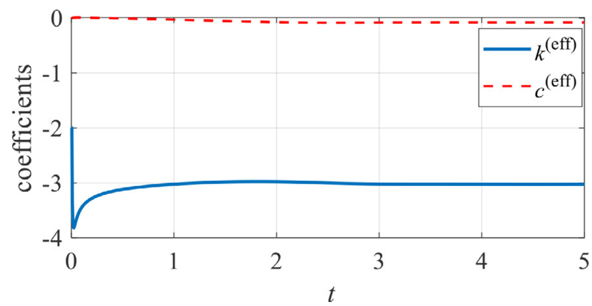


Fig. 4. Linear factors of the intrinsic drift coefficient in Example #2.

where $\sigma_l^2(t) = \varsigma_{ll}(t)$ is the variance of $Y_l(t)$, namely the (l, l)th element of $\mathbf{\Sigma}(t)$; $\mathrm{D}_1^\alpha f(\cdot,\cdot)$ is the αth order fractional partial derivative of the function $f(\cdot,\cdot)$ with respect to the first argument [44], $0 < \alpha < 1$, namely,

$$\mathrm{D}_1^\alpha f(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} \frac{\partial f(\xi,y)}{\partial \xi} \mathrm{d}\xi . \tag{64}$$

Substituting Eqs. (62) and (63) into Eq. (58) yields

$$\begin{aligned} a^{(\mathrm{eff})}(y,t) &= -\sum_{j=1}^{n} \left\{ c_{lj} \left[\mathrm{D}^\alpha \mu_j(t) + \frac{y-\mu_l(t)}{\sigma_l^2(t)} \mathrm{D}_1^\alpha r_{jl}(t,t) \right] + k_{lj} \left\{ \mu_j(t) + \frac{[y-\mu_l(t)]\varsigma_{lj}(t)}{\sigma_l^2(t)} \right\} \right\} \\ &= -\boldsymbol{c}_{(l,\cdot)} \left[\mathrm{D}^\alpha \boldsymbol{\mu}(t) + \frac{y-\mu_l(t)}{\sigma_l^2(t)} \mathrm{D}_1^\alpha \tilde{\boldsymbol{\varsigma}}_{(\cdot,l)}(t,t) \right] - \boldsymbol{k}_{(l,\cdot)} \left[\boldsymbol{\mu}(t) + \frac{y-\mu_l(t)}{\sigma_l^2(t)} \boldsymbol{\varsigma}_{(\cdot,l)}(t) \right] \\ &= -\frac{\mathbf{y}-\boldsymbol{\mu}_1(\mathbf{t})}{\boldsymbol{\sigma}_1^2(\mathbf{t})} \left[\mathbf{c}_{(1,\cdot)} \mathrm{D}_1^\alpha \tilde{\boldsymbol{\varsigma}}_{(\cdot,1)}(\mathbf{t},\mathbf{t}) + \mathbf{k}_{(1,\cdot)} \boldsymbol{\varsigma}_{(\cdot,1)}(\mathbf{t}) \right] - \mathbf{c}_{(1,\cdot)} \mathrm{D}^\alpha \boldsymbol{\mu}(\mathbf{t}) - \boldsymbol{k}_{(1,\cdot)} \boldsymbol{\mu}(\mathbf{t}). \end{aligned} \tag{65}$$

where $\boldsymbol{\varsigma}_{(\cdot,l)}(t)$ is the lth column vector of $\mathbf{\Sigma}(t)$; $\mathrm{D}_1^\alpha \tilde{\boldsymbol{\xi}}_{(\cdot,l)}(t,t)$ is the αth order fractional partial derivative of the lth column vectors of $\tilde{\mathbf{\Sigma}}(t_1,t_2)$ with respect to t_1 at $t_1 = t$ and $t_2 = t$. It can be seen that $a^{(\mathrm{eff})}(y,t)$ is also a linear function of y. Note that a one-dimensional Markov diffusion process $\tilde{Y}(t)$ can be constructed to satisfy the Itô SDE

$$\tilde{Y}(t) = a^{(\mathrm{eff})}\left[\tilde{Y}(t), t\right]\mathrm{d}t + \sqrt{b_{ll}}\mathrm{d}W(t), \tag{66}$$

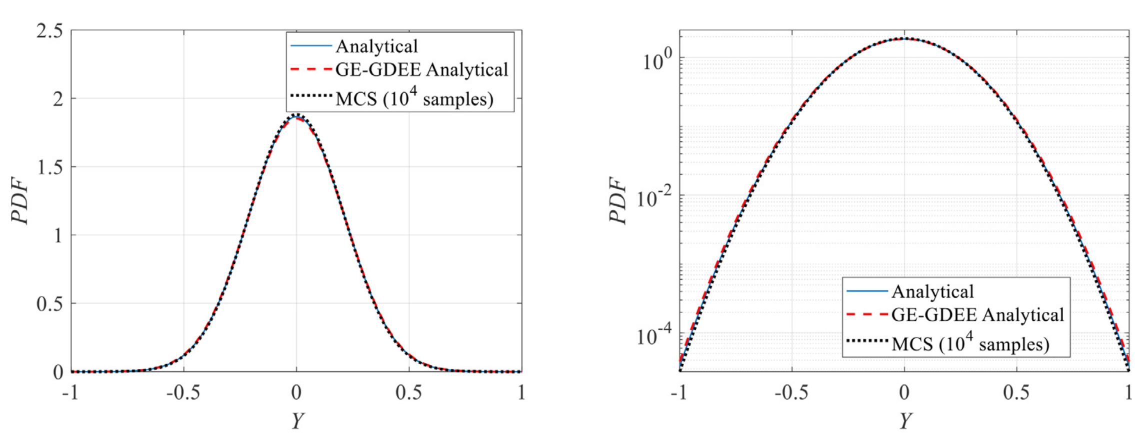
with the same initial condition $\tilde{Y}(0) = y_{0,l}$ as that of $Y_l(t)$, where $a^{(\mathrm{eff})}(\cdot)$ is given by Eq. (54) as a linear function with respect to y. Clearly, the TPD of $\tilde{Y}(t)$ during two different instants $t' < t$, denoted as $p_{\tilde{Y}}\left(y,t\middle|y',t'\right)$, satisfies the Fokker-Planck equation

$$\frac{\partial p_{\tilde{Y}}\left(y,t|y',t'\right)}{\partial t} = -\frac{\partial\left[a^{(\mathrm{eff})}(y,t)p_{\tilde{Y}}\left(y,t|y',t'\right)\right]}{\partial y} + \frac{b_{ll}}{2}\frac{\partial^2 p_{\tilde{Y}}\left(y,t|y',t'\right)}{\partial y^2}, \tag{67}$$

which is in the same form of PDE as GE-GDEE (54). Clearly, the transient PDF of the Markov diffusion process $\tilde{Y}(t)$ is identical to that of the non-Markovian process $Y_l(t)$. The preceding claim can be readily made when more than one component of $\boldsymbol{Y}(t)$ are of interest. To illustrate this point, an example in which two components of a multi-dimensional system are of concern simultaneously is given in the next paragraph.

Example #1: A one-dimensional linear stochastic fractional differential system Consider the one-dimensional linear stochastic fractional differential Eq. (1), where $\alpha = 0.5$, $k = 1$, $c = 1$, $y_0 = 0$, and $\sigma_W^2 = 0.2$. The transient PDF of process $Y(t)$ governed by Eq. (1) satisfies GE-GDEE (4), and the intrinsic drift coefficient, which is a linear function with respect to the state quantity y, can be obtained by Eq. (17). It can be seen that in Eq. (17), the constant coefficient $f^{(\mathrm{eff})}(t) = 0$, and the linear coefficient $k^{(\mathrm{eff})}(t)$ is shown in Fig. 1.

After obtaining the intrinsic drift coefficient, the GE-GDEE (4) can be used, and the alternative linear system (24) can be obtained. The analytical solution of the time-varying variance of the integer-order linear system subject to Gaussian white noise is available (See Appendix). Figure 2 shows a comparison of the time histories of the variance given by the analytical solution (Eq. (13)), the GE-GDEE scheme and the variance of system (24), and a pertinent Monte Carlo simulation (MCS) with 10^4 samples, respectively. The sample paths in MCS are obtained by a modified stochastic Runge-Kutta algorithm [29]. Since the response PDF is Gaussian, the transient PDF solution of $Y(t)$ can be readily obtained. A comparison of the transient PDFs at $t = 5$ is shown in Fig. 3. The con-



(a) $Y(t)$; (b) $Y(t)$ (log-coordinate);

Fig. 3. Transient PDFs of $Y(t)$ at $t = 5$ in Example #1.

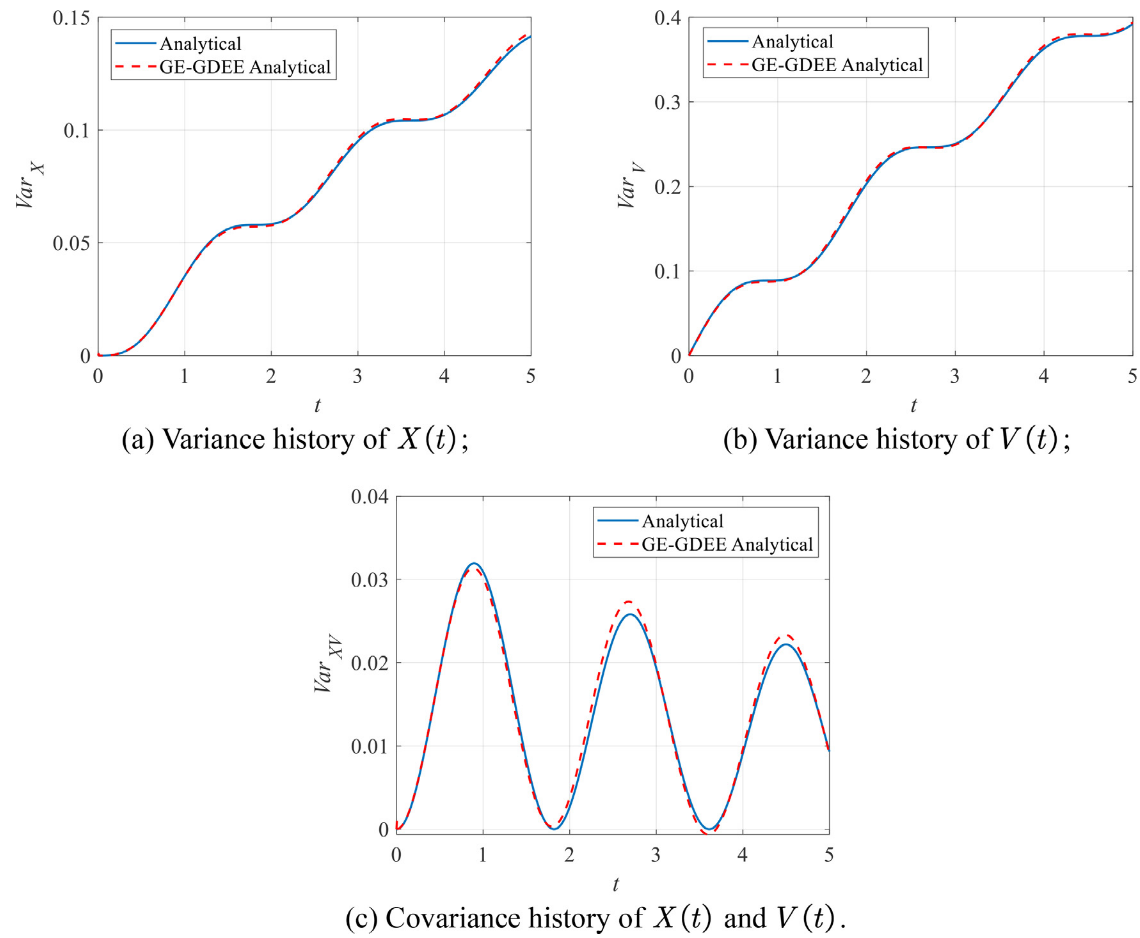


Fig. 5. Variance and Covariance histories of $X(t)$ and $V(t)$ in Example #2. (**a**) Variance history of $X(t)$; (**b**) Variance history of $V(t)$;(**c**) Covariance history of $X(t)$ and $V(t)$.

sistency of the results in Figs. 2 and 3 verifies the assertion in previous dicussuion. The minor inconsistency between the analytical solution and the GE-GDEE results is primarily due to the numerical integration.

Example #2 An SDOF linear fractional differential oscillator enforced by Gaussian white noise Consider an SDOF linear fractional differential oscillator subject to Gaussian white noise. Its motion is describe by Eq. (26), with $\alpha = 0.1$, $k = 2$, $c = 1$, $x_0 = 0$, $v_0 = 0$, and $\sigma_W^2 = 0.2$. The transient joint PDF of displacement and velocity, i.e., $X(t)$ and $V(t)$, satisfies GE-GDEE (46), and the intrinsic drift coefficient, which is a linear function with respect to state quantity and, can be obtained by Eq. (42). In Eq. (42), the coefficient $f^{(\mathrm{eff})}(t) = 0$, and the linear factors $k^{(\mathrm{eff})}(t)$ and $c^{(\mathrm{eff})}(t)$ are shown in Fig. 4.

Then, by substituting the closed-form expression of the intrinsic drift coefficient into GE-GDEE (46) and the alternative linear system (47), the time histories of the response covariances and the transient PDFs are obtained. The time histories of the covariances obtained by the GE-GDEE scheme and the analytical solution by Eq. (39) are shown in Fig. 5. The good agreement of the results in Fig. 5 confirms the conclusions drawn.

Example #3 A multi-degree-of-freedom (MDOF) linear fractional differential oscillator subject to Gaussian white noise Consider a 12-DOF linear fractional differential oscillator subject to Gaussian white noise. Its equation of motion is

$$\ddot{\boldsymbol{X}}(t) + \tilde{\mathbf{C}}\dot{\boldsymbol{X}}(t) + \tilde{\mathbf{C}}_\beta \mathrm{D}^\beta \boldsymbol{X}(t) + \tilde{\mathbf{K}}\boldsymbol{X}(t) = \tilde{\mathbf{L}}\xi(t), \tag{68}$$

where $\boldsymbol{X}(t)$, $\dot{\boldsymbol{X}}(t)$, and $\ddot{\boldsymbol{X}}(t)$ are the displacement, velocity, and acceleration vectors, respectively; $\beta = 0.6$; and

$$\tilde{\mathbf{C}} = \begin{pmatrix} 0.7902 & -0.3804 & & \\ -0.3804 & \ddots & \ddots & \\ & \ddots & 0.7902 & -0.3804 \\ & & -0.3804 & 0.4099 \end{pmatrix},$$

$$\tilde{\mathbf{C}}_\beta = \begin{pmatrix} 4 & -2 & & \\ -2 & \ddots & \ddots & \\ & \ddots & 4 & -2 \\ & & -2 & 2 \end{pmatrix}, \quad \tilde{\mathbf{K}} = \begin{pmatrix} 4 & -2 & & \\ -2 & \ddots & \ddots & \\ & \ddots & 4 & -2 \\ & & -2 & 2 \end{pmatrix}. \tag{69}$$

$\tilde{\mathbf{L}} = (1, \cdots, 1)^{\mathrm{T}}$; and $\xi(t)$ is a one-dimensional white noise process with intensity $\sigma_W^2 = 0.04$. The initial values of displacement and velocity take $\boldsymbol{x}_0 = (0, \cdots, 0)^{\mathrm{T}}$ and $\boldsymbol{v}_0 = (0, \cdots, 0)^{\mathrm{T}}$. Introducing the velocity process vector $\boldsymbol{V}(t) = \dot{\boldsymbol{X}}(t)$ and the state vector $\boldsymbol{Y}(t) = \left(\boldsymbol{X}^{\mathrm{T}}(t), \boldsymbol{V}^{\mathrm{T}}(t)\right)^{\mathrm{T}}$, Eq. (68) can be recast as the stochastic differential Eq. (49). Next consider the transient joint PDF of the displacement and velocity of the lth DOF, denoted as $p_{X_l V_l}(x, v, t)$, for arbitrary $1 \le l \le 12$. Following the derivation of previous dicussion, it is found that $p_{X_l V_l}(x, v, t)$ satisfies the following GE-GDEE

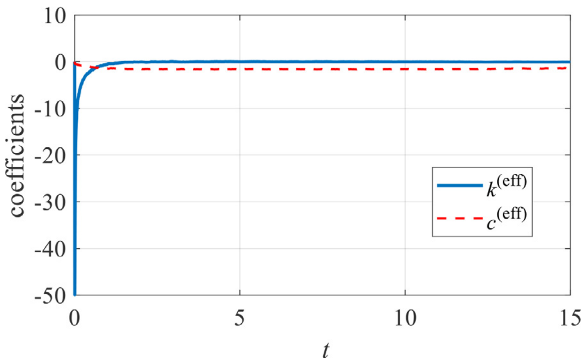


Fig. 6. Linear factors of the intrinsic drift coefficient in Example #3.

$$\frac{\partial p_{X_lV_l}(x,v,t)}{\partial t} = -v\frac{\partial p_{X_lV_l}(x,v,t)}{\partial x} - \frac{\partial\left[a_V^{(\mathrm{eff})}(x,v,t)p_{X_lV_l}(x,v,t)\right]}{\partial v} + \frac{\sigma_W^2}{2}\frac{\partial^2 p_{X_lV_l}(x,v,t)}{\partial v^2}, \tag{70}$$

where the intrinsic drift coefficient is

$$a_V^{(\mathrm{eff})}(x,v,t) = k^{(\mathrm{eff})}(t)x + c^{(\mathrm{eff})}(t)v, \tag{71}$$

in which the time-variant linear factors are given by

$$\left(k^{(\mathrm{eff})}(t), c^{(\mathrm{eff})}(t)\right) = \Big[\tilde{\mathbf{c}}_{(l,\cdot)}\big(\varsigma_{VX_l}(t), \varsigma_{VV_l}(t)\big) + \tilde{\mathbf{c}}_{\beta,(l,\cdot)}\mathrm{D}_1^{\beta}\big(\tilde{\varsigma}_{XX_l}(t,t), \tilde{\varsigma}_{XV_l}(t,t)\big) + \tilde{\mathbf{k}}_{(l,\cdot)}\big(\varsigma_{XX_l}(t), \varsigma_{XV_l}(t)\big)\Big]\mathbf{\Sigma}_{X_lV_l}^{-1}. \tag{72}$$

The symbol $\mathbf{\Sigma}_{X_lV_l}$ denotes the 2×2 covariance matrix of $X_l(t)$ and $V_l(t)$; $\varsigma_{XX_l}(t)$, $\varsigma_{XV_l}(t)$, $\varsigma_{VX_l}(t)$, and $\varsigma_{VV_l}(t)$ are the 12-dimensional covariance column vectors of the corresponding quantities denoted by their respective subscripts; $\tilde{\varsigma}_{XX_l}(t_1,t_2)$ and $\tilde{\varsigma}_{XV_l}(t_1,t_2)$ are the corresponding 12-dimensional cross-covariance column vectors; $\tilde{\mathbf{c}}_{(l,\cdot)}$, $\tilde{\mathbf{c}}_{\beta,(l,\cdot)}$, and $\tilde{\mathbf{k}}_{(l,\cdot)}$ are the lth row vectors of $\tilde{\mathbf{C}}$, $\tilde{\mathbf{C}}_\beta$, and $\tilde{\mathbf{K}}$, respectively. In this example, $l=1$. Note that it is quite cumbersome to analytically derive the covariance functions used in Eq. (72). Hence, the linear coefficients $k^{(\mathrm{eff})}(t)$ and $c^{(\mathrm{eff})}(t)$ in the intrinsic drift coefficient, given by Eq. (72), are determined numerically by a least-square algorithm based on the data from 200 samples via dynamical analyses of Eq. (26). The identified results of $k^{(\mathrm{eff})}(t)$ and $c^{(\mathrm{eff})}(t)$ are shown in Fig. 6.

Then, substituting the closed-form expression of the intrinsic drift coefficient in Eq. (71) with the identified linear factors into GE-GDEE (70), the transient PDF solutions of $X_l(t)$ and $V_l(t)$ can be captured. GE-GDEE (70) can be solved as a general two-dimensional PDE, such as path integral solution (PIS) [31]. The transient PDFs at $t=15$ are shown in Fig. 7. The covariance histories of $X_l(t)$ and $V_l(t)$ can also be obtained by relying on the transient PDF solution, which is shown in Fig. 8. Figures 7 and 8 also comprise the MCS results of transient PDF solutions and variance

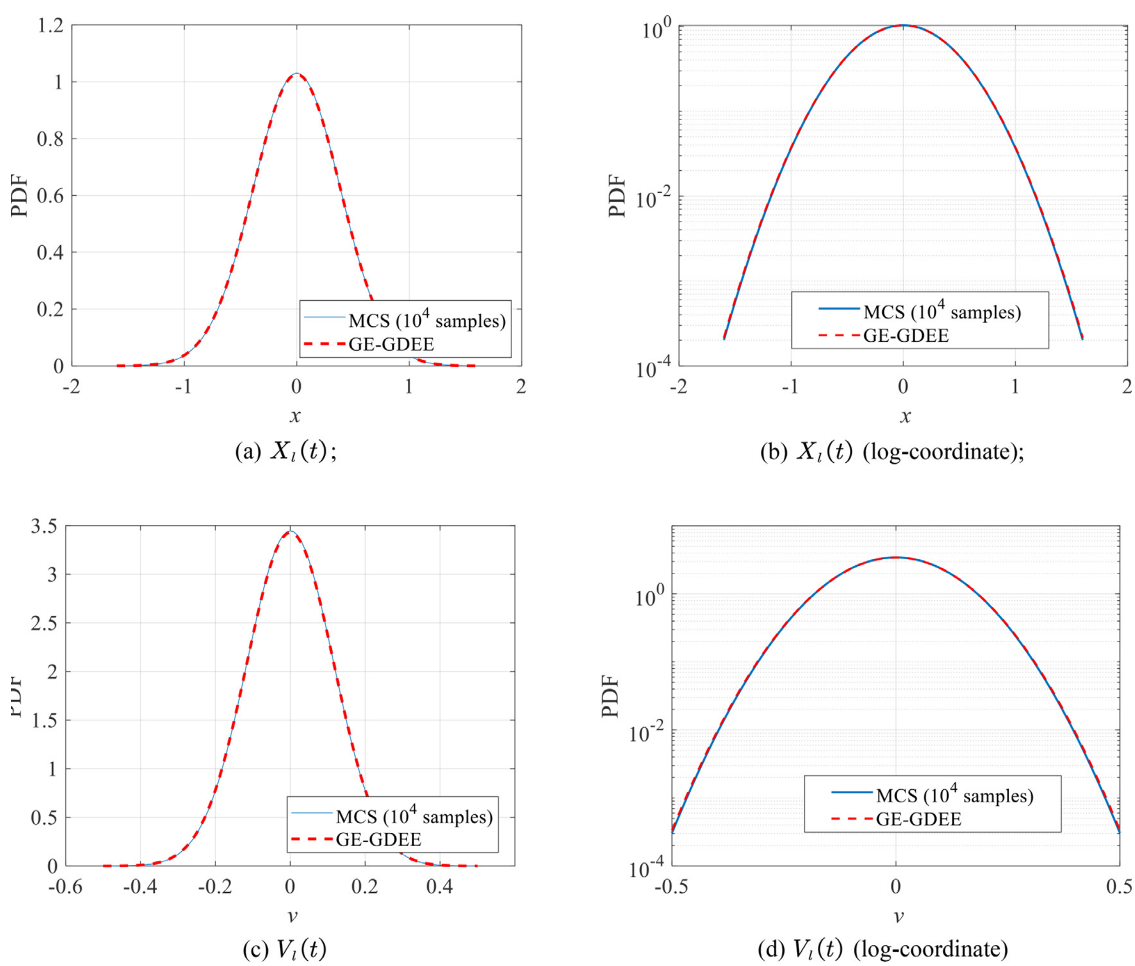


Fig. 7. Transient PDFs of $X_l(t)$ and $V_l(t)$ at $t=15$ in Example #3. (**a**) $X_l(t)$; (**b**) $X_l(t)$ (log-coordinate); (**c**) $V_l(t)$; (**d**) $V_l(t)$ (log-coordinate).

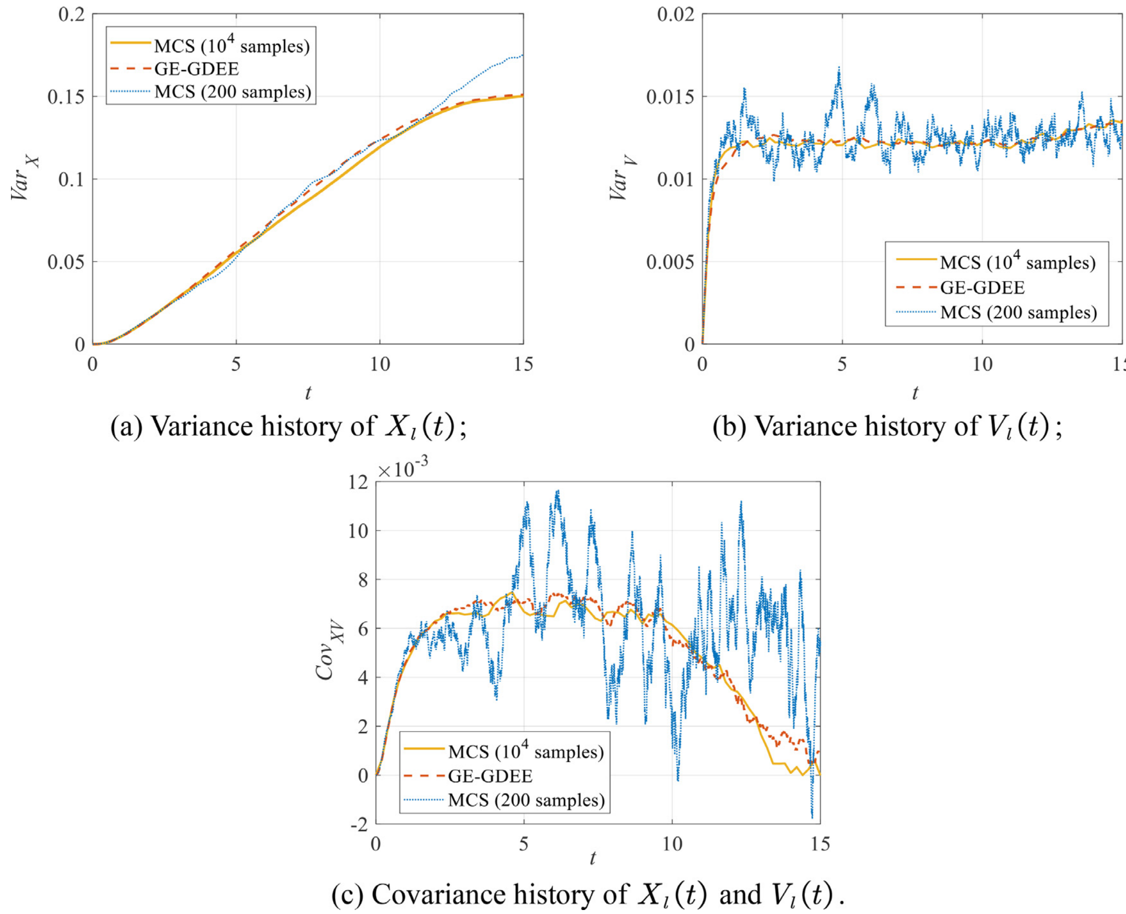


Fig. 8. Variance and Covariance histories of $X_l(t)$ and $V_l(t)$ in Example #3. (**a**) Variance history of $X_l(t)$; (**b**) Variance history of $V_l(t)$; (**c**) Covariance history of $X_l(t)$ and $V_l(t)$.

histories, respectively, as comparison. The consistency of the results in Figs. 7 and 8 supports the conclusions drawn in previous discussion.

It is also seen from Fig. 8 that the response statistics obtained by the GE-GDEE with intrinsic drift coefficient identified based on the data from 200 samples are quite more accurate than the MCS results directly estimated from the 200 samples. This indicates that the GE-GDEE has much higher accuracy and efficiency when using the same number of deterministic analyses. The detailed discussion on this numerical superiority can be found in Refs. [28,31].

In the paper, an exact low-dimensional PDE governing the transient PDF of any quantity of interest in a multi-dimensional linear fractional differential system subject to Gaussian white noise has been derived relying on the formalism of the GE-GDEE. The analytical expression of the intrinsic drift coefficient in the GE-GDEE has been discussed. In this context, the following conclusions may be drawn.

(1) If a process is governed by a one-dimensional linear stochastic fractional differential equation, or is a component of a vector process governed by a multi-dimensional linear stochastic fractional differential equation, its transient PDF satisfies a dimension-reduced GE-GDEE with analytical intrinsic drift coefficient. The intrinsic drift coefficient is a time-variant linear function of the process considered.

(2) For multi-dimensional linear fractional differential systems subject to Gaussian white noise, by estimating the inherently-linear intrinsic drift coefficient in the GE-GDEE, the stochastic response can be determined with high accuracy and efficiency. Specifically, the accuracy and robustness of the results of the GE-GDEE are much superior to that of the MCS based on the same sample data, and the accuracy of at least the order of magnitude of 10^{-4} can be achieved in the tail of the PDF.

Though the above investigation is limited to linear fractional differential systems subject to additive Gaussian white noise, it provides a set of remarkable rigorously proved examples on the existence and eligibility of GE-GDEE for the case that the original high-dimensional system itself is non-Markovian, and an informative demonstration for the physical-mechanism-informed determination of the intrinsic drift coefficient of GE-GDEE, and provides insights for tackling more complex nonlinear systems. The GE-GDEE and intrinsic drift coefficient for linear fractional differential systems subject to other stochastic excitations, such as multiplicative noise, colored noise or Poisson white noise, are worth studying in the future. Further, the ideas can be extended to the nonlinear fractional differential systems.

Data Availability Statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

Declaration of Competing Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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Appendix

Consider a n-dimensional linear integer-order differential system subject to Gaussian white noise governed by the equation

$$\mathrm{d}\boldsymbol{Y}(t) = \mathbf{A}(t)\boldsymbol{Y}(t)\mathrm{d}t + \mathbf{L}\mathrm{d}\boldsymbol{W}(t), \tag{A1}$$

where $\mathbf{A}(t)$ is an $n \times n$ time-varying matrix; $\mathbf{L}$ is an $n \times r$ matrix; and $\boldsymbol{W}(t)$ is an r -dimensional standard Wiener process vector, i.e., $\boldsymbol{W}(t) \sim \mathcal{N}(0_r, \mathbf{I}_r t)$, and $\mathrm{E}\left[\boldsymbol{W}(t)\boldsymbol{W}^{\mathrm{T}}(t+\tau)\right] = \mathbf{I}_r t$. The initial value of $\boldsymbol{Y}(t)$ satisfies

$$\boldsymbol{Y}(0) \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0). \tag{A2}$$

Then, the transient mean vector $\boldsymbol{\mu}(t)$ and covariance matrix $\boldsymbol{\Sigma}(t)$ of process $\boldsymbol{Y}(t)$ can be determined by the equation

$$\boldsymbol{\mu}(t) = \mathrm{E}[\boldsymbol{Y}(t)] = \exp\left(\int_0^{\mathrm{t}} \mathbf{A}(\tau)\mathrm{d}\tau\right)\boldsymbol{\mu}_0, \tag{A3}$$

and

$$\begin{aligned}\boldsymbol{\Sigma}(t) &= \mathrm{E}\left[\boldsymbol{Y}(t)\boldsymbol{Y}^{\mathrm{T}}(t)\right] - \boldsymbol{\mu}(t)\boldsymbol{\mu}^{\mathrm{T}}(t) \\ &= \exp\left(\int_0^{\mathrm{t}} \mathbf{A}(\tau)\mathrm{d}\tau\right)\boldsymbol{\Sigma}_0 \exp\left(\int_0^{\mathrm{t}} \mathbf{A}^{\mathrm{T}}(\tau)\mathrm{d}\tau\right) \\ &\quad + \int_0^t \exp\left(\int_\tau^{\mathrm{t}} \mathbf{A}(\mathrm{u})\mathrm{d}\mathrm{u}\right)\mathbf{L}\mathbf{L}^{\mathrm{T}} \exp\left(\int_\tau^{\mathrm{t}} \mathbf{A}^{\mathrm{T}}(\mathrm{u})\mathrm{d}\mathrm{u}\right)\mathrm{d}\tau.\end{aligned} \tag{A4}$$

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