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Nicolai maps in supersymmetric Yang–Mills theories

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Resources

In the course of the PhD, the following papers have been published. This thesis reuses material presented in these peer-reviewed works.

(A) O. Lechtenfeld and **M. Rupprecht**,
Universal form of the Nicolai map,
Phys. Rev. D **104** (2021) L021701 [arXiv:2104.00012 [hep-th]].

(B) O. Lechtenfeld and **M. Rupprecht**,
Construction method for the Nicolai map in supersymmetric Yang–Mills theories, *Phys. Lett. B* **819** (2021) 136413 [arXiv:2104.09654 [hep-th]].

(C) **M. Rupprecht**,
The coupling flow of $\mathcal{N}=4$ super Yang–Mills theory,
JHEP **04** (2022) 004 [arXiv:2111.13223 [hep-th]].

(D) O. Lechtenfeld and **M. Rupprecht**,
Is the Nicolai map unique?,
JHEP **09** (2022) 139 [arXiv:2207.09471 [hep-th]].

(E) O. Lechtenfeld and **M. Rupprecht**,
An improved Nicolai map for super Yang–Mills theory,
Phys. Lett. B **838** (2023) 137681 [arXiv:2211.07660 [hep-th]].

In particular, there is substantial overlap of

- Chapter 2 with A and D,
- Chapter 3 with D,
- Chapter 4 with B and E,
- Chapter 5 with C.

Zusammenfassung

Diese Dissertation präsentiert neue Ergebnisse für den „Nicolai map" Formalismus. Integriert man alle fermionischen Variablen einer beliebigen supersymmetrischen Feldtheorie aus, erhält man eine nichtlokale Theorie der verbleibenden bosonischen Felder. 1979 bewies Hermann Nicolai die Existenz einer nichtlokalen und nichtlinearen Transformation der bosonischen Felder, die es ermöglicht, Quantenkorrelatoren nur mit einem freien, rein bosonischen Funktionsmaß zu berechnen. Sie stimmen mit den bosonischen Korrelatoren der ursprünglichen supersymmetrischen Feldtheorie überein und somit ermöglicht der Formalismus ein völlig anderes Verständnis der Supersymmetrie. In dieser klassischen Konstruktion werden viele Informationen über Quantenkorrelatoren in die Konstruktion der Nicolai-Abbildung verlagert.

Ein zentrales Ergebnis dieser Arbeit ist eine neue universelle Formel für die Nicolai-Abbildung in Form eines pfadgeordneten Exponentials des sogenannten „coupling flow"-Differentialoperators. Daraus kann die Abbildung störungstheoretisch konstruiert werden, wenn die Supersymmetrie „off-shell" realisiert ist. Außerdem kann dies in Eichtheorien für beliebige Eichungen erreicht werden.

Ein zweites wiederkehrendes Thema ist die weitgehende Mehrdeutigkeit der Nicolai-Abbildung. Neben der bekannten Eichabhängigkeit gibt es mindestens zwei weitere Mehrdeutigkeiten: Eine Integrationspfad-Abhängigkeit für Theorien mit mehr als einer Kopplung und eine R-Symmetrie-Ambiguität für Theorien mit erweiterter Supersymmetrie. Trotz dieser Mehrdeutigkeiten sind die Korrelatoren, die mit der Nicolai-Abbildung erhalten werden, immer eindeutig. Dies ermöglicht eine Feinabstimmung der Abbildung, um möglichst einfach, physikalische Observablen zu bestimmen. Eine besonders nützliche Beobachtung ist, dass die Einbeziehung topologischer Terme in die Wirkung als Hinzufügen einer speziellen Art von Kopplung interpretiert werden kann, die den Formalismus erheblich vereinfacht.

In dieser Arbeit untersuchen wir, beginnend mit einem allgemeinen Überblick, als Nächstes die Nicolai-Abbildung in der supersymmetrischen Quantenmechanik, da sie uns erlaubt, viele Eigenschaften in einem relativ einfachen „Spielzeug"-Modell zu verstehen. Anschließend befassen wir uns mit $\mathcal{N}=1$ supersymmetrischen Yang–Mills-Theorien, vorwiegend in vier Raumzeitdimensionen. Durch die Kombination der Landau-Eichung mit der Feinabstimmung eines topologischen Terms entwickeln wir eine im Vergleich zu früheren Konstruktionen wesentlich einfachere Entwicklung der Nicolai-Abbildung. Schließlich wenden wir uns der $\mathcal{N}=4$ supersymmetrischen Yang–Mills-Theorie zu und entwickeln ein Verständnis für ihren R-kovarianten Kopplungsflussoperator.

Schlagwörter: Quantenfeldtheorie; Supersymmetrie; supersymmetrische Eichtheorien; Nicolai Abbildung.

Abstract

This dissertation presents new results for the Nicolai map formalism. Integrating out all the fermionic variables of any supersymmetric field theory, one obtains a non-local theory of the remaining bosonic fields. In 1979, Hermann Nicolai proved the existence of a nonlocal and nonlinear transformation of the bosonic fields, that enables the evaluation of quantum correlators using only a free, purely bosonic functional measure. They agree with the bosonic correlators of the original supersymmetric field theory and thus the formalism provides an entirely distinct understanding of supersymmetry. In this classical construction, a lot of information about quantum correlators is shifted to the construction of the Nicolai map.

A central result of this work is a new universal formula for the Nicolai map in terms of a path-ordered exponential of the so-called coupling flow differential operator. From this, the map can be constructed perturbatively whenever supersymmetry is realized off-shell. Moreover, in gauge theories, this can be achieved in any gauge.

A second recurrent theme is the broad ambiguity of the Nicolai map. Next to the known gauge dependence, there are at least two more ambiguities: An integration-path dependence for theories with multiple couplings and an R-symmetry ambiguity for theories with extended supersymmetry. Despite any ambiguities, correlators obtained by the Nicolai map formalism are always unique. This allows one to fine-tune the map for finding the most simple avenues towards physical observables. A particularly useful observation is that incorporating topological terms in the action can be interpreted as adding a special kind of coupling that provides significant simplifications in the formalism.

In this thesis, starting with a general overview, we next study the Nicolai map in supersymmetric quantum mechanics, as it allows us to understand many properties in a relatively simple 'toy' model. We then move on to $\mathcal{N}=1$ supersymmetric Yang–Mills theories, predominantly in four space-time dimensions. Combining the Landau gauge with the fine-tuning of a topological term, we develop a much simpler Nicolai map expansion compared to previous constructions. Lastly, we turn to $\mathcal{N}=4$ supersymmetric Yang–Mills theory and develop a framework for its R-covariant coupling flow operator.

Keywords: Quantum field theory; supersymmetry; supersymmetric gauge theories; Nicolai map.

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Contents

To my parents, Sabine and Thomas.

Chapter 1

Introduction

Fundamental particle physics is best understood in the language of quantum field theory. In fact, the standard model of particle physics is a particular quantum field theory, namely a $SU(3) \times SU(2) \times U(1)$ Yang–Mills theory. While this model has achieved tremendous empirical success, it is known to be incomplete. From a theoretical viewpoint, the most obvious shortcoming is its failure to incorporate gravity. Moreover, it describes only around 5% of the matter in our universe, ignoring dark energy and dark matter. Further, it contains 19 free parameters that have to be fine-tuned carefully in order to make the theory match with observations. Other issues include the hierarchy problem (related to the already mentioned problem of fine-tuning) or the unification of the three gauge couplings at high energies. Lastly, quite recently the muon $g-2$ collaboration [1] has revealed experimental deviations of the standard model with a high precision measurement of the muon's magnetic moment.

One possible tool to fix a number of these outstanding problems is the framework of supersymmetry (SUSY), which was first discovered in 1971 by Golfand and Likhtman [2]. It relates two fundamental types of particles, bosons (integer spin) and fermions (half-integer spin). More concretely, it states that there is a transformation of the bosonic fields into fermionic fields and vice versa, such that the theory-defining action is invariant.[1] One of the most promising candidates for a theory of everything are (super)string theories, which require SUSY as an essential ingredient. The concrete ways in which SUSY might be able to cure many problems of modern particle physics will not be discussed in this thesis. For this, see for example the review articles [3–5] or any standard textbook on supersymmetry instead. Despite its theoretical beauty and problem-solving potential, there have to date been no experimental observations confirming SUSY. In fact, in 2012 most simple supersymmetric extensions of the standard model were ruled out by experiments at the LHC. Nevertheless, superparticles could have very high or near-degenerate energies, that are not yet accessible to experiments. It is hoped that future colliders (such as the Chinese CEPC or the proposed FCC at CERN) can address this question. In any case, next to potential contributions to our understanding of fundamental physics, SUSY has many applications in other branches of physics and pure mathematics. Additionally, there is a multitude of non-phenomenological motivations for studying supersymmetry. For example, it can act as a theoretical toy model for strongly coupled

[1] In case that there is even more than one such transformation, one speaks of extended SUSY.

gauge dynamics. Due to powerful constraints from SUSY, the strong coupling regimes of non-abelian SUSY gauge theories often become accessible analytically. This allows for insights into the microscopic mechanisms behind e.g., confinement, transport properties, or chiral symmetry breaking.[2] These phenomena occur in the (non-supersymmetric) theory of the strong force, quantum chromodynamics (QCD), and can so far not be understood in that framework directly. Other important theoretical appeals of SUSY are related to SUSY localization, superconformal field theories, the AdS/CFT correspondence, and more.

This work investigates the Nicolai map, an approach to SUSY that so far has not received widespread attention. It is able to extract physical information of supersymmetric field theories in a completely distinct fashion compared to more traditional methods.

1.1 History

The Nicolai map is named after its discoverer Hermann Nicolai, who in 1979 proved the existence [6,7] of this special transformation of the bosonic fields in a wide range of supersymmetric field theories. The transformation has the remarkable properties that the full bosonic action is mapped to the free one and that its Jacobi determinant equals the nonlocal part of the action. Almost all of the scientific pursuit of the Nicolai map falls into one of two time periods. The first one spans its discovery in 1979 to around 1985 when the first superstring revolution took over. The second, modern time period started in 2020 and is currently ongoing.

The main developments in the first period came from Nicolai, Dietz, Flume, and Lechtenfeld [8–13] focusing on investigating the infinitesimal version of the Nicolai map, the so-called coupling flow. For the rare cases where stochastic variables exist, non-perturbative results were found in the works [14–19]. A pedagogical introduction is given by Ezawa and Klauder [20]. For a broader overview of this initial time period see Nicolai's review [21], where he demonstrates important results for supersymmetric quantum mechanics, the $\mathcal{N}=1$ Wess-Zumino model and four-dimensional $\mathcal{N}=1$ super Yang–Mills theory (SYM). Another key work is Lechtenfeld's Ph.D. thesis (in German) [11], where he develops a systematic way to construct the Nicolai map.

The second time period was induced by the three papers [22–24], collaborations of Nicolai himself and authors Ananth, Panday, Pant, Plefka, Lechtenfeld, and Malcha. They generalize the results for $\mathcal{N}=1$ SYM to all dimensions $D=3,4,6,10$ for which interacting SYM can exist [25]. This spawned a renewed interest in the approach and has led to many further developments [26–35]. Discoveries that are central to this thesis are the non-uniqueness of the map [26,31,33,35], a universal formula in terms of a path-ordered exponential [27], a general construction scheme for the map in SYM theories for arbitrary gauges [28,29] and a study of $\mathcal{N}=4$ SYM [31]. These will be outlined in Section 1.3. Other notable results that will not be discussed in

[2] See for example Matteo Bertolini's lecture notes on supersymmetry. (https://people.sissa.it/~bertmat/susycourse.pdf)

detail in this work concern the map in supermembrane and matrix theory by Lechtenfeld and Nicolai [30], a computation of the expectation value of the infinite straight line Maldacena-Wilson loop in $\mathcal{N}=4$ SYM to sixth order by Malcha [32] and a proof of a finite convergence radius of the map in supersymmetric quantum mechanics by Lechtenfeld [34]. It is hoped that the current trend of increasing interest continues so that the Nicolai map can reach its full potential.

1.2 The Nicolai map in a nutshell

Note: *This section is largely following the author's published work [27, 33].*

To demonstrate the basic features of the Nicolai map, we consider supersymmetric quantum mechanics (SQM) in one dimension [36–41] as a simple toy model. In fact, this theory will be discussed extensively in the second chapter, as one can extract a lot of valuable information about the Nicolai map from this simple setting already. The action is given by

$$S[x,\psi;g,\theta] = \int \mathrm{d}t\, \left\{ \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}V'(x)^2 + \bar{\psi}\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - V''(x)\right]\psi + \mathrm{i}\theta \tfrac{\mathrm{d}}{\mathrm{d}t}V(x) \right\} \quad (1.1)$$

with (0+1)-dimensional fermionic 'fields' $\bar{\psi}(t)$, $\psi(t)$, bosonic 'field' $x(t)$, and one or more couplings g are hiding in the superpotential $V(x)$. Here, we already added the topological θ-term, which will be studied in more detail in Chapter 3. Supersymmetry is realized via two fermionic operators δ and $\bar{\delta}$

$$\begin{aligned} \delta x &= \psi\,, & \delta\psi &= 0\,, & \delta\bar{\psi} &= \mathrm{i}\dot{x} - V'\,, \\ \bar{\delta} x &= \bar{\psi}\,, & \bar{\delta}\psi &= \mathrm{i}\dot{x} + V'\,, & \bar{\delta}\bar{\psi} &= 0\,. \end{aligned} \quad (1.2)$$

It is easy to confirm that indeed δS and $\bar{\delta} S$ are integrals of total derivatives. We now consider the path integral formulation of quantum field theory with the partition function

$$Z = \int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left(\tfrac{\mathrm{i}}{\hbar} S[x;g,\theta]\right)\,. \quad (1.3)$$

Since the fermions appear quadratically (and this seems to be a necessity for the Nicolai map formalism), they can be integrated out as a (Grassmann) Gaussian path integral

$$\begin{aligned} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left(\tfrac{\mathrm{i}}{\hbar}\bar{\psi}\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - V''(x)\right]\psi\right) &= \det M(g;x) \\ &\equiv \det M(g{=}0;x)\cdot \Delta_{\mathrm{MSS}}(g;x)\,, \end{aligned} \quad (1.4)$$

where

$$M(g;x) = \tfrac{\mathrm{i}}{\hbar}\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - V''(x)\right]\delta(t-t') \quad (1.5)$$

and

$$\Delta_{\mathrm{MSS}}(g;x) = \det\left[\delta(t-t') + \mathrm{i}\vartheta(t-t')V''(x(t'))\right] \quad (1.6)$$

with step-function $\vartheta(t-t')$ is the Matthews-Salam-Seiler (MSS) determinant. Here we used the standard distributional identity

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\vartheta(t-t') = \delta(t-t')\ . \tag{1.7}$$

This allows one to define a purely bosonic but nonlocal theory

$$S_g[x,\theta] = S_g^{\mathrm{b}}[x,\theta] + \hbar S_g^{\mathrm{f}}[x]\ , \tag{1.8}$$

with

$$S_g^{\mathrm{b}}[x,\theta] = \int \mathrm{d}t\ \Big\{ \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}V'(x)^2 + \mathrm{i}\theta \tfrac{\mathrm{d}}{\mathrm{d}t}V(x) \Big\} \tag{1.9}$$

and

$$S_g^{\mathrm{f}}[x] = S_0^{\mathrm{f}}[x] - \mathrm{i}\ln\Delta_{\mathrm{MSS}}(g;x)\ . \tag{1.10}$$

In the bosonic theory S_g expectation values of physical observables $\mathcal{O}[x]$ are computed with the path integral

$$\langle \mathcal{O}[x] \rangle_g = \int \mathcal{D}x\ \exp\big(\tfrac{\mathrm{i}}{\hbar} S_g[x,\theta]\big)\ \mathcal{O}[x]\ , \tag{1.11}$$

which at least perturbatively cannot depend on θ. When supersymmetry is unbroken and non-anomalous, the vacuum energy vanishes exactly [42]. The fermionic contributions come with a negative sign and cancel the bosonic ones. This implies that the functional (1.3) must be constant and we can normalize $\langle \mathbb{1} \rangle_g = 1$. With this setup we can now give a definition for the Nicolai map T_g in terms of expectation values.

> Definition of the Nicolai map (from [27]). For a given bosonic theory $S_g[x]$, that arises from integrating out the fermions of a supersymmetric field theory as above, a Nicolai map is a nonlinear and nonlocal field transformation
>
> $$T_g : x(t) \mapsto x'(t;g,x) \tag{1.12}$$
>
> invertible at least as a formal power series in g, which admits the key identity
>
> $$\langle \mathcal{O}[x] \rangle_g = \langle \mathcal{O}[T_g^{-1}x] \rangle_0\ , \qquad \forall\ \mathcal{O}\ , \tag{1.13}$$
>
> relating the interacting theory (at coupling g) to the free one (at coupling $g=0$).

With path integrals, we can equivalently write (1.13) as

$$\begin{aligned} \mathcal{D}x\ \exp\big\{\tfrac{\mathrm{i}}{\hbar} S_g[x,\theta]\big\} &= \mathcal{D}(T_g x)\ \exp\big\{\tfrac{\mathrm{i}}{\hbar} S_0[(T_g x),\theta]\big\} \\ &= \mathcal{D}x\ \exp\big\{\tfrac{\mathrm{i}}{\hbar} S_0[(T_g x),\theta] + \ln\det \tfrac{\delta T_g x}{\delta x}\big\}\ . \end{aligned} \tag{1.14}$$

Comparing powers of $\hbar$ in the exponential, this gives two conditions[3] for a Nicolai map. The

[3] necessary and sufficient (when both are satisfied).

free-action condition
$$S_g^{\mathrm{b}}[x,\theta] = S_0^{\mathrm{b}}[(T_g x),\theta] \tag{1.15}$$

and the

determinant-matching condition
$$\Delta_{\mathrm{MSS}}(g;x) = \det \frac{\delta T_g x}{\delta x}\,. \tag{1.16}$$

In fact, these were originally used to define the Nicolai map instead of (1.13).

Let us now return to our example of SQM and assume that the superpotential only depends on one coupling g with $V|_{g=0} = 0$. Observe that the maps T_g^+ and T_g^- given by

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} T_g^{\pm} x(t) = T_g^{\pm}\mathrm{i}\dot{x}(t) = \mathrm{i}\dot{x}(t) \mp V'(x(t)) \tag{1.17}$$

satisfy the property that they map the free ($g = 0$) bosonic action to the full bosonic action

$$\int \mathrm{d}t\, \tfrac{1}{2}[T_g^{\pm}\dot{x}(t)]^2 = \int \mathrm{d}t\, \left\{\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}V'(x)^2 \pm \mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}V(x)\right\}. \tag{1.18}$$

Even though the maps (1.17) are complex, they lead to real expectation values, as we will see explicitly in Chapter 3. The sign of the last term of (1.18) does not matter perturbatively as it is a total derivative. However, comparing (1.1) and (1.18), we can associate the maps $T_g^{\pm}$ with the theta values $\theta = \pm 1$. The determinant matching condition now requires that the Jacobian of the Nicolai map exactly equals the MSS determinant (1.5). By writing (1.17) in an integral form

$$T_g^{\pm} x(t) = x(t) \pm \mathrm{i}\int \mathrm{d}t'\, \vartheta(t-t')V'(x(t'))\,, \tag{1.19}$$

we immediately see that this is the case for $\theta = +1$. For $\theta = -1$, there is an involution symmetry of the action (1.1) that flips the sign of V'' [33]. This shows that both $T_g^{\pm}$ are in fact Nicolai maps for $\mathcal{N} = 1$ SQM. Later on, we will see that there is actually an infinite family of Nicolai maps for SQM for any value of θ.

At this point, we can remark that the maps $T_g^{\pm}$ (1.17) give a direct connection to instantons. The trajectories $\bar{x}(t)$ given by

$$T_g\bar{x}(t) = \text{constant} \qquad \Rightarrow \qquad \mathrm{i}\dot{\bar{x}}(t) \mp V'\big(\bar{x}(t)\big) = 0 \tag{1.20}$$

are instantons or antiinstantons (depending on the sign) moving between neighboring zeros of V', after a Wick rotation to imaginary time [43]. It is known that the instanton velocity $\dot{\bar{x}}$ is a zero mode of the fermion kernel (also known as the fluctuation operator)

$$\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - V''(\bar{x})\right]\dot{\bar{x}} = 0\,, \tag{1.21}$$

the Goldstone mode of broken time translation invariance. However, we have shown that the fluctuation operator agrees with the Jacobian of the Nicolai maps $T_g^{\pm}$ and thus this Jacobian becomes non-invertible for $x = \bar{x}$. We conclude that the invertibility of the Nicolai map can only be assumed for configurations near the vacuum $x \equiv 0$ and fails (for $\theta = \pm 1$) at the latest when x grows to an instanton $\bar{x}$. This agrees with the identification of the winding number of $T_g^{\pm}$ with the Witten index [21, 44]. It further indicates nonperturbative information in the full Nicolai map, although exact expressions for T_g can only be found for very special cases, such as SQM with $\theta = \pm 1$. In more complex theories, it is usually not possible to write down a Nicolai map in closed form. Instead, one constructs the (inverse) map to a certain order in the coupling(s). To that order, observables can be computed using the identity (1.13) with a free, purely bosonic measure.

1.3 Outline and summary of results

The main results presented in this dissertation, in roughly the order of appearance, are as follows:

- The discovery of a universal formula for the Nicolai map (in both supersymmetric scalar and gauge theories) in terms of a path-ordered exponential of the coupling flow operator. (Based on the author's work [27])
- The refinement of a canonical construction scheme for the coupling flow operator in theories with off-shell supersymmetry. (Based on Lechtenfeld's PhD thesis [11], and the author's modern works [29, 35])
- The generalization of the Nicolai map to multiple couplings, the discovery of an associated functional ambiguity, and a sufficient condition for the uniqueness and polynomiality of the map. (Based on the author's work [33])
- The computation of the one-, two-, and three-point function of an interacting SQM theory using the Nicolai map formalism, matching the results from Feynman perturbation theory and using a graphical Feynman-like notation ('Nicolai rules'). (Based on the author's work [33])
- The effects and simplifications resulting from adding a topological term to the action of off-shell theories. (Based on the author's works [33, 35])
- The analysis and explicit formulae for the Nicolai map and coupling flow operator of $\mathcal{N} = 4$ SYM. (Based on the author's work [31])

These findings were all (co-)published by this author in peer-reviewed journals. In more detail, the structure of the thesis is as follows. In the next chapter (Chapter 2), we investigate the Nicolai map from an abstract, general viewpoint. We start by defining the coupling flow operator, a functional differential operator that captures how expectation values vary with respect to the coupling. Naturally, since the Nicolai map connects expectation values at finite coupling to those at zero coupling, the knowledge of the coupling flow operator allows for the construction of the Nicolai map. This is achieved via the universal formula, a path-ordered exponential of the coupling flow operator. This formula holds for any kind of supersymmetric field theory, even if there is more than one coupling. Next, we discuss how, in a generic theory with off-shell supersymmetry, the coupling flow operator can be constructed canonically from the superfield formalism that any off-shell SUSY theory can be formulated in. We then focus on the case of multiple couplings, which creates a functional ambiguity in the Nicolai map, since it depends on the integration path in coupling space. However, a weak flatness condition guarantees the uniqueness of quantum correlators obtained through the Nicolai map formalism. At the end of the chapter, we discuss the uniqueness of the Nicolai map, in particular with respect to the functional ambiguity. We present a condition for uniqueness and polynomiality, where the latter also applies to the one-coupling case.

In Chapter 3, we consider supersymmetric quantum mechanics (SQM) as a toy model. As the ingredient for the canonical construction, we start by presenting its superfield description, including a topological θ-term. From that, it is easy to construct the coupling and θ flow operators for an arbitrary superpotential. Considering the easiest scenario, we can solve the free massive case exactly. We find that θ can be seen as another special kind of coupling that gives access to the functional ambiguity of the Nicolai map, which we investigate explicitly. We give the most general formula of the (g,θ) Nicolai map for free, massive SQM and compare special cases. As a first explicit demonstration of the Nicolai map, we show how it generates the massive boson propagator from the massless one. We continue with the first non-trivial example, a simple massive interacting SQM theory. Focusing on the g Nicolai map, with θ only as a fixed parameter, we develop Nicolai rules that allow a compact representation of the (inverse) Nicolai map to an arbitrary order. Lastly, we compute explicitly the one-, two- and three-point functions of the interacting SQM theory and find perfect agreement with the traditional Feynman approach, but only after adding 1PI and 1PR contributions. Notably, the special values $\theta = \pm 1$ simplify the whole formalism drastically.

Chapter 4 deals with $\mathcal{N}=1$ supersymmetric Yang–Mills (SYM) theory. Focusing on four spacetime dimensions, where we have an off-shell formalism available, we first construct an intermediate coupling flow operator, that cannot be used for perturbation theory. Only by suitably rescaling the fields with appropriate powers of the coupling, one arrives at the rescaled coupling flow operator, that can be inserted into the universal formula. A particular decomposition of the gauge field into longitudinal and transverse modes further simplifies the formulae. We show how to perturbatively construct the $(D=4)$ Nicolai map in any gauge. Most works on gauge theories to date have used the Landau gauge. We construct the map to second order in the axial gauge, demonstrating a significant increase in complexity. We argue that the main reason for the simplicity of the Landau gauge is the matching of the scalar and ghost propagator. We also briefly describe an alternative 'ad-hoc' construction scheme [24] for the map in Landau gauge, that works in all critical dimensions $D{=}3,4,6,10$. The remaining part of the chapter deals with the addition of a topological theta term to the $D{=}4$ action in Landau gauge, and drastic simplifications that follow from dialing the special 'chiral' theta values $\theta{=}\pm 1$, similar to the situation in SQM. This includes an explicit expansion of the chiral Nicolai map to fourth order.

Finally, in Chapter 5 we study the Nicolai map for the ubiquitous $\mathcal{N}{=}4$ $D{=}4$ SYM theory. We investigate two approaches for the construction of the coupling flow operator. On one hand, via the canonical construction, building up on the knowledge from the $\mathcal{N}{=}1$ case, and on the other hand, by dimensional reduction of the known $D{=}10$ coupling flow operator in Landau gauge. This results in two similar but distinct expressions, which can be explained through a unified R-symmetric framework. The $\mathrm{SU}(4)$ R-symmetry freedom of the action translates to a 15-dimensional $\mathfrak{su}(4)$ Lie algebra freedom in the coupling flow operator. The two construction methods are merely special points in $\mathfrak{su}(4)$. Lastly, we present five distinct Nicolai maps to second order, which can all be shown explicitly to satisfy the necessary conditions.

Chapter 6 gives an overview of possible future research directions, including renormalization, non-linear sigma models, gravity, and non-perturbative properties of the Nicolai map.

There are five appendices. Appendix A elaborates on the notation and conventions used in Chapters 5 and 6. Appendix B (adopted from the author's work [33]) supplements technical details of the computation of the SQM three-point amplitude using the Nicolai map and contains the traditional Feynman-graph computation of the one-, two- and three-point function for comparison. Appendices C, D and E (adopted from the author's work [31]) are quite technical and present the detailed construction of the $\mathcal{N}{=}4$ SYM action with an $\mathcal{N}{=}1$ superfield formalism, the construction of the $\mathcal{N}{=}4$ coupling flow operator and the cross-checks of its infinitesimal conditions respectively.

Chapter 2

Theory of the Nicolai map

Before we specialize to a particular field theory, there is a lot of mathematical structure of the Nicolai that can be studied abstractly. We begin by discussing the coupling flow operator that allows one to write down a universal formula for the Nicolai map (Section 2.1). Next, we elaborate on how to construct the flow operator in theories where supersymmetry is realized off-shell (Section 2.2). This can also be generalized to multiple couplings (Section 2.3). At the end of this chapter, we discuss the ambiguity of the Nicolai map and give a condition in terms of the flow operator for uniqueness and polynomiality of the map (Section 2.4).

2.1 Coupling flow operator and universal formula

Note: *This section is largely following the author's published work [27,33].*

In Section 1.2 the key identity (1.13) of the Nicolai map that serves as its definition was given. It allows one to compute correlators using a free bosonic measure. While the existence of such a map has been proved by Nicolai for supersymmetric scalar and gauge theories [6,7], we have not yet discussed how to construct it. Leaving aside theories with stochastic variables [14–19], this can only be achieved perturbatively. Naturally, instead of trying to find the Nicolai map directly, one usually first computes its infinitesimal version, the so-called coupling flow operator, first investigated by Dietz, Flume and Lechtenfeld [9–13]. For simplicity, we first consider the theory to only depend on one coupling g and generalize to multiple couplings later in Section 2.3. Taking the g-derivative of (1.13) defines the

coupling flow operator R_g

$$\partial_g \langle \mathcal{O}[x] \rangle_g = \langle (\partial_g + R_g[x]) \mathcal{O}[x] \rangle_g \,. \tag{2.1}$$

It is a functional differential operator that satisfies a Leibniz rule

$$R_g(\mathcal{O}_1 \mathcal{O}_2) = (R_g \mathcal{O}_1)\, \mathcal{O}_2 + \mathcal{O}_1\, (R_g \mathcal{O}_2) \,. \tag{2.2}$$

From (1.13) it follows

$$\begin{aligned} \partial_g \langle \mathcal{O}[x] \rangle_g &= \partial_g \langle \mathcal{O}[T_g^{-1} x] \rangle_0 \\ &= \langle (\partial_g \mathcal{O})[T_g^{-1} x] \rangle_0 + \left\langle \int \mathrm{d}t\ (\partial_g T_g^{-1} x(t)) \frac{\delta}{\delta T_g^{-1} x(t)} \mathcal{O}[T_g^{-1} x] \right\rangle_0 , \end{aligned} \tag{2.3}$$

where we can again use (1.13) 'in reverse' to formally express R_g in terms of T_g:

$$R_g[x] = \int \mathrm{d}t\, (\partial_g T_g^{-1}) \circ T_g\, x(t) \tfrac{\delta}{\delta x(t)} =: \int \mathrm{d}t\, K[x] \tfrac{\delta}{\delta x(t)}\,, \tag{2.4}$$

with kernel K. In practice it is however more useful to invert this and express T_g in terms of R_g. To do so, we set $\mathcal{O}[x] = T_g x$ in (1.13) and take its g-derivative. With the definition (2.1), we find

$$(\partial_g + R_g[x])T_g x = 0\,. \tag{2.5}$$

One might recognize this equation, e.g. from the time evolution in quantum mechanics. In fact, it is the differential equation for a path-ordered exponential. The solution is the

universal formula for the Nicolai map (one coupling, from [27])

$$\begin{aligned} T_g x &= \overrightarrow{\mathcal{P}} \exp\Big\{ - \int_0^g \mathrm{d}h\, R_h[x] \Big\}\, x \\ &= \sum_{s=0}^{\infty} (-1)^s \int_0^g \mathrm{d}h_s \ldots \int_0^{h_3} \mathrm{d}h_2 \int_0^{h_2} \mathrm{d}h_1\, R_{h_s}[x] \ldots R_{h_2}[x]\, R_{h_1}[x]\, x\,, \end{aligned} \tag{2.6}$$

where $\overrightarrow{\mathcal{P}}$ indicates standard ordering.

Note that since the Nicolai map is a (path-ordered) exponential of a derivative operator, it acts distributively

$$T_g(\mathcal{O}[x]) = \mathcal{O}[T_g x]\,. \tag{2.7}$$

The inversion of (2.6) is now formally trivial:

$$\begin{aligned} T_g^{-1} x &= \overleftarrow{\mathcal{P}} \exp\Big\{ \int_0^g \mathrm{d}h\, R_h[x] \Big\}\, x \\ &= \sum_{s=0}^{\infty} \int_0^g \mathrm{d}h_1 \int_0^{h_1} \mathrm{d}h_2 \ldots \int_0^{h_{s-1}} \mathrm{d}h_s\, R_{h_s}[x] \ldots R_{h_2}[x]\, R_{h_1}[x]\, x\,. \end{aligned} \tag{2.8}$$

We can utilize the perturbative expansion of the flow operator

$$R_g[x] = \sum_{k=1}^{\infty} g^{k-1} r_k[x] = r_1[x] + g r_2[x] + g^2 r_3[x] + \ldots \tag{2.9}$$

to compute the integrals in (2.6) and (2.8). To that end, we introduce a multi-index

$$\mathbf{n} = (n_1, n_2, \ldots, n_s) \qquad \text{with} \quad n_i \in \mathbb{N} \quad \text{and} \quad \sum_i n_i = n\,, \tag{2.10}$$

where $1 \leq s \leq n$ and the $n{=}0$ term is the identity. It allows one to write down the perturbative expansion of the (inverse) Nicolai map (one coupling, from [27])

$$T_g x = \sum_{\mathbf{n}} g^n \, c_{\mathbf{n}} \, r_{n_s}[x] \ldots r_{n_2}[x] \, r_{n_1}[x] \; x \,, \tag{2.11}$$

and

$$T_g^{-1} x = \sum_{\mathbf{n}} g^n \, d_{\mathbf{n}} \, r_{n_s}[x] \ldots r_{n_2}[x] \, r_{n_1}[x] \; x \,, \tag{2.12}$$

to any order in g, with the numerical coefficients

$$\begin{aligned} c_{\mathbf{n}} &= (-1)^s \int_0^1 \mathrm{d}p_s \; p_s^{n_s-1} \ldots \int_0^{p_3} \mathrm{d}p_2 \; p_2^{n_2-1} \int_0^{p_2} \mathrm{d}p_1 \; p_1^{n_1-1} \\ &= (-1)^s \big[n_1 \cdot (n_1+n_2) \cdots (n_1+n_2+\ldots+n_s) \big]^{-1} \,. \end{aligned} \tag{2.13}$$

$$\begin{aligned} d_{\mathbf{n}} &= \int_0^1 \mathrm{d}p_1 \; k_1^{n_1-1} \int_0^{p_1} \mathrm{d}p_2 \; p_2^{n_2-1} \ldots \int_0^{p_{s-1}} \mathrm{d}p_s \; p_s^{n_s-1} \\ &= \big[n_s \cdot (n_s+n_{s-1}) \cdots (n_s+n_{s-1}+\ldots+n_1) \big]^{-1} \,. \end{aligned} \tag{2.14}$$

To fourth order, the expansion of the map reads

$$\begin{aligned} T_g x = x &- g \, r_1 x - \tfrac{1}{2} g^2 (r_2 - r_1^2) x - \tfrac{1}{6} g^3 (2r_3 - r_1 r_2 - 2r_2 r_1 + r_1^3) x \\ &- \tfrac{1}{24} g^4 (6r_4 - 2r_1 r_3 - 3r_2 r_2 + r_1^2 r_2 \\ &\qquad - 6r_3 r_1 + 2r_1 r_2 r_1 + 3r_2 r_1^2 - r_1^4) x + \mathcal{O}(g^5) \,, \end{aligned} \tag{2.15}$$

and its inverse is

$$\begin{aligned} T_g^{-1} x = x &+ g \, r_1 x + \tfrac{1}{2} g^2 (r_2 + r_1^2) x + \tfrac{1}{6} g^3 (2r_3 + r_2 r_1 + 2r_1 r_2 + r_1^3) x \\ &+ \tfrac{1}{24} g^4 (6r_4 + 2r_3 r_1 + 3r_2 r_2 + r_2 r_1^2 \\ &\qquad + 6r_1 r_3 + 2r_1 r_2 r_1 + 3r_1^2 r_2 + r_1^4) x + \mathcal{O}(g^5) \,. \end{aligned} \tag{2.16}$$

These formulas are valid for both scalar and gauge theories, as the only input was the defining relation of the Nicolai map (1.13). All the physical information is contained in the flow operator R_g, that depends on the field content and interactions of the given theory. In the next section, we proceed with a method for constructing this operator.

However, first we conclude this section by investigating the infinitesimal versions of the free-action (1.15) and determinant matching condition (1.16), as first proposed by Lechtenfeld in [13]. To that end, we consider (1.11) and compare

$$\partial_g \langle \mathcal{O} \rangle_g = \langle \partial_g \mathcal{O} \rangle_g + \tfrac{\mathrm{i}}{\hbar} \langle \mathcal{O} \partial_g S_g \rangle_g \tag{2.17}$$

to

$$\begin{aligned} \partial_g \langle \mathcal{O} \rangle_g &= \langle \partial_g \mathcal{O} \rangle_g + \Big\langle \int \mathrm{d}t \; K[x] \tfrac{\delta \mathcal{O}}{\delta x(t)} \Big\rangle_g \\ &= \langle \partial_g \mathcal{O} \rangle_g - \tfrac{\mathrm{i}}{\hbar} \Big\langle \mathcal{O} \int \mathrm{d}t \; K[x] \tfrac{\delta S_g}{\delta x(t)} \Big\rangle_g - \Big\langle \mathcal{O} \int \mathrm{d}t \; \tfrac{\delta K[x]}{\delta x(t)} \Big\rangle_g \end{aligned} \tag{2.18}$$

where we used integration by parts. This has to hold for any observable $\mathcal{O}[x]$, so we can deduce that

$$(\partial_g + R_g) S_g = \mathrm{i}\hbar \int \mathrm{d}t \, \frac{\delta K[x]}{\delta x(t)} \, . \tag{2.19}$$

From the free action condition (1.15), it also follows that

$$(\partial_g + R_g) S_g^{\mathrm{b}}[x] = (\partial_g + R_g) S_0[T_g x] = 0 \tag{2.20}$$

with (2.5). Combining these, we have arrived at the

infinitesimal free-action condition

$$(\partial_g + R_g) S_g^{\mathrm{b}}[x] = 0 \tag{2.21}$$

and the

infinitesimal determinant-matching condition

$$(\partial_g + R_g) S_g^{\mathrm{f}}[x] = \mathrm{i} \int \mathrm{d}t \, \frac{\delta K[x]}{\delta x(t)} \, , \tag{2.22}$$

where the factor of $\hbar$ got absorbed in the definition of S_g (1.8).

2.2 Canonical construction

We now investigate how to construct the flow operator R_g in theories that exhibit off-shell supersymmetry. This means that the action is invariant under the supersymmetry transformation without the need of using the equations of motion. Usually that requires additional non-dynamical auxiliary degrees of freedom. Such theories can be formulated using superfields, see for example the standard textbook by Wess and Bagger [46]. As a very brief introduction to the superfield formalism, we consider the one dimensional superspace $(t, \vartheta, \bar{\vartheta})$ for SQM that is labeled by the usual coordinate t and two Grassmann parameters ϑ and $\bar{\vartheta}$. In this case the Grassmann parameters are scalar, while they are Weyl spinors later when we consider gauge theories. The scalar superfield that is needed for SQM is

$$\Phi(t, \vartheta, \bar{\vartheta}) = x(t) + \vartheta \psi(t) + \bar{\vartheta} \bar{\psi}(t) + \vartheta \bar{\vartheta} A(t) \, , \tag{2.23}$$

with the bosonic field $x(t)$, fermionic fields $\psi(t)$, $\bar{\psi}(t)$ and a bosonic auxiliary field $A(t)$. Any superfield transforms under supersymmetry as

$$\delta_\xi \Phi(t, \vartheta, \bar{\vartheta}) = (\xi Q + \bar{\xi} \bar{Q}) \Phi \, , \tag{2.24}$$

where ξ, $\bar{\xi}$ are (anticommuting) supersymmetry parameters and Q, $\bar{Q}$ are differential operators that take the form

$$Q = \partial_\vartheta + \mathrm{i}\bar{\vartheta}\partial_t \, , \qquad \bar{Q} = \partial_{\bar{\vartheta}} + \mathrm{i}\vartheta\partial_t \, , \tag{2.25}$$

when ϑ, $\bar{\vartheta}$ are scalar. They generate (part of) the supersymmetry algebra

$$\{Q, \bar{Q}\} = 2P = 2\mathrm{i}\partial_t \,, \tag{2.26}$$

where P is the generator of translations. Comparing the powers of ϑ and $\bar{\vartheta}$ in

$$\begin{aligned} \delta_\xi \Phi(t,\vartheta,\bar{\vartheta}) &= \delta_\xi x(t) + \mathrm{i}\vartheta\delta_\xi\psi(t) - \mathrm{i}\delta_\xi\bar{\psi}(t)\bar{\vartheta} + \bar{\vartheta}\vartheta\delta_\xi A(t) \\ &\equiv (\xi Q + \bar{\xi}\bar{Q})\Phi(t,\vartheta,\bar{\vartheta}) \end{aligned} \tag{2.27}$$

allows one to read off the supersymmetry transformations

$$\begin{aligned} &\delta_\xi x = \xi\psi + \bar{\xi}\bar{\psi} \,, \qquad \delta_\xi\psi = \mathrm{i}\bar{\xi}\dot{x} + \bar{\xi}A \,, \qquad \delta_\xi\bar{\psi} = \mathrm{i}\xi\dot{x} - \xi A \,, \\ &\delta_\xi A = -\mathrm{i}\partial_t(\xi\psi - \bar{\xi}\bar{\psi}) \end{aligned} \tag{2.28}$$

where one should remember that Grassmann numbers anti-commute, which in particular means $\vartheta\vartheta = 0$, and so on. We now note a key fact. Firstly, we divide (2.23) into three contributions. One without a Grassmann coordinate (x), one with one Grassmann coordinate ($\vartheta\psi + \bar{\vartheta}\bar{\psi}$), and one with two Grassmann coordinates ($-\bar{\vartheta}\vartheta A(t)$).[1] There cannot be any higher contributions as they would vanish due to $\vartheta^2 = 0 = \bar{\vartheta}^2$. By construction, taking the supervariation (2.28) of the first two contributions generates the next higher contribution, while the last component transforms into a total derivative. This holds for any superfield and combinations thereof! That is the reason why it is straightforward to define a supersymmetric action using superfields. For example, the off-shell action of SQM can be written in terms of superfields as

$$S = \int \mathrm{d}t \; [\tfrac{1}{2}|\mathrm{D}_\vartheta\Phi|^2 - W(\Phi)]_{\vartheta\bar{\vartheta}} \,, \tag{2.29}$$

where $\mathrm{D}_\vartheta = \partial_\vartheta - \mathrm{i}\bar{\vartheta}\partial_t$ is the superspace covariant derivative, $W(\Phi)$ is the superpotential in superspace (which is just another superfield) and $[\ldots]_{\vartheta\bar{\vartheta}}$ indicates the extraction of the $\vartheta\bar{\vartheta}$ contribution. The details of this theory are not important at this point and will be discussed in Chapter 3. However, since the square brackets make up a superfield, the action (2.29) is automatically invariant under supersymmetry transformations (2.28). Most importantly, this action can itself be written as a supervariation, as one can simply collect the penultimate contribution to the superfield in the square bracket. In other words, for theories with off-shell supersymmetry, there exists a fermionic quantity Δ, such that

$$S = \delta\Delta \,, \qquad \text{(up to total derivatives)} \tag{2.30}$$

where δ is the fermionic operator that generates the supervariations (2.28) with the supersymmetry parameter ξ stripped-off as $\delta_\xi =: \xi\delta + \bar{\xi}\bar{\delta}$. One could equally well use the operator $\bar{\delta}$. There are also two corresponding quantities Δ and $\bar{\Delta}$. Mixing these two possibilities of generating S is exactly where the free parameter θ that controls the topological term comes into play. This will be discussed in detail in Chapter 3. Eq. (2.30) still holds true for the g-derivative of the action and leads us to the following central fact:

[1] Later when ϑ and $\bar{\vartheta}$ are Weyl spinors, there will be more contributions.

Generation of the g-derivative of an off-shell supersymmetric action: For any off-shell supersymmetric action $S[x, \psi, A]$ that depends on some coupling g, there exists a fermionic functional $\Delta_g[x, \psi, A]$ such that

$$\partial_g S[x, \psi, A] = \delta \Delta_g[x, \psi, A] \, . \tag{2.31}$$

This relation can now be used to construct the coupling flow operator. At this point, we can integrate out the auxiliary field(s)[2] and rewrite the path integral (with the fermions still present and setting $\hbar = 1$)

$$\begin{aligned} \partial_g \int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathrm{e}^{\mathrm{i}S[x,\psi]} \; \mathcal{O}[x] &= \mathrm{i} \int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathrm{e}^{\mathrm{i}S[x,\psi]} \; (\partial_g + \delta\Delta[x,\psi])\mathcal{O}[x] \\ &= \mathrm{i} \int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathrm{e}^{\mathrm{i}S[x,\psi]} \; (\partial_g + \Delta[x,\psi]\delta)\mathcal{O}[x] \, , \end{aligned} \tag{2.32}$$

where we used the unbroken supersymmetric Ward identity[3]

$$\int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathrm{e}^{\mathrm{i}S[x,\psi]} \; \delta Y[x,\psi] = 0 \qquad \text{since} \qquad \delta S = 0 \, . \tag{2.33}$$

We now also integrate out the fermions in (2.32) and compare with the definition of the flow operator (2.1). This results in

the g-flow operator for scalar theories (from [27])

$$R_g[x] = \mathrm{i} \, \underbrace{\Delta_g[x] \, \delta} = \mathrm{i} \int \mathrm{d}x \; \underbrace{\Delta_g[x] \; \delta} x(t) \; \frac{\delta}{\delta x(t)} \, , \tag{2.34}$$

where the contraction signifies a fermionic propagator. It is important to realize here that $\Delta[x, \psi]$ and $\delta x(t)$ are both linear in the fermions, thus combining to a fermionic bilinear that becomes a propagtor when integrating out the fermions.

2.3 Multiple couplings

Note: *This section is largely following the author's published work [33].*

In this section, we generalize the above formulae to theories that depend on multiple couplings. We now consider

$$g = (g_1, \ldots, g_k) \tag{2.35}$$

to be local coordinates of a k-dimensional coupling space. In this setting, the Nicolai map T_g is in general a multivariate power series in g. It should be noted that for a theory with multiple couplings, it is in principle always possible to study the Nicolai map only for a subset of these couplings while keeping the rest fixed. This only changes the meaning of the 'free' theory

[2] This is not strictly necessary, but convenient. If one chooses to keep the auxiliary field(s), the Nicolai map will also act on them.

[3] assuming that there are no path integral anomalies.

in the key identity (1.13), i.e. free meaning the theory where this subset of couplings goes to zero. Instead of a single flow operator, there are now k such operators defined via

$$\partial_i \langle \mathcal{O}[x] \rangle_g = \big\langle \big(\partial_i + R^{(i)}(g)\big)\mathcal{O}[x] \big\rangle_g \qquad \text{for} \quad i = 1, \ldots, k \,, \tag{2.36}$$

where $\partial_i := \partial/(\partial g_i)$. For a simpler notation, we only indicate the dependence of $R^{(i)}$ on g, and leave the functional dependence on x implicit. The analogue of Eq. (2.4) is

$$R^{(i)}(g) \;\equiv\; R_{g_i}(g) = \int \mathrm{d}t \; (\partial_i T_g^{-1} \circ T_g) x(t) \, \tfrac{\delta}{\delta x(t)} \;\; =: \int \mathrm{d}x \; K_i[x;\, t] \, \tfrac{\delta}{\delta x(t)} \,. \tag{2.37}$$

With the exact same argument as for one coupling, i.e. setting $\mathcal{O}[x] = T_g x$ in (1.13) and taking the g_i-derivative, we arrive at k differential equations

$$\big(\partial_i + R^{(i)}(g)\big) T_g x = 0 \,. \tag{2.38}$$

The solution is again a path-ordered exponential, generalized to multiple variables

universal formula for the Nicolai map (multiple couplings, from [33])

$$T_g[h]x = \overrightarrow{\mathcal{P}} \exp\Big\{ - \int_0^1 \mathrm{d}s \; h_i'(s) \; R^{(i)}(h(s)) \Big\} \; x \,, \tag{2.39}$$

with a functional path dependence $h(s) = \big(h_1(s), \ldots h_k(s)\big)$ in coupling space with starting and endpoints

$$h_i(0) = 0 \qquad \text{and} \qquad h_i(1) = g_i \,. \tag{2.40}$$

Just like in the one-coupling case, a formal inversion can be achieved by reversing the path ordering and the sign in the exponential.

The characteristic properties of the Nicolai map, i.e., the (infinitesimal) free-action and determinant matching conditions, can easily be generalized to the multiple-coupling scenario. The (non-infinitesimal) conditions (1.15) and (1.16) are unchanged, while we have the

infinitesimal characteristic properties for multiple couplings (c.f. (2.21), (2.22), from [33])

$$\big(\partial_i + R^{(i)}(g)\big) \, S_g^{\mathrm{b}}[x] = 0 \quad \text{and} \quad \big(\partial_i + R^{(i)}(g)\big) \, S_g^{\mathrm{f}}[x] = \mathrm{i} \int \mathrm{d}t \; \frac{\delta K_i[x]}{\delta x(t)} \,, \tag{2.41}$$

with $i = 1, \ldots, k$.

One might now be worried about the path dependence of the Nicolai map. Of course, expectation values (1.13) should be independent of the choice of integration path h in (2.39). To understand this better, we observe

$$\partial_i \partial_j \langle X \rangle_g = \partial_j \partial_i \langle X \rangle_g \quad \Rightarrow \quad \left\langle \partial_i \big(R^{(j)} X \big) - \partial_j \big(R^{(i)} X \big) + [R^{(i)},\, R^{(j)}]\, X \right\rangle_g = 0\,, \tag{2.42}$$

which for $X = \mathbb{1}$ yields a

weak flatness condition

$$\left\langle \partial_i R^{(j)} - \partial_j R^{(i)} + [R^{(i)},\, R^{(j)}] \right\rangle_g = 0\,, \tag{2.43}$$

which we call 'weak' because it only holds within expectation values. Defining a one-form field in coupling space

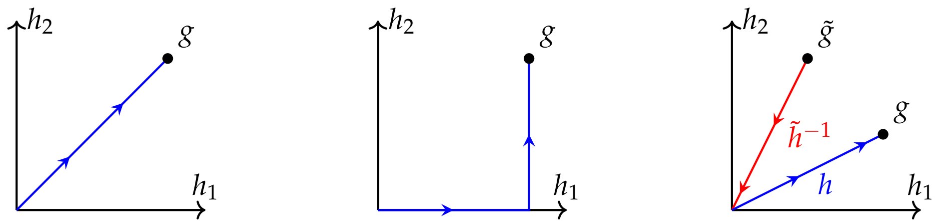
$$R(g) := \sum_{i=1}^{k} \mathrm{d}g_i\, R^{(i)}(g)\,, \tag{2.44}$$

a generalized Stokes theorem implies that the averaged holonomy of $R(g)$ is trivial. This means that expectation values (1.13) are indeed independent of the integration path (2.40). However, as we will see in the next chapter explicitly for SQM, the Nicolai maps themselves are strongly dependent on the integration contour! As two special cases, we define the straight and sequential contour, see Figs. 2.1a and 2.1b.

We remark that due to the functional ambiguity of the Nicolai map, one can also flow between any two points $\tilde{g}$ and g in coupling space using a 'partial Nicolai map' [33]

$$T_{g\tilde{g}}[\tilde{h}^{-1} \circ h]\; x := T_g[h] T_{\tilde{g}}^{-1}[\tilde{h}^{-1}]\; x \tag{2.45}$$

for $\tilde{h}_i(1) = \tilde{g}_i$ and $h_i(1) = g_i$, see Figure 2.1c. This is not used in the rest of



(A) Straight contour, from [33] (B) Sequential contour, from [33] (C) Partial Nicolai map

FIGURE 2.1: Various integration contours in coupling space.

this work, but it is an interesting observation that could be studied further in the future.

Let us next give explicit formulae for the power series of $T_g x$ for the straight and sequential integration contour, as done in [33]. First, we expand the universal formula (2.39) as

$$
\begin{aligned}
T_g[h]\,x &= \overrightarrow{\mathcal{P}}\exp\Big\{-\int_0^1 \mathrm{d}s\,\vec{h}'(s)\cdot\vec{R}\big(\vec{h}(s)\big)\Big\}\,x \\
&= \sum_{n=0}^{\infty}(-)^n\int_0^1 \mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1\Big[\vec{h}'(s_n)\cdot\vec{R}\big(\vec{h}(s_n)\big)\Big]\cdots\Big[\vec{h}'(s_1)\cdot\vec{R}\big(\vec{h}(s_1)\big)\Big]\,x\,,
\end{aligned}
\tag{2.46}
$$

with self-explanatory shorthand notations. In the following, for simplicity, we only consider two couplings, g_1 and g_2. The generalizations to more couplings are straightforward. As we did for just one coupling, it is useful to expand the operators (c.f. (2.9))

$$
\begin{aligned}
R^{(1)}(g) &= \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} g_1^{k-1}g_2^{l}r^{(1)}_{k,l} = r^{(1)}_{1,0} + g_1\,r^{(1)}_{2,0} + g_2\,r^{(1)}_{1,1} + g_1g_2r^{(1)}_{2,1} + \dots\,, \\
R^{(2)}(g) &= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty} g_1^{k}g_2^{l-1}r^{(2)}_{k,l} = r^{(2)}_{0,1} + g_2\,r^{(2)}_{0,2} + g_1\,r^{(2)}_{1,1} + g_1g_2r^{(2)}_{1,2} + \dots\,.
\end{aligned}
\tag{2.47}
$$

Investigating the straight flow

$$
h_1(s) = s\,g_1 \quad \text{and} \quad h_2(s) = s\,g_2 \tag{2.48}
$$

first, the formula for the two couplings reads

$$
T_g[\nearrow]\,x = \sum_{n=0}^{\infty}\sum_{i_1,\dots,i_n=1}^{2}\sum_{\alpha} c_\alpha\, g^\alpha\, r^{(i_n)}_{\alpha^n}\cdots r^{(i_1)}_{\alpha^1}\,x\,, \tag{2.49}
$$

where $\alpha = (\alpha^1,\dots,\alpha^n)$ is a multi-index of n pairs $(i = 1,\dots,n)$

$$
\alpha^i = \big((\alpha^i)_1, (\alpha^i)_2\big) \tag{2.50}
$$

that takes values

$$
(\alpha^p)_{i_q} \geq 0 \quad \text{for} \quad p \neq q\,, \qquad\qquad (\alpha^p)_{i_p} \geq 1\,, \tag{2.51}
$$

and

$$
g^\alpha = g^{\alpha^1}\cdots\, g^{\alpha^n} = g_1^{\sum_{l=1}^n(\alpha^l)_1}\, g_2^{\sum_{l=1}^n(\alpha^l)_2}\,. \tag{2.52}
$$

To second order, the power series is

$$
\begin{aligned}
&T_g[\nearrow]\,x = x - g_1r^{(1)}_{1,0}x - g_2r^{(2)}_{0,1}x - \tfrac{1}{2}g_1^2\big(r^{(1)}_{2,0} - r^{(1)}_{1,0}r^{(1)}_{1,0}\big)x \\
&-\tfrac{1}{2}g_2^2\big(r^{(2)}_{0,2} - r^{(2)}_{0,1}r^{(2)}_{0,1}\big)x - \tfrac{1}{2}g_1g_2\big(r^{(1)}_{1,1} + r^{(2)}_{1,1} - r^{(1)}_{1,0}r^{(2)}_{0,1} - r^{(2)}_{0,1}r^{(1)}_{1,0}\big)x + \dots\,.
\end{aligned}
\tag{2.53}
$$

In appendix A of [33], one can find the analogue for an arbitrary number of couplings in the case of the straight flow. When we instead consider the sequential flow (as in Figure 2.1b), we find

$$
\begin{aligned}
T_g[\,\lrcorner\!\uparrow]\,x &= x - g_1r^{(1)}_{1,0}x - g_2r^{(2)}_{0,1}x - \tfrac{1}{2}g_1^2\big(r^{(1)}_{2,0} - r^{(1)}_{1,0}r^{(1)}_{1,0}\big)x \\
&\quad - \tfrac{1}{2}g_2^2\big(r^{(2)}_{0,2} - r^{(2)}_{0,1}r^{(2)}_{0,1}\big)x \; - g_1g_2\big(r^{(2)}_{1,1} - r^{(2)}_{0,1}r^{(1)}_{1,0}\big)x + \dots\,.
\end{aligned}
\tag{2.54}
$$

Note that the expansions (2.53) and (2.54) start to differ at order $g_1 g_2$. From the weak flatness condition (2.42), however, we obtain

$$\left\langle (r^{(1)}_{1,0} r^{(2)}_{0,1} - r^{(2)}_{0,1} r^{(1)}_{1,0}) x \right\rangle_g = \left\langle (r^{(1)}_{1,1} - r^{(2)}_{1,1}) x \right\rangle_g \,, \tag{2.55}$$

implying that the expansions are equivalent inside expectation values.

2.4 Uniqueness

Note: *This section is largely following the author's published work [33].*

In general, there are now three known types of ambiguities for the Nicolai map:

1. **Integration path dependency** (in theories with multiple couplings): This is a functional ambiguity, as discussed in the previous section. A special case is the ambiguity arising by adding a topological term, since the θ-parameter can be interpreted as an additional (special) coupling (see Chapter 3).

2. **Gauge dependency** (in gauge theories): The Nicolai map depends on the chosen gauge, as we will see in Chapter 4.

3. **R-symmetry** (in theories with extended supersymmetry): There is an R-symmetry freedom, as we will discuss in the context of $\mathcal{N} = 4$ SYM in Chapter 5.

The various non-uniquenesses of the Nicolai map are a central theme of this dissertation. In this section, we want to focus on the first kind of ambiguity, the integration path dependency. In fact, we will give a condition that destroys this ambiguity, collapsing the Nicolai map to a unique, linear function in the couplings. To do so, we consider the first few terms of the expansion of the universal formula (2.46) for two couplings g_1, g_2:

$$\begin{aligned} T_g[h]x = x &- \int_0^1 \mathrm{d}s \left[h_1'(s) r^{(1)}_{1,0} + h_2'(s) r^{(2)}_{0,1} \right] x \\ &- \int_0^1 \mathrm{d}s \left[h_1'(s) \left(h_1(s) r^{(1)}_{2,0} + h_2(s) r^{(1)}_{1,1} \right) + h_2'(s) \left(h_2(s) r^{(2)}_{0,2} + h_1(s) r^{(2)}_{1,1} \right) \right] x \\ &+ \int_0^1 \mathrm{d}s \int_0^s \mathrm{d}s' \, \left[h_1'(s) h_1'(s') r^{(1)}_{1,0} r^{(1)}_{1,0} + h_2'(s) h_2'(s') r^{(2)}_{0,1} r^{(2)}_{0,1} \right. \\ &\qquad\qquad \left. + h_1'(s) h_2'(s') r^{(1)}_{1,0} r^{(2)}_{0,1} + h_2'(s) h_1'(s') r^{(2)}_{0,1} r^{(1)}_{1,0} \right] x + \ldots \end{aligned} \tag{2.56}$$

First note, that the s' integral can be carried out explicitly, for example

$$\int_0^s \mathrm{d}s' \, h_1'(s') = h_1(s) \,. \tag{2.57}$$

Now it is easy to see that the

uniqueness and polynomiality condition (from [33])

$$
\begin{aligned}
r^{(1)}_{k,l}\, r^{(1)}_{1,0}\, x &= r^{(1)}_{k+1,l}\, x \qquad \text{and} \qquad & r^{(1)}_{k,l}\, r^{(2)}_{0,1}\, x &= r^{(1)}_{k,l+1}\, x\,, \\
r^{(2)}_{k,l}\, r^{(1)}_{1,0}\, x &= r^{(2)}_{k+1,l}\, x \qquad \text{and} \qquad & r^{(2)}_{k,l}\, r^{(2)}_{0,1}\, x &= r^{(2)}_{k,l+1}\, x\,,
\end{aligned}
\tag{2.58}
$$

for all possible values of k and l (with a natural generalization to three or more couplings),

leads to the cancellation of all the equally colored terms. It is straightforward to verify this for higher orders. There will always be pairs of terms that end in the same structure as the colored terms in (2.56). Further, they have opposite signs due to the $(-)^n$ factor in (2.46), so they cancel each other. Hence, all terms higher than linear in the coupling cancel and we are left with

$$
T_g\, x = x \;-\; g_1 r^{(1)}_{1,0}\, x \;-\; g_2 r^{(2)}_{0,1}\, x\,. \tag{2.59}
$$

In most theories $R^{(i)}(g{=}0)$ is polynomial in the bosonic fields x, so the map becomes polynomial as well.

If we restrict ourselves to one coupling, there is no ambiguity of the Nicolai map from path dependency, but the condition

$$
r_k\, r_1\, x = r_{k+1}\, x \quad \text{for} \quad k \geq 1\,, \tag{2.60}
$$

still leads to a g-linear Nicolai map

$$
T_g\, x = x \;-\; g\, r_1\, x\,. \tag{2.61}
$$

The inverse map, however, never truncates, because all contributions come with the same sign. In the next chapter, we will see explicit examples of these scenarios.

Chapter 3

Supersymmetric quantum mechanics as a toy model

Note: *This whole chapter is largely based on the author's published work [33].*

3.1 Off-shell action

Note: *This section is largely following the author's published work [33].*

We have already outlined how to construct one-dimensional supersymmetric quantum mechanics in Section 2.2. Here, we repeat the first few steps given there, but we can generalize to D-dimensional SQM. Instead of only one (c.f. (2.23)), we introduce D scalar superfields

$$\Phi_i(t,\vartheta,\bar\vartheta) = x_i(t) + \vartheta\psi_i(t) + \bar\vartheta\bar\psi_i(t) + \vartheta\bar\vartheta A_i(t)\,, \quad \text{with} \quad i = 1,\ldots,D\,. \tag{3.1}$$

From the superfield structure, as we did in Section 2.2, one can extract the supersymmetry transformations $\delta_\xi x_i$ and so on (they are just (2.28) with an index i on each field). The supersymmetry parameters can be removed by defining two fermionic operators $\mathring{\delta}$, $\bar{\mathring{\delta}}$ via

$$\delta_\xi = \xi\mathring{\delta} + \bar\xi\bar{\mathring{\delta}}\,. \tag{3.2}$$

The supervariations become (c.f. (1.2))

$$\begin{aligned} &\mathring{\delta}x_i = \psi_i\,, \qquad \mathring{\delta}\psi_i = 0\,, \qquad \mathring{\delta}\bar\psi_i = \mathrm{i}\dot{x}_i - A_i\,, \qquad \mathring{\delta}A_i = \mathrm{i}\dot\psi_i\,, \\ &\bar{\mathring{\delta}}x_i = \bar\psi_i\,, \qquad \bar{\mathring{\delta}}\psi_i = \mathrm{i}\dot{x}_i + A_i\,, \qquad \bar{\mathring{\delta}}\bar\psi_i = 0\,, \qquad \bar{\mathring{\delta}}A_i = -\mathrm{i}\dot{\bar\psi}_i\,. \end{aligned} \tag{3.3}$$

The circle above the operators indicates that these variations are off-shell, i.e. that the auxiliary fields A_i are still present. Next to the superfields Φ_i, we also need a superfield W that contains the superpotential $V(x)$:

$$\begin{aligned} W(t,\vartheta,\bar\vartheta) = &-V(x) - \vartheta\psi_i(t)V_i(x) - \bar\vartheta\bar\psi_i(t)V_i(x) \\ &- \vartheta\bar\vartheta\big(A_i(t)V_i(x) + \tfrac{1}{2}\bar\psi_i(t)\psi_j(t)V_{ij}(x) - \tfrac{1}{2}\psi_i(t)\bar\psi_j(t)V_{ij}(x)\big)\,, \end{aligned} \tag{3.4}$$

where we use the shorthand notations

$$V_i(x) \equiv \tfrac{\partial}{\partial x_i}V(x)\,, \qquad V_{ij}(x) \equiv \tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j}V(x)\,. \tag{3.5}$$

As described in Section 2.2, the off-shell action $\mathring{S}$ can be obtained as the last component of a superfield

$$\mathring{S} = \int \mathrm{d}t \, [\tfrac{1}{2}|\mathrm{D}_\vartheta \Phi|^2 - W(\Phi)]_{\vartheta\bar{\vartheta}} \, , \tag{3.6}$$

with the superspace covariant derivative

$$\mathrm{D}_\vartheta = \partial_\vartheta - \mathrm{i}\bar{\vartheta}\partial_t \, . \tag{3.7}$$

It is a good exercise for the reader to compute the last and penultimate components of the superfield in (3.6) to find the action in components

$$\mathring{S} = \int \mathrm{d}t \, \left\{ \tfrac{1}{2}\dot{x}_i^2 + \tfrac{1}{2}A_i^2 - A_i V_i(x) + \bar{\psi}_i \left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\delta_{ij} - V_{ij}(x) \right] \psi_j \right\} , \tag{3.8}$$

up to total derivatives. Further, one can verify two ways of generating of the action

$$\mathring{S} \;=\; \mathring{\delta}\mathring{\bar{\Delta}} \;=\; \mathring{\bar{\delta}}\mathring{\Delta} \qquad \text{up to boundary terms} \tag{3.9}$$

as a supervariation with the (integrated) off-shell components

$$\begin{aligned} \mathring{\Delta} &= \int \mathrm{d}t \, \{ \tfrac{1}{2}(+A_i - \mathrm{i}\dot{x}_i) - V_i(x) \} \, \psi_i \qquad \text{and} \\ \mathring{\bar{\Delta}} &= \int \mathrm{d}t \, \{ \tfrac{1}{2}(-A_i - \mathrm{i}\dot{x}_i) + V_i(x) \} \, \bar{\psi}_i \, . \end{aligned} \tag{3.10}$$

3.2 Topological term

Note: *This section is largely following the author's published work [33].*

Notice that there are two ways (3.9) to generate the off-shell action (3.8) up to total derivatives. Although it is 'only' a boundary term, we can introduce the topological θ-term in the off-shell formalism via

$$\mathring{S} \;=\; \tfrac{1}{2}(\mathring{\delta}\mathring{\bar{\Delta}} + \mathring{\bar{\delta}}\mathring{\Delta}) \;+\; \tfrac{\theta}{2}(\mathring{\delta}\mathring{\bar{\Delta}} - \mathring{\bar{\delta}}\mathring{\Delta}) \;=\; \tfrac{1+\theta}{2}\,\mathring{\delta}\mathring{\bar{\Delta}} \;+\; \tfrac{1-\theta}{2}\,\mathring{\bar{\delta}}\mathring{\Delta} \, , \tag{3.11}$$

leading to

$$\mathring{S} = \int \mathrm{d}t \, \left\{ \tfrac{1}{2}\dot{x}_i^2 + \tfrac{1}{2}A_i^2 - A_i V_i(x) + \bar{\psi}_i \left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\delta_{ij} - V_{ij}(x) \right] \psi_j + \mathrm{i}\theta \tfrac{\mathrm{d}}{\mathrm{d}t} V\big(x(t)\big) \right\} . \tag{3.12}$$

The equation of motion of the auxiliary field is simply $A_i{=}V_i$, so we obtain the

on-shell SQM action with a topological θ-term

$$S = \int \mathrm{d}t \, \{ \tfrac{1}{2}\dot{x}_i^2 - \tfrac{1}{2}V_i(x)^2 + \bar{\psi}_i[\mathrm{i}\mathrm{d}_t\delta_{ij} - V_{ij}(x)]\psi_j + \mathrm{i}\theta\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(x(t)\big) \} \, . \tag{3.13}$$

This action (3.13) can only be obtained by taking the off-shell supervariations $\mathring{\delta}$, $\mathring{\bar{\delta}}$ of the off-shell integrals $\mathring{\Delta}$, $\mathring{\bar{\Delta}}$, and *afterwards* setting $A_i = V_i$. Vice versa, one finds an incorrect factor for the fermion term. However, if we isolate the θ term it can be generated with the on-shell variations δ and $\bar{\delta}$ (where

$A_i = V_i$) even containing an ambiguity parameter $\beta \in \mathbb{R}$:

$$\partial_\theta S = \tfrac{1}{2}\big(\mathring{\delta}\mathring{\bar{\Delta}} - \mathring{\bar{\delta}}\mathring{\Delta}\big) = \tfrac{1}{2}\big(\delta\bar{\Delta}^\beta - \bar{\delta}\Delta^\beta\big) = \int \mathrm{d}t\ V_i(x)\ \dot{x}_i = \mathrm{i}\,V\big(x(t)\big)\big|_{-\infty}^{+\infty}\,, \tag{3.14}$$

with

$$\begin{aligned} \Delta^\beta &:= \int \mathrm{d}t\ \{(\beta{-}1)\mathrm{i}\dot{x}_i - \beta V_i(x)\}\ \psi_i \qquad \text{and} \\ \bar{\Delta}^\beta &:= \int \mathrm{d}t\ \{(\beta{-}1)\mathrm{i}\dot{x}_i + \beta V_i(x)\}\ \bar{\psi}_i\,. \end{aligned} \tag{3.15}$$

The parameter β is a technical curiosity that will propagate to some results in the next sections. Note that

$$\Delta^{\beta=\frac{1}{2}} = \mathring{\Delta}\big|_{A_i=V_i} \qquad \text{and} \qquad \Delta^{\beta=1}\big|_{V=0} = 0\,. \tag{3.16}$$

3.3 General superpotential

***Note:** This section is largely following the author's published work [33].*

In this section, we construct the g- and θ-flow operators for an arbitrary superpotential V that depends on one coupling g. This can easily be generalized to multiple couplings by applying the arguments to each coupling distinctly. After taking the g-derivative of (3.11), it is possible to go on-shell:

$$\partial_g S = (\partial_g \mathring{S})\big|_{A_i=V_i} = \tfrac{1+\theta}{2}\,\delta\bar{\Delta}_g \;+\; \tfrac{1-\theta}{2}\,\bar{\delta}\Delta_g\,, \tag{3.17}$$

where

$$\Delta_g := \partial_g\mathring{\Delta} = -\int \mathrm{d}t\ \{\psi_i\,\partial_g V_i\} \qquad \text{and} \qquad \bar{\Delta}_g := \partial_g\mathring{\bar{\Delta}} = \int \mathrm{d}t\ \{\bar{\psi}_i\,\partial_g V_i\}\,. \tag{3.18}$$

To find the g-flow operator, we use the supersymmetric Ward identity as described in Section 2.2, which yields

$$\begin{aligned} R_g(g,\theta) &= \tfrac{1+\theta}{2}\,\mathrm{i}\bar{\Delta}_g\,\delta \;+\; \tfrac{1-\theta}{2}\,\mathrm{i}\Delta_g\,\bar{\delta} \\ &= \int \mathrm{d}t\,\mathrm{d}t'\ \tfrac{\mathrm{i}}{2}(\partial_g V_i)(t)\ \{(1{+}\theta)\ \bar{\psi}_i(t)\,\psi_j(t') - (1{-}\theta)\ \psi_i(t)\,\bar{\psi}_j(t')\}\ \tfrac{\delta}{\delta x_j(t')}\,, \end{aligned} \tag{3.19}$$

with the fermion propagators

$$\begin{aligned} \psi_i(t)\,\bar{\psi}_j(t') = \mathrm{i}\,S_{ij}(t,t') \quad &\text{and} \quad \bar{\psi}_i(t)\,\psi_j(t') = -\mathrm{i}\,S_{ij}(t',t)\,, \\ &\text{with} \quad \big(\mathrm{i}\delta_{ik}\tfrac{\mathrm{d}}{\mathrm{d}t} - V_{ik}\big)S_{kj}(t,t') = \delta(t{-}t')\delta_{ij}\,. \end{aligned} \tag{3.20}$$

Introducing the useful definition

$$\theta^\pm := \tfrac{1}{2}(1 \pm \theta)\,, \tag{3.21}$$

we can compactly express

the g-flow operator for SQM with an arbitrary superpotential

$$R_g(g,\theta) = \theta^+ R_g^+(g) \ + \ \theta^- \ R_g^-(g)$$
$$\text{with} \quad R_g^+(g) = \int \mathrm{d}t\, \mathrm{d}t' \ (\partial_g V_i)(t) \ S_{ij}(t',t) \ \frac{\delta}{\delta x_j(t')} \qquad (3.22)$$
$$\text{and} \quad R_g^-(g) = \int \mathrm{d}t\, \mathrm{d}t' \ (\partial_g V_i)(t) \ S_{ij}(t,t') \ \frac{\delta}{\delta x_j(t')} \ .$$

To derive the θ-flow operator, we follow the same procedure. From the generation of the theta term (3.14) we find

the θ-flow operator for SQM with an arbitrary superpotential

$$R_\theta^\beta(g) = \tfrac{\mathrm{i}}{2}\, \underbracket{\bar\Delta^\beta \delta} - \tfrac{\mathrm{i}}{2}\, \underbracket{\Delta^\beta \bar\delta} = \tfrac{\beta}{2} \int \mathrm{d}t\, \mathrm{d}t' \ V_i(t)\, [S_{ij}(t',t) - S_{ij}(t,t')] \ \frac{\delta}{\delta x_j(t')}$$
$$+ \ \tfrac{\beta-1}{2} \int \mathrm{d}t\, \mathrm{d}t' \ \mathrm{i}\dot x_i(t) \ [S_{ij}(t',t) + S_{ij}(t,t')] \ \frac{\delta}{\delta x_j(t')} \ , \qquad (3.23)$$

with the technical ambiguity $\beta \in \mathbb{R}$. Notably, it is completely independent of θ. It is straightforward to verify that R_g (3.22) and R_θ^β (3.23) satisfy their respective free-action (2.21) and determinant-matching condition (2.22).[1]

3.4 Free massive theory

Note: *This section is largely following the author's published work [33].*

It is quite instructive to consider the one-dimensional free massive theory with superpotential

$$V = \tfrac{1}{2} m x^2 \ , \qquad (3.24)$$

since it leads to analytical expressions for the flow operators and the Nicolai maps. In this case, the Lagrangian is given by

$$\mathcal{L}_0 = \tfrac{1}{2}\dot x^2 - \tfrac{1}{2} m^2 x^2 + \bar\psi \big[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - m\big] \psi + \mathrm{i}\theta m\, x \dot x \qquad (3.25)$$

and the fermion propagators follow from

$$\big[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - m\big] S_0(t,t') = \delta(t-t')$$
$$\Rightarrow \quad \underbracket{\psi(t)\, \bar\psi(t')} = \mathrm{i}S_0(t,t') = \tfrac{1}{2}\, \mathrm{sgn}(t-t')\, \mathrm{e}^{-\mathrm{i}m(t-t')} \ , \qquad (3.26)$$

where we fixed an integration constant by requiring antisymmetry for $m{=}0$. Inserting (3.24) into the formulae for the flow operators (3.22), (3.23) yields

$$R_m(m,\theta) = \theta^+ R_m^+(m) \ + \ \theta^- \ R_m^-(m) \ , \qquad \text{with}$$
$$R_m^+(m) = \int \mathrm{d}t\mathrm{d}t' \ x(t) S_0(t',t) \frac{\delta}{\delta x(t')} \quad \text{and} \quad R_m^-(m) = \int \mathrm{d}t\mathrm{d}t' \ x(t) S_0(t,t') \frac{\delta}{\delta x(t')} \ , \qquad (3.27)$$

[1]It is shown explicitly in appendix B of [33].

and

$$
\begin{aligned}
R_\theta^\beta(m) &= \tfrac{2\beta-1}{2} m \int \mathrm{d}t\, \mathrm{d}t'\; x(t)\, [S_0(t',t) - S_0(t,t')]\, \tfrac{\delta}{\delta x(t')} \\
&= \tfrac{2\beta-1}{2}\, m\, [R_m^+(m) - R_m^-(m)]\,.
\end{aligned}
\tag{3.28}
$$

For $R_\theta^\beta(m)$, we used integration by parts in the second line of (3.23), with the definition of the fermion propagators (3.26). Interestingly, for $\beta{=}\frac{1}{2}$ the θ-flow entirely vanishes. The expressions become even simpler once we Fourier transform to frequency space via

$$
x(t) = \int \tfrac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega t}\, \widetilde{x}(\omega) \quad \Rightarrow \quad \tfrac{\delta}{\delta x(t)} = \int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t}\, \tfrac{\delta}{\delta \widetilde{x}(\omega)}\,. \tag{3.29}
$$

It follows that the fermion propagators take the frequency representation

$$
S_0(t,t') = \int \tfrac{\mathrm{d}\omega}{2\pi}\, \widetilde{S}_0(\omega)\, \mathrm{e}^{\mathrm{i}\omega(t-t')} \qquad \Rightarrow \qquad \widetilde{S}_0(\omega) = -\tfrac{1}{\omega+m}\,, \tag{3.30}
$$

while the flow operators (3.27) and (3.28) can be read off from

$$
R_m^\pm(m) = \int \tfrac{\mathrm{d}\omega}{2\pi}\, \widetilde{x}(\omega)\, \widetilde{S}_0(\mp\omega)\, \tfrac{\delta}{\delta \widetilde{x}(\omega)}\,. \tag{3.31}
$$

They act on $\widetilde{x}(\omega)$ by simple multiplications,

$$
R_m(m,\theta)\, \widetilde{x}(\omega) = \tfrac{m-\theta\,\omega}{\omega^2-m^2}\, \widetilde{x}(\omega) \quad \text{and} \quad R_\theta^\beta(m)\, \widetilde{x}(\omega) = (1-2\beta)\tfrac{m\,\omega}{\omega^2-m^2}\, \widetilde{x}(\omega)\,. \tag{3.32}
$$

With these expressions, it is now easy to compute the path-ordered exponential (2.39) of the full (g,θ) Nicolai map. Since R_m and $R_\theta^\beta(m)$ simply act multiplicatively, they always commute with each other. Thus, the path-ordered exponential simplifies to a regular exponential

$$
T_{m,\theta}^{-1}[h]\, \widetilde{x}(\omega) = \exp\Big\{ \int_0^1 \mathrm{d}s\, \big[m'(s) R_m\big(m(s),\theta(s)\big) + \theta'(s) R_\theta^\beta\big(m(s),\theta(s)\big) \big] \Big\} \widetilde{x}(\omega)\,, \tag{3.33}
$$

with a generic integration path

$$
h(s) = \big(m(s),\theta(s)\big) \quad \text{with} \quad h(0) = (0,0) \quad \text{and} \quad h(1) = (m,\theta)\,. \tag{3.34}
$$

For simplicity, we assume monotonous parametrizations $m(s)$ and $\theta(s)$, allowing us to introduce global coordinates μ and ϑ such that we may write[2]

$$
R_m\big(\mu,\theta(\mu)\big) \quad \text{with} \quad \mu \in [0,m] \quad \text{and} \quad R_\theta^\beta\big(m(\vartheta),\vartheta\big) \quad \text{with} \quad \vartheta \in [0,\theta]\,. \tag{3.35}
$$

[2]Otherwise, one can decompose the integration path into piecewise monotonous parts.

Dealing with a regular exponential, we can evaluate

$$
\begin{aligned}
T^{-1}_{m,\theta}[h]\,\widetilde{x}(\omega) &= \exp\Big\{\int_0^m \mathrm{d}\mu\; R_m\big(\mu,\theta(\mu)\big)\Big\}\;\exp\Big\{\int_0^\theta \mathrm{d}\vartheta\; R^\beta_\theta\big(m(\vartheta),\vartheta\big)\Big\}\;\widetilde{x}(\omega)\\
&= \exp\Big\{\int_0^m \mathrm{d}\mu \tfrac{\mu}{\omega^2-\mu^2}\Big\}\exp\Big\{-\omega\int_0^m \mathrm{d}\mu \tfrac{\theta(\mu)}{\omega^2-\mu^2}\Big\}\exp\Big\{(1-2\beta)\omega\int_0^\theta \mathrm{d}\vartheta \tfrac{m(\vartheta)}{\omega^2-m(\vartheta)^2}\Big\}\widetilde{x}(\omega)\,.
\end{aligned}
\tag{3.36}
$$

It is easy to confirm that regardless of the exact form of integration path, the result will always take the form of

> the most general full (g,θ) inverse Nicolai map for free, massive SQM
>
> $$T^{-1}_{m,\theta}[h]\,\widetilde{x}(\omega) = \sqrt{\tfrac{\omega^2}{\omega^2-m^2}}\;\mathrm{e}^{\,\omega\,\theta\,f(\omega^2)}\;\widetilde{x}(\omega)\,, \tag{3.37}$$
>
> with some function f that depends on the integration contour.

We now compute this expression for three contours. If we follow the line 'first m then θ', we obtain

$$
\begin{aligned}
T^{-1}_{m,\theta}[\,\lrcorner\!\uparrow]\;\widetilde{x} &= \exp\Big\{(1-2\beta)\int_0^\theta \mathrm{d}\vartheta\;\tfrac{m\,\omega}{\omega^2-m^2}\Big\}\;\exp\Big\{\int_0^m \mathrm{d}\mu\;\tfrac{\mu}{\omega^2-\mu^2}\Big\}\;\widetilde{x}\\
&= \sqrt{\tfrac{\omega^2}{\omega^2-m^2}}\;\mathrm{e}^{(1-2\beta)\frac{\theta\,m\,\omega}{\omega^2-m^2}}\;\widetilde{x}\,.
\end{aligned}
\tag{3.38}
$$

Secondly, in the 'first θ then m' contour, the θ flow is trivial, giving

$$
\begin{aligned}
T^{-1}_{m,\theta}[\,\ulcorner\!\rightarrow]\;\widetilde{x} &= \exp\{\int_0^m \mathrm{d}\mu\;\tfrac{\mu-\theta\,\omega}{\omega^2-\mu^2}\}\;\widetilde{x} = (1+\tfrac{m}{\omega})^{-\frac{1+\theta}{2}}(1-\tfrac{m}{\omega})^{-\frac{1-\theta}{2}}\;\widetilde{x}\\
&= \sqrt{\tfrac{\omega^2}{\omega^2-m^2}}\;\big(\tfrac{\omega-m}{\omega+m}\big)^{\theta/2}\;\widetilde{x}\,.
\end{aligned}
\tag{3.39}
$$

Lastly, the symmetric path $m(\vartheta) = \frac{m}{\theta}\vartheta$ and $\theta(\mu) = \frac{\theta}{m}\mu$ yields

$$
\begin{aligned}
T^{-1}_{m,\theta}[\nearrow]\;\widetilde{x} &= \exp\Big\{\big[(1-\tfrac{\theta}{m}\omega)+(1-2\beta)\tfrac{\theta}{m}\omega\big]\int_0^m \mathrm{d}\mu\;\tfrac{\mu}{\omega^2-\mu^2}\Big\}\;\widetilde{x}\\
&= \sqrt{\tfrac{\omega^2}{\omega^2-m^2}}\;\big(\tfrac{\omega^2}{\omega^2-m^2}\big)^{-\beta\frac{\theta}{m}\omega}\;\widetilde{x}\,.
\end{aligned}
\tag{3.40}
$$

Clearly, the three results are not equivalent, demonstrating explicitly the functional ambiguity of the Nicolai map for multiple couplings. This is expected, since the condition for contour independence (2.58) is not satisfied. Moreover, Fourier transforming back to the time domain, one can explicitly check that the flatness condition (2.43) of R_m and R_θ is only weakly satisfied. That is, only the expectation value of the curvature is zero, not the curvature itself. This had to be the case, since we have shown explicit examples of the path dependency of the full Nicolai map. However, this dependence only affects terms that have a vanishing expectation value. Correlation functions of course are equivalent for any choice of Nicolai map, as we will see shortly. We further note that even for the special variables $\theta = \pm 1$ the Nicolai map depends on the integration contour h when θ is a variable coupling. The $\Rsh$ contour (3.39) is quite special, because the θ-flow performed at $m{=}0$ is trivial. Effectively, we have the equivalence

$$T_{m,\theta}[\,\Rsh\,] = T_m(\theta) \ , \tag{3.41}$$

where the right-hand side is the partial Nicolai map where θ is only a fixed external parameter. Here, for the two special theta values $\theta = \pm 1$, the Nicolai map simplifies:

$$T_m^{-1}(\pm 1)\ \widetilde{x}(\omega) = \tfrac{\omega}{\omega \pm m}\ \widetilde{x}(\omega) \quad \Rightarrow \quad T_m(\pm 1)\ \widetilde{x}(\omega) = \big(1 \pm \tfrac{m}{\omega}\big)\ \widetilde{x}(\omega)\ . \tag{3.42}$$

This agrees with our brief discussion in Chapter 1 around (1.17), once we Fourier transform back to the time domain

$$T_m(\theta{=}\pm 1)\ \mathrm{i}\dot{x}(t) = \mathrm{i}\dot{x}(t) \mp m\, x(t)\ . \tag{3.43}$$

Generally, for any Nicolai map (3.37), we can see that it connects the massless boson propagator

$$\big\langle \widetilde{x}(\omega)\ \widetilde{x}(\omega')\big\rangle_{0,0} = 2\pi\delta(\omega+\omega')/\omega^2 \tag{3.44}$$

to the massive boson propagator

$$\begin{aligned}
&\big\langle T^{-1}_{m,\theta}[h]\widetilde{x}(\omega)\ T^{-1}_{m,\theta}[h]\widetilde{x}(\omega')\big\rangle_{0,0} \\
&\qquad = \sqrt{\tfrac{\omega^2\omega'^2}{(\omega^2-m^2)(\omega'^2-m^2)}}\ \mathrm{e}^{\theta[\omega f(\omega^2)+\omega' f(\omega'^2)]}\big\langle \widetilde{x}(\omega)\ \widetilde{x}(\omega')\big\rangle_{0,0} \\
&\qquad\qquad = \tfrac{2\pi\delta(\omega+\omega')}{\omega^2-m^2} = \big\langle \widetilde{x}(\omega)\ \widetilde{x}(\omega')\big\rangle_{m,\theta}\ ,
\end{aligned} \tag{3.45}$$

for any function f, since the delta-function enforces energy conservation $\omega' {=} -\omega$. By Wick's theorem[3], this generalizes to arbitrary N-point functions.

3.5 Interacting theory

Note: *This section is largely following the author's published work [33].*

3.5.1 Setup

To investigate a less trivial theory, we can add a cubic term to the superpotential,

$$V = \tfrac{1}{2}mx^2 + \tfrac{1}{3}gx^3\ , \tag{3.46}$$

yielding the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}m^2x^2 - mgx^3 - \tfrac{1}{2}g^2x^4 + \bar{\psi}\big[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - m - 2gx\big]\psi + \mathrm{i}\theta(mx+gx^2)\,\dot{x}\ . \tag{3.47}$$

[3] Also known as Isserlis' theorem.

The fermion propagators now interact with the bosonic field and follow from

$$\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} - m - 2gx(t)\right] S(t,t') = \delta(t-t')\,, \tag{3.48}$$

whereas the coupling flow operators become

$$\begin{aligned} R_g(g,m,\theta) &= \theta^+\, R_g^+(g,m) \;+\; \theta^-\, R_g^-(g,m)\,, \\ R_m(g,m,\theta) &= \theta^+\, R_m^+(g,m) \;+\; \theta^-\, R_m^-(g,m)\,, \end{aligned} \tag{3.49}$$

with $R_g^\pm(g,m)$ and $R_m^\pm(g,m)$ as in (3.22) but for $D{=}1$. We can trivialize the θ-flow in the following by setting $\beta{=}1$, moving to a finite θ value (at $g{=}m{=}0$) first and only then starting to move towards the other couplings along some contour h in (g,m) space. We thus leave out the θ subscript on $T_{g,m}$. Choosing $\theta{=}\pm 1$ implies the collapse condition (2.58), so that the full Nicolai map becomes linear in g and m for any contour h,

$$T_{g,m}[h,\theta{=}\pm 1]\,\mathrm{i}\dot{x}(t) = \mathrm{i}\dot{x}(t) \;\mp\; m\,x(t) \;\mp\; g\,x(t)^2\,, \tag{3.50}$$

while the inverse map is always an infinite power series. It is a good exercise to verify the generation of the bosonic Lagrangian via $\frac{1}{2}(T_{m,g}\dot{x})^2$ (free-action condition) from (3.50).

It is more instructive though to start with the massive free theory and work with the one-variable g-flow map, because that sidesteps any infrared divergences of the massless propagators. In this case, we are dealing with the (inverse) partial map $T_g(m,\theta)$ at fixed m and θ, without any contour ambiguities (since there is only one variable coupling). Again, the map becomes linear for $\theta = \pm 1$:

$$\begin{aligned} &T_g(m,\theta{=}\pm 1)\;\mathrm{i}\dot{x}(t) = \mathrm{i}\dot{x}(t) \mp g x(t)^2 - gm \int \mathrm{d}t'\;\tfrac{1}{2\mathrm{i}}\mathrm{sgn}(t-t')\mathrm{e}^{\pm\mathrm{i}m(t-t')}x(t')^2 \\ \Leftrightarrow\qquad &T_g(m,\theta{=}\pm 1)\;\left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} \pm m\right] x(t) = \left[\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} \mp m\right] x(t) \;\mp\; g\,x(t)^2\,. \end{aligned} \tag{3.51}$$

As a double check, we see that the $\Rsh$-composition of (3.51) with $T_m(\theta{=}\pm 1)$ from (3.43) is equivalent to (3.50),

$$\begin{aligned} T_{g,m}[\,\Rsh\,,\theta{=}\pm 1]\;\mathrm{i}\dot{x}(t) &= T_g(m,\theta{=}\pm 1)\; T_m(g{=}0,\theta{=}\pm 1)\;\mathrm{i}\dot{x}(t) \\ &= \mathrm{i}\dot{x}(t) \;\mp\; m\,x(t) \;\mp\; g\,x(t)^2\,. \end{aligned} \tag{3.52}$$

Settling for the one-variable g-flow, we can now set up the perturbative expansion of the inverse Nicolai map, according to the universal formula (2.12). This requires to expand the g-flow operator. In frequency space, the $O(g^{k-1})$

contribution is

$$
\begin{aligned}
r_k = \theta^+ 2^{k-1} \int \tfrac{\mathrm{d}\nu_1}{2\pi} \cdots \tfrac{\mathrm{d}\nu_{k+1}}{2\pi}\, \widetilde{x}(\nu_1)[\widetilde{x}(-\nu_1+\nu_2)\widetilde{S}_0(+\nu_2)] \\
\cdots [\widetilde{x}(-\nu_k+\nu_{k+1})\widetilde{S}_0(+\nu_{k+1})]\, \tfrac{\delta}{\delta\widetilde{x}(\nu_{k+1})} \\
+\ \theta^- 2^{k-1} \int \tfrac{\mathrm{d}\nu_1}{2\pi} \cdots \tfrac{\mathrm{d}\nu_{k+1}}{2\pi}\, \widetilde{x}(\nu_1)[\widetilde{x}(-\nu_1+\nu_2)\widetilde{S}_0(-\nu_2)] \\
\cdots [\widetilde{x}(-\nu_k+\nu_{k+1})\widetilde{S}_0(-\nu_{k+1})]\, \tfrac{\delta}{\delta\widetilde{x}(\nu_{k+1})}\,,
\end{aligned}
\tag{3.53}
$$

which naturally contains k factors of $\widetilde{x}$. For completeness, we recall the expansions of the (inverse) Nicolai map in terms of the coupling flow operator contributions

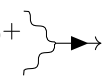
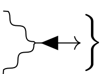
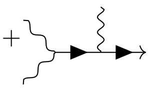
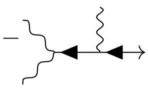
$$
\begin{aligned}
T_g\widetilde{x}(\omega) &= \widetilde{x}(\omega) - g r_1 \widetilde{x}(\omega) - \tfrac{1}{2} g^2 (r_2 - r_1^2)\widetilde{x}(\omega) - \tfrac{1}{6} g^3 (2r_3 - 2r_2 r_1 - r_1 r_2 + r_1^3)\widetilde{x}(\omega) \\
&\quad - \tfrac{1}{24} g^4 (6r_4 - 6r_3 r_1 - 3r_2 r_2 + 3r_2 r_1^2 - 2r_1 r_3 + 2r_1 r_2 r_1 + r_1^2 r_2 - r_1^4)\widetilde{x}(\omega) + \mathcal{O}(g^5)\,,
\end{aligned}
\tag{3.54}
$$

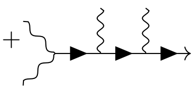
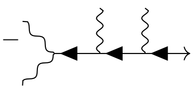
$$
\begin{aligned}
T_g^{-1}\widetilde{x}(\omega) &= \widetilde{x}(\omega) + g r_1 \widetilde{x}(\omega) + \tfrac{1}{2} g^2 (r_2 + r_1^2)\widetilde{x}(\omega) + \tfrac{1}{6} g^3 (2r_3 + 2r_1 r_2 + r_2 r_1 + r_1^3)\widetilde{x}(\omega) \\
&\quad + \tfrac{1}{24} g^4 (6r_4 + 6r_1 r_3 + 3r_2 r_2 + 3r_1^2 r_2 + 2r_3 r_1 + 2r_1 r_2 r_1 + r_2 r_1^2 + r_1^4)\widetilde{x}(\omega) + \mathcal{O}(g^5)\,,
\end{aligned}
\tag{3.55}
$$

to fourth order. Instead of working with (3.53) directly, we can introduce a systematic graphical notation ('Nicolai rules') that gives a much clearer view of the computations.

3.5.2 Nicolai rules and maps

In the following, we will always work in frequency space and drop the tildes for a simpler notation. The 'Nicolai rules' for writing down the maps (3.54), (3.55) are given in Figure 3.1.[4] With these rules, the g-flow operator to third order takes the simple form (c.f. (3.53))

$R_g(g,m,\theta) = \Big\{\theta^+$  $+\ \theta^-$  $\Big\} + g\,\Big\{\theta^+$  $+\ \theta^-$  $\Big\}$

$+\, g^2\,\Big\{\theta^+$  $+\ \theta^-$  $\Big\} + \mathcal{O}(g^4)\,,$ (3.59)

with the obvious continuations at higher orders. The arrows at the end of the fermion lines are the functional derivative with respect to x. In the map (3.54), the individual graphs from (3.59) act on each other. This leads to a

[4] All the Feynman-like graphs in this thesis are generated with Joshua Ellis' 'TikZ-Feynman' package [45].

FIGURE 3.1: Nicolai rules for interacting massive SQM (with superpotential (3.46), taken from [33])

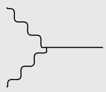
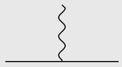
- External boson lines with a frequency ω give a factor $x(\omega)$.
- Free fermion propagators are

$$S_0(\omega) \; = $$  $$= \frac{-1}{\omega+m}\,, \qquad S_0(-\omega) \; = $$  $$= \frac{1}{\omega-m}\,. \qquad (3.56)$$

- Free boson propagators are

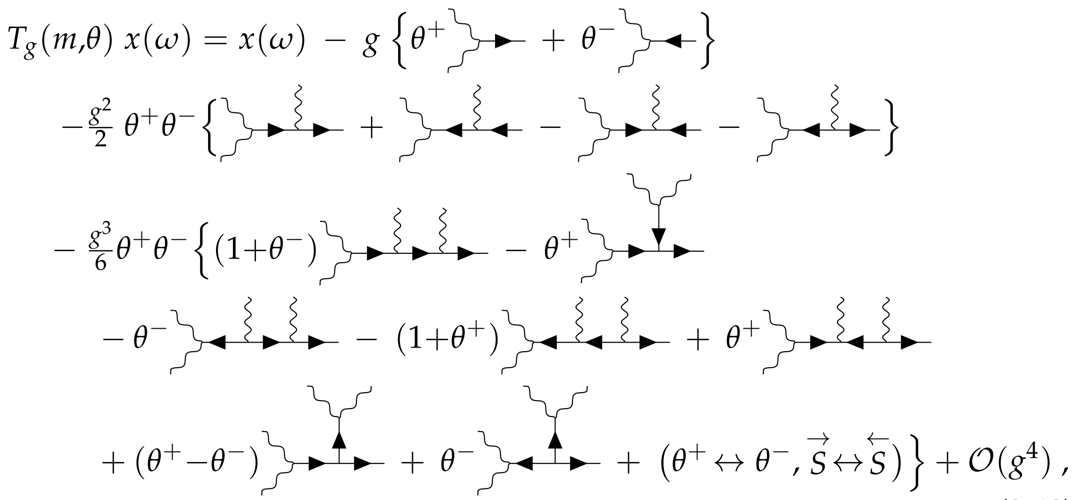
$$G_0(\omega) \; = $$ $$= \frac{1}{\omega^2-m^2}\,. \qquad (3.57)$$

- Vertices (implicit of order g) give factors

 $= 1\,,$  $= 2\,,$  $= 2\,.$ (3.58)

- At every vertex, energy conservation is enforced. We take all frequencies to be oriented towards the root of the tree, which carries the frequency ω of the transformed field $T_g\, x(\omega)$. Each remaining frequency ν comes with an integral $\int \frac{d\nu}{2\pi}$.

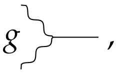
mixing of the two kinds of fermion propagators. The map to third order is



(3.60)

where $(\theta^+ \leftrightarrow \theta^-, \overrightarrow{S} \leftrightarrow \overleftarrow{S})$ contains the same terms of the third order that are already written out explicitly, but with the arrows on the fermion propagator and $\theta^\pm$ reversed. We note that all possible tree topologies and all possible combinations of fermion arrows appear in the expansion. Including the bosonic lines, the kind of trees that appear in (3.60) are known as 'strictly binary' or 'Otter trees', because seen from the root, each node has either zero or two children. Ignoring the bosonic lines, they are called 'weakly binary'. The number of such trees with n vertices (excluding the root) is the $(n+1)$th

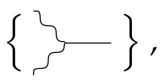
'Wedderburn–Etherington number' $\mathrm{WE}(n+1)$, as observed by Lechtenfeld in [34]. He also worked out a general rule for the weight of each tree. It is always a factor of $1/(2n)!!$ times a polynomial of order n in θ with integral coefficients. Moreover, every tree of second or higher order is multiplied with a factor $\theta^+\theta^- = \theta^+(1-\theta^+) = \theta^-(1-\theta^-)$. This shows that for the special values $\theta = \pm 1$, the map truncates after the first order. We are left with a map that is linear in g and quadratic in x:

$$T_g(m,\pm 1)\; x \;=\; x \;-\; g\, r_1\, x \;=\; x \;-\; g \tag{3.61}$$


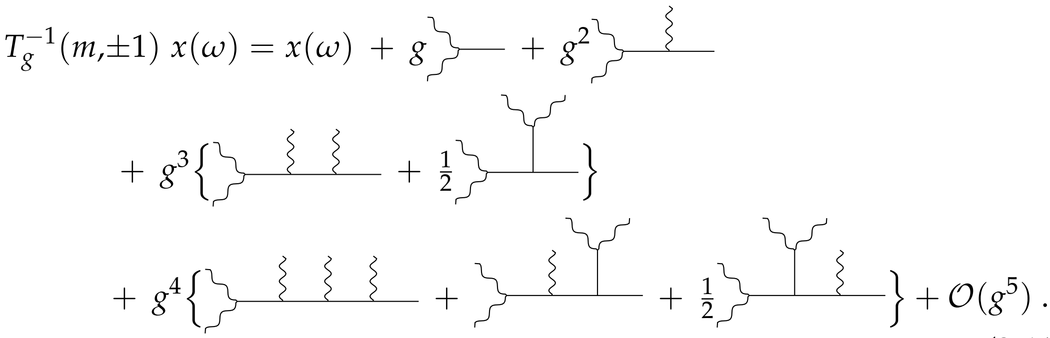
where one needs to add an arrow on the fermion propagator to the right (or left) for $\theta = +1$ (or -1). We could have also deduced this immediately by verifying the polynomiality and uniqueness condition (2.60) from Section 2.4:

$$r_k r_1 x = r_{k+1} x \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \theta = \pm 1\,. \tag{3.62}$$

If we stick with $\theta = \pm 1$ for a moment, by inverting (3.61) iteratively[5]

$$T_g^{-1}(m,\pm 1)\; x \;=\; x \;+\; g\, T_g^{-1}(m,\pm 1) \tag{3.63}$$


one finds for the inverse Nicolai map the expansion

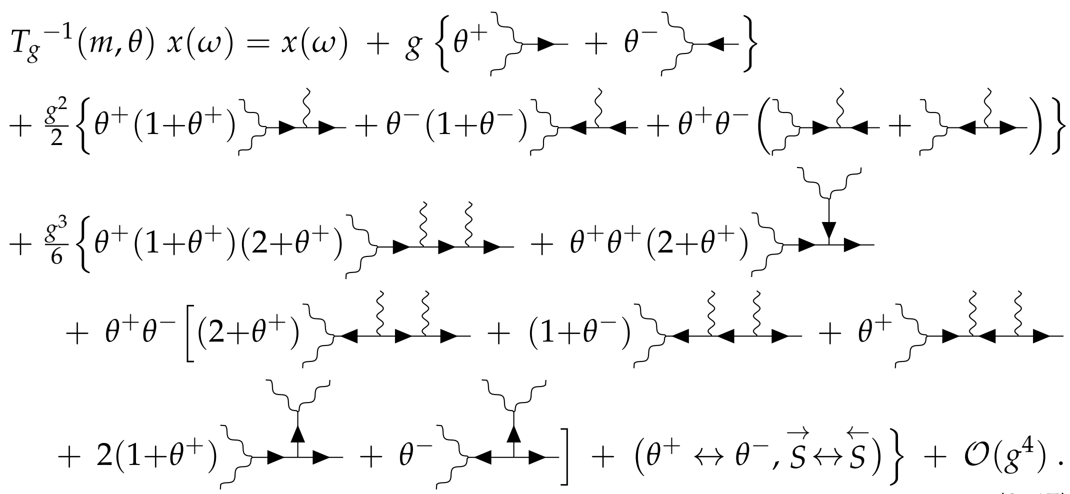


(3.64)

From the iterative construction (recalling the factors of the vertices from the Nicolai rules Figure 3.1), one can prove that every topology comes with a weight of unity times a symmetry factor. For example, the diagrams with the coefficients $\frac{1}{2}$ in (3.64) are symmetric under the exchange of two subtrees. If

[5]Using the fact that the (inverse) Nicolai map acts distributively on each bosonic line.

we allow any value for θ, the inverse map becomes significantly more complicated:



(3.65)

This expansion will allow us to compute various correlators in the following subsection, via the defining relation (1.13), which implies for the n-point function

$$\big\langle x(\omega_1)\ \ldots\ x(\omega_n)\big\rangle_g = \big\langle T_g^{-1} x(\omega_1)\ \ldots\ T_g^{-1} x(\omega_n)\big\rangle_0\,. \tag{3.66}$$

Before that, we give one more remark about the expansion of the (inverse) Nicolai map trees (3.60), (3.65). By estimating the number of tree graphs in a given order n with the Wedderburn-Etherington numbers (times 2^n taking into account all the possibilities of the orientation of fermion propagators) and finding bounds for a functional norm of a generic tree graph, Lechtenfeld showed in [34] that

$$\|T_g(m,\theta)x\|_2 \lessapprox \Big(1+\gamma\sum_{n=1}^{\infty} n^{-\beta}\big(\alpha\|x\|_2/\sqrt{m}\big)^n g^n\Big)\|x\|_2\,, \quad \text{for} \quad \theta=\mathcal{O}(1)\,, \tag{3.67}$$

where $\|\ldots\|_2$ is the usual L^2 norm and α,β,γ are numerical constants of roughly order one. Importantly, this demonstrates that for sufficiently small couplings g, the radius of convergence of the Nicolai map is finite (for $m{>}0$ and $\|x\|_2{<}\infty$). This must also hold for the inverse map, which is obtained through a formal power series inversion. It is interesting that the large order growth of tree diagrams

$$2^n\,\mathrm{WE}(n+1) \sim n^{-3/2}\cdot 4.967^n \qquad \text{for} \qquad n\to\infty \tag{3.68}$$

is not faster than exponential. This is in contrast to the Feynman diagram expansion for correlators, which is known to grow factorially with the order of the coupling. In the Nicolai map approach, the factorial growth is only reproduced when computing correlators via (3.66), after contracting inverse Nicolai maps with each other using Wick's theorem.

3.5.3 Amplitudes

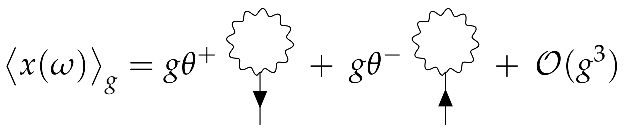
We now proceed with the computation of the bosonic one-, two-, and three-point function in the interacting theory. Naturally, loop integrals will appear, that are most conveniently evaluated by Wick rotating to Euclidean space ($\omega \to \mathrm{i}\omega$). We will only consider one loop at maximum. The free Euclidean propagators are

$$S_0(\omega) \;=\; \longrightarrow \;=\; \tfrac{\mathrm{i}\omega-m}{\omega^2+m^2}\,, \qquad S_0(-\omega) \;=\; \longleftarrow \;=\; \tfrac{-\mathrm{i}\omega-m}{\omega^2+m^2}\,, \tag{3.69}$$

and

$$G_0(\omega) \;=\; \sim\!\sim\!\sim \;=\; \tfrac{-1}{\omega^2+m^2}\,. \tag{3.70}$$

One-point function. In this simplest case, we just need to contract the open boson lines of (3.65). This results in two one-loop diagrams,



$$= 2\pi\delta(\omega)\, g\,(\theta^+{+}\theta^-)\ \tfrac{-m}{m^2}\int \tfrac{\mathrm{d}l}{2\pi}\,\tfrac{-1}{l^2+m^2} \;+\; \mathcal{O}(g^3) = g\,\tfrac{\pi\delta(\omega)}{m^2} \;+\; \mathcal{O}(g^3)\,, \tag{3.71}$$

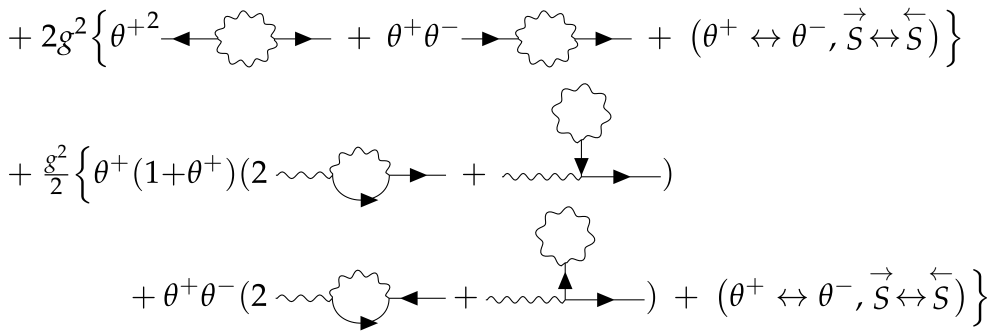
with the loop integral

$$\int_l G_0(l) = \int \tfrac{\mathrm{d}l}{2\pi}\,\tfrac{-1}{m^2+l^2} = -\tfrac{1}{2m}\,. \tag{3.72}$$

It is instructive to see that the θ dependence drops out of the final result as must be the case for all correlators. Further, (3.71) matches exactly the result on finds via traditional Feynman diagram techniques, see Appendix B.

Two-point function. To find the two-point function, we need to consider the square of the inverse Nicolai map and afterwards contract the open boson lines. Considering only connected and at most one-loop diagrams, we find

$$\langle x(\omega)\, x(\omega')\rangle_g = 2\pi\delta(\omega{+}\omega')\, G_0(\omega)$$



$$+ \tfrac{g^2}{2}\Big\{\omega \leftrightarrow \omega'\Big\} \;+\; \mathcal{O}(g^4)\,. \tag{3.73}$$

Here, two kinds of loop integrals appear:

$$\int_l G_0(l)G_0(\omega-l) = \int \tfrac{\mathrm{d}l}{2\pi}\,\tfrac{-1}{m^2+l^2}\,\tfrac{-1}{m^2+(\omega-l)^2} = \tfrac{1}{4m^3+m\omega^2}\,, \tag{3.74}$$

$$\int_l G_0(l)S_0(\omega+l) = \int \tfrac{\mathrm{d}l}{2\pi}\,\tfrac{-1}{m^2+l^2}\,\tfrac{\mathrm{i}(\omega+l)-m}{m^2+(\omega+l)^2} = \tfrac{1}{2}\,\tfrac{2m-\mathrm{i}\omega}{4m^3+m\omega^2}\,. \tag{3.75}$$

Note that the second kind of loop (3.75) could never appear in an ordinary Feynman diagram expansion, since it is a mixed loop consisting of one fermion and one boson propagator. The evaluation of (3.73) is still relatively short, so we present it here explicitly. We obtain

$$\begin{aligned}
\big\langle x(\omega)\,x(\omega')\big\rangle_g \;=\;& 2\pi\delta(\omega+\omega')\,G_0(\omega) \\
&+ 2g^2\tfrac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\Big\{\tfrac{{\theta^+}^2(\mathrm{i}\omega'-m)(\mathrm{i}\omega-m)\,+\,\theta^+\theta^-(-\mathrm{i}\omega'-m)(\mathrm{i}\omega-m)}{4m^3+m\omega^2} + (\theta^+\leftrightarrow\theta^-,\omega\leftrightarrow\omega')\Big\} \\
&+ \tfrac{g^2}{2}\tfrac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\Big\{\theta^+(1+\theta^+)(-4\,\tfrac{1}{2}\tfrac{(\mathrm{i}\omega-m)(2m-\mathrm{i}\omega)}{4m^3+m\omega^2} \;-\; 2\tfrac{\mathrm{i}\omega-m}{2m^2}) \\
&\qquad + \theta^+\theta^-(-4\,\tfrac{1}{2}\tfrac{(-\mathrm{i}\omega-m)(2m-\mathrm{i}\omega)}{4m^3+m\omega^2} \;-\; 2\tfrac{\mathrm{i}\omega-m}{2m^2}) + (\theta^+\leftrightarrow\theta^-,\omega\leftrightarrow\omega)\Big\} \\
&+ \tfrac{g^2}{2}\tfrac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\Big\{\omega\to-\omega\Big\} \;+\; \mathcal{O}(g^4)\,,
\end{aligned} \tag{3.76}$$

which can be simplified to

$$\begin{aligned}
\big\langle x(\omega)\,x(\omega')\big\rangle_g =\;& 2\pi\delta(\omega+\omega')G_0(\omega) \\
&+ 2g^2\tfrac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\,\tfrac{({\theta^+}^2+{\theta^-}^2)(m^2+\omega^2)\,+\,2\theta^+\theta^-(m^2-\omega^2)}{4m^3+m\omega^2} \\
&+ \tfrac{g^2}{2}\tfrac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\Big\{[1+{\theta^+}^2+{\theta^-}^2](4\,\tfrac{2m^2-\omega^2}{4m^3+m\omega^2} + \tfrac{2}{m}) \\
&\qquad + 2\theta^+\theta^-(4\,\tfrac{2m^2+\omega^2}{4m^3+m\omega^2} + \tfrac{2}{m})\Big\} + \mathcal{O}(g^4)\,.
\end{aligned} \tag{3.77}$$

Finally, with $\theta^+ + \theta^- = 1$ and $\theta^+ - \theta^- = \theta$, the theta dependence cancels and we arrive at

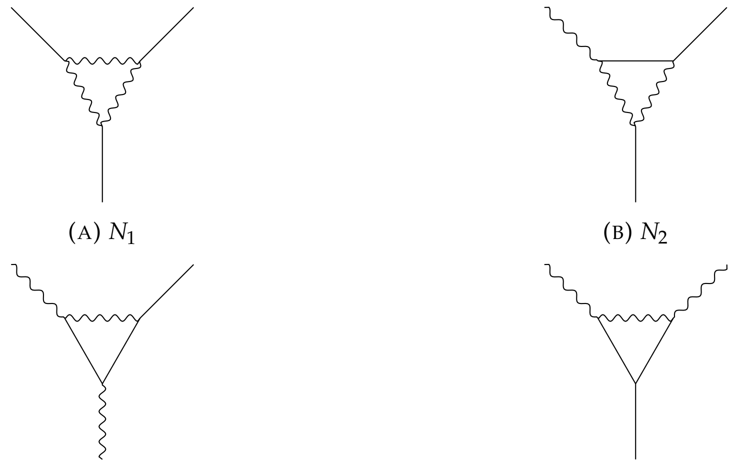
$$\big\langle x(\omega)\,x(\omega')\big\rangle_g = 2\pi\delta(\omega+\omega')G_0(\omega) + g^2\,\tfrac{2\pi\delta(\omega+\omega')}{(m^2+\omega^2)^2}\,\tfrac{18m^2}{4m^3+m\omega^2} + \mathcal{O}(g^4)\,, \tag{3.78}$$

again surviving the cross-check from the traditional Feynman diagram approach, see Appendix B.

Three-point function. As we have seen above, for arbitrary θ, the calculations become quite involved. Having verified generally and explicitly that correlators do not depend on θ, for the three-point function we allow ourselves to work with the special values $\theta = \pm1$ (still retaining the sign ambiguity). Thus, we can use the simpler expansion (3.64) for the inverse map, where all the fermion propagators are given by $S_0(\pm\omega)$ (3.69) for $\theta = \pm1$. We evaluate the three-point function with the usual argument

$$\big\langle x(\omega_1)\,x(\omega_2)\,x(\omega_3)\big\rangle_g = \big\langle T_g^{-1}x(\omega_1)\,T_g^{-1}x(\omega_2)\,T_g^{-1}x(\omega_3)\big\rangle_0\,. \tag{3.79}$$

From (3.65), we see that at $\mathcal{O}(g^3)$, we obtain correlators of six bosonic fields.



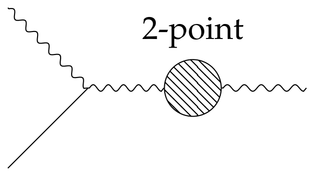
(A) N_1 (B) N_2

(C) N_3 (D) N_4

FIGURE 3.2: Irreducible 1-loop diagrams for the SQM 3-point function.

By Wick's theorem, such a correlator gives 15 diagrams[6]. We ignore any disconnected diagrams and can combine many equivalent diagrams. Since (3.79) is totally symmetric in $(\omega_1, \omega_2, \omega_3)$, our final result must be as well. Modulo permutations of the frequencies, there remain 11 distinct diagrams. Of these, four fall within the usual notion of 1-particle-irreducible[7] (1PI). We collect them in Figure 3.2. Moreover, five diagrams can be combined to generate the 2-point function on one leg, see (3.73) for $\theta = \pm 1$.[8] Naturally, they are one-particle reducible (1PR). Schematically, we represent them as

$$N_5 = $$



. (3.80)

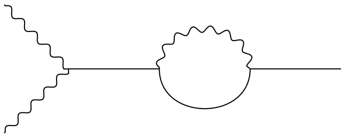
While we have drawn two boson lines for the bosonic two-point function in (3.80), in the actual Nicolai graphs, the external lines can also be fermion lines, as we have seen in (3.73). There are in fact two more 'reducible' diagrams to consider, shown in Figure 3.3. The reason why these should strictly not be called reducible is that the Nicolai rules do not allow fermion lines to run all the way through a single diagram. That means one could argue that N_6 and N_7 should be 1PI as well. The explicit evaluation of all these diagrams is straightforward but quite technical. It can be found in Appendix B. The general idea is to express each contribution in terms of the elementary symmetric poynomials of the external frequencies

$$t_1 = \omega_1 + \omega_2 + \omega_3 \equiv 0\,, \quad t_2 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3\,, \quad t_3 = \omega_1\omega_2\omega_3\,, \tag{3.81}$$

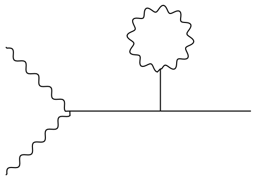
[6]In general, a free n-point correlator (with n even, otherwise it will vanish) reduces to $(n-1)!! = (n-1)\cdot(n-3)\cdot\ldots\cdot 1$ terms $n/2$ two-point correlators.

[7]That is, no internal line can be cut such that one is left with two distinct diagrams.

[8]These five diagrams for e.g. $\theta = +1$ are the first, third and fourth diagram of (3.73), plus the third and fourth diagram, but with the external lines exchanged.



(A) N_6



(B) N_7

FIGURE 3.3: Two more 'reducible' 1-loop diagrams for the SQM 3-point function.

where t_1 vanishes due to energy (frequency) conservation. It emerges from the calculation that the irreducible contributions

$$N_{1\text{PI}} = N_1 + N_2 + N_3 + N_4\,, \tag{3.82}$$

and the reducible contributions

$$N_{1\text{PR}} = N_5 + N_6 + N_7 \tag{3.83}$$

are θ-independent[9] separately. This was not expected a priori and is quite a curious fact. Furthermore, the θ-dependent terms are all imaginary and proportional to an odd power of t_3. Comparing the Nicolai diagram approach with the Feynman diagram approach (see Appendix B), we find that the notion of 1PI differs, since the results only match after adding all (1PI and 1PR) contributions in the two methods (even if we count N_6 and N_7 as 1PI).

[9]Recall that we still allow the two different signs $\theta = \pm 1$.

Chapter 4

$\mathcal{N} = 1$ super Yang–Mills theories

Note: *This whole chapter is largely based on the author's published work [29, 35].*

We begin this chapter by discussing the action of pure $\mathcal{N} = 1$ super Yang–Mills theories. The field content (A, λ, D) consists of the gauge (Yang–Mills) fields A_μ^a, their supersymmetric partners the gaugini λ^a, as well as potentially auxiliaries D^a, when considering off-shell theories. The redundant degrees of freedom of this gauge theory can be fixed by the Faddeev-Popov procedure, which incorporates a gauge fixing function $\mathcal{G}^a(A)$ and ghost fields $\bar{C}, C$. For simplicity, we consider as gauge group $\mathrm{SU}(N)$ with real antisymmetric structure constants f^{abc} such that

$$f^{abc} f^{abd} = N\delta^{cd} \,, \qquad a, b, \ldots = 1, 2, \ldots, N^2-1 \,. \tag{4.1}$$

All fields are in the adjoint representation of the gauge group. As first discovered by Brink, Schwarz and Scherk in 1977 [25], pure $\mathcal{N}=1$ super Yang–Mills theories can only exist[1] in $D=3, 4, 6, 10$ spacetime dimensions. To quickly recount their argument, recall that in D dimensions the (on-shell) gauge field has $D-2$ degrees of freedom, while the Dirac spinor has $2^{\lfloor D/2 \rfloor}$. To match these two numbers, in even dimensions, the Weyl condition can be imposed, while in $D \equiv 1, 2, 3, 4 \mod 8$, the Majorana condition can be imposed. Both conditions reduce the number of degrees of freedom by a factor of one-half. Additionally, in $D \equiv 2 \mod 8$, the Majorana and Weyl conditions can be imposed simultaneously, i.e. reducing the degrees of freedom by a factor of $1/4$. An overview of the four remaining cases where the dimensions exactly match can be found in Table 4.1. In this entire chapter, we work in D-dimensional

TABLE 4.1: Matching of the bosonic and fermionic degrees of freedom (dofs) in on-shell $\mathcal{N}=1$ super Yang–Mills theories.

D	bosonic dofs	spinor	fermionic dofs
3	1	Majorana	$2^1/2 = 1$
4	2	Majorana or Weyl	$2^2/2 = 2$
6	4	Weyl	$2^3/2 = 4$
10	8	Majorana-Weyl	$2^5/4 = 8$

Minkowski spacetime, although this can equally well be done in Euclidean

[1] i.e. be invariant under SUSY transformations.

spacetime. The action can be written in simplified form as

$$S_{\text{SUSY}}[A,\lambda,D,C,\bar{C}] = \int \mathrm{d}^D x \,\Big\{ -\tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} - \tfrac{1}{2\xi} \mathcal{G}^a(A)^2 \\ + \text{fermions} + \text{ghosts} + \text{auxiliaries} \Big\}\,, \tag{4.2}$$

with gauge fixing function $\mathcal{G}^a(A)$ and Yang–Mills field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,. \tag{4.3}$$

We introduce extra notation to distinguish the expectation values obtained with the original action S_{SUSY} from those obtained from the action S_g, where the fermions and auxiliaries have been integrated out:

$$\begin{aligned} \langle\!\langle X[A] \rangle\!\rangle_g &:= \int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}D\mathcal{D}c\mathcal{D}\bar{c}\; \mathrm{e}^{\mathrm{i}S_{\text{SUSY}}[A,\lambda,D,c,\bar{c}]}\, X[A]\,, \\ \langle X[A] \rangle_g &:= \int \mathcal{D}A\; \mathrm{e}^{\mathrm{i}S_g[A]}\, \Delta_{\text{MSS}}[A]\, \Delta_{\text{FP}}[A]\, X[A]\,. \end{aligned} \tag{4.4}$$

Here, $\Delta_{\text{FP}}[A]$ the Faddeev–Popov determinant, produced by integrating out the ghosts, and $\Delta_{\text{MSS}}[A]$ is the Matthews–Salam–Seiler determinant, obtained from integrating out the gaugini.[2] In $\mathcal{N}=1$ SYM, integrating out the auxiliaries D is trivial since they have an equation of motion $D=0$. The distinction (4.4) is only for convenience in equations later on, since of course $\langle\!\langle X[A]\rangle\!\rangle_g = \langle X[A]\rangle_g$ by construction. Further, it should be noted that (just like for scalar theories) the expectation values can be normalized to $\langle\!\langle \mathbb{1}\rangle\!\rangle_g = \langle \mathbb{1}\rangle_g = 1$ by the vanishing of the vacuum energy. We can define the Nicolai map for gauge theories in the exact same way we defined it for scalar theories (1.13). It is a nonlinear and nonlocal mapping

$$T_g:\; A^a_\mu(x) \;\mapsto\; A'^a_\mu(x;g,A) \tag{4.5}$$

of the Yang–Mills fields A^a_μ ($\mu = 0,1,\ldots,D-1$). Its inverse exists at least as a formal power series in g near the identity, so that the Nicolai map can be defined by the key identity

$$\langle\!\langle X[A] \rangle\!\rangle_g = \langle X[A] \rangle_g = \langle X[T_g^{-1}A] \rangle_0 \quad \forall\, X\,. \tag{4.6}$$

As for scalar theories, this gives access to quantum correlators in the interacting theory via a free, purely bosonic functional measure.

There are two known ways for constructing the Nicolai map in $\mathcal{N}=1$ super Yang–Mills theories:

1. **Canonical construction** via the off-shell formalism in $D=4$ for general gauges: This construction makes use of the off-shell formulation of $\mathcal{N}=1$ SYM in $D=4$. It will be described in detail in Section 4.1. This approach was suggested by Dietz and Lechtenfeld in the 1980s [10–13] and was further developed by Lechtenfeld and this author in 2021 [29] and at the same time by Malcha and Nicolai in [28]. As a particular

[2] Technically it is a Pfaffian and not a determinant for Majorana fermions.

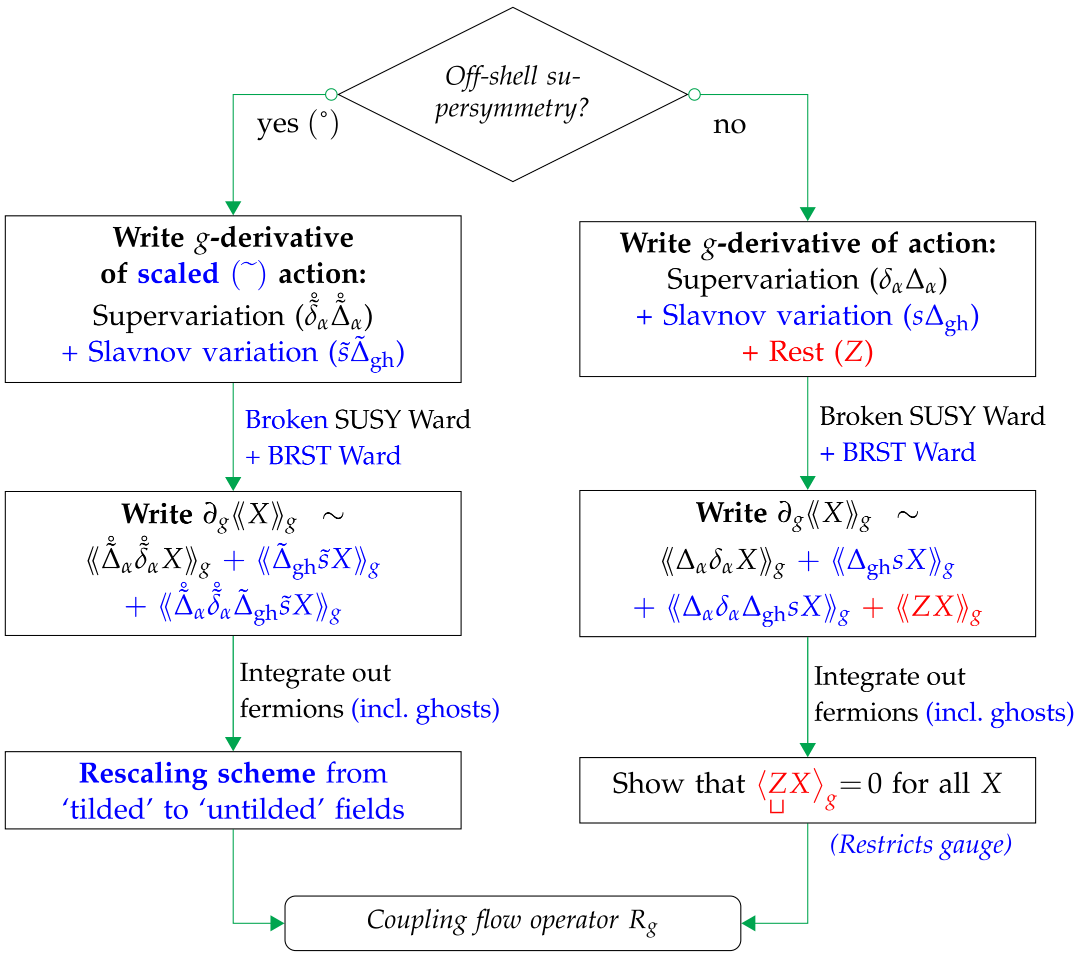


FIGURE 4.1: Diagram for the construction of the coupling flow operator. Everything written in blue applies only to gauge theories. The left side is the canonical construction scheme (Section 4.1) for theories with off-shell supersymmetry. The right side is the ad-hoc construction (Section 4.3) that does not require off-shell supersymmetry, but it is not guaranteed that the final step works. The relations in the second step are schematic and only hold up to prefactors.

example, the map in axial gauge will be computed to second order in Section 4.2. In four dimensions, the action can also be amended by a topological θ-term. This leads to a simplified construction [35], that will be discussed in Section 4.4.

2. **Ad-hoc construction** in $D = 3, 4, 6, 10$ in the Landau gauge: For $D=6$ and $D=10$, there are no known off-shell formulations of the SYM action. Here, an ad-hoc construction must be used that seems to only work in the Landau gauge $\mathcal{G}(A) = \partial_\mu A^\mu$. This construction was proposed by Ananth, Lechtenfeld, Malcha, Nicolai, Pandey, and Pant in 2020 [24] and will be outlined in Section 4.3.

The technical details of these two methods are summarized as a diagram in Figure 4.1. The various steps will be described in detail in the following.

4.1 Canonical construction for general gauges (D=4)

Note: *This section is largely following the author's published work [29].*

4.1.1 Off-shell action

The introduction of the gauge fixing and ghost terms has important consequences for the construction of the Nicolai map. It breaks the manifest supersymmetry and reduces gauge symmetry to BRST symmetry (with transformations given in due course). Therefore, the action cannot be written as a supervariation anymore. To circumvent this problem, a particular scaling[3] of the field content (notated by tildes above the fields) can be chosen, where

$$\begin{aligned} S_{\text{SUSY}}[\widetilde{A},\widetilde{\lambda},\widetilde{\bar{\lambda}},\widetilde{D},\widetilde{c},\widetilde{\bar{c}}] &= S_{\text{inv}}[\widetilde{A},\widetilde{\lambda},\widetilde{\bar{\lambda}},\widetilde{D}] \;+\; S_{\text{gf}}[\widetilde{A},\widetilde{c},\widetilde{\bar{c}}]\,, \\ S_{\text{inv}}[\widetilde{A},\widetilde{\lambda},\widetilde{\bar{\lambda}},\widetilde{D}] &= \tfrac{1}{g^2}\int \mathrm{d}^4x\,\left\{-\tfrac14 \widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu} - \tfrac{\mathrm{i}}{2}\widetilde{\bar{\lambda}}\widetilde{\not{D}}\widetilde{\lambda} \;+\tfrac12 \widetilde{D}^2\right\}, \\ S_{\text{gf}}[\widetilde{A},\widetilde{c},\widetilde{\bar{c}}] &= \tfrac{1}{g^2}\int \mathrm{d}^4x\,\left\{-\tfrac{1}{2\xi}\mathcal{G}(\widetilde{A})^2 + g\,\widetilde{\bar{c}}\tfrac{\partial\mathcal{G}(\widetilde{A})}{\partial\widetilde{A}_\mu}\widetilde{\mathrm{D}}_\mu\widetilde{c}\right\}, \end{aligned} \tag{4.7}$$

working in four spacetime dimensions.[4] Here, the coupling only appears as an overall factor g^{-2} in front of the integrals, besides one factor of g multiplying the ghost term. As the subscripts indicate, the action splits into a SUSY-invariant part S_{inv} and a gauge fixing part S_{gf}. Before continuing with the construction, we give a few remarks about our conventions and notation. Note that we have left the color indices implicit in (4.7), adopting the notation first used by Lechtenfeld in his thesis [11]. For example, the covariant derivative in this scaling is

$$\widetilde{\mathrm{D}}_\mu = \partial_\mu + \tilde{A}_\mu\times \qquad \Longleftrightarrow \qquad (\widetilde{\mathrm{D}}_\mu ...)^a = \partial_\mu(...)^a + f^{abc}\widetilde{A}^b_\mu(...)^c\,, \tag{4.8}$$

while the field strength is

$$\widetilde{F}_{\mu\nu} = \partial_\mu\widetilde{A}_\nu - \partial_\nu\widetilde{A}_\mu + \widetilde{A}_\mu\times\widetilde{A}_\nu \qquad \Longleftrightarrow \qquad \widetilde{F}^a_{\mu\nu} = \partial_\mu\widetilde{A}^a_\nu - \partial_\nu\widetilde{A}^a_\mu + f^{abc}\widetilde{A}^b_\mu\widetilde{A}^c_\nu\,. \tag{4.9}$$

We sum over colors in products, e.g.

$$\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} \equiv \widetilde{F}^a_{\mu\nu}\widetilde{F}^{a\,\mu\nu}\,, \tag{4.10}$$

except when we indicate an explicit cross-product, e.g.

$$\widetilde{\bar{\lambda}}\;\widetilde{A}_\mu\times\widetilde{\lambda} \equiv f^{abc}\widetilde{\bar{\lambda}}^a\widetilde{A}^b_\mu\widetilde{\lambda}^c\,. \tag{4.11}$$

[3] sometimes known as the 'canonical' or 'geometric' scaling in modern high energy physics theory literature.

[4] There are no known off-shell formulations of $\mathcal{N}=1$ SYM in D=3, 6, 10 (with finitely many auxiliary fields). However, a dimensional reduction of the $\mathcal{N}=1$ D=4 case would lead to a Nicolai map for $\mathcal{N}=2$ D=3 SYM.

Furthermore, throughout this thesis, we work in Minkowski space using the mostly-plus metric

$$\eta^{\mu\nu} = \mathrm{diag}(-1,\ +1,\ +1,\ +1)\,. \tag{4.12}$$

Our spinor- and gamma-matrix conventions are mostly adopted from [46] and are listed in Appendix A.

The scaling of the fields in (4.7) is only used intermediately for the construction of the coupling flow operator. In the end, we want to do perturbation theory in g. For now, it allows us to generate the invariant Lagrangian through an off-shell superfield formalism

$$\mathcal{L}_{\mathrm{inv}} = \tfrac{1}{16g^2N}\mathrm{tr}\big(W^\alpha W_\alpha\big|_{\vartheta\vartheta} + \bar{W}^{\dot\alpha}\bar{W}_{\dot\alpha}\big|_{\bar\vartheta\bar\vartheta}\big)\,, \tag{4.13}$$

where the trace is over colors and W_α is the non-abelian supersymmetric field strength (with conjugate $\bar{W}_{\dot\alpha}$). In (4.13), α and $\dot\alpha$ are Weyl spinor indices. For details on the superfield formulation, see Appendix C, where the construction in terms of the superfields is carried out in detail for the more extensive $\mathcal{N}{=}4$ SYM (using an $\mathcal{N}{=}1$ superfield formalism), but this only adds more fields to the $\mathcal{N}{=}1$ SYM field content. As mentioned in Section 2.2, the superfield structure gives us the form of the (off-shell) supervariations

$$\begin{aligned}
\mathring{\delta}_\alpha \widetilde{A}_\nu &= -\mathrm{i}(\widetilde{\bar\lambda}\gamma_\nu)_\alpha\,,\\
\mathring{\delta}_\alpha \widetilde{\lambda}_\beta &= -\tfrac{1}{2}(\gamma^{\mu\nu})_{\beta\alpha}\widetilde{F}_{\mu\nu} + \widetilde{D}(\gamma_5)_{\beta\alpha}\,,\\
\mathring{\delta}_\alpha \widetilde{D} &= -\mathrm{i}(\widetilde{\mathrm{D}}_\mu\widetilde{\bar\lambda}\gamma_5\gamma^\mu)_\alpha\,,
\end{aligned} \tag{4.14}$$

where we have switched from the Weyl basis to a Majorana basis[5], which we find more convenient. This means α is a four-component Majorana spinor index.

4.1.2 Intermediate coupling flow operator

This construction is mostly following Lechtenfeld's Ph.D. thesis [11], only that he remained in the Weyl basis. We write the invariant action as a supervariation

$$S_{\mathrm{inv}}[\widetilde{A},\widetilde{\lambda},\widetilde{\bar\lambda},\widetilde{D}] = \int \mathrm{d}^4x\ \mathcal{L}_{\mathrm{inv}} = \tfrac{1}{2g^2}\mathring{\delta}_\alpha\mathring{\Delta}_\alpha\,, \tag{4.15}$$

with the superfield component

$$\mathring{\Delta}_\alpha = \tfrac{1}{4}\int \mathrm{d}^4x\big\{-\widetilde{D}\gamma_5\widetilde{\lambda} - \tfrac{1}{2}\widetilde{F}_{\mu\nu}\gamma^{\mu\nu}\widetilde{\lambda}\big\}_\alpha\,. \tag{4.16}$$

From (4.15) it follows that also the g-derivative of the invariant action can be generated by a supervariation. On the other hand, the g-derivative of the gauge fixing part of the action can be generated by a Slavnov variation [11]. The Slavnov (or BRST) transformations are generated by a fermionic operator

[5] Recall that in four dimensions, we can choose the spinors to be either Majorana or Weyl spinors.

s and take the form[6]

$$\begin{aligned} s\widetilde{A}_\mu &= \sqrt{g}\,\widetilde{\mathrm{D}}_\mu \widetilde{c}\,, & s\widetilde{\lambda} &= \sqrt{g}\,\widetilde{\lambda} \times \widetilde{c}\,, & s\widetilde{\bar{\lambda}} &= \sqrt{g}\,\widetilde{\bar{\lambda}} \times \widetilde{c}\,, \\ s\widetilde{D} &= \sqrt{g}\,\widetilde{D} \times \widetilde{c}\,, & s\widetilde{c} &= -\tfrac{\sqrt{g}}{2}\,\widetilde{c} \times \widetilde{c}\,, & s\widetilde{\bar{c}} &= \tfrac{1}{\sqrt{g}}\tfrac{1}{\xi}\,\mathcal{G}(\widetilde{A})\,. \end{aligned} \tag{4.17}$$

Introducing a ghost contribution

$$\Delta_{\mathrm{gh}}[\widetilde{\bar{c}}, \widetilde{A}] = \int \mathrm{d}^4x \{\widetilde{\bar{c}}\mathcal{G}(\widetilde{A})\}\,, \tag{4.18}$$

it is easy to confirm that

$$\partial_g S_{\mathrm{gf}}[\widetilde{A}, \widetilde{c}, \widetilde{\bar{c}}] = g^{-5/2}\, s\Delta_{\mathrm{gh}}[\widetilde{\bar{c}}, \widetilde{A}]\,. \tag{4.19}$$

Here, we have assumed a g-independent linear gauge

$$\mathcal{G}(\widetilde{A}) = \partial^\mu \widetilde{A}_\mu \qquad \text{or} \qquad n^\mu \widetilde{A}_\mu\,, \tag{4.20}$$

but this is only for convenience. Any g-dependence of the gauge will cancel out in the construction of the coupling flow operator, and nonlinear gauges can be implemented with $\mathcal{G}(A) = \widetilde{\mathcal{G}}(gA)/g$ for an arbitrary function $\widetilde{\mathcal{G}}$ [27]. In total, the g-derivative of the action can be written as

$$\partial_g S_{\mathrm{SUSY}} = -g^{-3}\{\mathring{\delta}_\alpha \mathring{\Delta}_\alpha - \sqrt{g}\, s\,\Delta_{\mathrm{gh}}\}\,. \tag{4.21}$$

We can use (4.21) to construct the coupling flow operator. Taking the g-derivative of expectation values (with respect to the full action) gives

$$\partial_g \langle\!\langle X \rangle\!\rangle_g = \langle\!\langle \partial_g X \rangle\!\rangle_g + \mathrm{i}\big\langle\!\big\langle X\big[-g^{-3}\mathring{\delta}_\alpha \mathring{\Delta}_\alpha + g^{-5/2}\, s\Delta_{\mathrm{gh}}\big]\big\rangle\!\big\rangle_g\,. \tag{4.22}$$

Assuming that there are no anomalies in the path integration, for the term that is a supervariation, we can use the Ward identity for BRST invariance

$$\langle\!\langle sY \rangle\!\rangle = 0\,, \tag{4.23}$$

for any observable Y, while the SUSY Ward identity gets modified to

$$\langle\!\langle \delta_\alpha Y \rangle\!\rangle = \mathrm{i}g^{-3/2}\big\langle\!\big\langle \delta_\alpha \Delta_{\mathrm{gh}}\, sY \big\rangle\!\big\rangle\,, \quad \text{since} \quad \mathring{\delta}_\alpha S_{\mathrm{gf}} = -\sqrt{g}\, s(\delta_\alpha \Delta_{\mathrm{gh}})\,, \tag{4.24}$$

meaning the full action is not invariant under supersymmetry (c.f. the scalar case (2.32)). Together with the graded Leibniz rules for the fermionic operators δ_α and s, we find

$$\begin{aligned} \partial_g \langle\!\langle X \rangle\!\rangle_g = {} & \langle\!\langle \partial_g X \rangle\!\rangle_g - \mathrm{i}g^{-3}\langle\!\langle \mathring{\Delta}_\alpha \mathring{\delta}_\alpha X \rangle\!\rangle \\ & - \mathrm{i}g^{-3}\big\langle\!\big\langle X\big[-\mathrm{i}g^{-3/2}\mathring{\Delta}_\alpha \mathring{\delta}_\alpha \Delta_{\mathrm{gh}} - g^{1/2}\Delta_{\mathrm{gh}}\big]sX\big\rangle\!\big\rangle_g\,. \end{aligned} \tag{4.25}$$

[6] In principle, one could multiply all the Slavnov variations with an arbitrary power of g, but in the end, this does not change the coupling flow operator, so we simply adopted the conventions from [11].

Next, we integrate out the fermions and the auxiliary field with equation of motion $\widetilde{D}$=0. When doing so, we obtain propagators of the gaugini

$$\widetilde{S} = \widetilde{\not{D}}^{-1} = -\underbracket{\widetilde{\lambda}\ \widetilde{\bar{\lambda}}}\,, \tag{4.26}$$

and the ghosts

$$\widetilde{G} = \Big(\tfrac{\partial \mathcal{G}(\widetilde{A})}{\partial \widetilde{A}_\mu}\widetilde{\mathrm{D}}_\mu\Big)^{-1} = -\mathrm{i}\,\underbracket{\widetilde{c}\ \widetilde{\bar{c}}}\,. \tag{4.27}$$

Note that there are extra factors of $-\mathrm{i}/g^2$ in the gaugino term and $-\mathrm{i}/g$ in the ghost term in the exponential of the first line in (4.4), which we have not inserted in the propagators above. They are taken care of separately in the coupling flow operator below. In this 'tilde' scaling, we define an intermediate coupling flow operator $\widetilde{R}_g$ via

$$\partial_g \langle \mathcal{O}[\widetilde{A}]\rangle_g = \langle (\partial_g + \tfrac{1}{g}\widetilde{R}_g[\widetilde{A}])\mathcal{O}[\widetilde{A}]\rangle_g\,. \tag{4.28}$$

Comparing this expression with (4.25), we deduce the (see also [10,11])

intermediate coupling flow operator

$$\widetilde{R}_g[\widetilde{A}] = -\mathrm{i}\,\underbracket{\Delta_\alpha[\widetilde{A}]\ \delta_\alpha} + \tfrac{\mathrm{i}}{\sqrt{g}}\,\underbracket{\Delta_{\mathrm{gh}}[\widetilde{A}]\ s} - \tfrac{1}{\sqrt{g}}\,\underbracket{\Delta_\alpha[\widetilde{A}]\ \big(\delta_\alpha}\underbracket{\Delta_{\mathrm{gh}}[\widetilde{A}]\big)\ s}\,, \tag{4.29}$$

where the 'on-shell versions' Δ_α and δ_α are obtained from $\mathring{\Delta}_\alpha$ and $\mathring{\delta}_\alpha$ by setting $\widetilde{D}$=0. Since $s\widetilde{A}_\mu = \sqrt{g}\widetilde{\mathrm{D}}_\mu\widetilde{c}$, we note that $\widetilde{R}_g[\widetilde{A}]$ is completely independent of the coupling g [27]. We further remark that

$$\underbracket{\Delta_{\mathrm{gh}}[\widetilde{A}]\ s}\,\mathcal{G}(\widetilde{A}) = -\mathrm{i}\sqrt{g}\,\mathcal{G}(\widetilde{A})\,, \tag{4.30}$$

and thus the gauge condition

$$\widetilde{R}_g[\widetilde{A}]\mathcal{G}(\widetilde{A}) = \mathcal{G}(\widetilde{A}) \qquad \Rightarrow \qquad (\partial_g + \tfrac{1}{g}\widetilde{R}_g[\widetilde{A}])\tfrac{1}{g}\mathcal{G}(\widetilde{A}) = 0 \tag{4.31}$$

holds. It shows that the gauge class is invariant under the coupling constant flow and therefore a fixed point of the Nicolai map [27]. In the original definition of the Nicolai map, this was an additional requirement next to the free action (1.15) and determinant matching condition (1.16), but here it holds by construction. In fact, it follows from the free action condition since the parameter ξ in S_{gf} is arbitrary.[7]

4.1.3 Rescaled coupling flow operator

Instead of working with the operator $\widetilde{R}_g[\widetilde{A}]$, we would now like to find an operator that allows us to set up perturbation theory in g. To that end, we rescale the gauge fields $\widetilde{A}$=gA, following closely the paper [27] by Lechtenfeld and this author. From the defining relation for the Nicolai map

[7] In the original definition, the free action condition was only referring to the invariant part of the action.

(c.f. (1.13))

$$\langle \mathcal{O}[\widetilde{A}] \rangle_g = \langle \mathcal{O}[T_g^{-1} \widetilde{A}] \rangle_0 \tag{4.32}$$

and (4.28), we can deduce a formal expansion of the inverse map acting on $\widetilde{A}$

$$T_g^{-1} \widetilde{A} = \exp\Big\{ g \, \big(\partial_{g'} + \tfrac{1}{g'} \widetilde{R}_{g'}[\widetilde{A}] \big) \Big\} \, \widetilde{A} \, \Big|_{g'=0} \, , \tag{4.33}$$

but the $g' \to 0$ limit is ill-defined. We have to rescale first, yielding the prescription[8]

$$T_g^{-1} A = \tfrac{1}{g} \exp\Big\{ g \, \big(\partial_{g'} + \tfrac{1}{g'} \widetilde{R}_{g'}[\widetilde{A}] \big) \Big\} \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \, , \tag{4.34}$$

where we will see explicitly that the $g' \to 0$ limit is nonsingular. Remembering that $\widetilde{R}_{g'}[\widetilde{A}]$ is actually independent of g', we can execute the g' derivatives

$$\begin{aligned}
T_g^{-1} A &= \tfrac{1}{g} \sum_{n=0}^{\infty} \tfrac{g^n}{n!} \, \big(\partial_{g'} + \tfrac{1}{g'} \widetilde{R}_{g'}[\widetilde{A}] \big)^n \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \\
&= \tfrac{1}{g} \sum_{n=0}^{\infty} \tfrac{g^n}{n!} \, \Big((g')^{-\widetilde{R}_{g'}[\widetilde{A}]} \, \partial_{g'} \, (g')^{\widetilde{R}_{g'}[\widetilde{A}]} \Big)^n \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \\
&= \tfrac{1}{g} \sum_{n=0}^{\infty} \tfrac{g^n}{n!} \, (g')^{-\widetilde{R}_{g'}[\widetilde{A}]} \, \partial^n_{g'} \, (g')^{\widetilde{R}_{g'}[\widetilde{A}]} \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \\
&= \tfrac{1}{g} \, (g')^{-\widetilde{R}_{g'}[\widetilde{A}]} \, \exp\{ g \, \partial_{g'} \} \, (g')^{\widetilde{R}_{g'}[\widetilde{A}]} \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \\
&= \tfrac{1}{g} \, (g')^{-\widetilde{R}_{g'}[\widetilde{A}]} \, (g' + g)^{\widetilde{R}_{g'}[\widetilde{A}]} \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} \\
&= \tfrac{1}{g} \Big(1 + \tfrac{g}{g'} \Big)^{\widetilde{R}_{g'}[\widetilde{A}]} \, \widetilde{A} \, \Big|_{\widetilde{A}=g'A} \Big|_{g'=0} = \tfrac{g'}{g} \Big(1 + \tfrac{g}{g'} \Big)^{\widetilde{R}_{g'}[g'A]} \, A \, \Big|_{g'=0} \\
&= \sum_{n=1}^{\infty} \tfrac{1}{n!} \Big(\tfrac{g}{g'} \Big)^{n-1} \, \widetilde{R}_{g'}[g'A] \, \big(\widetilde{R}_{g'}[g'A] - 1 \big) \cdots \big(\widetilde{R}_{g'}[g'A] - n{+}1 \big) \, A \, \Big|_{g'=0} \, ,
\end{aligned} \tag{4.35}$$

employing a Taylor expansion around $g/g' = 0$ in the last step. Still, it is not clear that $g' \to 0$ is well defined. To show that this is the case, we isolate the degree-zero part of $\widetilde{R}$,

$$\widetilde{R}_g[\widetilde{A}] = \sum_{k=0}^{\infty} r_k[\widetilde{A}] =: r_0[A] + g \, R_g[A] \, , \tag{4.36}$$

with the degree-k parts r_k satisfying

$$E \, r_k[\widetilde{A}] \;\equiv\; \int \mathrm{d}^4 x \; \widetilde{A}_\mu(x) \, \tfrac{\delta}{\delta \widetilde{A}_\mu(x)} \, r_k[\widetilde{A}] = k \, r_k[\widetilde{A}] \, , \tag{4.37}$$

[8] Malcha and Nicolai propose a slightly different prescription in [28], but upon closer inspection, one finds that it is actually equivalent to (4.34).

where we defined the functional Euler operator E. For the following computation, we need two ingredients. Firstly, we can use for any functional F the equivalence

$$g\,\partial_g\,F[\widetilde{A}] = 0 \qquad \Leftrightarrow \qquad \big(g\,\partial_g - E\big)\,F[gA] = 0 \tag{4.38}$$

and secondly, the commutator

$$\left[g\partial_g, \tfrac{1}{g}\right] = -\tfrac{1}{g}\,. \tag{4.39}$$

These allow us to rearrange the inverse Nicolai map as

$$\begin{aligned}
T_g^{-1}A &= \frac{1}{g}\sum_{n=0}^{\infty}\frac{g^n}{n!}\left[\frac{1}{g'}\left(g'\partial_{g'}+\widetilde{R}_{g'}[\widetilde{A}]\right)\right]^n\widetilde{A}\,\Big|_{\widetilde{A}=g'A}\Big|_{g'=0}\\
&\overset{(4.35)}{=}\frac{1}{g}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{g}{g'}\right)^n\left(\widetilde{R}_{g'}[g'A]-n+1\right)\cdots\left(\widetilde{R}_{g'}[g'A]-1\right)\widetilde{R}_{g'}[g'A]\,g'A\,\Big|_{g'=0}\\
&\overset{(4.38)}{=}\frac{1}{g}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{g}{g'}\right)^n\left(g'\partial_{g'}-E+\widetilde{R}_{g'}[g'A]-n+1\right)\\
&\qquad\qquad\qquad\cdots\left(g'\partial_{g'}-E+\widetilde{R}_{g'}[g'A]\right)g'A\,\Big|_{g'=0}\\
&\overset{(4.39)}{=}\frac{1}{g}\sum_{n=1}^{\infty}\frac{g^n}{n!}\left[\frac{1}{g'}\left(g'\partial_{g'}-E+\widetilde{R}_{g'}[g'A]\right)\right]^n g'A\,\Big|_{g'=0}\\
&\overset{(4.36)}{=}\frac{1}{g}\sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\partial_{g'}+\frac{1}{g'}(r_0[A]-E)+R_{g'}[A]\right)^n g'A\,\Big|_{g'=0}\,.
\end{aligned}\tag{4.40}$$

Now it is important to note that, since the Nicolai map is an expansion around the identity

$$T_g^{-1}A = A + g\,r_1[A]\,A + \mathcal{O}(g^2)\,, \tag{4.41}$$

it is a necessity that

$$r_0[A] = E = \textstyle\int A\,\frac{\delta}{\delta A}\,. \tag{4.42}$$

This will also be seen in the explicit constructions of the operator later. It means that (4.40) further simplifies and we can write

$$\begin{aligned}
T_g^{-1}A &= \frac{1}{g}\sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\partial_{g'}+R_{g'}[A]\right)^n g'A\,\Big|_{g'=0}\\
&= \frac{1}{g}\sum_{n=1}^{\infty}\frac{g^n}{n!}\,n\left(\partial_{g'}+R_{g'}[A]\right)^{n-1}A\,\Big|_{g'=0}\,,
\end{aligned}\tag{4.43}$$

where we can set $g' = 0$ without generating any divergences. This shows that, once we know the intermediate coupling flow operator (4.29), we can compute the rescaled (perturbative) coupling flow operator

$$R_g[A] = \tfrac{1}{g}\big(\widetilde{R}_g[gA] - E\big)\,, \tag{4.44}$$

from which we can construct the (inverse) Nicolai map with the same universal formula (2.6) that holds for scalar theories.

4.1.4 Explicit form of the coupling flow operator

Next, we derive an explicit expression for $R_g[A]$ in terms of the fields. We start from (4.29) and write

$$\begin{aligned} \mathring{\delta}_\alpha X[\widetilde{A}] &= -\mathrm{i}\int \mathrm{d}^4x\, \widetilde{\bar{\lambda}}_\beta(\gamma_\mu)_{\beta\alpha}\tfrac{\delta X}{\delta \widetilde{A}_\mu}\,, \\ sX[\widetilde{A}] &= \sqrt{g}\int \mathrm{d}^4x\, \widetilde{\mathrm{D}}_\mu \widetilde{c}\,\tfrac{\delta X}{\delta \widetilde{A}_\mu}\,. \end{aligned} \tag{4.45}$$

This allows us to express the intermediate operator acting to the left as

$$\overleftarrow{\widetilde{R}_g}\,[\widetilde{A}] = -\tfrac{1}{8}\,\overleftarrow{\tfrac{\delta}{\delta \widetilde{A}_\mu}}\,\widetilde{P}_\mu{}^\nu[\widetilde{A}]\,\mathrm{tr}\big(\gamma_\nu \widetilde{S}[\widetilde{A}]\,\gamma^{\rho\lambda}\big)\widetilde{F}_{\rho\lambda} + \overleftarrow{\tfrac{\delta}{\delta \widetilde{A}_\mu}}\,\widetilde{\mathrm{D}}_\mu \widetilde{G}[\widetilde{A}]\,\mathcal{G}(\widetilde{A})\,, \tag{4.46}$$

where we combined the first and third term in (4.29) using the covariant projector

$$\widetilde{P}_\mu{}^\nu[\widetilde{A}] := \delta_\mu{}^\nu - \mathrm{D}_\mu \widetilde{G}\tfrac{\partial \mathcal{G}(\widetilde{A})}{\partial \widetilde{A}_\nu}\,. \tag{4.47}$$

Here, we adopt the compact notation from Section 4 in [24]. Color indices and position labels are suppressed, with implicit integrations convoluted with insertions of A. This notation is used throughout the rest of the thesis and is described in detail in Appendix A. To obtain the rescaled operator $\widetilde{R}$, we need to split off the Euler operator from (4.46). This can be achieved by means of the identity

$$\gamma^{\rho\lambda}\widetilde{F}_{\rho\lambda} = 2\widetilde{\not{\mathrm{D}}}\widetilde{\not{A}} + 2\partial\cdot\widetilde{A} - \widetilde{\not{A}}\times\widetilde{\not{A}}\,, \tag{4.48}$$

$$\overleftarrow{\widetilde{R}_g}\,[\widetilde{A}] = \overleftarrow{\tfrac{\delta}{\delta \widetilde{A}_\mu}}\,\widetilde{A}_\mu - \tfrac{1}{8}\,\overleftarrow{\tfrac{\delta}{\delta \widetilde{A}_\mu}}\,\widetilde{P}_\mu{}^\nu[\widetilde{A}]\,\mathrm{tr}\big(\gamma_\nu \widetilde{S}[\widetilde{A}]\,[2\partial\cdot\widetilde{A} - \widetilde{\not{A}}\times\widetilde{\not{A}}]\big)\,. \tag{4.49}$$

This yields the rescaled coupling flow operator (4.44)

$$\overleftarrow{R_g}\,[A] = -\tfrac{1}{8g}\,\overleftarrow{\tfrac{\delta}{\delta A_\mu}}\,P_\mu{}^\nu[A]\,\mathrm{tr}\big(\gamma_\nu S[A]\,[2\partial\cdot A - g\not{A}\times\not{A}]\big)\,, \tag{4.50}$$

where the rescaled fermion propagator is

$$S = \not{\mathrm{D}}^{-1} = -\underbrace{\lambda\,\bar{\lambda}}\,, \quad \text{with} \quad \mathrm{D}_\mu = \partial_\mu + gA_\mu\times\,, \tag{4.51}$$

and the rescaled ghost propagator is

$$G = \big(\tfrac{\partial\mathcal{G}(A)}{\partial A_\mu}\mathrm{D}_\mu\big)^{-1} = -\mathrm{i}\,\underbrace{c\,\bar{c}}\,. \tag{4.52}$$

Later, we will use the perturbative expansions

$$\begin{aligned} S &= S_0 - gS_0 \not{A} S = \sum_{l=0}^{\infty} (-gS_0 \not{A})^l S_0 \,, \\ G &= G_0 - gG_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\mu} A_\mu G = \sum_{k=0}^{\infty} \Big(-gG_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\mu} A_\mu\Big)^k G_0 \,, \end{aligned} \tag{4.53}$$

in terms of their free ($g = 0$) versions

$$G_0 = \big(\tfrac{\partial \mathcal{G}(A)}{\partial A_\mu} \partial_\mu\big)^{-1} \,, \qquad S_0 = \not{\partial}^{\,-1} = -\not{\partial} C \,, \qquad C = \Box^{-1} \,. \tag{4.54}$$

The rescaled coupling flow operator (4.50) is well defined at $g = 0$, because

$$P_\mu{}^\nu[A] \mathrm{tr}\big(\gamma_\nu S[A]\big)\Big|_{g=0} \propto \Pi_\mu{}^\nu \mathrm{tr}\big(\gamma_\nu \gamma_\rho\big) \partial^\rho C \propto \Pi_\mu{}^\nu \partial_\nu C = 0 \,, \tag{4.55}$$

where we introduced the free ($g = 0$) projector

$$\Pi_\mu{}^\nu := \delta_\mu{}^\nu - \partial_\mu G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\nu} \quad \text{with} \quad G_0 := \big(\tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} \partial_\rho\big)^{-1} \,. \tag{4.56}$$

While one could stop right here and construct the Nicolai map by perturbatively expanding (4.50) and inserting into the universal formula (2.6), one can express the coupling flow operator in a more symmetrical way. To do so, we decompose the gauge field into transversal and longitudinal components [29] in the next subsection.

4.1.5 Gauge field decomposition

Introducing the standard transversal projector

$$\amalg_\mu{}^\nu := \delta_\mu{}^\nu - \partial_\mu C \partial^\nu \,, \tag{4.57}$$

we can split the Yang–Mills fields as follows

$$A_\mu = A^{\mathrm{T}}_\mu + A^{\mathrm{L}}_\mu \,, \qquad A^{\mathrm{T}}_\mu = \amalg_\mu{}^\nu A_\nu \,, \qquad A^{\mathrm{L}}_\mu = (\delta_\mu{}^\nu - \amalg_\mu{}^\nu) A_\nu = \partial_\mu C \, \partial \cdot A \,. \tag{4.58}$$

Instead of using the transversal and longitudinal components of the Yang–Mills fields, we can equally well introduce a 'conjugate' gauge field

$$A^*_\mu := A^{\mathrm{T}}_\mu - A^{\mathrm{L}}_\mu = A_\mu - 2\partial_\mu C \, \partial \cdot A \,, \tag{4.59}$$

and use A and A^* as a 'basis' for expressions where one does not use the Landau gauge to set $\mathcal{G}(A) = \partial \cdot A = 0$. Using

$$2\, S_0 \, \partial \cdot A = -2\not{A}^{\mathrm{L}} = \not{A}^* - \not{A} \,, \tag{4.60}$$

this allows us to rewrite (4.44) as [29]

$$
\begin{aligned}
g \overleftarrow{R}_g[A] &= -\tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left(\gamma_\nu S \left[2\partial \cdot A - g \not{A} \times \not{A}\right]\right) \\
&= -\tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu \left[\sum_{l=0}^{\infty} (-g S_0 \not{A})^l \times (\not{A}^* - \not{A}) - \sum_{l=0}^{\infty} (-g S_0 \not{A})^l S_0 g \not{A} \times \not{A}\right]\right\} \\
&= -\tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu \left[\sum_{l=0}^{\infty} (-g S_0 \not{A})^l \times (\not{A}^* - \not{A}) + \sum_{l=1}^{\infty} (-g S_0 \not{A})^l \times \not{A}\right]\right\} \\
&= -\tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu \left[\sum_{l=1}^{\infty} (-g S_0 \not{A})^l \times (\not{A}^* - \not{A}) + \sum_{l=1}^{\infty} (-g S_0 \not{A})^l \times \not{A}\right]\right\} \\
&\quad + \tfrac{1}{4} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \left\{\gamma_\nu \not{A}^{\mathrm{L}}\right\} \\
&= -\tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu \sum_{l=1}^{\infty} (-g S_0 \not{A})^l \times \not{A}^*\right\} - \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu A^{\mathrm{L}}_\nu \\
&= +\tfrac{g}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu \sum_{l=0}^{\infty} (-g S_0 \not{A})^l S_0 \not{A} \times \not{A}^*\right\} - \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu A^{\mathrm{L}}_\nu \\
&= +\tfrac{g}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \ \left\{\gamma_\nu S \ \not{A} \times \not{A}^*\right\} - \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu A^{\mathrm{L}}_\nu \, .
\end{aligned} \tag{4.61}
$$

Here, we observe that the $g \to 0$ limit of R_g is well defined from the fact that $\Pi_\mu{}^\nu A^{\mathrm{L}}_\nu = 0$.

4.1.6 Perturbative construction

In order to construct the Nicolai map perturbatively, we need the expansion of $R_g[A]$ in orders of g. To that end, we first expand the covariant projector (4.47) to find

$$
\begin{aligned}
P_\mu{}^\nu A^{\mathrm{L}}_\nu &= \Pi_\mu{}^\sigma \left\{\delta_\sigma{}^\nu - g A_\sigma \sum_{k=0}^{\infty} \left(-g G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} A_\rho\right)^k G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\nu}\right\} A^{\mathrm{L}}_\nu \\
&= -g \Pi_\mu{}^\sigma A_\sigma \sum_{k=0}^{\infty} \left(-g G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} A_\rho\right)^k \times (C \partial \cdot A) \, .
\end{aligned} \tag{4.62}
$$

We deduce that we can write $R_g[A]$ compactly as

$$
\overleftarrow{R}_g \ [A] = \tfrac{1}{8} \ \frac{\overleftarrow{\delta}}{\delta A_\mu} \ P_\mu{}^\nu \ \mathrm{tr} \left\{\gamma_\nu S \ \not{A} \times \not{A}^*\right\} + \frac{\overleftarrow{\delta}}{\delta A_\mu} \ \Pi_\mu{}^\sigma A_\sigma \ G \ \tfrac{\partial \mathcal{G}(A)}{\partial A_\nu} A^{\mathrm{L}}_\nu \, , \tag{4.63}
$$

or more practically we can write down the full perturbative expansion of the $\mathcal{N}=1$ $D=4$ SYM coupling flow operator in any gauge (from [29])

$$
\begin{aligned}
\overleftarrow{R}_g\,[A] = \tfrac{1}{8}\ \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\sigma \Big\{ \delta_\sigma{}^\nu - g A_\sigma G_0 \sum_{k=0}^{\infty} \Big(-g \tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} A_\rho G_0 \Big)^k \tfrac{\partial \mathcal{G}(A)}{\partial A_\nu} \Big\} \\
\cdot\, \mathrm{tr}\Big\{ \gamma_\nu S_0 \sum_{l=0}^{\infty} (-g \not{A} S_0)^l \not{A} \times \not{A}^* \Big\} \\
+ \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\sigma A_\sigma \sum_{k=0}^{\infty} \Big(-g G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} A_\rho \Big)^k \times (C\partial\!\cdot\! A)\,.
\end{aligned}
\tag{4.64}
$$

Each order can be extracted, for example

$$
\begin{aligned}
\overleftarrow{r}_1\,[A] &= \tfrac{1}{8}\ \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\nu\ \mathrm{tr}\Big\{ \gamma_\nu S_0 \not{A} \times \not{A}^* \Big\} + \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\sigma A_\sigma \times (C\partial\!\cdot\! A)\,, \\
\overleftarrow{r}_2\,[A] &= -\tfrac{1}{8}\ \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\nu\ \mathrm{tr}\Big\{ \gamma_\nu S_0 \not{A} S_0 \not{A} \times \not{A}^* \Big\} \\
&\quad - \tfrac{1}{8}\ \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\sigma\ A_\sigma G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\nu}\ \mathrm{tr}\Big\{ \gamma_\nu S_0 \not{A} \times \not{A}^* \Big\} \\
&\quad - \frac{\overleftarrow{\delta}}{\delta A_\mu}\,\Pi_\mu{}^\sigma A_\sigma G_0 \tfrac{\partial \mathcal{G}(A)}{\partial A_\rho} A_\rho \times (C\partial\!\cdot\! A)\,,
\end{aligned}
\tag{4.65}
$$

and so on. From the universal formula (2.6) (or for the first few orders (2.15)), the Nicolai map immediately follows. We emphasize again that this holds for any gauge fixing function $\mathcal{G}(A)$ and in the full gauge-field configuration space [29]. It is a generalization of the previous formula for the Landau gauge hypersurface given in [24].

4.2 Axial gauge ($D=4$)

Note: *This section is largely following the author's published work [29].*

The simplest choice of gauge seems to be the Landau gauge

$$
\mathcal{G}(A) = \partial^\mu A_\mu\,. \tag{4.66}
$$

We argue that one of the central reasons for its simplicity is the fact that the two projectors that are relevant for the perturbative construction, i.e., the gauge dependent projector $\Pi_\mu{}^\nu$ (4.56) and the transversal projector $\text{П}_\mu{}^\nu$ (4.57) coincide. Furthermore, on its gauge hypersurface, the longitudinal component of the gauge field vanishes $A^{\mathrm{L}}_\mu = 0$. We will discuss explicit results for the Landau gauge later. Here, for the sake of exploring other options, we instead want to consider a different gauge, namely the axial gauge

$$
\mathcal{G}(A) = n^\mu A_\mu\,, \tag{4.67}
$$

with an arbitrary constant four-vector n^μ. This also includes the light-cone gauge where $n^2 = 0$ in Minkowskian signature. Inserting this into (4.65) and using $S_0 = -\partial\!\!\!/ C$, gives

$$\begin{aligned}
\overleftarrow{r}_1 &= \frac{\overleftarrow{\delta}}{\delta A_\mu}\, \Pi_\mu{}^\sigma \Big[A_\sigma \times (C\partial\cdot A) - \tfrac{1}{8} \mathrm{tr}\left(\gamma_\sigma\, \partial\!\!\!/ C\, A\!\!\!/ \times A\!\!\!/^*\right)\Big] \,, \\
\overleftarrow{r}_2 &= \frac{\overleftarrow{\delta}}{\delta A_\mu}\, \Pi_\mu{}^\sigma \Big[-A_\sigma G_0 n\cdot A \times (C\partial\cdot A) + \tfrac{1}{8} A_\sigma G_0 n_\nu \mathrm{tr}\left(\gamma^\nu\, \partial\!\!\!/ C\, A\!\!\!/ \times A\!\!\!/^*\right) \\
&\qquad - \tfrac{1}{8} \mathrm{tr}\left(\gamma_\sigma\, \partial\!\!\!/ C\, A\!\!\!/\, \partial\!\!\!/ C\, A\!\!\!/ \times A\!\!\!/^*\right)\Big] \,.
\end{aligned} \tag{4.68}$$

The traces can be evaluated using standard gamma matrix techniques (see Appendix A for relevant formulae). We find

$$r_1 A_\mu = -\Pi_\mu{}^\nu A^{(1)}_\nu \qquad \text{with} \quad A^{(1)}_\nu \;:=\; C^\rho A_{[\rho} A^*_{\nu]} + \tfrac{1}{2} C_\nu A^\rho A^*_\rho - A_\nu (C\partial\cdot A) \,, \tag{4.69}$$

where we use the usual notation of square brackets around indices to indicate anti-symmetrizations with weight one, e.g. $A_{[\mu} B_{\nu]} := \frac{1}{2}(A_\mu B_\nu - B_\mu A_\nu)$. Moreover, we understand the last object in each term to be a color vector instead of a matrix and write the shorthand $\partial^\rho C \equiv C^\rho$. For all details regarding the notation, see Appendix A. The next higher order is

$$\begin{aligned}
r_2 A_\mu = \Pi_\mu{}^\nu \Big[& A_\nu G_0 n \cdot A^{(1)} - 3 C^\rho A^\lambda C_{[\nu} A_\rho A^*_{\lambda]} + 2 C^\rho A_{[\rho} A^{(1)}_{\nu]} \\
& + 2 C^\rho A_{[\rho} A_{\nu]} (C\partial\cdot A) \Big] \,.
\end{aligned} \tag{4.70}$$

In the second order of the Nicolai map, there also appears $R_1^2 A_\mu$, which we can simplify by introducing the conjugate projector

$$\Pi^{*\,\nu}_\mu := 2\, \amalg_\mu{}^\nu - \Pi_\mu{}^\nu \,, \tag{4.71}$$

and using the identity

$$A_\rho - A^*_\rho = 2 A^{\mathrm{L}}_\rho = 2\partial_\rho (C\partial\cdot A) \,, \tag{4.72}$$

giving

$$\begin{aligned}
r_1^2 A_\mu &= \Pi_\mu{}^\nu \Big[(C\partial\cdot A) \Pi_\nu{}^\lambda A^{(1)}_\lambda - A_\nu C^\sigma \Pi_\sigma{}^\lambda A^{(1)}_\lambda + C^\rho A_{[\rho} \Pi^*_{\nu]}{}^\lambda A^{(1)}_\lambda \\
&\qquad + C^\rho A^*_{[\rho} \Pi_{\nu]}{}^\lambda A^{(1)}_\lambda \Big] \\
&= \Pi_\mu{}^\nu \Big[(C\partial\cdot A) \Pi_\nu{}^\lambda A^{(1)}_\lambda - A_\nu C^\sigma \Pi_\sigma{}^\lambda A^{(1)}_\lambda + 2 C^\rho A_{[\rho}\, \amalg_{\nu]}{}^\lambda\, A^{(1)}_\lambda \\
&\qquad - 2 C^\rho \partial_{[\rho} (C\partial\cdot A)\, \Pi_{\nu]}{}^\lambda A^{(1)}_\lambda \Big] \\
&= \Pi_\mu{}^\nu \Big[-A_\nu C^\sigma \Pi_\sigma{}^\lambda A^{(1)}_\lambda + 2 C^\rho A_{[\rho}\, \amalg_{\nu]}{}^\lambda\, A^{(1)}_\lambda + 2 C^\rho (C\partial\cdot A) \partial_{[\rho} A^{(1)}_{\nu]} \Big] \,.
\end{aligned} \tag{4.73}$$

In the last step, we used the identities

$$\Pi_\mu{}^\nu C_\nu = 0 \,, \qquad C^\rho{}_\rho = 1 \,, \qquad \partial_{[\rho} \Pi_{\nu]}{}^\lambda = \partial_{[\rho} \delta_{\nu]}{}^\lambda \,, \tag{4.74}$$

and integration by parts. We must be careful with signs when integrating by parts. For example, with explicit position labels, we find

$$\begin{aligned}
&\int \mathrm{d}^4 y\, C^\rho(x-y)\partial^y_{[\rho}(C\partial\cdot A)(y)\, \Pi_{\nu]}{}^\lambda(y-z)A^{(1)}_\lambda(z) \\
&\quad = \int \mathrm{d}^4 y\, \partial^x_{[\rho} C^\rho(x-y)(C\partial\cdot A)(y)\, \Pi_{\nu]}{}^\lambda(y-z)A^{(1)}_\lambda(z) \\
&\quad - \int \mathrm{d}^4 y\, C^\rho(x-y)(C\partial\cdot A)(y)\, \partial^y_{[\rho}\Pi_{\nu]}{}^\lambda(y-z)A^{(1)}_\lambda(z)\,.
\end{aligned} \tag{4.75}$$

As a general rule, one can remember that there is an extra minus sign when derivatives move to the left, while there is no extra sign when they move to the right. With

$$C^\sigma \Pi_\sigma{}^\lambda = C^\lambda - G_0 n^\lambda\,, \tag{4.76}$$

we find the relevant combination of (4.70) and (4.73) to be

$$\begin{aligned}
\left(r_1^2 A - r_2 A\right)_\mu = \Pi_\mu{}^\nu \Big[&- A_\nu C^\lambda A^{(1)}_\lambda - 2C^\rho A_{[\rho} C_{\nu]}{}^\lambda A^{(1)}_\lambda + 2C^\rho (C\partial\cdot A)\partial_{[\rho} A^{(1)}_{\nu]} \\
&+ 3C^\rho A^\lambda C_\nu A_\rho A^*_\lambda - 2C^\rho A_{[\rho}A_{\nu]}(C\partial\cdot A)\Big]\,.
\end{aligned} \tag{4.77}$$

It is interesting that the influence of the ghosts lies only in the projector Π in front of the square bracket. Inserting the expression for $A^{(1)}_\lambda$ (4.69) and with more identities such as

$$C_\lambda C^\lambda = C C_\lambda{}^\lambda = C\,, \qquad C^{[\rho\lambda]} = 0\,, \tag{4.78}$$

we arrive at the following expression for

the Nicolai map in the axial gauge for $\mathcal{N}=1$ $D=4$ SYM (from [29])

$$\begin{aligned}
T_g A_\mu = A_\mu &+ g\Pi_\mu{}^\nu\{C^\rho A_{[\rho}A^*_{\nu]} - A_\nu(C\partial\cdot A)\} \\
&+\tfrac{g^2}{2}\Pi_\mu{}^\nu\{3C^\rho A^\lambda C_{[\nu}A_\rho A^*_{\lambda]} - 2C^\rho A_{[\rho}A_{\nu]}(C\partial\cdot A) \\
&\qquad -2C^\rho(C\partial\cdot A)\partial_{[\rho}(A_{\nu]}(C\partial\cdot A)) + 2C^\rho A_{[\rho}C_{\nu]}{}^\lambda A_\lambda(C\partial\cdot A) \\
&\qquad -C^\rho A_{[\rho}C_{\nu]}A^\lambda A^*_\lambda + A_\nu C^\lambda A_\lambda(C\partial\cdot A) - \tfrac{1}{2}A_\nu C A^\lambda A^*_\lambda\} \\
&+g^2\Pi_\mu{}^{[\nu}C^{\rho]}(C\partial\cdot A)C_\rho{}^\sigma A_{[\sigma}A^*_{\nu]} \;+\; \mathcal{O}(g^3)\,.
\end{aligned} \tag{4.79}$$

An equivalent expression was derived by Malcha and Nicolai in [28], but they do not make use of the conjugate gauge field A^*, thus resulting in a higher number of terms. They explicitly verify the free-action (1.15) and determinant-matching condition (1.16). Curiously, while constructed only in four dimensions D=4, the map (4.79) seems to generalize to the other critical dimensions, because their checks are valid also for D=3, 6, 10, at least to second order and even outside of the gauge hypersurface. It remains to be seen whether this holds for higher orders. Interestingly, from (4.79) we can also recover the known expression on the Landau gauge hypersurface (see e.g. [24])

$$T_g A_\mu = A_\mu + g\Pi_\mu{}^\nu C^\rho A_{[\rho} A_{\nu]} + \tfrac{g^2}{2}\Pi_\mu{}^\nu 3 C^\rho A^\lambda C_{[\nu} A_\rho A_{\lambda]} \; + \; \mathcal{O}(g^3) \,, \tag{4.80}$$

where $\partial\cdot A = 0$ and $A = A^*$.

4.3 Ad-hoc construction for the Landau gauge ($D{=}3, 4, 6, 10$)

In theories without an off-shell superfield formalism, such as $\mathcal{N}{=}1$ SYM in $D{=}3, 6, 10$, there is an alternative construction scheme for the Nicolai map, at least for the Landau gauge. This was discovered before the start of this Ph.D. project, so it will only be outlined briefly, following the main work on the topic [24]. First, we note that for $D = 3, 4, 6, 10$, the corresponding Clifford algebras have dimensions $r = 2, 4, 8, 16$ which can be summarized by the discrete relation

$$r = 2(D-2) \,. \tag{4.81}$$

As opposed to the canonical construction, in the ad-hoc construction, one does not need to start with the geometric scaling of the fields (as in (4.7)), but can instead directly work in the perturbative scaling where the on-shell action takes the form

$$\begin{aligned} S_{\text{SUSY}}[A,\lambda,\bar\lambda,C,\bar c] &= S_{\text{inv}}[A,\lambda,\bar\lambda] \; + \; S_{\text{gf}}[A,C,\bar c] \,, \\ S_{\text{inv}}[A,\lambda,\bar\lambda] &= \int \mathrm{d}^D x \, \left\{ -\tfrac14 F^{\mu\nu}F_{\mu\nu} - \tfrac{\mathrm{i}}{2}\bar\lambda \not{D}\lambda \right\} , \\ S_{\text{gf}}[A,C,\bar c] &= \int \mathrm{d}^D x \, \left\{ -\tfrac{1}{2\xi}\mathcal{G}(A)^2 + \bar c \tfrac{\partial \mathcal{G}(A)}{\partial A_\mu} \mathrm{D}_\mu C \right\} , \end{aligned} \tag{4.82}$$

The gaugino propagator is

$$S = \not{D}^{-1} = -\lambda \bar\lambda \,, \quad \text{with} \quad \mathrm{D}_\mu = \partial_\mu + g A_\mu \times \,, \tag{4.83}$$

and with the Landau gauge $\mathcal{G}(A) = \partial^\mu A_\mu$, the ghost propagator is

$$G = \left(\partial^\mu \mathrm{D}_\mu\right)^{-1} = -\mathrm{i}\, c\, \bar c \,. \tag{4.84}$$

Further, the covariant projector reads

$$P_\mu{}^\nu[A] = \delta_\mu{}^\nu - \mathrm{D}_\mu G \partial^\nu \,. \tag{4.85}$$

We will also need the free versions of the quantities above

$$S_0 = \not{\partial}^{-1} = -\not{\partial} C \,, \qquad G_0 = \Box^{-1} = C \,, \qquad \Pi_\mu{}^\nu = \delta_\mu{}^\nu - \partial_\mu C \partial^\nu \,. \tag{4.86}$$

The first part of the action S_{inv} is invariant under the on-shell supervariations

$$\delta_\alpha A_\nu = -\mathrm{i}(\bar\lambda\gamma_\nu)_\alpha \,, \qquad \delta_\alpha \lambda_\beta = -\tfrac12 (\gamma^{\mu\nu})_{\beta\alpha} F_{\mu\nu} \,, \tag{4.87}$$

while the full action is invariant under the on-shell Slavnov variations

$$\begin{aligned} sA_\mu &= \mathrm{D}_\mu c\,, \qquad & s\lambda &= g\,\lambda\times c\,, \qquad & s\bar\lambda &= g\,\bar\lambda\times c\,, \\ & & sC &= -\tfrac{g}{2}\,c\times c\,, \qquad & s\bar c &= \tfrac{1}{\xi}\,\mathcal{G}(A)\,. \end{aligned} \tag{4.88}$$

To find a coupling flow operator, as usual, we investigate the g-derivative of expectation values

$$\partial_g\langle\!\langle X\rangle\!\rangle_g = \langle\!\langle \partial_g X\rangle\!\rangle_g + \mathrm{i}\,\langle\!\langle \partial_g S_{\text{SUSY}}\,X\rangle\!\rangle_g\,. \tag{4.89}$$

The goal is to trade the g-derivative of the action for a functional differential operator acting on X. In the canonical construction, we made use of the off-shell superfield formalism to write $\partial_g S_{\text{SUSY}}$ as a supervariation up to a Slavnov variation. Here, this is not possible. This is where one introduces the ad-hoc quantity (c.f. (4.16))

$$\Delta_\alpha = -\tfrac{1}{2r}\int \mathrm{d}^D x\;(\gamma^{\mu\nu}\lambda)_\alpha\;A_\mu\times A_\nu\,. \tag{4.90}$$

Its supervariation generates a part of the g-derivative of S_{inv} [24]

$$\partial_g S_{\text{inv}} = -\delta_\alpha\Delta_\alpha - \mathrm{i}\Big(\tfrac{1}{2} - \tfrac{D-1}{r}\Big)\int \mathrm{d}^D\;\bar\lambda\gamma^\mu A_\mu\times\lambda\,. \tag{4.91}$$

Note that the second term in (4.91) will lead to a multiplicative contribution in the coupling flow operator, which we will discuss shortly. For the first term, just like in the canonical construction, the idea is to use the SUSY Ward identity (c.f. (4.24))

$$\langle\!\langle \delta_\alpha Y\rangle\!\rangle_g = \mathrm{i}\,\langle\!\langle \delta_\alpha\Delta_{\text{gh}}\,sY\rangle\!\rangle_g\,,\quad \text{since}\quad \delta_\alpha S_{\text{gf}} = -\,s\delta_\alpha\Delta_{\text{gh}}\,, \tag{4.92}$$

where the ghost component is defined as

$$\Delta_{\text{gh}}[\bar c, A] := \int \mathrm{d}^4 x\;\{\bar c\mathcal{G}(A)\}\,. \tag{4.93}$$

We apply this to $Y = \Delta_\alpha X$ in order to rewrite

$$\begin{aligned} \langle\!\langle (\delta_\alpha\Delta_\alpha)X\rangle\!\rangle_g &= \langle\!\langle \Delta_\alpha\delta_\alpha X\rangle\!\rangle_g + \mathrm{i}\,\langle\!\langle \delta_\alpha\Delta_{\text{gh}}\,s(\Delta_\alpha X)\rangle\!\rangle_g \\ &= \langle\!\langle \Delta_\alpha\delta_\alpha X\rangle\!\rangle_g + \mathrm{i}\,\langle\!\langle \delta_\alpha\Delta_{\text{gh}}\,s(\Delta_\alpha)X\rangle\!\rangle_g - \mathrm{i}\,\langle\!\langle \delta_\alpha\Delta_{\text{gh}}\,\Delta_\alpha\,sX\rangle\!\rangle_g\,. \end{aligned} \tag{4.94}$$

Unlike in the canonical construction, the on-shell superfield component Δ_α is not gauge invariant, so that $s\Delta_\alpha$ does not vanish and gives another multiplicative contribution

$$s\Delta_\alpha = \tfrac{1}{r}\int \mathrm{d}^D x\;(\gamma^{\rho\lambda}\lambda)_\alpha\partial_\rho c\times A_\lambda\,. \tag{4.95}$$

Furthermore, in this scaling, the g-derivative of the gauge fixing part of the action cannot be written as a Slavnov variation, and hence it leads to yet another multiplicative part

$$\partial_g S_{\mathrm{gf}} = \int \mathrm{d}^D x \, \bar{c} \, \partial^\mu \big(A_\mu \times c \big) \, . \tag{4.96}$$

In total, we find that the g-derivative of expectation values becomes

$$\partial_g \langle\!\langle X \rangle\!\rangle_g = \langle\!\langle \partial_g X \rangle\!\rangle_g - \mathrm{i} \langle\!\langle \Delta_\alpha \delta_\alpha X \rangle\!\rangle_g - \langle\!\langle \delta_\alpha \Delta_{\mathrm{gh}} \, \Delta_\alpha \, sX \rangle\!\rangle_g + \langle\!\langle Z(r,D) \; X \rangle\!\rangle_g \, , \tag{4.97}$$

where $Z(r,D)$ collects the multiplicative contributions resulting from (4.91), (4.95), and (4.96). It depends on the spinor dimension r and the spacetime dimension D. To obtain the coupling flow operator, the fermions and ghost fields must be integrated out, giving rise to gaugino and ghost propagators. In principle, R_g would then acquire a multiplicative contribution. This destroys the derivative property of the operator and hence, the distributivity of the Nicolai map, which is essential for satisfying the necessary conditions (free-action and determinant matching). It turns out, however, that for $\mathcal{G}(A) = \partial \cdot A \to 0$, it can be shown that

$$\big\langle \underset{\sqcup}{Z} X \big\rangle_g = 0 \qquad \forall X \, , \tag{4.98}$$

exactly when (4.81) holds, as is worked out in detail in [24]. This cures the derivation property of the coupling flow operator, thus making the construction of the Nicolai map possible in the critical dimensions $D = 3, 4, 6, 10$ (on the Landau gauge hypersurface). It is quite intriguing that the Nicolai map is in that sense aware of the critical dimensions. Ignoring the last term in (4.97), we can read off

> the ad-hoc coupling flow operator of $\mathcal{N}=1$ $D=3,4,6,10$ SYM on the Landau gauge hypersurface (from [24])
>
> $$R_g[A] = -\mathrm{i} \, \Delta_\alpha[A] \, \delta_\alpha - \Delta_\alpha[A] \, \big(\delta_\alpha \Delta_{\mathrm{gh}}[A] \big) \, s \, , \tag{4.99}$$

which we can compare to the intermediate coupling flow operator (4.29) from the canonical construction. It misses the second term of (4.29) and Δ_α is of a different form here, compare (4.16) and (4.90). Otherwise, (4.29) and (4.99) have the same general structure. Written in terms of explicit fields, the ad-hoc coupling flow operator reads

$$\overleftarrow{R_g} \, [A] = \tfrac{1}{2r} \; \frac{\overleftarrow{\delta}}{\delta A_\mu} \, P_\mu{}^\nu \, \mathrm{tr} \Big\{ \gamma_\nu S \, \not{A} \times \not{A} \Big\} \, , \tag{4.100}$$

which can be compared to (4.63). For $D = 4$, i.e. $r = 4$, the expressions fully agree once we enforce $\partial \cdot A = 0$. In four dimensions one can use the more general (4.63), while in the other critical dimensions $D = 3, 6, 10$, we have to resort to (4.99) on the Landau gauge hypersurface. For the construction of the Nicolai map, we need the full perturbative expansion of the $\mathcal{N}=1$ $D=3,4,6,10$ SYM coupling flow operator on the Landau gauge hypersurface (from [24])

$$\overleftarrow{R}_g\,[A] = \tfrac{1}{2r}\,\overleftarrow{\tfrac{\delta}{\delta A_\mu}}\,\Pi_\mu{}^\sigma\Big\{\delta_\sigma{}^\nu - gA_\sigma G_0\sum_{k=0}^{\infty}\Big(-gA\cdot\partial G_0\Big)^k\partial^\nu\Big\} \\ \cdot\,\mathrm{tr}\Big\{\gamma_\nu S_0\sum_{l=0}^{\infty}(-g\not{A}S_0)^l\not{A}\times\not{A}\Big\}\,, \tag{4.101}$$

where the partial derivatives act on everything to the right.

4.4 Topological term ($D=4$)

Note: *This section is largely following the author's published work [35].*

4.4.1 Off-shell action

In $D = 4$, the canonical construction from Section 4.1 offers an additional freedom: Adding a topological term. The setup is exactly the same as in Section 4.1.1, only that we add a topological term to the invariant action (c.f. (4.7))

$$S_{\mathrm{inv}}[\widetilde{A},\widetilde{\lambda},\widetilde{\bar{\lambda}},\widetilde{D}] = \tfrac{1}{g^2}\int \mathrm{d}^4x\,\Big\{-\tfrac{1}{4}\widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu} + \mathrm{i}\tfrac{g^2\theta}{32\pi^2}\widetilde{F}^{\mu\nu\star}\widetilde{F}_{\mu\nu} - \tfrac{\mathrm{i}}{2}\widetilde{\bar{\lambda}}\widetilde{\not{D}}\widetilde{\lambda} + \tfrac{1}{2}\widetilde{D}^2\Big\}\,, \tag{4.102}$$

with the dual field strength

$${}^\star\widetilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}\widetilde{F}^{\rho\lambda}\,. \tag{4.103}$$

The additional term is a total derivative, since

$$\widetilde{F}^{\mu\nu\star}\widetilde{F}_{\mu\nu} = \partial_\mu H^\mu \quad\text{with}\quad H^\mu \propto \epsilon^{\mu\nu\rho\lambda}\big(A_\nu\partial_\rho + \tfrac{1}{3}A_\nu A_\rho\times A_\lambda\big)\,. \tag{4.104}$$

It has non-perturbative consequences, e.g. for instanton configurations. However, later we return to the untilded fields and set up perturbation theory in g. Since we expand around the vacuum, where A is pure gauge, through perturbation theory we will not leave the topologically trivial sector where the additional term in the action vanishes. This means that our correlators cannot depend on θ, so that we can arbitrarily chose any constant for it.[9] Nevertheless, the coupling flow operator and the Nicolai map do depend on θ (as they did in SQM with $\theta \neq 0$, see Section 3.2). In particular, we set

$$\theta' := \tfrac{g^2\theta}{8\pi^2} \tag{4.105}$$

to a complex number and thus consider a flow in the (g,θ) parameter space along the curves $\theta = \frac{8\pi^2}{g^2}\theta'$, see Figure 4.2. Note that in our free $g=0$ theory θ is infinite. This is not a problem though, because in the topologically trivial

[9] Of course, the drawback is that we cannot access non-perturbative effects, but this is expected.

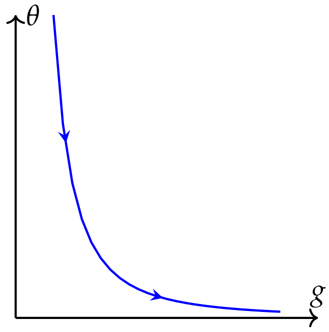


FIGURE 4.2: Flow in the (g, θ) parameter space.

sector that term drops out anyway. Just like in SQM (see Section 3.2), we can make use of choosing θ' freely. For $\theta' = \pm 1$, we obtain a chiral version of the Nicolai map. Unlike in SQM, it does not truncate to a linear function in g, but there are still considerable simplifications compared to the construction with $\theta = 0$.

4.4.2 Coupling flow operator

The construction can be done in exactly the same way as without the topological term (Section 4.1). The only difference is, that the superfield component $\mathring{\Delta}_\alpha$ (4.16) gets modified to[10]

$$\mathring{\Delta}'_\alpha := \mathring{\Delta}_\beta \big[\mathbb{1} + \theta' \, \mathrm{i}\gamma_5\big]_{\beta\alpha} \,. \tag{4.106}$$

It is straightforward to verify that the supervariation of the additional contribution generates the topological term. We can now skip ahead and replace $\Delta_\alpha[\widetilde{A}]$ by $\Delta'_\alpha[\widetilde{A}]$ in the intermediate coupling flow operator (4.29) (where $\Delta'[\widetilde{A}]$ is obtained from $\mathring{\Delta}'[\widetilde{A}, \widetilde{D}]$ by integrating out $\widetilde{D}$=0). For simplicity, we restrict ourselves to the Landau gauge hypersurface, where $\mathcal{G}(A)$=$\partial{\cdot}A$=0. Following the rescaling scheme of Section 4.1, we end up with the coupling flow operator (c.f. (4.63))

$$\overleftarrow{R}_g\,[A] = -\tfrac{1}{8}\;\frac{\overleftarrow{\delta}}{\delta A_\mu}\;P_\mu{}^\nu\;\mathrm{tr}\big\{\gamma_\nu S \gamma^{\rho\lambda}\big[\mathbb{1} + \theta'\,\mathrm{i}\gamma^5\big]\big\}\;A_\rho \times A_\lambda \,. \tag{4.107}$$

Observe that for the values $\theta' = \pm 1$, a chiral projector[11]

$$\mathrm{P}^\pm = \tfrac{1}{2}\big[\mathbb{1} \pm \mathrm{i}\gamma_5\big] \tag{4.108}$$

enters the trace, which is why we refer to this as a chiral version of the Nicolai map formalism. As usual, we expand the coupling flow operator in orders

[10] We remark that while we follow [35], here we use a different gamma-matrix convention to work with the same conventions throughout the thesis. This is why some signs are different. Furthermore, we differ by a factor of $-\mathrm{i}$ in our definition of θ.

[11] recall that in our conventions $(\mathrm{i}\gamma_5)^2 = \mathbb{1}$.

of g

$$R_g[A] = \sum_{k=1}^{\infty} g^{k-1} r_k[A] = r_1[A] + g\, r_2[A] + g^2 r_3[A] + \dots\,, \tag{4.109}$$

to construct the first few orders of the Nicolai map

$$\begin{aligned} T_g A = A \;-\; g\, r_1 A \;-\; \tfrac{1}{2} g^2 (r_2 - r_1^2) A \;-\; \tfrac{1}{6} g^3 (2r_3 - 2r_2 r_1 - r_1 r_2 + r_1^3) A \\ -\; \tfrac{1}{24} g^4 (6r_4 - 6r_3 r_1 - 2r_1 r_3 + 2r_1 r_2 r_1 - 3r_2 r_2 + 3r_2 r_1^2 + r_1^2 r_2 - r_1^4) A + \mathcal{O}(g^5) \end{aligned} \tag{4.110}$$

via the universal formula (2.6). For the following construction, we find it very useful to decompose the covariant projector

$$P_\mu{}^\nu = \delta^\mu{}_\nu - \mathrm{D}_\mu (\partial \cdot \mathrm{D})^{-1} \partial^\nu = \underbrace{\delta_\mu{}^\nu}_{\text{inv}} \; \underbrace{-\,\partial_\mu C \partial^\nu}_{\text{lgt}} \; \underbrace{-\,g [A_\mu - C_\mu A \cdot \partial]\, G\, \partial^\nu}_{\text{gh}} \tag{4.111}$$

into three parts: An ‘invariant’, a ‘longitudinal’, and a ‘ghost’ part. Accordingly, we split up the coupling flow operator (4.107) into three contributions,

$$R_g = R_g^{\text{inv}} + R_g^{\text{lgt}} + R_g^{\text{gh}} = \sum_{k=1}^{\infty} g^{k-1} (r_k^{\text{inv}} + r_k^{\text{lgt}} + r_k^{\text{gh}})\,. \tag{4.112}$$

Abbreviating

$$E_\mu[A;x] := \tfrac{1}{8}\, \mathrm{tr}\big\{ \gamma_\mu S \gamma^{\rho\lambda} \big[\mathbb{1} + \theta' \,\mathrm{i}\gamma^5 \big] \big\}\; A_\rho \times A_\lambda = E_\mu^{(1)} + g\, E_\mu^{(2)} + g^2 E_\mu^{(3)} + \dots\,, \tag{4.113}$$

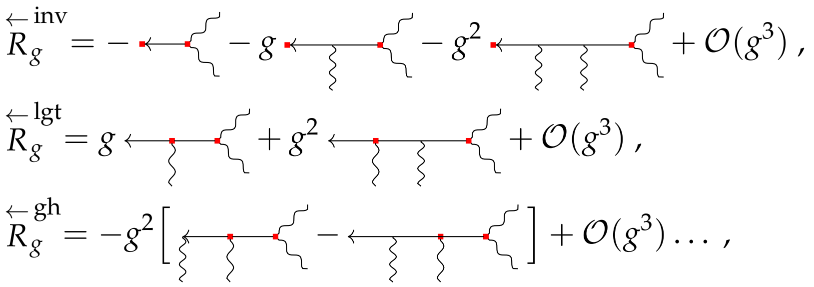
and by the important fact

$$\mathrm{D}^\nu E_\nu = 0 \qquad \Longleftrightarrow \qquad \partial^\nu E_\nu = -g A^\nu \times E_\nu\,, \tag{4.114}$$

the decomposition of R_g can be put into the compact form

$$\begin{aligned} \overleftarrow{R}_g^{\text{inv}} &= -\, \frac{\overleftarrow{\delta}}{\delta A_\mu}\, E_\mu\,, \qquad \overleftarrow{R}_g^{\text{lgt}} = g\, \frac{\overleftarrow{\delta}}{\delta A_\mu}\, \partial_\mu C A^\nu \times E_\nu\,, \\ \overleftarrow{R}_g^{\text{gh}} &= -\, g^2\, \frac{\overleftarrow{\delta}}{\delta A_\mu}\, [A_\mu - C_\mu A \cdot \partial]\, G\, A^\nu \times E_\nu\,. \end{aligned} \tag{4.115}$$

For a better overview, we can represent the expansion of the three contributions to the coupling flow operator graphically in a schematic form



(4.116)

similar to the representation of trees in Section 3.5.2, but with additional implicit color and spinor structure[12]. The red square dots indicate the internal trace over gamma matrices. It involves all propagators in between the square dots and all external bosonic lines attached to the square dots or vertices in between. Note that the longitudinal contributions only start at $\mathcal{O}(g)$ and the ghost contributions at $\mathcal{O}(g^2)$. Due to

$$\gamma_{\mu\alpha} = \gamma_\mu \gamma_\alpha + \eta_{\mu\alpha} \,, \tag{4.117}$$

the invariant and longitudinal parts of a given order can be combined by an antisymmetrization of indices:

$$\begin{aligned} \overleftarrow{r}_k^{\text{inv}} &= \tfrac{1}{8} \frac{\overleftarrow{\delta}}{\delta A_\mu} \operatorname{tr}\{\gamma_\mu \gamma_\alpha \ldots \gamma^{\rho\lambda}[\ldots]\}\, C^\alpha \ldots A_\rho \times A_\lambda \,, \\ \overleftarrow{r}_k^{\text{inv+lgt}} &= \tfrac{1}{8} \frac{\overleftarrow{\delta}}{\delta A_\mu} \operatorname{tr}\{\gamma_{\mu\alpha} \ldots \gamma^{\rho\lambda}[\ldots]\}\, C^\alpha \ldots A_\rho \times A_\lambda \,, \end{aligned} \tag{4.118}$$

which for $k = 1$ is trivial with $r_1^{\text{inv}} = r_1^{\text{inv+lgt}}$.

4.4.3 Simplifications

We now focus on the chiral case, where $\theta' = 1$ (the other sign is analogous). The essential simplifications in the chiral formulation follow from a Fierz identity

$$[\gamma^\nu(\mathbb{1}+\mathrm{i}\gamma^5)]_{\alpha\beta}\, [\gamma_\nu(\mathbb{1}-\mathrm{i}\gamma^5)]_{\gamma\delta} = -2\,(\mathbb{1}-\mathrm{i}\gamma^5)_{\alpha\delta}\,(\mathbb{1}+\mathrm{i}\gamma^5)_{\gamma\beta} \,, \tag{4.119}$$

implying the

> key simplification for the chiral SYM Nicolai map
>
> $$r_{k-1}^{\text{inv}}\, r_1\, A = (r_k^{\text{inv}} + r_k^{\text{lgt}})\, A \qquad \text{for} \quad \theta' = \pm 1 \quad \text{and} \quad k \geq 2 \,. \tag{4.120}$$

For example, we can manipulate the traces in $r_1^{\text{inv}} r_1^{\text{inv}} A_\mu$ as follows

$$\begin{aligned} &\tfrac{1}{32}\operatorname{tr}\{\gamma_\mu\gamma_\alpha\gamma^{\rho\lambda}[\mathbb{1}+\mathrm{i}\gamma^5]\}\, \operatorname{tr}\{\gamma_\lambda\gamma_\sigma\gamma^{\alpha\beta}[\mathbb{1}+\mathrm{i}\gamma^5]\} \\ &\quad = +\tfrac{1}{32}\operatorname{tr}\{\gamma_{\mu\alpha}\gamma^\rho\gamma^\lambda[\mathbb{1}+\mathrm{i}\gamma^5]\}\, \operatorname{tr}\{\gamma_\lambda[\mathbb{1}-\mathrm{i}\gamma^5]\gamma_\sigma\gamma^{\alpha\beta}\} \\ &\quad = -\tfrac{1}{16}\operatorname{tr}\{\gamma_{\mu\alpha}\gamma^\rho[\mathbb{1}-\mathrm{i}\gamma^5]\gamma_\sigma\gamma^{\alpha\beta}[\mathbb{1}+\mathrm{i}\gamma^5]\} \\ &\quad = -\tfrac{1}{8}\operatorname{tr}\{\gamma_{\mu\alpha}\gamma^\rho\gamma_\sigma\gamma^{\alpha\beta}[\mathbb{1}+\mathrm{i}\gamma^5]\} \,, \end{aligned} \tag{4.121}$$

which yields the expression for $r_2 A_\mu = r_2^{\text{inv+lgt}} A_\mu$. For higher orders it works in the same way, only that there are more gamma matrices in the traces. The simplification (4.120) looks similar to the polynomiality condition (2.60), but it is restricted to the invariant and longitudinal parts of the coupling flow operator. Therefore, the map does not truncate to a linear function in g, but there are still significant simplifications. In fact, the second order completely

[12] Note also that we draw these trees with the root on the very left, because it complies better with the internal color indices.

vanishes, and writing out the map to fourth order we find

$$
\begin{aligned}
T_g A_\mu = A_\mu \;&-\; g\, r_1 A_\mu \;-\; \tfrac{1}{3} g^3 \big(r_3^{\text{gh}} - r_2^{\text{lgt}} r_1\big) A_\mu \\
&\quad-\; \tfrac{1}{12} g^4 \big(3 r_4^{\text{gh}} - 3(r_3^{\text{lgt}} + r_3^{\text{gh}})\, r_1 - r_1 (r_3^{\text{gh}} - r_2^{\text{lgt}} r_1)\big) A_\mu \;+\; \mathcal{O}(g^5) \\
= A_\mu \;&-\; \tfrac{1}{8} g \;\mathrm{tr}\big\{\gamma_{\mu\alpha}\gamma^{\rho\lambda}\big[\mathbb{1}+\mathrm{i}\gamma^5\big]\big\}\, C^\alpha A_\rho \times A_\lambda \\
&-\; \tfrac{1}{24} g^3 \left[A_\mu - C_\mu A\!\cdot\!\partial\right] C\, A^\nu \;\mathrm{tr}\big\{\gamma_\nu\gamma_\beta\gamma^{\rho\lambda}\big[\mathbb{1}+\mathrm{i}\gamma^5\big]\big\}\, C^\beta A_\rho \times A_\lambda \\
&+\; \tfrac{1}{96} g^3 \;\mathrm{tr}\big\{\gamma_{\mu\alpha}\gamma^{\nu\beta}\big[\mathbb{1}+\mathrm{i}\gamma^5\big]\big\}\;\mathrm{tr}\big\{\gamma_{\sigma\gamma}\gamma^{\rho\lambda}\big[\mathbb{1}+\mathrm{i}\gamma^5\big]\big\}\, C^\alpha A_\nu C_\beta A^\sigma C^\gamma A_\rho \times A_\lambda \\
&-\; \tfrac{1}{12} g^4 \big(3 r_4^{\text{gh}} - 3(r_3^{\text{lgt}} + r_3^{\text{gh}})\, r_1 - r_1 (r_3^{\text{gh}} - r_2^{\text{lgt}} r_1)\big) A_\mu \;+\; \mathcal{O}(g^5)\,,
\end{aligned}
\tag{4.122}
$$

where we spelled out the first three orders explicitly. In the next subsection, we will evaluate the traces and write out the map explicitly to fourth order. We can be more systematic. Considering the universal formula (2.11) with the $c_{\mathbf{n}}$ coefficients (2.13), we notice that we can pair up terms to write

$$
\begin{aligned}
T_g A &= A \;-\; g\, r_1 A \;+\; {\sum_{\mathbf{n}}}' g^n\, c_{\mathbf{n}}\, r_{n_s} \ldots r_{n_2}\, (r_k - r_{k-1} r_1)\; A \\
&= A \;-\; g\, r_1 A \;+\; {\sum_{\mathbf{n}}}' g^n\, c_{\mathbf{n}}\, r_{n_s} \ldots r_{n_2}\, (r_k^{\text{gh}} - r_{k-1}^{\text{lgt+gh}} r_1)\; A\,,
\end{aligned}
\tag{4.123}
$$

where the primed sum is restricted to the multi-indices $\mathbf{n}$ with $n \geq 3$ and $n_1 \equiv k > 1$,

$$
\mathbf{n} = (k, n_2, \ldots, n_s)\,. \tag{4.124}
$$

The unrestricted sum over the (ordered) partitions contains 2^{n-1} compositions. It is a nice exercise to prove this by induction. In the process, one realizes that restricting the sum to those compositions with $n_1 \equiv k > 1$ cuts their number in half, as should be the case when we pair up the terms. Applying (4.120) to these pairs leads to the simplification in the second line of (4.123). Each diagram begins with either r_1 or with a ghost contribution r_k^{gh}. A term involving $r_{i>1}^{\text{inv}}$ can only occur in higher iterations ($s > 1$) of the coupling flow operator. We will also see explicitly, that this reduces the number of terms in the expansion significantly, compared to the construction without the topological term. There could be even more simplifications in higher-order actions of r_k^{inv}. As we can see from the graphical representation (4.116), whenever some r_k^{inv} acts on a bosonic line that is part of the spin trace of any other graph, there is the possibility of fusing the two resulting traces together with the Fierz identity (4.119). This remains to be investigated in detail.

4.4.4 Chiral map to fourth order

For the traces in (4.122), we use standard techniques for gamma matrices

$$
\tfrac{1}{4}\,\mathrm{tr}\big(\gamma_\nu\gamma_\beta\gamma_{\rho\lambda}[\mathbb{1}+\mathrm{i}\gamma^5]\big) = 2\eta_{\nu[\rho}\eta_{\lambda]\beta} + \mathrm{i}\,\epsilon_{\nu\beta\rho\lambda}\,, \tag{4.125}
$$

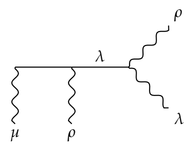
$$
\begin{aligned}
&\tfrac{1}{4}\,\mathrm{tr}\big(\gamma_\nu\gamma_\beta\gamma_\sigma\gamma_\gamma\gamma_{\rho\lambda}[\mathbb{1}+\mathrm{i}\gamma^5]\big) \\
&\qquad = -4\big(\eta_{\nu[\beta}\eta_{\sigma][\rho}\eta_{\lambda]\gamma} + \eta_{\gamma[\nu}\eta_{\beta][\rho}\eta_{\lambda]\sigma} + \eta_{\nu[\rho}\eta_{\lambda][\beta}\eta_{\gamma]\sigma}\big) \\
&\qquad\quad -\mathrm{i}\big(\eta_{\nu\beta}\epsilon_{\sigma\gamma\rho\lambda} - \eta_{\nu\sigma}\epsilon_{\beta\gamma\rho\lambda} + \eta_{\beta\sigma}\epsilon_{\nu\gamma\rho\lambda} - 2\eta_{\gamma[\rho}\epsilon_{\lambda]\nu\beta\sigma}\big)\,,
\end{aligned}
\tag{4.126}
$$

where the trace with six gamma matrices is only needed for the fourth order. The resulting map reads

$$
\begin{aligned}
T_g A_\mu = A_\mu \; &- \; g \left\{ C_\lambda A_\mu \times A^\lambda + \tfrac{\mathrm{i}}{2} \epsilon_{\mu\alpha\rho\lambda} C^\alpha A^\rho \times A^\lambda \right\} \\
&- \tfrac{g^3}{3} \left[A_\mu - C_\mu A \cdot \partial \right] C A_\rho C_\lambda A^\rho \times A^\lambda \; + \; \tfrac{2g^3}{3} \, C^\alpha A_{[\mu} C_{\alpha]} A_\rho C_\lambda A^\rho \times A^\lambda \\
&+ 4g^3 \, C^\nu A^\alpha C^\beta A_{[\mu} C_\nu A_\alpha \times A_{\beta]} - \tfrac{\mathrm{i}g^3}{6} \epsilon_{\nu\sigma\rho\lambda} [A_\mu - C_\mu A \cdot \partial] C A^\nu C^\sigma A^\rho \times A^\lambda \\
&+ \tfrac{\mathrm{i}g^3}{3} \epsilon_{\nu\sigma\rho\lambda} C^\alpha A_{[\mu} C_{\alpha]} A^\nu C^\sigma A^\rho \times A^\lambda - \tfrac{\mathrm{i}g^3}{3} \epsilon_{\mu\nu\alpha\beta} C^\alpha A^\nu C^\beta A_\rho C_\lambda A^\rho \times A^\lambda \\
&+ g^4 \, T_g A_\mu \big|_{\mathcal{O}(g^4)} \; + \; \mathcal{O}(g^5) \, .
\end{aligned}
\tag{4.127}
$$

Similar to the presentation of Nicolai maps before, we use a very compact notation here. More explicitly, there are two types of implicit color and position ordering here:

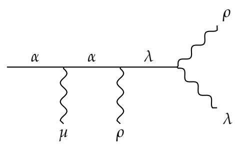
$$
(A_\mu C A_\rho C_\lambda A^\rho \times A^\lambda)^a(x) \qquad \equiv
$$



$$
\equiv \; f^{abc} f^{cde} f^{efg} \int_{y_1, y_2} A^b_\mu(x) C(x - y_1) A^d_\rho(y_1) C_\lambda(y_1 - y_2) A^{f\,\rho}(y_2) A^{g\,\lambda}(y_2)
\tag{4.128}
$$

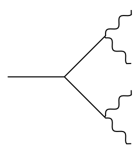
and

$$
(C_\alpha A_\mu C^\alpha A_\rho C_\lambda A^\rho \times A^\lambda)^a(x) \qquad \equiv
$$



$$
\begin{aligned}
&\equiv \; f^{abc} f^{cde} f^{efg} \int_{y_1, y_2, y_3} C_\alpha(x - y_1) A^b_\mu(y_1) C^\alpha(y_1 - y_2) \\
&\qquad\qquad \cdot A^d_\rho(y_2) C_\lambda(y_2 - y_3) A^{f\,\rho}(y_3) A^{g\,\lambda}(y_3) \, ,
\end{aligned}
\tag{4.129}
$$

where we also introduced a self-explanatory graphical notation. Note that the first branched tree



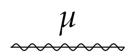
(4.130)

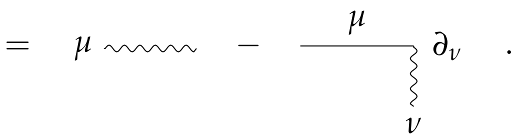
drops out by the Fierz identity, while it is present in the non-chiral map, being generated from $r_1^3 A$. In the fourth order, we abbreviate the following six position and color structures

$$\begin{aligned}
(G^{1A}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{cde}f^{efg}f^{ghi}\int_{y_1\ldots y_4} C_\mu(x-y_1)A^b_\nu(y_1)C_\alpha(y_1-y_2)\\
&\quad\cdot A^d_\beta(y_2)C_\gamma(y_2-y_3)A^f_\delta(y_3)C_\sigma(y_3-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,,\\
(G^{1B}_{\mu\nu\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{cde}f^{efg}f^{ghi}\int_{y_1\ldots y_4}[A^b_\mu(x)\delta(x-y_1) - C_\mu(x-y_1)A^b(y_1)\cdot\partial]\\
&\quad\cdot C(y_1-y_2)A^d_\nu(y_2)C_\gamma(y_2-y_3)A^f_\delta(y_3)C_\sigma(y_3-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,,\\
(G^{1C}_{\mu\nu\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{cde}f^{efg}f^{ghi}\int_{y_1\ldots y_4} C_\mu(x-y_1)A^b_\nu(y_1)\\
&\quad\cdot\ [A^d_\gamma(y_1)\delta(y_1-y_2) - C_\gamma(y_1-y_2)A^d(y_2)\cdot\partial]\\
&\quad\cdot C(y_2-y_3)A^f_\delta(y_3)C_\sigma(y_3-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,,\\
(G^{2A}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{cde}f^{dfg}f^{ehi}\int_{y_1\ldots y_4} C_\mu(x-y_1)A^b_\nu(y_1)C_\alpha(y_1-y_2)\\
&\quad\cdot C_\beta(y_2-y_3)A^f_\gamma(y_3)A^g_\delta(y_3)C_\sigma(y_2-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,,\\
(G^{2B}_{\mu\beta\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{cde}f^{dfg}f^{ehi}\int_{y_1\ldots y_4}[A^b_\mu(x)\delta(x-y_1) - C_\mu(x-y_1)A^b(y_1)\cdot\partial]\\
&\quad\cdot C(y_1-y_2)C_\beta(y_2-y_3)A^f_\gamma(y_3)A^g_\delta(y_3)C_\sigma(y_2-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,,\\
(G^{3}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda})^a(x) &= f^{abc}f^{bde}f^{cfg}f^{ghi}\int_{y_1\ldots y_4} C_\mu(x-y_1)C_\nu(y_1-y_2)\\
&\quad\cdot A^d_\alpha(y_2)A^e_\beta(y_2)C_\gamma(y_1-y_3)A^f_\delta(y_3)C_\sigma(y_3-y_4)A^h_\rho(y_4)A^i_\lambda(y_4)\,.
\end{aligned} \tag{4.131}$$

We can also represent them graphically as diagrams, see Figure 4.3, where we abbreviate the square brackets from (4.131) as

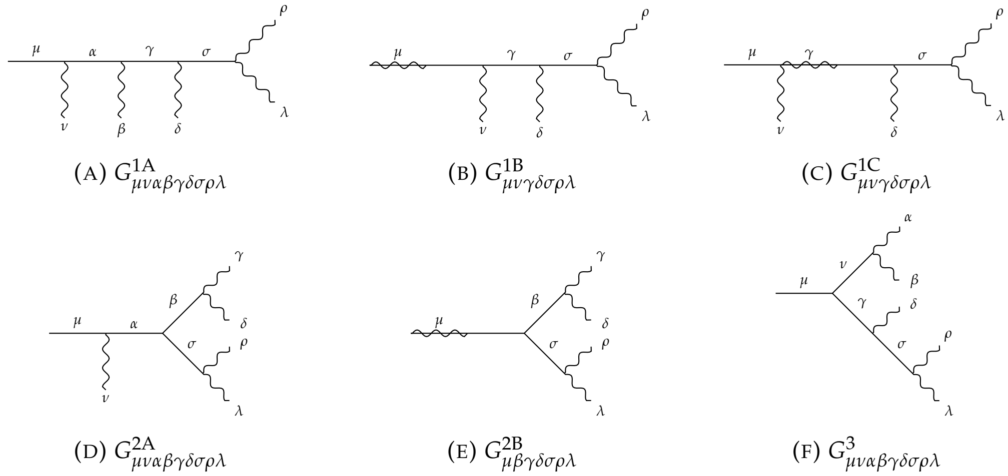
$$\int_y [A_\mu(x)\delta(x-y) - C_\mu(x-y)A(y)\cdot\partial] \quad\equiv$$





(4.132)

These square brackets are contributions of the ghosts, as they appear in the expansion of the ghost part of the coupling flow operator, see (4.115). With



(A) $G^{1A}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda}$ (B) $G^{1B}_{\mu\nu\gamma\delta\sigma\rho\lambda}$ (C) $G^{1C}_{\mu\nu\gamma\delta\sigma\rho\lambda}$

(D) $G^{2A}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda}$ (E) $G^{2B}_{\mu\beta\gamma\delta\sigma\rho\lambda}$ (F) $G^{3}_{\mu\nu\alpha\beta\gamma\delta\sigma\rho\lambda}$

FIGURE 4.3: Diagrams in the fourth order chiral Nicolai map.

the abbreviations (4.131), we can express the fourth order relatively compactly as

$$
\begin{aligned}
&T_g A_\mu\big|_{\mathcal{O}(g^4)} = -G^{1A}_{\nu[\mu\nu]\beta[\beta\rho]\lambda\rho\lambda} - G^{1A}_{\nu[\mu\nu][\beta\rho]\lambda\beta\rho\lambda} - G^{1A}_{\nu[\mu\nu]\rho[\lambda|\beta|\beta]\rho\lambda} - \tfrac{1}{3}G^{1A}_{\nu[\mu\nu][\beta\rho]\beta\lambda\rho\lambda} \\
&+\tfrac{1}{2}G^{1A}_{\alpha[\mu\alpha]\nu\sigma\rho[\nu\sigma\rho]} - 3G^{1A}_{\nu\sigma\epsilon\delta\delta[\mu\nu\sigma\epsilon]} + 4G^{1A}_{\nu\sigma\epsilon\delta[\mu|\delta|\nu\sigma\epsilon]} - 4G^{1A}_{\nu\sigma\epsilon[\mu|\delta\delta|\nu\sigma\epsilon]} - 8G^{1A}_{\nu\sigma\epsilon[\mu\nu\sigma|\delta\delta|\epsilon]} \\
&-\tfrac{\mathrm{i}}{2}\epsilon_{\mu\nu\alpha\beta}\big(G^{1A}_{\nu\alpha\beta\sigma[\sigma\rho]\lambda\rho\lambda} + G^{1A}_{\nu\alpha\beta[\sigma\rho]\lambda\sigma\rho\lambda} + G^{1A}_{\nu\alpha\beta\rho[\lambda|\sigma|\sigma]\rho\lambda} + \tfrac{1}{3}G^{1A}_{\nu\alpha\beta[\rho\lambda]\rho\gamma\lambda\gamma} - \tfrac{1}{2}G^{1A}_{\nu\alpha\beta\sigma\delta\rho[\sigma\delta\rho]}\big) \\
&-\tfrac{\mathrm{i}}{12}\epsilon_{\alpha\beta\rho\lambda}\big(3G^{1A}_{\nu[\mu\nu]\delta\delta\alpha\beta\rho\lambda} - 4G^{1A}_{\nu[\mu\nu][\delta\alpha]\delta\beta\rho\lambda} + 4G^{1A}_{\nu[\mu\nu]\alpha\beta\rho\delta\delta\lambda}\big) \\
&+\tfrac{1}{2}G^{1B}_{\mu\beta\beta\rho\lambda\rho\lambda} - \tfrac{1}{4}G^{1B}_{\mu\beta\rho\beta\lambda\rho\lambda} + \tfrac{1}{2}G^{1B}_{\mu[\beta\rho]\lambda\beta\rho\lambda} + \tfrac{1}{2}G^{1B}_{\mu\rho[\lambda|\beta|\beta]\lambda\rho\lambda} \\
&-\tfrac{1}{6}G^{1B}_{\mu[\beta\rho]\beta\lambda\rho\lambda} - \tfrac{1}{4}G^{1B}_{\mu\nu\rho\lambda[\nu\rho\lambda]} + \tfrac{\mathrm{i}}{12}\epsilon_{\alpha\beta\rho\lambda}\big(3G^{1B}_{\mu\nu\nu\alpha\beta\rho\lambda} + 4G^{1B}_{\mu[\alpha\nu]\nu\beta\rho\lambda} + 2G^{1B}_{\mu\alpha\beta\rho\nu\nu\lambda}\big) \\
&-\tfrac{1}{2}G^{1C}_{\lambda[\lambda\mu]\sigma\delta\sigma\delta} + 3G^{1C}_{\nu\rho\lambda[\mu\nu\rho\lambda]} + \tfrac{\mathrm{i}}{4}\epsilon_{\mu\nu\rho\lambda}G^{1C}_{\nu\rho\lambda\sigma\delta\sigma\delta} + \tfrac{\mathrm{i}}{4}\epsilon_{\sigma\delta\epsilon\gamma}G^{1C}_{\lambda[\mu\lambda]\sigma\delta\epsilon\gamma} \\
&-2G^{2A}_{\nu\alpha\beta[\mu\nu\alpha|\gamma|\beta]\gamma} + \tfrac{\mathrm{i}}{12}\epsilon_{\nu\rho\lambda\delta}G^{2B}_{\mu\alpha\nu\alpha\rho\lambda\delta} \\
&-\tfrac{1}{6}G^{3}_{\alpha\gamma[\mu|\gamma|\alpha]\rho\lambda\rho\lambda} - G^{3}_{\nu\gamma\alpha\gamma\beta[\mu\nu\alpha\beta]} - \tfrac{1}{4}G^{3}_{\alpha[\mu\alpha\beta]\beta\rho\lambda\rho\lambda} + \tfrac{1}{2}G^{3}_{\alpha[\sigma\rho\lambda|\alpha|\mu]\sigma\rho\lambda} - \tfrac{1}{2}G^{3}_{\alpha[\sigma\rho\lambda|\mu|\alpha]\sigma\rho\lambda} \\
&+\tfrac{\mathrm{i}}{12}\epsilon_{\mu\nu\alpha\beta}G^{3}_{\alpha\gamma\nu\gamma\beta\rho\lambda\rho\lambda} - \tfrac{\mathrm{i}}{12}\epsilon_{\nu\sigma\rho\lambda}G^{3}_{\alpha\gamma[\mu|\gamma|\alpha]\nu\sigma\rho\lambda} \\
&-\tfrac{\mathrm{i}}{12}\epsilon_{[\mu|\nu\alpha\beta}G^{3}_{\gamma\nu\alpha\beta|\gamma]\rho\lambda\rho\lambda} - \tfrac{\mathrm{i}}{8}\epsilon_{\nu\sigma\rho\lambda}G^{3}_{\alpha[\mu\alpha\gamma]\gamma\nu\sigma\rho\lambda}\,.
\end{aligned}
\tag{4.133}
$$

We use the usual square bracket notation, where indices within vertical lines are omitted from antisymmetrization. For readability, we wrote all indices downstairs, although pairs of indices are still contracted with the Minkowski metric. We can compare our chiral map (4.127) and (4.133) with the non-chiral map to fourth order from [28]. In particular we can count the number of terms in each order, factoring in the antisymmetrizations of indices and the intrinsic (anti-)symmetries of the diagrams. For example, an antisymmetrization of the last two indices of G^{1A} (Figure 4.3a) does not generate an extra term, because due to the color structure, the diagram is intrinsically antisymmetric under the exchange of those indices anyway. The numbers of terms are listed in Table 4.2. Note that in the non-chiral map, there are no epsilon symbols present. They lead to a considerable reduction of terms in the case of the chiral map. For an independent check of our result, in the next subsection, we explicitly prove that the gauge (4.31), free-action (1.15), and

TABLE 4.2: Number of terms in the chiral [35] and non-chiral [28] Nicolai maps of $\mathcal{N}=1$ $D=4$ SYM.

order	1	2	3	4
non-chiral map	1	3	34	380
chiral map	2	0	21	224

determinant matching (1.16) conditions for our result (4.127) hold to third order.

4.4.5 Tests to third order

Note: *The following tests follow the appendix of Lechtenfeld's and this author's published work [35].*

Gauge condition (4.31)*:* By construction of the Nicolai map (4.127), it has to satisfy

$$\partial^\mu (T_g A)_\mu = \partial^\mu A_\mu = 0 \tag{4.134}$$

on the Landau gauge hypersurface. By simple symmetry arguments, it is clear that this holds for the first order. For the third order, we use similar symmetry arguments (e.g. $C_{\mu\alpha}\epsilon^{\mu\alpha\rho\lambda} = 0$) to remove most terms. The ghost contributions vanish by

$$\partial^\mu [A_\mu - C_\mu A\cdot\partial] \ldots = [A\cdot\partial - A\cdot\partial] \ldots = 0\,. \tag{4.135}$$

Free-action condition (1.15)*:* Next, we verify the free-action condition

$$S_0[T_g A] = S_g[A]\,, \tag{4.136}$$

where the bosonic action is $S_g[A] = \frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu}$. The condition at first order in the coupling is

$$-\int \mathrm{d}^4x\; A_\mu \Box T_g A^\mu|_{O(g)} \stackrel{!}{=} \int \mathrm{d}^4x\; \partial_\mu A_\nu (A^\mu \times A^\nu)\,. \tag{4.137}$$

The contribution from the ϵ vanishes as an integral over a total derivative and integration by parts of the other term immediately generates the desired right-hand side. For the second order, we need (recalling that the second order of the chiral Nicolai map drops out)

$$-\tfrac{1}{2}\int \mathrm{d}^4x\; T_g A_\mu|_{O(g)} \Box T_g A^\mu|_{O(g)} \stackrel{!}{=} \tfrac{1}{4}\int \mathrm{d}^4x\; (A_\mu \times A_\nu)(A^\mu \times A^\nu)\,. \tag{4.138}$$

Factorizing the left-hand side gives four terms. The two terms proportional to one ϵ symbol cancel each other. We can rewrite the term involving two ϵ symbols with the identity

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\sigma\rho\lambda} = -\delta_\nu{}^\sigma \delta^{\rho\lambda}_{\alpha\beta} + \delta_\alpha{}^\sigma \delta^{\rho\lambda}_{\nu\beta} - \delta_\beta{}^\sigma \delta^{\rho\lambda}_{\nu\alpha}\,, \tag{4.139}$$

generating three contributions. One of them is exactly the desired term on the r.h.s. of (4.138), while the other two combine to cancel the remaining term on the l.h.s. of (4.138). There are no terms of order g^3 or higher on the r.h.s. of (4.136), so we need to show that

$$\int \mathrm{d}^4x \; A_\mu \Box T_g A^\mu |_{O(g^3)} \stackrel{!}{=} 0 \,. \tag{4.140}$$

Firstly, with integration by parts, the ghost contributions

$$\begin{aligned} \int (\Box A^\mu) \times [A_\mu - C_\mu A \cdot \partial] \ldots = &- \int (\partial^\nu A^\mu) \times (\partial_\nu A_\mu) \ldots \\ &+ \int \partial{\cdot}A \times A{\cdot}\partial \ldots = 0 \end{aligned} \tag{4.141}$$

vanish by symmetry and $\partial{\cdot}A = 0$. For the first term in the third line of (4.127) we find

$$(\partial^\nu A^\mu \times A^\alpha) C^\beta A_{[\mu} C_\nu A_\alpha \times A_{\beta]} = -\tfrac{1}{2} (A^\mu \times A^\alpha) C^{\nu\beta} A_{[\mu} C_\nu A_\alpha \times A_{\beta]} = 0 \,, \tag{4.142}$$

after integration by parts. In the same way we can remove the last term in the third order of (4.127). Including antisymmetrization, we are left with four terms that cancel in pairs.

Determinant matching (1.16)*:* Lastly, we need to check that

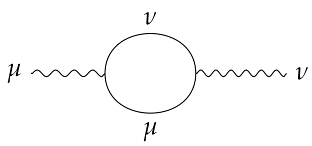
$$\det\left(\tfrac{\delta T_g A}{\delta A}\right) = \Delta_{\mathrm{MSS}}[A] \, \Delta_{\mathrm{FP}}[A] \,, \tag{4.143}$$

with the Matthews–Salam–Seiler determinant (Pfaffian for Majorana fermions) $\Delta_{\mathrm{MSS}}[A]$ and the Faddeev–Popov determinant $\Delta_{\mathrm{FP}}[A]$. It is the most difficult condition to show. For practical purposes, the logarithm of both sides is taken. For the evaluation of the right-hand side, one identifies the MSS and FP kernels from the action (4.82) and expands in orders of g. This procedure is described in detail for example in [22] and [24]. Instead of rederiving this here, we take the result for the right-hand side to third order from there, only adjusting for different metric conventions and setting $D = r = 4$. The condition at $\mathcal{O}(g)$ is trivial as the right-hand side is zero and the (color) trace on the left-hand side makes it vanish by $f^{aac} = 0$. At $\mathcal{O}(g^2)$, the condition reads

$$-\tfrac{1}{2}\mathrm{tr}\left[\tfrac{\delta A'}{\delta A}\Big|_{\mathcal{O}(g)} \tfrac{\delta A'}{\delta A}\Big|_{\mathcal{O}(g)} \right] \stackrel{!}{=} -\tfrac{1}{2} g^2 [5 \,\mathrm{tr}(C_\mu A^\mu C_\nu A^\nu) - 2\,\mathrm{tr}(C_\mu A_\nu C^\mu A^\nu)] \,, \tag{4.144}$$

with the traces over color, position and Lorentz indices on the left-hand side. Generally, the trace in the determinant matching condition generates loops. For example, graphically we would identify

$$\mathrm{tr}(C_\mu A^\mu C_\nu A^\nu) = \tag{4.145}$$



.

Using the identity

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\nu\mu\rho\lambda} = 2\delta_\alpha{}^\rho\delta_\beta{}^\lambda - 2\delta_\alpha{}^\lambda\delta_\beta{}^\rho\,, \tag{4.146}$$

we compute

$$\begin{aligned} \mathrm{tr}\Big[\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\Big] &= (D{-}1)\ \mathrm{tr}(C_\mu A^\mu C_\nu A^\nu) - \epsilon_{\mu\nu\alpha\beta}\epsilon^{\nu\mu\rho\lambda}\ \mathrm{tr}(C^\alpha A^\beta C_\rho A_\lambda) \\ &\qquad + 2\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{tr}(C^\mu A^\nu C^\alpha A^\beta) \\ &= (D{+}1)\ \mathrm{tr}(C_\mu A^\mu C_\nu A^\nu) - 2\ \mathrm{tr}(C_\mu A_\nu C^\mu A^\nu) + 2\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{tr}(C^\mu A^\nu C^\alpha A^\beta)\,. \end{aligned} \tag{4.147}$$

For $D = 4$, we find the desired terms of (4.144). The remaining term vanishes using $C^\mu(x{-}y) = -C^\mu(y{-}x)$, so that

$$\epsilon_{\mu\nu\alpha\beta}\ \mathrm{tr}(C^\mu A^\nu C^\alpha A^\beta) = \epsilon_{\mu\nu\alpha\beta}\ \mathrm{tr}(C^\alpha A^\nu C^\mu A^\beta) = 0\,. \tag{4.148}$$

We have only left to show the $\mathcal{O}(g^3)$ part of the determinant matching condition (4.143). For readability, we use the same color coding of the right-hand side as in [24], so the condition is

$$\begin{aligned} \mathrm{tr}\Big[\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g^3)}\Big] + \tfrac{1}{3}\mathrm{tr}\Big[\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\Big] \overset{!}{=} &+4\ \mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) \\ &- \tfrac{5}{3}\ \mathrm{tr}(C_\mu A_\rho C^\rho A_\lambda C^\lambda A^\mu) - 2\ \mathrm{tr}(C_\mu A_\rho C^\rho A^\mu C_\lambda A^\lambda) \\ &+ \tfrac{2}{3}\ \mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\rho A^\lambda) - 2\ \mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\lambda A^\rho)\,. \end{aligned} \tag{4.149}$$

We start by evaluating

$$\begin{aligned} &\tfrac{1}{3}\mathrm{tr}\Big[\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\tfrac{\delta A'}{\delta A}\big|_{\mathcal{O}(g)}\Big] \\ &\quad = \tfrac{1}{3}\mathrm{tr}(C^\rho A_\lambda C^\alpha A_\beta C^\sigma A_\delta)\ \big[-\delta^{\nu\lambda}_{\mu\rho}\delta^{\gamma\beta}_{\nu\alpha}\delta^{\mu\delta}_{\gamma\sigma} - \mathrm{i}\epsilon_\mu{}^\nu{}_\rho{}^\lambda\epsilon_\nu{}^\gamma{}_\alpha{}^\beta\epsilon_\gamma{}^\mu{}_\sigma{}^\delta \\ &\qquad\qquad + 3\delta^{\nu\lambda}_{\mu\rho}\epsilon_\nu{}^\gamma{}_\alpha{}^\beta\epsilon_\gamma{}^\mu{}_\sigma{}^\delta + 3\mathrm{i}\delta^{\nu\lambda}_{\mu\rho}\delta^{\gamma\beta}_{\nu\alpha}\epsilon_\gamma{}^\mu{}_\sigma{}^\delta\big]\,. \end{aligned} \tag{4.150}$$

The four parts in the square bracket lead to the following contributions:

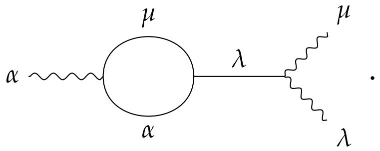
$$\begin{aligned} -\tfrac{1}{3}\delta^{\nu\lambda}_{\mu\rho}\delta^{\gamma\beta}_{\nu\alpha}\delta^{\mu\delta}_{\gamma\sigma} \ &\to\ \tfrac{3-D}{3}\ \mathrm{tr}(C_\mu A_\rho C^\rho A_\lambda C^\lambda A^\mu) - \mathrm{tr}(C_\mu A_\rho C^\rho A^\mu C_\lambda A^\lambda) \\ &\qquad + \tfrac{1}{3}\ \mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\rho A^\lambda)\,, \\ -\tfrac{1}{3}\mathrm{i}\epsilon_\mu{}^\nu{}_\rho{}^\lambda\epsilon_\nu{}^\gamma{}_\alpha{}^\beta\epsilon_\gamma{}^\mu{}_\sigma{}^\delta \ &\to\ \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\big\{\mathrm{tr}(C^\mu A^\alpha C^\nu A_\alpha C^\rho A^\lambda) + \mathrm{tr}(C^\alpha A^\mu C_\alpha A^\nu C^\rho A^\lambda) \\ &\qquad - \mathrm{tr}(C^\mu A^\alpha C_\alpha A^\nu C^\rho A^\lambda) - \mathrm{tr}(C^\alpha A^\mu C^\nu A_\alpha C^\rho A^\lambda)\big\}\,, \\ \delta^{\nu\lambda}_{\mu\rho}\epsilon_\nu{}^\gamma{}_\alpha{}^\beta\epsilon_\gamma{}^\mu{}_\sigma{}^\delta \ &\to\ -\ \mathrm{tr}(C_\mu A_\rho C^\rho A_\lambda C^\lambda A^\mu) + (3{-}D)\ \mathrm{tr}(C_\mu A_\rho C^\rho A^\mu C_\lambda A^\lambda) \\ &\qquad + (D{-}1)\ \mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) - \mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\lambda A^\rho)\,, \\ \mathrm{i}\delta^{\nu\lambda}_{\mu\rho}\delta^{\gamma\beta}_{\nu\alpha}\epsilon_\gamma{}^\mu{}_\sigma{}^\delta \ &\to\ -\mathrm{i}\epsilon_{\mu\nu\rho\lambda}\big\{2\mathrm{tr}(C^\alpha A_\alpha C^\mu A^\nu C^\rho A^\lambda) + \mathrm{tr}(C^\alpha A^\mu C^\nu A_\alpha C^\rho A^\lambda)\big\}\,. \end{aligned} \tag{4.151}$$

The other term on the left-hand side of (4.149) involves the third order of the chiral map. We list the contributions in one line per term of the $\mathcal{O}(g^3)$ part of (4.127). When doing so, we separate the antisymmetrization $[\mu\,\alpha]$ into two lines (3rd+4th and 8th+9th) and combine the antisymmetrization $[\mu\,\nu\,\alpha\,\beta]$ in one line (5th):

$$
\begin{aligned}
\mathrm{tr}\left[\frac{\delta A'}{\delta A}\Big|_{\mathcal{O}(g^3)}\right] = &- \tfrac{N}{3} A_\mu(C)C_\lambda A^\mu \times A^\lambda - \tfrac{2}{3}\mathrm{tr}(A^\mu C A_{[\mu} C_{\lambda]} A^\lambda) \\
&+ \tfrac{N}{3} A^\alpha (C_\mu C_\alpha) C_\lambda A^\mu \times A^\lambda + \tfrac{1}{3}\,\mathrm{tr}(C_\mu A_\rho C^\rho A^\mu C_\lambda A^\lambda) - \tfrac{1}{3}\,\mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) \\
&+ \tfrac{N}{3} A_\mu (C^\alpha C_\alpha) C_\lambda A^\mu \times A^\lambda - \tfrac{1}{3}\,\mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\lambda A^\rho) + \tfrac{1}{3}\,\mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) \\
&- \tfrac{N}{3} A^\alpha (C_\mu C_\alpha) C_\lambda A^\mu \times A^\lambda - \tfrac{1}{3}\,\mathrm{tr}(C_\mu A_\rho C^\rho A_\lambda C^\lambda A^\mu) + \tfrac{1}{3}\,\mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) \\
&- \tfrac{1}{3}\,\mathrm{tr}(C_\mu A_\rho C^\rho A^\mu C_\lambda A^\lambda) - \tfrac{2}{3}\,\mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\lambda A^\rho) + \tfrac{2}{3}\,\mathrm{tr}(C_\mu A^\mu C_\rho A_\lambda C^\rho A^\lambda) \\
&+ \tfrac{1}{3}\,\mathrm{tr}(C_\mu A_\rho C_\lambda A^\mu C^\rho A^\lambda) \\
&- \mathrm{i}\tfrac{N}{6}\epsilon_{\mu\nu\rho\lambda} A^\mu (C) C^\nu A^\rho \times A^\lambda - \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(A^\mu C A^\nu C^\rho A^\lambda) \\
&+ \mathrm{i}\tfrac{N}{6}\epsilon_{\mu\nu\rho\lambda} A_\alpha (C^\alpha C^\mu) C^\nu A^\rho \times A^\lambda + \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\,\mathrm{tr}(C^\mu A^\alpha C_\alpha A^\nu C^\rho A^\lambda) \\
&+ \mathrm{i}\tfrac{N}{6}\epsilon_{\mu\nu\rho\lambda} A^\mu (C^\alpha C_\alpha) C^\nu A^\rho \times A^\lambda + \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\,\mathrm{tr}(C_\alpha A^\mu C^\alpha A^\nu C^\rho A^\lambda) \\
&- \mathrm{i}\tfrac{N}{6}\epsilon_{\mu\nu\rho\lambda} A_\alpha (C^\alpha C^\mu) C^\nu A^\rho \times A^\lambda - \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\,\mathrm{tr}(C_\alpha A^\alpha C^\mu A^\nu C^\rho A^\lambda) \\
&- \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\,\mathrm{tr}(C_\alpha A^\alpha C^\mu A^\nu C^\rho A^\lambda) - \tfrac{\mathrm{i}}{3}\epsilon_{\mu\nu\rho\lambda}\,\mathrm{tr}(C^\mu A^\alpha C^\nu A^\rho C^\lambda A_\alpha)\,.
\end{aligned}
\tag{4.152}
$$

Here we use a very brief notation where round brackets indicate a loop. For example, graphically we translate

$$A^\alpha (C_\mu C_\alpha) C_\lambda A^\mu \times A^\lambda =$$



(4.153)

Let us now consider the various terms in (4.152). Note that there are six gray terms, which we can put in groups of three, one group without an ϵ symbol, and one group with an ϵ symbol. Using a calculation that was already performed in [24], both groups vanish. This requires the Jacobi identity in color space, with the ϵ symbols playing no role in the calculation. Next, let us investigate the black terms. We can read the traces 'backwards', e.g.

$$\mathrm{tr}(C^\alpha A^\beta C^\mu A^\nu C^\rho A^\lambda) = \mathrm{tr}(C^\alpha A^\lambda C^\rho A^\nu C^\mu A^\beta)\,, \tag{4.154}$$

with $C^\alpha(x-y) = -C^\alpha(y-x)$ and $f^{abc}A^b_\mu = -f^{cba}A^b_\mu$ (giving us six minus signs, i.e. a plus in total). Using additionally the cyclicity of the trace and symmetry, we find

$$
\begin{aligned}
&\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\mu A^\alpha C^\nu A_\alpha C^\rho A^\lambda) = \epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\rho A_\alpha C^\nu A^\alpha C^\mu A^\lambda) = -\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\mu A^\alpha C^\nu A_\alpha C^\rho A^\lambda) = 0\,,\\
&\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\alpha A^\mu C_\alpha A^\nu C^\rho A^\lambda) = \epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C_\alpha A^\mu C^\alpha A^\lambda C^\rho A^\nu) = -\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\alpha A^\mu C_\alpha A^\nu C^\rho A^\lambda) = 0\,,\\
&\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\mu A^\alpha C_\alpha A^\nu C^\rho A^\lambda) = \epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C_\alpha A^\alpha C^\mu A^\lambda C^\rho A^\nu) = -\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C_\alpha A^\alpha C^\mu A^\nu C^\rho A^\lambda)\,.
\end{aligned}
\tag{4.155}
$$

Next, we can make use of the

Schouten identity

$$\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\lambda} + \eta_{\alpha\mu}\epsilon_{\nu\rho\lambda\beta} + \eta_{\alpha\nu}\epsilon_{\rho\lambda\beta\mu} + \eta_{\alpha\rho}\epsilon_{\lambda\beta\mu\nu} + \eta_{\alpha\lambda}\epsilon_{\beta\mu\nu\rho} = 0\,, \tag{4.156}$$

for D=4 spacetime dimensions,

which implies that

$$\begin{aligned}
&\epsilon_{\mu\nu\rho\lambda}[\mathrm{tr}(C^\alpha A_\alpha C^\mu A^\nu C^\rho A^\lambda) - \mathrm{tr}(C^\alpha A^\mu C_\alpha A^\nu C^\rho A^\lambda) + \mathrm{tr}(C^\alpha A^\mu C^\nu A_\alpha C^\rho A^\lambda) \\
&\qquad\qquad - \mathrm{tr}(C^\alpha A^\mu C^\nu A^\rho C_\alpha A^\lambda) + \mathrm{tr}(C^\alpha A^\mu C^\nu A^\rho C^\lambda A_\alpha)] \\
&= 2\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\alpha A_\alpha C^\mu A^\nu C^\rho A^\lambda) + \epsilon_{\mu\nu\rho\lambda}\mathrm{tr}(C^\alpha A^\mu C^\nu A_\alpha C^\rho A^\lambda) = 0\,.
\end{aligned} \tag{4.157}$$

This can be used to show that all the black terms in (4.151) and (4.152) vanish after applying (4.155). Lastly, the remaining colored terms add up to the same factors as on the right-hand side of (4.149). This concludes the check of all Nicolai map conditions for our map (4.127) to third order.

Chapter 5

$\mathcal{N} = 4$ super Yang–Mills theory

Note: *This whole chapter is largely based on the author's published work [31].*

To be able to discuss all aspects of $\mathcal{N}=4$ SYM properly, we have to introduce more notation, including a whole range of indices, listed in Table 5.1. It is well known, and we will demonstrate explicitly, that the action of $\mathcal{N}=4$

TABLE 5.1: Types of indices that appear in this chapter. Color and spinor indices are often left implicit. Abbreviations: u.c. - uppercase, l.c. - lowercase. (Table taken from [31].)

Name	Representation	Range	Alphabet
R-symmetry	**4** of $\mathrm{SU}(4)$	1 to 4	1st half of u.c. Latin $(A, B, C, \ldots)$
R-symmetry (broken)	**3** of $\mathrm{SU}(3)$	1 to 3	2nd half of u.c. Latin $(I, J, K, \ldots)$
R-symmetry	**6** of $\mathrm{SU}(4) \cong \mathrm{SO}(6)$	1 to 6	2nd half of l.c. Latin $(i, j, k, \ldots)$
Color	Adjoint of $\mathrm{SU}(N)$	1 to N^2-1	1st half of l.c. Latin $(a, b, c, \ldots)$
Lorentz (4-dim.)	Spin 1 of $\mathrm{SO}(1,3)$	0 to 3	2nd half of l.c. Greek $(\mu, \nu, \rho, \ldots)$
Lorentz (10-dim.)	Spin 1 of $\mathrm{SO}(1,9)$	0 to 9	u.c. Greek $(\Sigma, \Theta, \Gamma, \ldots)$
Spinor	Spin $\frac{1}{2}$	1 to 4	1st half of l.c. Greek $(\alpha, \beta, \gamma, \ldots)$

D=4 SYM can be obtained by dimensional reduction of $\mathcal{N}=1$ D=10 SYM. For that, we use the four- and ten-dimensional mostly plus metrics

$$\eta^{\mu\nu} = \mathrm{diag}(-1,\ +1,\ +1,\ +1)\,, \qquad \eta^{\Sigma\Theta} = \mathrm{diag}(-1,\ +1,\ \ldots,\ +1)\,. \tag{5.1}$$

Our four-dimensional gamma and sigma matrix conventions are unchanged from the previous chapters. They are adopted from Wess and Bagger [46], and can be found in Appendix A. We collect the ten bosonic fields of $\mathcal{N}=1$ D=10 SYM in the symbol

$$\mathscr{A}_\Gamma = (A_\mu,\ \varphi_i)\,, \tag{5.2}$$

with $\mu = 0, 1, 2, 3$ and $i = 1, \ldots, 6$. We will reuse all the quantities that were relevant for the coupling flow operator in Chapter 4. These are the gaugino propagator S (4.83) and ghost propagator G (4.84), the covariant projector $P^\mu{}_\nu$ (4.85), as well as the gauge field decomposition from Section 4.1.5 with the free projector $\Pi^\mu{}_\nu$ (4.56), and transversal projector $\amalg^\mu{}_\nu$ (4.57). We sometimes extend the free projector to capital Greek indices

$$\Pi_\Gamma{}^\Sigma = \delta_\Gamma{}^\Sigma - \partial_\Gamma G_0 \frac{\partial \mathcal{G}(\mathscr{A})}{\partial \mathscr{A}_\Sigma}\,, \tag{5.3}$$

requiring $\partial_{3+i} = 0$ for $i = 1, \ldots, 6$ in the reduced four-dimensional theory.

5.1 Action

Note: *This section is largely following the author's published work [31].*

The $\mathcal{N} = 4$ SYM invariant action is commonly written in a form [25]

$$S_{\text{inv}} = \int \mathrm{d}^4x \Big\{ -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\mathrm{D}_\mu\varphi_i\mathrm{D}^\mu\varphi_i - \tfrac{\mathrm{i}}{2}\bar{\chi}_A\not{D}\mathrm{P}^+\chi^A - \tfrac{\mathrm{i}}{2}\bar{\tilde{\chi}}^A\not{D}\mathrm{P}^-\tilde{\chi}_A \\ - \mathrm{i}g\, t^i{}_{AB}\bar{\tilde{\chi}}^A\mathrm{P}^+\varphi_i\times\chi^B + \mathrm{i}g\, t^{iAB}\bar{\chi}_A\mathrm{P}^-\varphi_i\times\tilde{\chi}_B - \tfrac{g^2}{4}(\varphi_i\times\varphi_j)^2 \Big\}\,, \tag{5.4}$$

formulated using Weyl spinors χ^A, $\tilde{\chi}_A$. They are related via the charge conjugation operator C in four dimensions as $\tilde{\chi}_A = \mathrm{C}(\bar{\chi}^A)^{\mathrm{T}}$. Exactly like in Chapter 4, all fields are in the adjoint representation of the gauge group with implicit color indices. Further, the chiral projectors $\mathrm{P}^\pm$ (4.108) appear. As a theory of extended supersymmetry, the action (5.4) possesses a global R-symmetry, which 'rotates' the supersymmetries. It is given by the Lie group $\mathrm{SU}(4) \cong \mathrm{SO}(6)$. The bosonic fields φ_i transform as a $\mathbf{6}$, while the Weyl spinors χ^A and $\tilde{\chi}_A$ transform as a $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively. Furthermore, there appear the Clebsch-Gordon coefficients $t^i{}_{AB} = (t^{iAB})^*$ (the structure constants of the R-symmetry), that couple two $\mathbf{4}$'s to a $\mathbf{6}$ [47]. With them, we can define anti-symmetric complex scalars

$$\varphi_{AB} = t^i{}_{AB}\varphi_i\,, \qquad \varphi^{AB} = t^{iAB}\varphi_i = (\varphi_{AB})^*\,. \tag{5.5}$$

In four spacetime dimensions, we can freely choose whether to work with Weyl or Majorana spinors. Here, we find it more comfortable to use the latter, so we define

$$\psi^A = \mathrm{P}^+\chi^A + \mathrm{P}^-\tilde{\chi}_A\,, \qquad \bar{\psi}_A = \bar{\chi}_A\mathrm{P}^- + \bar{\tilde{\chi}}^A\mathrm{P}^+\,. \tag{5.6}$$

From the property $\mathrm{C}\gamma_5 = \gamma_5\mathrm{C}$, it follows that $\psi^A = \mathrm{C}(\bar{\psi}_A)^{\mathrm{T}}$, proving that ψ^A are indeed Majorana spinors. A downside of using Majorana spinors is that the R-symmetry transformations become less transparent. As one can see from the fact that the index positions in (5.6) do not match up on the left- and right-hand sides of the equations, the spinors ψ^A and $\bar{\psi}_A$ transform neither as a $\mathbf{4}$ nor a $\bar{\mathbf{4}}$, but are mixed quantities, namely a $\mathbf{4} \oplus \bar{\mathbf{4}}$ and $\bar{\mathbf{4}} \oplus \mathbf{4}$ respectively. It is easiest to deduce the R transformations by translating the spinors back to the Weyl formulation with (5.6). The upside is that we can write very compactly the

invariant action of $\mathcal{N} = 4$ SYM in Majorana notation

$$S_{\text{inv}} = \int \mathrm{d}^4x \left\{ -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\mathrm{D}_\mu\varphi_i\mathrm{D}^\mu\varphi_i - \tfrac{\mathrm{i}}{2}\bar{\psi}_A\mathscr{D}^A{}_B\psi^B - \tfrac{g^2}{4}(\varphi_i\times\varphi_j)^2 \right\}, \tag{5.7}$$

with a matrix-valued field

$$\Phi^A{}_B := 2\big[t^i{}_{AB}\mathrm{P}^+ - t^{iAB}\mathrm{P}^-\big]\varphi_i \equiv (c^i)^A{}_B\varphi_i\,, \tag{5.8}$$

obtained from matrix-valued coefficients $(c^i)^A{}_B$ and a generalization of the covariant derivative

$$\mathscr{D}^A{}_B := \not{D}\delta^A{}_B + g\Phi^A{}_B\times\,. \tag{5.9}$$

To construct an $\mathcal{N}=4$ coupling flow operator, we consider two options. The first one is the canonical construction via an $\mathcal{N} = 1$ superfield formalism and the second one is the reduction of the known $\mathcal{N}=1$ $D=10$ operator to four dimensions. We dedicate the next two subsections to the construction of the action (5.7) in these two approaches.

5.1.1 $\mathcal{N} = 1$ superfield formalism

There is no formulation of $\mathcal{N} = 4$ SYM, where all four supersymmetries are realized off-shell. However, for the canonical construction of the coupling flow operator, it is sufficient to have just one off-shell supersymmetry. This is very much possible, using an $\mathcal{N} = 1$ superfield formalism [47]. The field components come from one vector superfield V and three chiral superfields Φ_I

$$V = (A_\mu,\, \lambda,\, D)\,, \qquad \Phi_I = (\phi_I,\, \psi_I,\, F_I) \quad \text{with} \quad I = 1,2,3\,, \tag{5.10}$$

with explicit expansions given in Appendix C. The dynamical fields are the gauge field A_μ, four Weyl- (or Majorana-) spinors ψ^A (A=1,2,3,4, with λ=ψ^4) and three complex scalars ϕ_I. Additionally, there is one real scalar auxiliary field D and three complex scalar auxiliary fields F_I. Just like in our setup of $\mathcal{N}=1$ SYM (c.f. (4.13)), the $\mathcal{N}=4$ Lagrangian is the last component of a superfield:

$$\begin{aligned}\mathcal{L} = \tfrac{1}{g^2N}\mathrm{tr}\Big[\tfrac{1}{16}\big(W^\alpha W_\alpha\big|_{\vartheta\vartheta} + \text{h.c.}\big) \;+\; \mathrm{e}^{-2V}\Phi_I^\dagger \mathrm{e}^{2V}\Phi_I\big|_{\vartheta\vartheta\bar\vartheta\bar\vartheta} \\ + \;\mathrm{i}\tfrac{\sqrt{2}}{3!}\big(\epsilon_{IJK}\Phi_I[\Phi_J,\Phi_K]\big|_{\vartheta\vartheta} + \text{h.c.}\big)\Big]\,,\end{aligned} \tag{5.11}$$

As usual, there is a trace over color space. There appear the non-abelian supersymmetric field strength W_α and its conjugate, with the superspace covariant derivatives D_α, $\bar{\mathrm{D}}^{\dot\alpha}$, which all can be found explicitly in Appendix C. This is the geometric scaling, where the coupling only appears as an overall factor $1/g^2$ in front of (5.11). The perturbative scaling can be recovered by $V \to gV$ and $\Phi_I \to g\Phi_I$. With the technical details given in the appendix, the Lagrangian in the Majorana basis comes out as

$$\begin{aligned}g^2\mathcal{L} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{\mathrm{i}}{2}\bar\lambda\gamma^\mu \mathrm{D}_\mu\lambda + \tfrac{1}{2}D^2 - \tfrac{1}{\sqrt{2}}\epsilon_{IJK}\big(F_I\phi_J\times\phi_K + F_I^\dagger\phi_J^\dagger\times\phi_K^\dagger\big) \\ &- \mathrm{D}_\mu\phi_I^\dagger \mathrm{D}^\mu\phi_I - \tfrac{\mathrm{i}}{2}\bar\psi_I\gamma^\mu\mathrm{D}_\mu\psi_I + F_I^\dagger F_I + \tfrac{1}{\sqrt{2}}\epsilon_{IJK}\big(\phi_I\bar\psi_J\mathrm{P}^+\times\psi_K + \phi_I^\dagger\bar\psi_J\mathrm{P}^-\times\psi_K\big) \\ &- \sqrt{2}\big(\bar\psi_I\mathrm{P}^-\lambda\times\phi_I + \bar\psi_I\mathrm{P}^+\lambda\times\phi_I^\dagger\big) - \mathrm{i}\phi_I^\dagger D\times\phi_I\,,\end{aligned} \tag{5.12}$$

and the off-shell supervariations can be deduced from the superfield structure:

$$
\begin{aligned}
&\mathring{\delta}_\alpha \phi_I = \sqrt{2}(\bar{\psi}_I \mathrm{P}^+)_\alpha \,, \\
&\mathring{\delta}_\alpha \phi_I^\dagger = \sqrt{2}(\bar{\psi}_I \mathrm{P}^-)_\alpha \,, \\
&\mathring{\delta}_\alpha (\mathrm{P}^+\psi_I)_\beta = -\mathrm{i}\sqrt{2}(\mathrm{P}^+\gamma^\mu)_{\beta\alpha}(\mathrm{D}_\mu\phi_I) - \sqrt{2}(\mathrm{P}^+)_{\beta\alpha} F_I \,, \\
&\mathring{\delta}_\alpha (\mathrm{P}^-\psi_I)_\beta = -\mathrm{i}\sqrt{2}(\mathrm{P}^-\gamma^\mu)_{\beta\alpha}(\mathrm{D}_\mu\phi_I^\dagger) - \sqrt{2}(\mathrm{P}^-)_{\beta\alpha} F_I^\dagger \,, \\
&\mathring{\delta}_\alpha F_I = -\mathrm{i}\sqrt{2}(\mathrm{D}_\mu \bar{\psi}_{I\beta})(\gamma^\mu \mathrm{P}^-)_{\beta\alpha} - 2\phi_I \times (\bar{\lambda}\mathrm{P}^-)_\alpha \,, \\
&\mathring{\delta}_\alpha F_I^\dagger = -\mathrm{i}\sqrt{2}(\mathrm{D}_\mu \bar{\psi}_{I\beta})(\gamma^\mu \mathrm{P}^+)_{\beta\alpha} - 2\phi_I^\dagger \times (\bar{\lambda}\mathrm{P}^+)_\alpha \,, \\
&\mathring{\delta}_\alpha A_\nu = -\mathrm{i}(\bar{\lambda}\gamma_\nu)_\alpha \,, \\
&\mathring{\delta}_\alpha D = -\mathrm{i}(\mathrm{D}_\mu \bar{\lambda}_\beta)(\gamma_5\gamma^\mu)_{\beta\alpha} \,, \\
&\mathring{\delta}_\alpha \lambda_\beta = -\tfrac{1}{2}(\gamma^{\mu\nu})_{\beta\alpha} F_{\mu\nu} + D(\gamma_5)_{\beta\alpha} \,.
\end{aligned} \tag{5.13}
$$

We can extract the penultimate superfield component, with respect to the supersymmetry that is realized off-shell. It consists of those terms in the superspace expansion of (5.11) that have one less power of ϑ or $\bar{\vartheta}$ than maximal. In Majorana notation, it is given by

$$
\begin{aligned}
\mathring{\Delta}_\alpha = \tfrac{1}{4}\int \mathrm{d}^4x \Big\{ &- D\gamma_5\lambda - \tfrac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\lambda + 2\epsilon_{IJK}\big[\mathrm{P}^+\psi_I\phi_J\times\phi_K + \mathrm{P}^-\psi_I\phi_J^\dagger\times\phi_K^\dagger\big] \\
&+ 2\mathrm{i}\gamma_5\phi_I^\dagger\lambda\times\phi_I + \mathrm{i}\sqrt{2}\big[\gamma^\mu\mathrm{P}^-\psi_I\mathrm{D}_\mu\phi_I + \gamma^\mu\mathrm{P}^+\psi_I\mathrm{D}_\mu\phi_I^\dagger\big] \\
&- \sqrt{2}\big[\mathrm{P}^+\psi_I F_I^\dagger + \mathrm{P}^-\psi_I F_I\big]\Big\}_\alpha \,.
\end{aligned} \tag{5.14}
$$

As usual, the invariant action is generated from its supervariation:

$$
S_{\text{inv}} = \int \mathrm{d}^4x \;\; \mathcal{L} = \tfrac{1}{2g^2}\,\delta_\alpha\mathring{\Delta}_\alpha \,. \tag{5.15}
$$

We can eliminate the auxiliary fields to obtain an on-shell invariant action. To do so, we insert the equations of motion of the auxiliary fields

$$
\mathcal{D} = -\mathrm{i}\phi_I^\dagger\times\phi_I \,, \qquad F_I = \tfrac{1}{\sqrt{2}}\epsilon_{IJK}\phi_J^\dagger\times\phi_K^\dagger \,, \tag{5.16}
$$

which yields

$$
\begin{aligned}
S_{\text{inv}} = \tfrac{1}{g^2}\int \mathrm{d}^4x \;\Big\{&-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \mathrm{D}_\mu\phi_I^\dagger\mathrm{D}^\mu\phi_I - \tfrac{\mathrm{i}}{2}\bar{\psi}_A\not{\mathrm{D}}\psi^A \\
&+ \tfrac{1}{\sqrt{2}}\epsilon_{IJK}\big(\phi_I\bar{\psi}_J\mathrm{P}^+\times\psi_K + \phi_I^\dagger\bar{\psi}_J\mathrm{P}^-\times\psi_K\big) - \sqrt{2}\big(\bar{\psi}_I\mathrm{P}^-\lambda\times\phi_I + \bar{\psi}_I\mathrm{P}^+\lambda\times\phi_I^\dagger\big) \\
&+ \tfrac{1}{2}(\phi_I^\dagger\times\phi_I)^2 - \tfrac{1}{2}\epsilon_{IJK}\epsilon_{ILM}(\phi_J\times\phi_K)(\phi_L^\dagger\times\phi_M^\dagger)\Big\} \,.
\end{aligned} \tag{5.17}
$$

In order to compare this expression to (5.7), we need to express the three complex fields ϕ_I in terms of six real fields φ_i. The identification

$$
\phi_I = \tfrac{1}{\sqrt{2}}(\varphi_{I+3} + \mathrm{i}\varphi_I) \,, \qquad \phi_I^\dagger = \tfrac{1}{\sqrt{2}}(\varphi_{I+3} - \mathrm{i}\varphi_I) \tag{5.18}
$$

allows one to rewrite the action as

$$S_{\text{inv}} = \tfrac{1}{g^2}\int \mathrm{d}^4x\ \Big\{-\tfrac14 F^{\mu\nu}F_{\mu\nu} - \tfrac12 \mathrm{D}_\mu\varphi_i \mathrm{D}^\mu\varphi_i - \tfrac{\mathrm{i}}{2}\bar\psi_A \not{D}\psi^A \\ + \tfrac12\epsilon_{IJK}\big(\bar\psi_I\varphi_{J+3}\times\psi_K - \bar\psi_I\varphi_J\gamma_5\times\psi_K\big) + \bar\psi_I\varphi_{I+3}\times\lambda + \bar\psi_I\varphi_I\gamma_5\times\lambda - \tfrac14(\varphi_i\times\varphi_j)^2\Big\}\,, \tag{5.19}$$

where we used the Jacobi identity in color space for the last term. This expression allows one to write down explicitly the coefficients $(c^i)^A{}_B$ from (5.8):

$$\begin{aligned} (c^I)^J{}_4 &= \mathrm{i}\delta_{IJ}\gamma_5\,, \qquad & (c^{I+3})^J{}_4 &= \mathrm{i}\delta_{IJ}\mathbb{1}_4\,, \\ (c^I)^J{}_K &= \mathrm{i}\epsilon_{IJK}\gamma_5\,, \qquad & (c^{I+3})^J{}_K &= -\mathrm{i}\epsilon_{IJK}\mathbb{1}_4\,. \end{aligned} \tag{5.20}$$

They are antisymmetric under the exchange of the outer two indices and all others are zero. In the next subsection, we show that the coefficients resulting from the dimensional reduction of the ten-dimensional theory to four dimensions are exactly the same. For the construction of the coupling flow operator in Section 5.2, we will also make use of the following two facts. Firstly, integrating out the auxiliaries in the penultimate component, we can write it as

$$\Delta_\alpha = \tfrac14\int \mathrm{d}^4x\ \Big\{-\tfrac12 F_{\mu\nu}\gamma^{\mu\nu}\lambda - (\Phi^4{}_A)^\dagger \not{\mathscr{D}}^A{}_B\psi^B + \tfrac12(\Phi^4{}_A)^\dagger\Phi^A{}_B\times\psi^B\Big\}\,, \tag{5.21}$$

and secondly, we note that the supervariations of the six real scalars are

$$\delta_\alpha\varphi_i = -\mathrm{i}\bar\psi_J (c^i)^J{}_4\,. \tag{5.22}$$

5.1.2 Dimensional reduction

We now show how to obtain the on-shell action of $\mathcal{N} = 4$ $D = 4$ SYM by dimensionally reducing the on-shell $\mathcal{N} = 1$ $D = 10$ SYM action [25]

$$S^{(10)} = \tfrac{1}{g^2}\int \mathrm{d}^{10}x\ \Big\{-\tfrac14 F^{\Sigma\Theta}F_{\Sigma\Theta} - \tfrac{\mathrm{i}}{2}\bar\lambda\,\Gamma^\Sigma\,\mathrm{D}_\Sigma\,\lambda\Big\}\,, \tag{5.23}$$

where capital Greek indices label run from 0 to 9, and Γ^Σ span the ten-dimensional Clifford algebra. The spinor λ is a Majorana-Weyl spinor, see the discussion at the beginning of Chapter 4 and Table 4.1. The gauge field can be reduced by

$$A_\Sigma = (A_\mu,\ \varphi_i)\,, \tag{5.24}$$

so the Yang–Mills term decomposes as

$$-\tfrac14 F^{\Sigma\Theta}F_{\Sigma\Theta} \quad\longrightarrow\quad -\tfrac14 F_{\mu\nu}F^{\mu\nu} - \tfrac12\mathrm{D}_\mu\varphi_i\mathrm{D}^\mu\varphi_i - \tfrac14(\varphi_i\times\varphi_j)^2\,, \tag{5.25}$$

where we imposed that all partial derivatives ∂_{3+i} with $i = 1,\ldots,6$ vanish. For the Dirac term, we have to choose an appropriate representation of the gamma matrices

$$\Gamma^\mu = \mathbb{1}_8\otimes\gamma^\mu\,, \qquad \Gamma^{AB} = \begin{pmatrix} 0 & \rho^{AB} \\ \rho_{AB} & 0 \end{pmatrix}\otimes \mathrm{i}\gamma_5\,, \qquad A,B = 1,2,3,4\,, \tag{5.26}$$

where we define the antisymmetric 4×4 matrices

$$(\rho^{AB})_{CD} := \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}\,, \qquad (\rho_{AB})_{CD} := \tfrac{1}{2}\epsilon_{ABFG}(\rho^{FG})_{CD} = \epsilon_{ABCD}\,. \tag{5.27}$$

Moreover, at this point, we can introduce the antisymmetric complex scalars

$$\varphi_{I4} = \tfrac{1}{2}(\varphi_I + \mathrm{i}\varphi_{I+3})\,, \qquad \varphi^{AB} = \tfrac{1}{2}\epsilon^{ABCD}\varphi_{CD} = (\varphi_{AB})^*\,, \tag{5.28}$$

where the prefactor of $1/2$ is conventional. This fixes the Clebsch-Gordon coefficients $t^i{}_{AB}$ from (5.5) to be

$$\begin{aligned} &(t^I)_{J4} = \tfrac{1}{2}\delta_{IJ} = (t^I)^{J4}\,, && (t^{I+3})_{J4} = \tfrac{\mathrm{i}}{2}\delta_{IJ} = -(t^{I+3})^{J4}\,, \\ &(t^I)_{JK} = \tfrac{1}{2}\epsilon_{IJK} = (t^I)^{JK}\,, && (t^{I+3})_{JK} = -\tfrac{\mathrm{i}}{2}\epsilon_{IJK} = -(t^{I+3})^{JK}\,. \end{aligned} \tag{5.29}$$

The Majorana-Weyl spinor λ can be decomposed as

$$\lambda = (\mathrm{P}^+\chi^1, \dots, \mathrm{P}^+\chi^4, \mathrm{P}^-\tilde{\chi}_1, \dots, \mathrm{P}^-\tilde{\chi}_4)^{\mathrm{T}}\,, \quad \text{with} \quad \tilde{\chi}_A = \mathrm{C}\bar{\chi}^{A\,\mathrm{T}}\,, \tag{5.30}$$

with four Weyl spinors χ^A as in (5.4). Instead of Weyl spinors, we can use the Majorana spinors ψ^A from (5.6). In terms of these, the Dirac term simply becomes

$$-\tfrac{\mathrm{i}}{2}\bar{\lambda}\,\Gamma^\Sigma\,\mathrm{D}_\Sigma\,\lambda \quad\longrightarrow\quad -\tfrac{\mathrm{i}}{2}\bar{\psi}_A\,\not{\mathscr{D}}^A{}_B\,\psi^B\,, \tag{5.31}$$

with the generalized covariant derivative (5.9) and the matrix-valued (c.f. (5.8))

$$\Phi^A{}_B = (c^i)^A{}_B\varphi_i = \big[(\rho^{CD})_{AB}\mathrm{P}^+ - (\rho_{CD})_{AB}\mathrm{P}^-\big]\varphi_{CD} = 2\big[t^i{}_{AB}\mathrm{P}^+ - t^{iAB}\mathrm{P}^-\big]\varphi_i\,. \tag{5.32}$$

This fixes the coefficients

$$(c^i)^A{}_B = 2\big[t^i{}_{AB}\mathrm{P}^+ - t^{iAB}\mathrm{P}^-\big]\,, \tag{5.33}$$

which are equivalent to those found via the superfield formalism (5.20).

5.2 Construction of the flow operator

Note: *This section is largely following the author's published work [31].*

We present two methods of constructing the coupling flow operator. First, in Section 5.2.1, we use the canonical construction via the $\mathcal{N}=1$ superfield formalism, and secondly, in Section 5.2.2, we reduce the known result from ten-dimensions (see Section 4.3) in the Landau gauge down to four dimensions.

5.2.1 Canonical construction

We can follow the procedure developed in Section 4.1. The only difference is that the superfield expansion (5.11) is more complicated. We could also add the same topological term as in Section 4.4, but this only affects the gauge field A_μ and not the full bosonic sector. For simplicity, here we set θ=0. We start with the on-shell invariant action (5.19), so the full action

$S_{\text{SUSY}}{=}S_{\text{inv}}{+}S_{\text{gf}}$ is

$$S_{\text{inv}} = \tfrac{1}{g^2}\int \mathrm{d}^4x \,\left\{-\tfrac14 \widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu} - \tfrac12 \widetilde{\mathrm{D}}_\mu\widetilde{\varphi}_i\widetilde{\mathrm{D}}^\mu\widetilde{\varphi}_i - \tfrac{\mathrm{i}}{2}\widetilde{\overline{\psi}}_A\widetilde{\mathscr{D}}^A{}_B\widetilde{\psi}^B - \tfrac14(\widetilde{\varphi}_i{\times}\widetilde{\varphi}_j)^2\right\},$$
$$S_{\text{gf}} = \tfrac{1}{g^2}\int \mathrm{d}^4x\,\left\{-\tfrac{1}{2\xi}\mathcal{G}(\widetilde{A},\widetilde{\varphi})^2 + g\,\widetilde{\bar{c}}\tfrac{\partial\mathcal{G}(\widetilde{A},\widetilde{\varphi})}{\partial\widetilde{A}_\mu}\widetilde{\mathrm{D}}_\mu\widetilde{c} + g\,\widetilde{\bar{c}}\tfrac{\partial\mathcal{G}(\widetilde{A},\widetilde{\varphi})}{\partial\widetilde{\varphi}_i}\widetilde{\varphi}_i{\times}\widetilde{c}\right\}, \tag{5.34}$$

where as in Chapter 4, the tildes indicate the geometric scaling of the fields. Later, for perturbation theory, we will rescale by appropriate powers of g, in particular $\widetilde{A}{=}gA$ and $\widetilde{\varphi}{=}g\varphi$. We use the by now familiar fact that, due to the off-shell supersymmetry, we can generate the g-derivative of the action as a supervariation (given by (5.13) with tildes everywhere) up to a Slavnov variation

$$\partial_g S_{\text{SUSY}} = -\tfrac{1}{g^3}\{\mathring{\delta}_\alpha\mathring{\Delta}_\alpha - \sqrt{g}\,s\,\Delta_{\text{gh}}\}\,, \tag{5.35}$$

with the penultimate superfield component $\mathring{\Delta}$ as in (5.14) but with tildes on all fields, the standard ghost component

$$\Delta_{\text{gh}} = \int \mathrm{d}^4x\,\left\{\widetilde{\bar{c}}\,\mathcal{G}(\widetilde{A},\widetilde{\varphi})\right\}, \tag{5.36}$$

and the Slavnov variations

$$\begin{aligned}
s\widetilde{A}_\mu &= \sqrt{g}\,\widetilde{\mathrm{D}}_\mu\widetilde{c}\,, & s\widetilde{\lambda} &= \sqrt{g}\,\widetilde{\lambda}\times\widetilde{c}\,, & s\widetilde{\bar{\lambda}} &= \sqrt{g}\,\widetilde{\bar{\lambda}}\times\widetilde{c}\,,\\
s\widetilde{D} &= \sqrt{g}\,\widetilde{D}\times\widetilde{c}\,, & s\widetilde{c} &= -\tfrac{\sqrt{g}}{2}\,\widetilde{c}\times\widetilde{c}\,, & s\widetilde{\bar{c}} &= \tfrac{1}{\sqrt{g}}\tfrac{1}{\xi}\,\mathcal{G}(\widetilde{A},\widetilde{\varphi})\,,\\
s\widetilde{\varphi}_i &= \sqrt{g}\,\widetilde{\varphi}_i\times\widetilde{c}\,, & s\widetilde{\psi}_I &= \sqrt{g}\,\widetilde{\psi}_I\times\widetilde{c}\,, & s\widetilde{F} &= \sqrt{g}\,\widetilde{F}_I\times\widetilde{c}\,.
\end{aligned} \tag{5.37}$$

Since (5.35) and (5.36) are of the exact same form as for $\mathcal{N}{=}1$ $D{=}4$ SYM, the construction of the coupling flow operator is completely analogous. The intermediate coupling flow operator is (c.f. (4.29) from Section 4.1.2)

$$\widetilde{R}_g[\widetilde{\mathscr{A}}] = -\mathrm{i}\,\underbracket{\Delta_\alpha[\widetilde{\mathscr{A}}]\;\delta_\alpha} + \tfrac{\mathrm{i}}{\sqrt{g}}\,\underbracket{\Delta_{\text{gh}}[\widetilde{\mathscr{A}}]\;s} - \tfrac{1}{\sqrt{g}}\,\underbracket{\Delta_\alpha[\widetilde{\mathscr{A}}]\;\big(\delta_\alpha}\underbracket{\Delta_{\text{gh}}[\widetilde{\mathscr{A}}]\big)\;s}\,, \tag{5.38}$$

where $\widetilde{\mathscr{A}}{\equiv}(\widetilde{A}_\mu,\widetilde{\varphi}_i)$, we integrated out the auxiliary fields, so we use the on-shell $\Delta_\alpha[\widetilde{\mathscr{A}}]$ (5.21), and, as usual, contractions indicate gaugino or ghost propagators. We can rescale $\widetilde{\mathscr{A}}{=}g\mathscr{A}$ and write this in terms of the explicit fields, in a calculation that can be found in Appendix D. The result has the same structure as (4.63), but with an additional R-symmetry index structure:

> The $\mathcal{N}{=}4$ $D{=}4$ SYM coupling flow operator from the canonical construction
>
> $$\begin{aligned}\overleftarrow{R}_g\,[\mathscr{A}] = \tfrac18\,\overleftarrow{\tfrac{\delta}{\delta\mathscr{A}_\Gamma}}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\{(\mathscr{C}_\Sigma)^4{}_A S^A{}_B\mathscr{A}^B{}_C\times\mathscr{A}^{*C}{}_4\}&\\ + \overleftarrow{\tfrac{\delta}{\delta\mathscr{A}_\Gamma}}\,\Pi_\Gamma{}^\Sigma\mathscr{A}_\Sigma G\tfrac{\partial\mathcal{G}(\mathscr{A})}{\partial A_\nu}A^{\mathrm{L}}_\nu\,,&\end{aligned} \tag{5.39}$$

where quite a few new quantities were introduced, which will be explained in the following. At the place where in the $\mathcal{N}{=}1$ case, there only was a gamma matrix, we now have the more general object

$$(\mathscr{C}_\Sigma)^A{}_B := \begin{cases} \delta^A{}_B \gamma_\mu & \text{for} \quad \Sigma = \mu = 0,1,2,3 \\ (c^i)^A{}_B & \text{for} \quad \Sigma = 3+i = 4,5,\ldots,9 \end{cases} , \tag{5.40}$$

that also contains the matrix valued coefficients $(c^i)^A{}_B$ (5.20). We also use the natural shorthand notations

$$\mathscr{D}_\Gamma \equiv (\mathrm{D}_\mu \, , \, g\varphi_i \times \,) \, , \qquad \not{\mathscr{D}}^A{}_B \equiv \mathscr{D}^\Sigma (\mathscr{C}_\Sigma)^A{}_B = \not{D}\delta^A{}_B + g\Phi^A{}_B \times \; , \tag{5.41}$$

which allow to express the gaugino and ghost propagators $S^A{}_B$, G, as

$$\psi^A(x)\bar{\psi}_B(y) = -S^A{}_B(x,y;\mathscr{A}) \, , \qquad \not{\mathscr{D}}^A{}_C \, S^C{}_B(x,y;\mathscr{A}) = \delta^A{}_B \delta(x-y) \, , \tag{5.42}$$

and

$$\mathrm{i}c(x)\bar{c}(y) = G(x,y;\mathscr{A}) \, , \qquad \frac{\partial \mathcal{G}(\mathscr{A})}{\partial \mathscr{A}_\Gamma} \mathscr{D}_\Gamma \, G(x,y;\mathscr{A}) = \delta(x-y) \, . \tag{5.43}$$

We also introduced a generalization of the conjugate gauge field (c.f. (4.59))

$$\not{\mathscr{A}}^A{}_B = \mathscr{A}^\Sigma (\mathscr{C}_\Sigma)^A{}_B = \not{A}\delta^A{}_B + \Phi^A{}_B \, , \qquad \not{\mathscr{A}}^{*A}{}_B := \not{A}^* \delta^A{}_B + (\Phi^A{}_B)^\dagger \, , \tag{5.44}$$

of the covariant projector (c.f. (4.85))

$$P_\Gamma{}^\Sigma = \delta_\Gamma{}^\Sigma - \mathscr{D}_\Gamma G \frac{\partial \mathcal{G}(\mathscr{A})}{\partial \mathscr{A}_\Sigma} \, , \tag{5.45}$$

and its free version

$$\Pi_\Gamma{}^\Sigma = P_\Gamma{}^\Sigma \big|_{g=0} \, . \tag{5.46}$$

It might seem quite peculiar that in the formula for the coupling flow operator (5.39) there appears an explicit '4' instead of a summation index. This is an artifact from having chosen one of the four supersymmetries (the 'fourth' one) for the $\mathcal{N}=1$ superfield formalism. It will become much clearer once we consider an R-symmetric framework in Section 5.3.

5.2.2 Dimensional reduction of the ad-hoc construction

We can take the known (ad-hoc, see Section 4.3) expression for the coupling flow operator in ten dimensions on the Landau gauge hypersurface (c.f. (4.100))

$$R_g[\mathscr{A}] = \frac{1}{32} \frac{\overleftarrow{\delta}}{\delta \mathscr{A}_\Gamma} \, P^{(10)}{}_\Gamma{}^\Sigma \, \mathrm{tr}^{(32)} \left\{ \Gamma_\Sigma S^{(10)} \not{\mathscr{A}} \times \not{\mathscr{A}} \right\} , \tag{5.47}$$

and dimensionally reduce it to four dimensions. The trace in (5.47) is over 32×32 spinor space with

$$\not{\mathscr{A}} = \Gamma^\Sigma \mathscr{A}_\Sigma \, . \tag{5.48}$$

With the same methods as in Section 5.1.2, we carry out the dimensional reduction. In the following, when leaving out the superscript that indicates the number of dimensions, we always assume the four-dimensional quantity, e.g., $S^{(4)} \equiv S$. From the fact that the partial derivatives ∂_{3+i} for i=1,...6 vanish in the reduced theory, and setting $\mathscr{A}=(A_\mu, \varphi_i)$, it immediately follows that

$$P^{(10)}{}_{\Gamma}{}^{\Sigma} \qquad \longrightarrow \qquad P_{\Gamma}{}^{\Sigma} \quad . \tag{5.49}$$

The gaugino propagator $S^{(10)}$ is a 32×32 matrix given by the contraction of the spinor fields. Using the dimensional reduction of λ (5.30), we can decompose the propagator as

$$S^{(10)} = -\underbrace{\lambda\bar{\lambda}} = \left(\begin{array}{c|c} (\mathrm{P}^+ S^A{}_B \mathrm{P}^-)_{\alpha\beta} & (\mathrm{P}^+ S^{AB} \mathrm{P}^+)_{\alpha\beta} \\ \hline (\mathrm{P}^- S_{AB} \mathrm{P}^-)_{\alpha\beta} & (\mathrm{P}^- S_A{}^B \mathrm{P}^+)_{\alpha\beta} \end{array}\right) , \tag{5.50}$$

into four 16×16 blocks with 'inner' (R-symmetry) indices A, B and 'outer' (Majorana) indices α, β, all ranging from one to four. Here, thanks to the chiral projectors, the position of the inner indices complies with the transformations under R-symmetry. This means, upper and lower indices transform as a $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively. In the same block notation, we represent the gamma matrices as (c.f. (5.26))

$$\Gamma^\mu = \left(\begin{array}{c|c} (\gamma^\mu)_{\alpha\beta}\delta^A{}_B & 0 \\ \hline 0 & (\gamma^\mu)_{\alpha\beta}\delta_A{}^B \end{array}\right) , \quad \Gamma^{AB} = \left(\begin{array}{c|c} 0 & (\mathrm{i}\gamma_5)_{\alpha\beta}\rho^{AB} \\ \hline (\mathrm{i}\gamma_5)_{\alpha\beta}\rho_{AB} & 0 \end{array}\right) , \tag{5.51}$$

and thus

$$\mathscr{A}\!\!\!/ = \Gamma^\Sigma \mathscr{A}_\Sigma = \left(\begin{array}{c|c} A\!\!\!/_{\alpha\beta}\delta^A{}_B & (\mathrm{i}\gamma_5)_{\alpha\beta}\varphi^{AB} \\ \hline (\mathrm{i}\gamma_5)_{\alpha\beta}\varphi_{AB} & A\!\!\!/_{\alpha\beta}\delta_A{}^B \end{array}\right) . \tag{5.52}$$

When multiplying two block matrices, we contract equal types of indices with each other, which for example yields

$$\mathscr{A}\!\!\!/ \times \mathscr{A}\!\!\!/ = \left(\begin{array}{c|c} (A\!\!\!/\times A\!\!\!/)_{\alpha\beta}\delta^A{}_B + (\mathbb{1}_4)_{\alpha\beta}\varphi^{AC}\times\varphi_{CB} & 2(A\!\!\!/\mathrm{i}\gamma_5)_{\alpha\beta}\times\varphi^{AB} \\ \hline 2(\mathrm{i}\gamma_5 A\!\!\!/)_{\alpha\beta}\times\varphi_{AB} & (A\!\!\!/\times A\!\!\!/)_{\alpha\beta}\delta_A{}^B + (\mathbb{1}_4)_{\alpha\beta}\varphi_{AC}\times\varphi^{CB} \end{array}\right) . \tag{5.53}$$

Through dimensional reduction, we want to bring (5.47) to a form similar to the first term of (5.39), which contains a trace over 4×4 matrices. To that end, we interpret $S^{(10)}$, Γ_Σ, and $\mathscr{A}\!\!\!/$ as 8×8 matrices with 4×4 matrix-valued entries. The latter matrices are spanned by the outer indices α, β. We multiply all the matrices in the trace of (5.47) with each other, and subsequently take a partial trace in the 8×8 matrix space. A sample contribution is

$$\begin{aligned}
&\mathrm{tr}^{(32)}\left\{\Gamma_\mu\, S^{(10)}\, \mathbb{1}_8 \otimes (A\!\!\!/\times A\!\!\!/)\right\} \\
&= \mathrm{tr}^{(32)}\left(\begin{array}{c|c} (\gamma_\mu)_{\alpha\gamma}(\mathrm{P}^+S^A{}_B\mathrm{P}^-)_{\gamma\delta}(A\!\!\!/\times A\!\!\!/)_{\delta\beta} & (\gamma_\mu)_{\alpha\gamma}(\mathrm{P}^+S^{AB}\mathrm{P}^+)_{\gamma\delta}(A\!\!\!/\times A\!\!\!/)_{\delta\beta} \\ \hline (\gamma_\mu)_{\alpha\gamma}(\mathrm{P}^-S_{AB}\mathrm{P}^-)_{\gamma\delta}(A\!\!\!/\times A\!\!\!/)_{\delta\beta} & (\gamma_\mu)_{\alpha\gamma}(\mathrm{P}^-S_A{}^B\mathrm{P}^+)_{\gamma\delta}(A\!\!\!/\times A\!\!\!/)_{\delta\beta} \end{array}\right) \\
&= \mathrm{tr}^{(4)}\{\gamma_\mu\mathrm{P}^+S^A{}_A\mathrm{P}^-A\!\!\!/\times A\!\!\!/\} \;+\; \mathrm{tr}^{(4)}\{\gamma_\mu\mathrm{P}^-S_A{}^A\mathrm{P}^+A\!\!\!/\times A\!\!\!/\} \\
&= \mathrm{tr}^{(4)}\{\gamma_\mu S^A{}_A A\!\!\!/\times A\!\!\!/\} ,
\end{aligned} \tag{5.54}$$

where in the last step the cyclicity of the trace allows one to commute the chiral projectors. From the third to the last line, the positions of the indices of the gaugino propagator in the first vs. second term do not match up, so the R-symmetry transformation properties become nontransparent again. This is due to the definition of our Majorana spinors (5.6), where the same issue appears. However, to recover the transformations, one can always express the composite objects in terms of those in Weyl notation, where the index positions agree with the R-symmetry transformation properties. By computations analogous to (5.54), one finally establishes

$$\mathrm{tr}^{(32)}\left\{\Gamma_\Sigma S^{(10)}\mathscr{A}\times\mathscr{A}\right\}=\mathrm{tr}^{(4)}\left\{(\mathscr{C}_\Sigma)^A{}_B S^B{}_C \mathscr{A}^C{}_D\times\mathscr{A}^{*D}{}_A\right\}, \tag{5.55}$$

with the same quantities as introduced in Section 5.2.1. By construction, this trace is invariant under R-symmetry transformations for $\Sigma{=}\mu$, and transforms as a **6** of SU(4) for $\Sigma{=}3{+}i$, which can also be checked explicitly.[1] We have found

the $\mathcal{N}{=}4$ $D{=}4$ SYM coupling flow operator from dimensional reduction

$$\overleftarrow{R_g}\,[\mathscr{A}]=\tfrac{1}{32}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\left\{(\mathscr{C}_\Sigma)^A{}_B S^B{}_C \mathscr{A}^C{}_D\times\mathscr{A}^{*D}{}_A\right\}, \tag{5.56}$$

which is valid on the Landau gauge hypersurface.

5.3 R-symmetry

Note: *This section is largely following the author's published work [31].*

We have found two results for the coupling flow operator, one from the canonical construction (5.39) and one from dimensional reduction (5.56). In the following, we will restrict to the Landau gauge hypersurface, because the latter only exists there. Comparing the two results,

$$\overleftarrow{R_g}\,[\mathscr{A}]=\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\left\{(\mathscr{C}_\Sigma)^4{}_B S^B{}_C \mathscr{A}^C{}_D\times\mathscr{A}^{*D}{}_4\right\}\qquad\text{from can. con.}, \tag{5.57}$$

$$\overleftarrow{R_g}\,[\mathscr{A}]=\tfrac{1}{32}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\left\{(\mathscr{C}_\Sigma)^A{}_B S^B{}_C \mathscr{A}^C{}_D\times\mathscr{A}^{*D}{}_A\right\}\qquad\text{from dim. red.}, \tag{5.58}$$

we conclude that they are not equal, because they come with slightly different index structures. To reconcile this, we first need to make two observations. From the definition of the coupling flow operator R_g (2.1), we can read off that real observables are mapped to real observables. That is, at least up to potential imaginary terms that drop out in expectation values. Such terms will be disregarded in the following. Thus, we demand that the kernel K in

$$\overleftarrow{R_g}\,[\mathscr{A}]=\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,K_\Gamma\,, \tag{5.59}$$

is real. The second observation is a principle of superposition for coupling flow operators. Consider the infinitesimal conditions (2.21), (2.22) and the

[1] By decomposing into chiral (Weyl) contributions. The contributions that do not seem to transform correctly vanish due to properties of the chiral projectors.

gauge condition (4.31)

$$
\begin{aligned}
&(\partial_g + R_g[\mathscr{A}])S^{\mathrm{b}}_g[\mathscr{A}] = 0\,, \qquad (\partial_g + R_g[\mathscr{A}])S^{\mathrm{f}}_g[\mathscr{A}] = \int \mathrm{d}x\, \frac{\delta K_\Gamma[\mathscr{A};\,x]}{\delta \mathscr{A}_\Gamma(x)}\,,\\
&(\partial_g + R_g[\mathscr{A}])\mathcal{G}(\mathscr{A}) = 0\,,
\end{aligned}
\tag{5.60}
$$

for our bosonic fields $\mathscr{A}$ (although this works for any supersymmetric field theory). If we now assume that we are given two operators $R^{(1)}_g$ and $R^{(2)}_g$ that satisfy these conditions, it is straightforward to verify the

principle of superposition of coupling flow operators:
If $R^{(1)}_g$ and $R^{(2)}_g$ are coupling flow operators, then

$$
R'_g := pR^{(1)}_g + qR^{(2)}_g \quad \text{with} \quad p,q \in \mathbb{R} \quad \text{and} \quad p+q=1 \tag{5.61}
$$

also satisfies the three conditions (5.60), i.e. it is also a coupling flow operator.

The index 4 in (5.57) is a reminiscence of choosing the 'fourth' supersymmetry for setting up the superfield formalism. Of course, we could have equally well chosen any of the three others, which would result in indices 1,2 or 3 instead of 4 in (5.57). The principle of superposition then allows us to superimpose those four cases with real coefficients, keeping the same normalization. We make the anticipatory ansatz

$$
\overleftarrow{R}_g[\mathscr{A}] = \tfrac{1}{32}\,\tfrac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\, P_\Gamma{}^\Sigma\, \mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^A{}_B S^B{}_C \mathscr{A}^C{}_D \times \mathscr{A}^{*D}{}_E(\delta^E{}_A + \mathrm{L}^E{}_A)\Big\}\,, \tag{5.62}
$$

where we inserted the identity plus a traceless matrix L. We can identify the two results (5.57), (5.58) with

$$
\mathrm{L} = \mathrm{diag}(-1,-1,-1,+3) \quad \text{and} \quad \mathrm{L}=0\,, \tag{5.63}
$$

respectively. Applying the principle of superposition to our ansatz, we find that any matrix from the set

$$
\Big\{\mathrm{L} = \mathrm{diag}(q_1,q_2,q_3,q_4) \quad \text{with} \quad \sum_i q_i = 0\Big\} =: \mathfrak{h} \tag{5.64}
$$

leads to a valid coupling flow operator. Note that the set $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{su}(4)$, the space of traceless and hermitian 4×4 matrices. Moreover, $\mathfrak{h}$ is a commutative subalgebra, since the commutator of any two elements vanishes. Additionally, given a matrix $x \in \mathfrak{su}(4)$, if

$$
[h,x] \in \mathfrak{h} \qquad \forall \quad h \in \mathfrak{h}\,, \tag{5.65}
$$

then also $x \in \mathfrak{h}$.[2] This shows that $\mathfrak{h}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{su}(4)$. While this already shows that there is a whole family of $\mathcal{N}=4$ coupling flow operators, the space $\mathfrak{h}$ is not exhaustive for possible choices of

[2] That is, $\mathfrak{h}$ is equal to its normalizer.

L. We know that the trace in (5.62) is invariant under SU(4) R transformations for $\Sigma = \mu$ and transforms as a **6** for $\Sigma = 3 + i$. This property has to be conserved when including the matrix L in the trace. By investigating the R symmetry index structure, one finds that exactly like the Majorana spinors (5.6), L consists of two chiral (Weyl) contributions

$$\mathrm{L} = L\,\mathrm{P}^- + L^*\,\mathrm{P}^+ \,, \tag{5.66}$$

where L^* is the complex conjugate of L. To preserve the R symmetry properties, L has to transform in the adjoint **15** of SU(4)

$$L \quad \longrightarrow \quad ULU^\dagger \,, \qquad \text{with} \quad U \in \mathrm{SU}(4) \,. \tag{5.67}$$

Let us now apply an SU(4) transformation to

$$\mathrm{L}_0 = L_0\,\mathrm{P}^- + L_0^*\,\mathrm{P}^+ \in \mathfrak{h} \,. \tag{5.68}$$

It is clear that zero trace and hermiticity are preserved

$$\mathrm{tr}(UL_0U^\dagger) = 0 \qquad \text{and} \qquad (UL_0U^\dagger)^\dagger = UL_0U^\dagger \,. \tag{5.69}$$

This, together with the fact that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{su}(4)$, implies that the group action of SU(4) on $\mathfrak{h}$ generates the entire Lie algebra $\mathfrak{su}(4)$. Thus, any $L \in \mathfrak{su}(4)$ represents a valid coupling flow operator. With the inverse argument, for any $L \in \mathfrak{su}(4)$, there is a matrix $U \in \mathrm{SU}(4)$ such that $ULU^\dagger$ is diagonal. Hence, we can characterize any L by its four eigenvalues q_i. Note that $\mathrm{tr}(L^m)$ is invariant under (5.67) for any integer $m \geq 1$, but only

$$\mathrm{tr}\, L = \sum_i q_i = 0 \,, \quad \mathrm{tr}\, L^2 = \sum_i q_i^2 \,, \quad \mathrm{tr}\, L^3 = \sum_i q_i^3 \,, \quad \mathrm{tr}\, L^4 = \sum_i q_i^4 \tag{5.70}$$

are functionally independent.[3] For example,

$$\mathrm{tr}\, L^5 = \tfrac{5}{6}\, \mathrm{tr}\, L^2 \cdot \mathrm{tr}\, L^3 \,. \tag{5.71}$$

We can characterize a given L by three real parameters, either its eigenvalues (q_1, q_2, q_3, q_4) with $\sum q_i = 0$ or by $(\mathrm{tr}\, L^2, \mathrm{tr}\, L^3, \mathrm{tr}\, L^4)$. It is sensible to differentiate the degrees of degeneracy in the eigenvalues. If all the eigenvalues are distinct, the stabilizer of the adjoint action[4] on L is the maximal torus $\mathrm{S}(\mathrm{U}(1)^4) \cong \mathrm{U}(1)^3$. In that case, the orbit under the action is given by SU(4) modulo the stabilizer, resulting in the 12-dimensional flag manifold

$$\mathrm{SU}(4)\big/\mathrm{U}(1)^3 \,. \tag{5.72}$$

If there is degeneracy in the eigenvalues, the orbit is smaller since the stabilizer is larger. All cases are listed in Table 5.2. The dimensional reduction led to $L = 0$, the fully degenerate case with no free parameters for the coupling flow operator. It is the most symmetric scenario, which can be expected, since all four supersymmetries were treated equally in the dimensional reduction.

[3]This can be proved by Newton's identities for symmetric polynomials.

[4]i.e. the space spanned by the group elements that keep L invariant.

TABLE 5.2: Stabilizers $X \subset \mathrm{SU}(4)$ acting on $L \in \mathfrak{su}(4)$, by degeneracy of the eigenvalues q_i. The last column lists the number of free parameters for the coupling flow operator. It is given by the sum of degrees of freedom (dofs) in the choice of the q_i's and the dimension of the orbit $\mathrm{SU}(4)/X$. (Table taken from [31].)

Degeneracy	dofs	Stabilizer X	$\dim(X)$	# free param.
all q_i distinct	3	$\mathrm{S}(\mathrm{U}(1)^4)$	3	15
two q_i equal	2	$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1)^2)$	5	12
two equal pairs	1	$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))$	7	9
three q_i equal	1	$\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(1))$	9	7
all $q_i = 0$	0	$\mathrm{SU}(4)$	15	0

The canonical construction on the other side led to an L where three of the eigenvalues are equal to zero, with stabilizer $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(1))$. This prompts the conclusion that any L with three degenerate eigenvalues originates from an off-shell formalism, where any gauge fixing function is admissible. For gauges other than the Landau gauge, the second term in (5.39) must be added to the general formula. To conclude this section, we write down the general formula for

the general $\mathcal{N}=4$ $D=4$ SYM coupling flow operator on the Landau gauge hypersurface

$$\overleftarrow{R}_g\,[\mathscr{A}] = \tfrac{1}{32}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^A{}_B S^B{}_C\mathscr{A}^C{}_D \times \mathscr{A}^{*D}{}_E(\delta^E{}_A + \mathrm{L}^E{}_A)\Big\}\,, \tag{5.73}$$

where $\mathrm{L} = L\mathrm{P}^- + L^*\mathrm{P}^+$ and L is any element of the Lie algebra $\mathfrak{su}(4)$.

We have shown that there is at least a 15-dimensional ambiguity for the coupling flow operator on the Landau gauge hypersurface and at least a 7-dimensional ambiguity for an arbitrary gauge. In Appendix E, we prove that (5.73) satisfies the necessary and sufficient infinitesimal conditions for the coupling flow operator, providing a double check of its validity.

5.4 Maps

The Nicolai map for $\mathcal{N} = 4$ SYM was first investigated by Nicolai and Plefka in [23]. They assumed the $\mathcal{N}=1$ $D=10$ map in the Landau gauge (to second order)[5]

$$T_g\mathscr{A}_\Sigma = \mathscr{A}_\Sigma - gC^\Theta\mathscr{A}_\Sigma\mathscr{A}_\Theta + \tfrac{3}{2}g^2C^\Theta\mathscr{A}^\Gamma C_{[\Sigma}\mathscr{A}_\Theta\mathscr{A}_{\Gamma]} + \mathcal{O}(g^3)\,, \tag{5.74}$$

and reduced it to four dimensions with the simple prescription $\mathscr{A} = (A_\mu, \varphi_i)$ and $\partial_{3+i} \equiv 0$. This yields

$$T_gA_\mu = A_\mu - gC^\rho A_\mu A_\rho + \tfrac{3}{2}g^2C^\rho A^\lambda C_{[\mu}A_\rho A_{\lambda]} + g^2C^\rho\varphi_i C_{[\mu}A_{\rho]}\varphi_i + \mathcal{O}(g^3)\,, \tag{5.75}$$

$$T_g\varphi_i = \varphi_i - gC^\rho\varphi_i A_\rho + g^2C^{[\rho}A^{\lambda]}C_\lambda\varphi_i A_\rho + \tfrac{1}{2}g^2C^\rho\varphi_j C_\rho\varphi_j\varphi_i + \mathcal{O}(g^3)\,. \tag{5.76}$$

[5] which we write with the usual compact notation, see Appendix A.

This agrees, as expected, with the result one obtains, when constructing the Nicolai map from the coupling flow operator (5.73), with the maximally symmetric L=0. Here, we want to investigate the ambiguity of the $\mathcal{N} = 4$ map by additionally computing the four distinct maps to second order on the Landau gauge hypersurface with

$$L = \text{diag}(+3,-1,-1,-1)\,, \ldots\,,\ \text{diag}(-1,-1,-1,+3)\,. \tag{5.77}$$

They correspond to the canonical construction of the coupling flow operator from the superfield formalism, where supersymmetry 'number' 1, 2, 3, 4 is realized off-shell, respectively. In these cases, the $\mathcal{N}=4$ coupling flow operator becomes

$$\overleftarrow{R}_g{}^{(\mathbf{A})}[\mathscr{A}] = \frac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\text{tr}\Big\{(\mathscr{C}_\Sigma)^{\mathbf{A}}{}_B S^B{}_C \mathscr{A}^C{}_D \times \mathscr{A}^{*D}{}_{\mathbf{A}}\Big\}\,. \tag{5.78}$$

with no sum over $\mathbf{A} = 1,2,3,4$. As always, the Nicolai map is given by the universal formula (2.11), which to second order yields

$$T_g\mathscr{A} = \mathscr{A} \ - \ g\,r_1\mathscr{A} \ - \ \tfrac{1}{2}g^2\big(r_2 - r_1^2\big)\mathscr{A} \ + \ \mathcal{O}(g^3)\,. \tag{5.79}$$

with the $\mathcal{O}(g^0)$ and $\mathcal{O}(g^1)$ components r_1 and r_2 of the coupling flow operator respectively. The computations that lead to the following explicit Nicolai maps are straightforward and can be found in detail in Appendix D of [31]. The Nicolai map for $\mathbf{A} = 4$ is given by

$$\begin{aligned} T_g^{(4)}A_\mu = A_\mu - gC^\rho A_\mu A_\rho + \tfrac{3}{2}g^2 C^\rho A^\lambda C_{[\mu}A_\rho A_{\lambda]} + g^2 C^\rho \varphi_i C_{[\mu}A_{\rho]}\varphi_i& \\ -\tfrac{1}{2}g^2\Pi_\mu{}^\nu \epsilon_{\nu\lambda\rho\sigma}\sum_{J=1}^{3}[C^\lambda\varphi_J C^\rho\varphi_{J+3}A^\sigma - C^\lambda\varphi_{J+3}C^\rho\varphi_J A^\sigma& \\ +C^\lambda A^\rho C^\sigma\varphi_{J+3}\varphi_J] + \mathcal{O}(g^3)\,,& \end{aligned} \tag{5.80}$$

$$\begin{aligned} T_g^{(4)}\varphi_I = \varphi_I - gC^\rho\varphi_I A_\rho + g^2 C^{[\rho}A^{\lambda]}C_\lambda\varphi_I A_\rho + \tfrac{1}{2}g^2 C^\rho\varphi_j C_\rho\varphi_j\varphi_I& \\ -\tfrac{1}{4}g^2\epsilon_{\mu\nu\rho\lambda}[C^\mu\varphi_{I+3}C^\nu A^\rho A^\lambda + 2C^\mu A^\nu C^\rho\varphi_{I+3}A^\lambda]& \\ -\tfrac{1}{2}g^2 C^\rho\sum_{J=1}^{3}[\varphi_{I+3}C_\rho\varphi_{J+3}\varphi_J + \varphi_J C_\rho\varphi_{I+3}\varphi_{J+3} - \varphi_{J+3}C_\rho\varphi_{I+3}\varphi_J] + \mathcal{O}(g^3)\,,& \end{aligned} \tag{5.81}$$

$$\begin{aligned} T_g^{(4)}\varphi_{I+3} = \varphi_{I+3} - gC^\rho\varphi_{I+3}A_\rho + g^2 C^{[\rho}A^{\lambda]}C_\lambda\varphi_{I+3}A_\rho + \tfrac{1}{2}g^2 C^\rho\varphi_j C_\rho\varphi_j\varphi_{I+3}& \\ +\tfrac{1}{4}g^2\epsilon_{\mu\nu\rho\lambda}[C^\mu\varphi_I C^\nu A^\rho A^\lambda + 2C^\mu A^\nu C^\rho\varphi_I A^\lambda]& \\ +\tfrac{1}{2}g^2 C^\rho\sum_{J=1}^{3}[\varphi_I C_\rho\varphi_{J+3}\varphi_J + \varphi_J C_\rho\varphi_I\varphi_{J+3} - \varphi_{J+3}C_\rho\varphi_I\varphi_J] + \mathcal{O}(g^3)\,,& \end{aligned} \tag{5.82}$$

where the black terms completely match with (5.75) and (5.76), while the blue terms are new. Similarly, the Nicolai maps for $\mathbf{A} = \mathbf{K} = 1, 2, 3$ are

$$\begin{aligned} T_g^{(\mathbf{K})} A_\mu = A_\mu - g C^\rho A_\mu A_\rho + \tfrac{3}{2} g^2 C^\rho A^\lambda C_{[\mu} A_\rho A_{\lambda]} + g^2 C^\rho \varphi_i C_{[\mu} A_{\rho]} \varphi_i \\ + \tfrac{1}{2} g^2 \Pi_\mu{}^\nu \epsilon_{\nu\lambda\rho\sigma} \sum_{J=1}^{3} (-)^{\delta_{\mathbf{K}J}} [C^\lambda \varphi_J C^\rho \varphi_{J+3} A^\sigma - C^\lambda \varphi_{J+3} C^\rho \varphi_J A^\sigma \\ + C^\lambda A^\rho C^\sigma \varphi_{J+3} \varphi_J] + \mathcal{O}(g^3) \,, \end{aligned} \tag{5.83}$$

$$\begin{aligned} T_g^{(\mathbf{K})} \varphi_I = \varphi_I - g C^\rho \varphi_I A_\rho + g^2 C^{[\rho} A^{\lambda]} C_\lambda \varphi_I A_\rho + \tfrac{1}{2} g^2 C^\rho \varphi_j C_\rho \varphi_j \varphi_I \\ + \tfrac{1}{4} g^2 \epsilon_{\mu\nu\rho\lambda} (-)^{\delta_{I\mathbf{K}}} [C^\mu \varphi_{I+3} C^\nu A^\rho A^\lambda + 2 C^\mu A^\nu C^\rho \varphi_{I+3} A^\lambda] \\ - \tfrac{1}{2} g^2 C^\rho (-)^{\delta_{I\mathbf{K}}} \sum_{J=1}^{3} [\varphi_{I+3} C_\rho \varphi_{J+3} \varphi_J + \varphi_J C_\rho \varphi_{I+3} \varphi_{J+3} - \varphi_{J+3} C_\rho \varphi_{I+3} \varphi_J] \\ + g^2 C^\rho [\varphi_{I+3} C_\rho \varphi_{\mathbf{K}+3} \varphi_{\mathbf{K}} + \varphi_{\mathbf{K}} C_\rho \varphi_{I+3} \varphi_{\mathbf{K}+3} - \varphi_{\mathbf{K}+3} C_\rho \varphi_{I+3} \varphi_{\mathbf{K}}] + \mathcal{O}(g^3) \,, \end{aligned} \tag{5.84}$$

$$\begin{aligned} T_g^{(\mathbf{K})} \varphi_{I+3} = \varphi_{I+3} - g C^\rho \varphi_{I+3} A_\rho + g^2 C^{[\rho} A^{\lambda]} C_\lambda \varphi_{I+3} A_\rho + \tfrac{1}{2} g^2 C^\rho \varphi_j C_\rho \varphi_j \varphi_{I+3} \\ - \tfrac{1}{4} g^2 \epsilon_{\mu\nu\rho\lambda} (-)^{\delta_{I\mathbf{K}}} [C^\mu \varphi_I C^\nu A^\rho A^\lambda + 2 C^\mu A^\nu C^\rho \varphi_I A^\lambda] \\ + \tfrac{1}{2} g^2 C^\rho (-)^{\delta_{I\mathbf{K}}} \sum_{J=1}^{3} [\varphi_I C_\rho \varphi_{J+3} \varphi_J + \varphi_J C_\rho \varphi_I \varphi_{J+3} - \varphi_{J+3} C_\rho \varphi_I \varphi_J] \\ - g^2 C^\rho [\varphi_I C_\rho \varphi_{\mathbf{K}+3} \varphi_{\mathbf{K}} + \varphi_{\mathbf{K}} C_\rho \varphi_I \varphi_{\mathbf{K}+3} - \varphi_{\mathbf{K}+3} C_\rho \varphi_I \varphi_{\mathbf{K}}] + \mathcal{O}(g^3) \,. \end{aligned} \tag{5.85}$$

Not that the novel blue terms in all four versions of the Nicolai map only differ in the signs. Actually, when symmetrically superimposing the four coupling flow operators $R_g^{(\mathbf{A})}$ as

$$R_g := \tfrac{1}{4} (R_g^{(1)} + R_g^{(2)} + R_g^{(3)} + R_g^{(4)}) \,, \tag{5.86}$$

the resulting Nicolai map exactly reduces to the black terms (5.75) and (5.76). It should be highlighted though that we cannot add the four Nicolai maps themselves, as they are not linear in R_g. By construction, the necessary and sufficient conditions for a Nicolai map are satisfied by all four maps presented in this section. This is also shown explicitly in Appendix E of [31].

Chapter 6

Outlook

An important topic that we did not address in this thesis is renormalization. It is important to recall that the Nicolai map only consists of tree diagrams. Loops only come into play when computing correlators in the free theory via $\langle X[\phi]\rangle_g = \langle T_g^{-1} X[\phi]\rangle_0$, where the open bosonic lines of the trees are contracted with each other. For simple, pure supersymmetric theories, such as the four-dimensional Wess-Zumino model or ϕ^4 theory, renormalization is almost trivial. There, the coupling flow operator and Nicolai map simply acquire a global renormalization factor. For gauge theories, this is more complicated, because regularization breaks supersymmetry. Here, each divergent integral has to be taken care of individually. This was already investigated for $\mathcal{N}=1$ SYM by Lechtenfeld and Dietz in 1985 [12]. They were able to restrict to gauge-invariant observables, such that there was no need to fix a gauge. With dimensional regularization, they rederived the universality of the gauge coupling to 1-loop order, in a computational effort similar to the usual Feynman diagram method. The gauge independence of this approach to the quantization of non-abelian gauge theories is quite special and should be investigated more deeply. Furthermore, it would be interesting to see how the addition of a topological θ-term affects this procedure. As a toy model, this could first be studied for our interacting SQM model from Chapter 3.

In Chapter 5, we studied the Nicolai map formalism for $D=4$ $\mathcal{N}=4$ SYM and uncovered a 15-dimensional ambiguity for the coupling flow operator. This allows one to search for the most simple Nicolai map construction. In principle, one can further add the same topological term that we also added to the $\mathcal{N}=1$ theory, but this only affects the gauge field and not the six real scalars of the $\mathcal{N}=4$ theory. Perhaps, it is possible to construct a topological term that affects all bosonic fields, such that we achieve similar drastic simplifications as we have seen in Section 4.4. This is equivalent to finding a topological term for $D=10$ $\mathcal{N}=1$ SYM. Since it does not have an off-shell superspace formulation, there is no canonical way of finding such a topological term. Other open questions regarding the $\mathcal{N}=4$ theory are how exactly the ambiguity from the coupling flow operator propagates to the Nicolai map. Further, it would be interesting to see whether the Nicolai map somehow knows about the vanishing of the beta function and the renormalization properties of $\mathcal{N}=4$ SYM. The ultimate goal in this direction would be to find hints of an integrable structure.

It would be quite remarkable to formulate a Nicolai map for supergravity. As a toy model, supersymmetric nonlinear sigma models could be studied. A main complication is that they (and formulations of supergravity) typically contain terms that are higher than quadratic in the fermions. This is a major complication for the formalism since it requires integrating out all fermionic fields. There are special manifolds and particular formulations of nonlinear sigma models that can indirectly be written with only fermion bilinears, bypassing this problem. It would be a good first step to investigate these cases. For supergravity, there are also special formulations, such as the 'first order' or '1.5 order' formulation (see e.g. [48]) that have an only-quadratic action in the fermions.

In the context of SQM, we have started to investigate the diagrammatics of the Nicolai map in Chapter 3. We discovered that the notion of 1-particle-irreducible diagrams is distinct from the situation in standard Feynman perturbation theory. It would be illuminating to find a precise definition of 1PI Nicolai diagrams and a corresponding effective action.

It is believed that the Nicolai map can be helpful in studying supersymmetry breaking. This was prompted by the identification of the winding number of the Nicolai map with the Witten index [21, 38, 44, 51], which allows formulating a necessary criterion for supersymmetry breaking in terms of the geometry of the Nicolai map. This connection should be further explored (e.g., for the O'Raifeartaigh model), coming from the modern viewpoint on the formalism.

Another big open question is how to apply the Nicolai map non-perturbatively. We have several reasons to believe that there is non-perturbative information contained in the Nicolai map. The computations of Lechtenfeld for SQM [34] prove a finite radius of convergence of the map expansion. This is expected to apply to gauge theories as well. Thus, one can assume that the Nicolai map does indeed exist non-perturbatively. As we have demonstrated for SQM, the special theta values $\theta = \pm 1$ make the expansion collapse into a linear function in the coupling, giving access to (non-perturbative) instanton configurations. Already in the 80s, there have been attempts to extract non-perturbative information from the Nicolai map [14–16, 49, 50] using stochastic variables, but for $\mathcal{N} = 1$ SYM, Nicolai pointed out complications at fourth order [51]. It would be worth reinvestigating these approaches, utilizing the modern insights into the formalism. Lastly, the universal formula gives, at least formally, an exact non-perturbative form of the map. It would be enlightening to discover more cases of exact Nicolai maps or find ways to extract information directly from the universal formula.

Appendix A

Notation and conventions

Here, we elaborate on our notation and conventions in Chapter 4 and Chapter 5, where we work in four, or sometimes more generally D-dimensional Minkowski space, with the mostly plus metric

$$\eta^{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1) \; . \tag{A.1}$$

Sigma and gamma matrices: Our sigma and gamma matrix conventions are adopted from the textbook of Wess and Bagger [46], but for completeness, we list them here again. The (four-dimensional) gamma matrices are written as

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \,, \qquad = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \,, \tag{A.2}$$

in terms of the sigma matrices

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \,, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \tag{A.3}$$

and $\bar{\sigma}^0 = \sigma^0$, $\bar{\sigma}^{1,2,3} = -\sigma^{1,2,3}$. The Clifford algebra is

$$\{\gamma^{\mu}, \gamma^{\nu}\} = -2\eta^{\mu\nu} \tag{A.4}$$

and we introduce the 'fifth' gamma matrix as

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \; . \tag{A.5}$$

Often, we use the chiral projectors given by

$$\mathrm{P}^{+} = \tfrac{1}{2}(1 + \mathrm{i}\gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \,, \qquad \mathrm{P}^{-} = \tfrac{1}{2}(1 - \mathrm{i}\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; . \tag{A.6}$$

Further, we use standard Feynman slash notation[1]

$$\not{a} = \gamma^{\mu} a_{\mu} \; . \tag{A.7}$$

There often appears the antisymmetrization of two gamma matrices

$$\gamma^{\mu\nu} = \tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) \tag{A.8}$$

[1] apart for the slashed script letters $\not{\mathscr{A}}$ and $\not{\mathscr{D}}$ from Chapter 5, with similar but distinct definitions (5.44) and (5.9).

and general antisymmetrizations of indices with weight one, indicated by square brackets, e.g.

$$a^{[\mu}b^{\nu]} = \tfrac{1}{2}(a^\mu b^\nu - a^\nu b^\mu)\ . \tag{A.9}$$

To evaluate traces of $2n$ gamma matrices (traces of an odd number of gamma matrices vanish), one can use (D=4)

$$\mathrm{tr}\ \gamma^\mu \gamma^\nu = -4\eta^{\mu\nu} \tag{A.10}$$

and the recursive relation

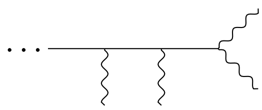
$$\mathrm{tr}\big(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}\big) = -\eta^{\mu_1\mu_2}\mathrm{tr}\big(\gamma^{\mu_3}\cdots\gamma^{\mu_{2n}}\big) + \eta^{\mu_1\mu_3}\mathrm{tr}\big(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_{2n}}\big) \mp \ldots\ , \tag{A.11}$$

with $2n-1$ terms on the right-hand side. If they involve the 'fifth' gamma matrix, we can use

$$\mathrm{tr}\ \gamma^\mu\gamma^\nu\gamma^5 = 0\ , \qquad \mathrm{tr}\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^5 = 4\epsilon^{\mu\nu\rho\lambda}\ . \tag{A.12}$$

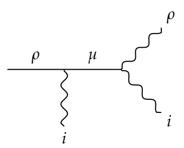
For more notation and identities involving spinors, we refer to appendices A and B of [46].

Notation for the coupling flow operator and Nicolai maps: In Chapter 4 and Chapter 5 we use a very compact notation for writing down coupling flow operators or Nicolai maps. Usually, we leave color and position labels implicit, as described around (4.8) and the following equations. Moreover, for Nicolai graphs that have a 'linear' tree structure (usually graphs to second order or expansions of propagators)


(A.13)

we use a special shorthand notation, where integration kernels are convoluted with insertions of bosonic fields A_μ (or φ_i). For example, we write a compact linear tree as

$$C^\rho \varphi_i C_\mu A_\rho \times \varphi_i \ \equiv$$



, (A.14)

which translates to

$$\int \mathrm{d}^4y\ \mathrm{d}^4z\ \partial^\rho C(x-y)\ (f^{abc}\varphi_i^b)(y)\ \partial_\mu C(y-z)\ (f^{cde}A_\rho^d)(z)\varphi_i^e(z)\ . \tag{A.15}$$

For more complicated 'branched' trees, one has to be more precise. In these cases, the exact expressions are always given in the main text.

Appendix B

Computation of SQM amplitudes

Note: *This appendix is adopted from the author's published work [33].*

B.1 Nicolai computation

We give the explicit results for the seven contributions $N_1, \ldots, N_7$ to the three-point function computed with the $\theta = \pm 1$ Nicolai map from Section 3.5.3. Pulling out an overall factor of

$$g^3\, 2\pi\delta(\omega_1+\omega_2+\omega_3) \prod_{i=1,2,3} (m^2+\omega_i^2)^{-2}(4m^2+\omega_i^2)^{-1}\,, \tag{B.1}$$

they are (with the symmetric polynomials (3.81)) the 1PI contributions

$$\begin{aligned}
N_1 \quad &\to \quad 96m^{10} - 296m^8 t_2 + 312m^6 t_2^2 - 120m^4 t_2^3 + 8m^2 t_2^4 \\
&\quad \pm 96\mathrm{i}m^7 t_3 \mp 200\mathrm{i}m^5 t_2 t_3 \pm 112\mathrm{i}m^3 t_2^2 t_3 \mp 8\mathrm{i}m t_2^3 t_3 \\
&\quad +96m^4 t_3^2 - 104m^2 t_2 t_3^2 + 8t_2^2 t_3^2 \pm 96\mathrm{i}m t_3^3 \mp 8\mathrm{i}m^{-1} t_2 t_3^3\,, \\
N_2 \quad &\to \quad 576m^{10} - 1200m^8 t_2 + 672m^6 t_2^2 - 48m^4 t_2^3 \\
&\quad \mp 288\mathrm{i}m^7 t_3 \pm 600\mathrm{i}m^5 t_2 t_3 \mp 336\mathrm{i}m^3 t_2^2 t_3 \pm 24\mathrm{i}m t_2^3 t_3 \\
&\quad +576m^4 t_3^2 - 48m^2 t_2 t_3^2 \mp 288\mathrm{i}m t_3^3 \pm 24\mathrm{i}m^{-1} t_2 t_3^3\,, \\
N_3 \quad &\to \quad 384m^{10} - 608m^8 t_2 + 48m^6 t_2^2 + 192m^4 t_2^3 - 16m^2 t_2^4 \\
&\quad \pm 96\mathrm{i}m^7 t_3 \mp 200\mathrm{i}m^5 t_2 t_3 \pm 112\mathrm{i}m^3 t_2^2 t_3 \mp 8\mathrm{i}m t_2^3 t_3 \\
&\quad +384m^4 t_3^2 + 160m^2 t_2 t_3^2 - 16t_2^2 t_3^2 \pm 96\mathrm{i}m t_3^3 \mp 8\mathrm{i}m^{-1} t_2 t_3^3\,, \\
N_4 \quad &= \quad N_3\,,
\end{aligned} \tag{B.2}$$

and the 1PR contributions

$$\begin{aligned}
N_5 \quad &\to \quad 3456m^{10} - 5760m^8 t_2 + 2952m^6 t_2^2 - 720m^4 t_2^3 + 72m^2 t_2^4 \\
&\quad \mp 2160\mathrm{i}m^7 t_3 \pm 1620\mathrm{i}m^5 t_2 t_3 \mp 360\mathrm{i}m^3 t_2^2 t_3 \pm 36\mathrm{i}m t_2^3 t_3 \\
&\quad -1080m^4 t_3^2 + 288m^2 t_2 t_3^2 \pm 108\mathrm{i}m t_3^3\,, \\
N_6 \quad &\to \quad 384m^{10} - 128m^8 t_2 - 312m^6 t_2^2 + 48m^4 t_2^3 + 8m^2 t_2^4 \\
&\quad \pm 1392\mathrm{i}m^7 t_3 \mp 980\mathrm{i}m^5 t_2 t_3 \pm 184\mathrm{i}m^3 t_2^2 t_3 \mp 20\mathrm{i}m t_2^3 t_3 \\
&\quad -1128m^4 t_3^2 + 352m^2 t_2 t_3^2 - 16t_2^2 t_3^2 \mp 120\mathrm{i}m t_3^3 \pm 4\mathrm{i}m^{-1} t_2 t_3^3\,, \\
N_7 \quad &\to \quad 384m^{10} - 448m^8 t_2 + 24m^6 t_2^2 + 48m^4 t_2^3 - 8m^2 t_2^4 \\
&\quad \pm 768\mathrm{i}m^7 t_3 \mp 640\mathrm{i}m^5 t_2 t_3 \pm 176\mathrm{i}m^3 t_2^2 t_3 \mp 16\mathrm{i}m t_2^3 t_3 \\
&\quad -378m^4 t_3^2 + 188m^2 t_2 t_3^2 - 26t_2^2 t_3^2 \pm 12\mathrm{i}m t_3^3 \mp 4\mathrm{i}m^{-1} t_2 t_3^3 - 6m^{-2} t_3^4\,.
\end{aligned} \tag{B.3}$$

As mentioned in the main text, the θ-dependence cancels independently in the two respective sums

$$\begin{aligned} N_{1\mathrm{PI}} &= N_1 + N_2 + N_3 + N_4 \\ &\quad \to \quad 1440m^{10} - 2712m^8 t_2 + 1080m^6 t_2^2 + 216m^4 t_2^3 - 24m^2 t_2^4 \\ &\quad\quad +1440m^4 t_3^2 + 168m^2 t_2 t_3^2 - 24 t_2^2 t_3^2 \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} N_{1\mathrm{PR}} &= N_5 + N_6 + N_7 \\ &\quad \to \quad 4224m^{10} - 6336m^8 t_2 + 2664m^6 t_2^2 - 624m^4 t_2^3 + 72m^2 t_2^4 \\ &\quad\quad -2586m^4 t_3^2 + 828m^2 t_2 t_3^2 - 42 t_2^2 t_3^2 - 6m^{-2} t_3^4 \,. \end{aligned} \tag{B.5}$$

In total, we find

$$\begin{aligned} N &= N_{1\mathrm{PI}} + N_{1\mathrm{PR}} \\ &\quad \to \quad 5664m^{10} - 9048m^8 t_2 + 3744m^6 t_2^2 - 408m^4 t_2^3 + 48m^2 t_2^4 \\ &\quad\quad -1146m^4 t_3^2 + 996m^2 t_2 t_3^2 - 66 t_2^2 t_3^2 - 6m^{-2} t_3^4 \,. \end{aligned} \tag{B.6}$$

B.2 Feynman computation

After a Wick rotation to Euclidean space, we are working with the following Feynman rules in frequency space.

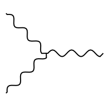
- The free fermion propagator is

$$= \frac{\mathrm{i}\omega - m}{\omega^2 + m^2} \,. \tag{B.7}$$

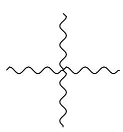
- The free boson propagator is

$$= \frac{-1}{\omega^2 + m^2} \,. \tag{B.8}$$

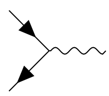
- The vertices are



$= 3!\, mg \,,$



$= \frac{4!}{2}\, g^2 \,,$



$= 2\, g \,.$ (B.9)

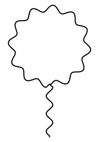
- Standard Feynman rules dictate momentum conservation at every vertex. Additionally, each fermion loop introduces a factor of -1. Every loop frequency l has to be integrated over with $\int \frac{\mathrm{d}l}{2\pi}$. Lastly, one has to divide by the symmetry factor of the diagram which captures the number of permutations of internal lines that leave the diagram invariant.

- We choose all external frequencies to be outgoing.

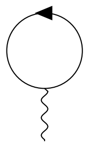
One-point function. At one loop order, we have to consider two connected diagrams contributing to the bosonic one-point function. Pulling out the prefactor

$$-g\frac{2\pi\delta(\omega)}{m^2}\,, \tag{B.10}$$

they are



$$\rightarrow \quad \tfrac{3!}{2}m\int\frac{\mathrm{d}l}{2\pi}\frac{-1}{l^2+m^2} = -\tfrac{3}{2}\,, \tag{B.11}$$



$$\rightarrow \quad -2\int\frac{\mathrm{d}l}{2\pi}\frac{-m}{l^2+m^2} = 1\,, \tag{B.12}$$

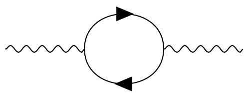
so that

$$\left\langle \widetilde{x}(\omega)\right\rangle_g = g\frac{\pi\delta(\omega)}{m^2} \;+\; \mathcal{O}(g^3)\,. \tag{B.13}$$

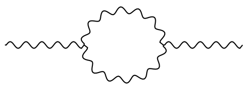
Two-point function. There are five connected diagrams contributing to the bosonic two-point function at one loop order. Again, pulling out a prefactor

$$g^2\frac{2\pi\delta(\omega+\omega')}{(\omega^2+m^2)^2}\,, \tag{B.14}$$

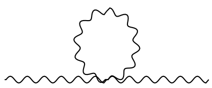
we have three 1PI contributions,



$$\rightarrow \quad 0\,, \tag{B.15}$$

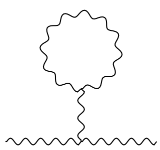


$$\rightarrow \quad \tfrac{3!\,3!}{2}m^2\int\frac{\mathrm{d}l}{2\pi}\frac{1}{l^2+m^2}\frac{1}{(l+\omega)^2+m^2} = \frac{18m^2}{m(4m^2+\omega^2)}\,, \tag{B.16}$$

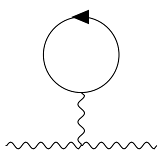


$$\rightarrow \quad \tfrac{4!}{2\cdot 2}\int\frac{\mathrm{d}l}{2\pi}\frac{-1}{l^2+m^2} = -\tfrac{3}{m}\,, \tag{B.17}$$

and two 1PR contributions known from the one-point function,



$$\rightarrow \quad \tfrac{3!\,3!}{2}m^2\frac{-1}{m^2}\int\frac{\mathrm{d}l}{2\pi}\frac{-1}{l^2+m^2} = \tfrac{9}{m}\,, \tag{B.18}$$

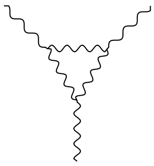


$$\rightarrow \quad -\tfrac{3!\,2}{1}m\frac{-1}{m^2}\int\frac{\mathrm{d}l}{2\pi}\frac{-m}{l^2+m^2} = -\tfrac{6}{m}\,. \tag{B.19}$$

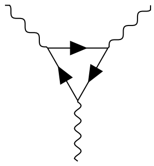
Summing all the contributions, we end up with

$$\left\langle \widetilde{x}(\omega)\widetilde{x}(\omega')\right\rangle_g = 2\pi\delta(\omega+\omega')G_0(\omega) \;+\; g^2\,\frac{2\pi\delta(\omega+\omega')}{(m^2+\omega^2)^2}\frac{18m^2}{4m^3+m\omega^2} \;+\; \mathcal{O}(g^3)\,. \tag{B.20}$$

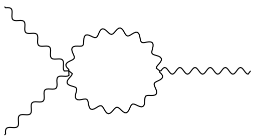
Three-point function. For the bosonic three-point function at one loop order, we have to compute three 1PI connected diagrams. The amputated diagrams are

$F_1 =$ 

$$\longrightarrow \quad (3!m)^3 \int_l G_0(l)G_0(l-\omega_3)G_0(l-\omega_3-\omega_2) \tag{B.21}$$
$$= -\frac{216m^2[12m^2+\frac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2)]}{(4m^2+\omega_1^2)(4m^2+\omega_2^2)(4m^2+\omega_3^2)}\,,$$

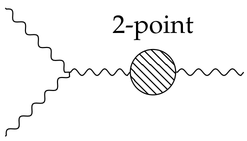
$F_2 =$ 

$$\longrightarrow \quad -2^3 \int_l S_0(l)S_0(l-\omega_3)S_0(l-\omega_3-\omega_2) = 0\,, \tag{B.22}$$

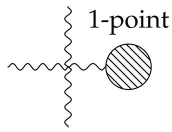
$F_3 =$ 

$$\longrightarrow \quad m\frac{1}{2}\frac{4!}{2}3! \int_l G_0(l)G_0(l-\omega_3) \;+\; (\omega_3 \to \omega_2) \;+\; (\omega_3 \to \omega_1)$$
$$= 36\,\frac{48m^4+8m^2(\omega_1^2+\omega_2^2+\omega_3^2)+\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2}{(4m^2+\omega_1^2)(4m^2+\omega_2^2)(4m^2+\omega_3^2)}\,. \tag{B.23}$$

Building up on the results for the two- and three-point functions, the 1PR contributions read

$F_4 =$ 

$$\longrightarrow \quad 3!m\frac{-18m^2}{(4m^3+m\omega_3^2)(m^2+\omega_3^2)} + (\omega_3\to\omega_2) + (\omega_3\to\omega_1)\,, \tag{B.24}$$

$F_5 =$ 

$$\longrightarrow \quad \frac{6}{m^2}\,. \tag{B.25}$$

We write down the contributions in the same way as we did for the Nicolai calculation, pulling out the prefactor (B.1) and expressing the remaining contributions in terms of the symmetric polynomials t_2 and t_3 (3.81):

$$\begin{aligned}
F_1 &\to 2592m^{10} - 5400m^8 t_2 + 3024m^6 t_2^2 \\
&\quad -216m^4 t_2^3 + 2592m^4 t_3^2 - 216m^2 t_2 t_3^2 \,, \\
F_2 &\to 0 \,, \\
F_3 &\to -1728m^{10} + 4032m^8 t_2 - 2916m^6 t_2^2 \\
&\quad +648m^4 t_2^3 - 36m^2 t_2^4 - 1728m^4 t_3^2 + 576m^2 t_2 t_3^2 - 36t_2^2 t_3^2 \,, \\
F_4 &\to 5184m^{10} - 8640m^8 t_2 + 4428m^6 t_2^2 \\
&\quad -1080m^4 t_2^3 + 108m^2 t_2^4 - 1620m^4 t_3^2 + 432m^2 t_2 t_3^2 \,, \\
F_5 &\to -384m^{10} + 960m^8 t_2 - 792m^6 t_2^2 \\
&\quad +240m^4 t_2^3 - 24m^2 t_2^4 - 390m^4 t_3^2 + 204m^2 t_2 t_3^2 - 30t_2^2 t_3^2 - 6m^{-2} t_3^4 \,.
\end{aligned} \tag{B.26}$$

The 1PI and 1PI sums become

$$\begin{aligned}
F_{1\mathrm{PI}} &= F_1 + F_2 + F_3 \\
&\to 864m^{10} - 1368m^8 t_2 + 108m^6 t_2^2 + 432m^4 t_2^3 - 36m^2 t_2^4 + 864m^4 t_3^2 \\
&\quad +360m^2 t_2 t_3^2 - 36t_2^2 t_3^2 \,,
\end{aligned} \tag{B.27}$$

$$\begin{aligned}
F_{1\mathrm{PR}} &= F_4 + F_5 \\
&\to 4800m^{10} - 7680m^8 t_2 + 3636m^6 t_2^2 - 840m^4 t_2^3 + 84m^2 t_2^4 - 2010m^4 t_3^2 \\
&\quad +636m^2 t_2 t_3^2 - 30t_2^2 t_3^2 - 6m^{-2} t_3^4 \,,
\end{aligned} \tag{B.28}$$

respectively. Modulo the prefactor, the final expression is

$$\begin{aligned}
F &= F_{1\mathrm{PI}} + F_{1\mathrm{PR}} \\
&\to 5664m^{10} - 9048m^8 t_2 + 3744m^6 t_2^2 - 408m^4 t_2^3 + 48m^2 t_2^4 - 1146m^4 t_3^2 \\
&\quad +996m^2 t_2 t_3^2 - 66t_2^2 t_3^2 - 6m^{-2} t_3^4 \,.
\end{aligned} \tag{B.29}$$

This completely agrees with the result obtained from the Nicolai computation (B.6).

Appendix C

$\mathcal{N}=4$ SYM via an $\mathcal{N}=1$ superfield formalism

Note: *This appendix is adopted from the author's published work [31].*

We present the details of the $\mathcal{N}=1$ superfield formalism that is used to construct the $\mathcal{N}=4$ SYM action from Chapter 5. The $\mathcal{N}=1$ action from Chaper 4 can be obtained by the same procedure, setting the chiral superfields $\Phi_I = \Phi_I^\dagger$ below to zero. Using the conventions of [46][1] and the Wess-Zumino (WZ) gauge, the vector superfield ($V^\dagger = V$) takes the form

$$\begin{aligned} V &= \vartheta\sigma^\mu\bar{\vartheta}A_\mu(x) - \mathrm{i}\vartheta^2\bar{\vartheta}\bar{\lambda}(x) + \mathrm{i}\bar{\vartheta}^2\vartheta\lambda(x) - \tfrac{1}{2}\vartheta^2\bar{\vartheta}^2 D(x) \\ &= \vartheta\sigma^\mu\bar{\vartheta}A_\mu(y\ ) - \mathrm{i}\vartheta^2\bar{\vartheta}\bar{\lambda}(y\ ) + \mathrm{i}\bar{\vartheta}^2\vartheta\lambda(y\ ) - \tfrac{1}{2}\vartheta^2\bar{\vartheta}^2[D(y\ ) - \mathrm{i}\mathrm{D}^\mu A_\mu(y\ )] \\ &= \vartheta\sigma^\mu\bar{\vartheta}A_\mu(y^\dagger) - \mathrm{i}\vartheta^2\bar{\vartheta}\bar{\lambda}(y^\dagger) + \mathrm{i}\bar{\vartheta}^2\vartheta\lambda(y^\dagger) - \tfrac{1}{2}\vartheta^2\bar{\vartheta}^2[D(y^\dagger) + \mathrm{i}\mathrm{D}^\mu A_\mu(y^\dagger)] \end{aligned} \tag{C.1}$$

where $y = x + \mathrm{i}\vartheta\sigma\bar{\vartheta}$ and $y^\dagger = x - \mathrm{i}\vartheta\sigma\bar{\vartheta}$ parameterize (anti-)chiral superspace. The WZ eliminates gauge degrees of freedom. In particular, it leads to

$$V^2 = -\tfrac{1}{2}\vartheta^2\bar{\vartheta}^2 A_\mu A^\mu\ , \tag{C.2}$$

while all higher powers vanish. Thus, the exponential

$$\mathrm{e}^{2V} = 1 + 2V + 2V^2 \tag{C.3}$$

truncates at the second order. The non-abelian supersymmetric field strength W_α and its conjugate are given by

$$\begin{aligned} W_\alpha &= -\tfrac{1}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{e}^{-2V}\mathrm{D}_\alpha\mathrm{e}^{2V} \\ &= +2\mathrm{i}\lambda_\alpha(y\ ) - 2\left[\delta_\alpha{}^\beta D(y\ ) - \mathrm{i}\sigma^{\mu\nu}{}_\alpha{}^\beta F_{\mu\nu}(y\ )\right]\vartheta_\beta - 2\vartheta^2 \not{\mathrm{D}}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(y\ )\ , \\ \bar{W}^{\dot{\alpha}} &= -\tfrac{1}{4}\mathrm{D}\mathrm{D}\mathrm{e}^{-2V}\bar{\mathrm{D}}^{\dot{\alpha}}\mathrm{e}^{2V} \\ &= -2\mathrm{i}\bar{\lambda}^{\dot{\alpha}}(y^\dagger) - 2\left[\delta^{\dot{\alpha}}{}_{\dot{\beta}} D(y^\dagger) + \mathrm{i}\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} F_{\mu\nu}(y^\dagger)\right]\bar{\vartheta}^{\dot{\beta}} + 2\bar{\vartheta}^2 \bar{\not{\mathrm{D}}}^{\dot{\alpha}\alpha}\lambda_\alpha(y^\dagger)\ , \end{aligned} \tag{C.4}$$

in chiral superspace, with the superspace covariant derivatives

$$\begin{aligned} \mathrm{D}_\alpha &= +\tfrac{\partial}{\partial\vartheta^\alpha} + \mathrm{i}\sigma_{\alpha\dot{\alpha}}{}^\mu\bar{\vartheta}^{\dot{\alpha}}\partial_\mu\ , \\ \bar{\mathrm{D}}_{\dot{\alpha}} &= -\tfrac{\partial}{\partial\bar{\vartheta}^{\dot{\alpha}}} - \mathrm{i}\vartheta^\alpha\sigma_{\alpha\dot{\alpha}}{}^\mu\partial_\mu\ . \end{aligned} \tag{C.5}$$

[1] up to a global sign to recover a plus sign in the field strength and covariant derivative instead of a minus sign.

The chiral superfields ($\bar{\mathrm{D}}_{\dot\alpha}\Phi_I = 0$, $\mathrm{D}_\alpha \Phi_I^\dagger = 0$) contain the extra fields compared to the $\mathcal{N}=1$ action and have expansions

$$\Phi_I = \phi_I(y) + \sqrt{2}\vartheta\psi_I(y) + \vartheta^2 F_I(y)\,, \quad \Phi_I^\dagger = \phi_I^\dagger(y^\dagger) + \sqrt{2}\bar\vartheta\bar\psi_I(y^\dagger) + \bar\vartheta^2 F_I^\dagger(y^\dagger)\,, \tag{C.6}$$

in chiral superspace. In order to couple them to the SUSY field strengths, one also needs their full superspace expansions

$$\begin{aligned}\Phi_I &= \phi_I(x) + \mathrm{i}\vartheta\sigma^\mu\bar\vartheta\partial_\mu\phi_I(x) + \tfrac{1}{4}\vartheta^2\bar\vartheta^2\Box\phi_I(x) + \sqrt{2}\vartheta\psi_I(x)\\ &\qquad\qquad - \tfrac{\mathrm{i}}{\sqrt{2}}\vartheta^2\partial_\mu\psi_I(x)\sigma^\mu\bar\vartheta + \vartheta^2 F_I(x)\,,\\ \Phi_I^\dagger &= \phi_I^\dagger(x) - \mathrm{i}\vartheta\sigma^\mu\bar\vartheta\partial_\mu\phi_I^\dagger(x) + \tfrac{1}{4}\vartheta^2\bar\vartheta^2\Box\phi_I^\dagger(x) + \sqrt{2}\bar\vartheta\bar\psi_I(x)\\ &\qquad\qquad + \tfrac{\mathrm{i}}{\sqrt{2}}\bar\vartheta^2\vartheta\sigma^\mu\partial_\mu\bar\psi_I(x) + \bar\vartheta^2 F_I^\dagger(x)\,.\end{aligned} \tag{C.7}$$

To compute the action, we need the penultimate components of the various contributions to (5.11). Firstly, we have

$$\begin{aligned}\tfrac{1}{4}W^\alpha W_\alpha &= -\lambda^2 + \left[-2\mathrm{i}D\lambda - 2F_{\mu\nu}\lambda\sigma^{\mu\nu}\right]\vartheta\\ &\quad + \left[-2\mathrm{i}\lambda\sigma^\mu \mathrm{D}_\mu\bar\lambda - \tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^2 + \tfrac{\mathrm{i}}{4}F^{\mu\nu}F^{\rho\lambda}\epsilon_{\mu\nu\rho\lambda}\right]\vartheta^2\,,\\ \tfrac{1}{4}\bar W_{\dot\alpha}\bar W^{\dot\alpha} &= -\bar\lambda^2 + \left[+2\mathrm{i}D\bar\lambda - 2F_{\mu\nu}\bar\lambda\bar\sigma^{\mu\nu}\right]\bar\vartheta\\ &\quad + \left[+2\mathrm{i}\mathrm{D}_\mu\lambda\sigma^\mu\bar\lambda - \tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^2 - \tfrac{\mathrm{i}}{4}F^{\mu\nu}F^{\rho\lambda}\epsilon_{\mu\nu\rho\lambda}\right]\bar\vartheta^2\,.\end{aligned} \tag{C.8}$$

Secondly, one needs

$$\epsilon_{IJK}\ \mathrm{tr}\ \Phi_I[\Phi_J,\Phi_K] = \mathrm{i}\epsilon_{IJK}f^{abc}\Big[\phi_I^a\phi_J^b\phi_K^c + 3\sqrt{2}\ \vartheta\ \psi_I^a\phi_J^b\phi_K^c + 3\vartheta^2\big(F_I^a\phi_J^b\phi_K^c - \phi_I^a\psi_J^b\psi_K^c\big)\Big]\,, \tag{C.9}$$

and the hermitian conjugate analogously. Lastly, we find

$$\begin{aligned}\tfrac{1}{N}\mathrm{tr}\ \mathrm{e}^{-2V}\Phi_I^\dagger \mathrm{e}^{2V}\Phi_I &= \Phi_I^{a\dagger}\Phi_I^a + \tfrac{2}{N}\mathrm{tr}\ [T^a,T^b]T^c\ \Phi_I^{a\dagger}V^b\Phi_I^c\\ &\qquad + \tfrac{2}{N}\mathrm{tr}\ [T^a,T^b][T^c,T^d]\ \Phi_I^{a\dagger}V^bV^c\Phi_I^d\\ &= \ldots\ +\vartheta^2\bar\vartheta\left[-\mathrm{i}\sqrt{2}\bar\sigma^\mu\psi_I^a\mathrm{D}_\mu\phi_I^{a\dagger} + \sqrt{2}F_I^a\bar\psi_I^a + 2f^{abc}\phi_I^{a\dagger}\bar\lambda^b\phi_I^c\right]\\ &\qquad +\bar\vartheta^2\vartheta\left[-\mathrm{i}\sqrt{2}\sigma^\mu\bar\psi_I^a\mathrm{D}_\mu\phi_I^a + \sqrt{2}F_I^{a\dagger}\psi_I^a - 2f^{abc}\phi_I^{a\dagger}\lambda^b\phi_I^c\right]\\ &\qquad +\vartheta^2\bar\vartheta^2\big[-\mathrm{D}_\mu\phi_I^{a\dagger}\mathrm{D}^\mu\phi_I^a + F_I^{a\dagger}F_I^a + \mathrm{i}\mathrm{D}_\mu\bar\psi_I^a\bar\sigma^\mu\psi_I\\ &\qquad\qquad - f^{abc}\big(\mathrm{i}\phi_I^{a\dagger}D^b\phi_I^c - \sqrt{2}\phi_I^{a\dagger}\lambda^b\psi_I^c + \sqrt{2}\bar\psi_I^a\bar\lambda^b\phi_I^c\big)\big]\\ &\qquad + \text{ total derivatives}\,,\end{aligned} \tag{C.10}$$

where we have left out terms of power 2 or less in ϑ (including $\bar\vartheta$) as they are not relevant for our purposes. The traces over the SU(N) generators were evaluated with

$$[T^a,T^b] = \mathrm{i}f^{abc}T^c\,, \qquad \mathrm{tr}\ T^aT^b = N\delta^{ab}\,. \tag{C.11}$$

We now have all the ingredients (C.8), (C.9), (C.10) for the Lagrangian (5.11), so we can read off its component formulation in Weyl notation

$$\begin{aligned} g^2\mathcal{L} = &-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \mathrm{i}\lambda^a\sigma^\mu \mathrm{D}_\mu\bar{\lambda}^a + \tfrac{1}{2}D^2 - \tfrac{1}{\sqrt{2}}\epsilon_{IJK}f^{abc}\big(F^a_I\phi^b_J\phi^c_K + F^{a\dagger}_I\phi^{b\dagger}_J\phi^{c\dagger}_K\big) \\ &- \mathrm{D}_\mu\phi^{a\dagger}_I\mathrm{D}^\mu\phi^a_I - \mathrm{i}\psi^a_I\sigma^\mu\mathrm{D}_\mu\bar{\psi}_I + F^{a\dagger}_I F^a_I + \tfrac{1}{\sqrt{2}}\epsilon_{IJK}f^{abc}\big(\phi^a_I\psi^b_J\psi^c_K + \phi^{a\dagger}_I\bar{\psi}^b_J\bar{\psi}^c_K\big) \\ &- \sqrt{2}f^{abc}\big(\psi^a_I\lambda^b\phi^{c\dagger}_I + \bar{\psi}^a_I\bar{\lambda}^b\phi^c_I\big) - \mathrm{i}f^{abc}\phi^{a\dagger}_I D^b\phi^c_I\,, \end{aligned} \tag{C.12}$$

up to total derivatives. Further, from the superspace expansions follow the supersymmetry transformations

$$\begin{aligned} &\delta\phi_I = \sqrt{2}\vartheta\psi_I\,, && \delta\psi_I = \mathrm{i}\sqrt{2}\sigma^\mu\bar{\vartheta}\mathrm{D}_\mu\phi_I + \sqrt{2}\vartheta F_I\,, \\ &\delta F_I = \mathrm{i}\sqrt{2}\bar{\vartheta}\bar{\sigma}^\mu\mathrm{D}_\mu\psi_I - 2\phi_I{\times}\bar{\lambda}\bar{\vartheta}\,, && \delta A^\mu = -\mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\vartheta + \mathrm{i}\bar{\vartheta}\bar{\sigma}^\mu\lambda\,, \\ &\delta\lambda = \sigma^{\mu\nu}\vartheta F_{\mu\nu} + \mathrm{i}\vartheta D\,, && \delta D = -\vartheta\sigma^\mu\mathrm{D}_\mu\bar{\lambda} - \mathrm{D}_\mu\lambda\sigma^\mu\bar{\vartheta}\,. \end{aligned} \tag{C.13}$$

For convenience, we translate the superfield formalism to a four-component Majorana basis using

$$\begin{aligned} &\lambda^{(\mathrm{M})} = \begin{pmatrix}\lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}}\end{pmatrix}, \qquad \bar{\lambda}^{(\mathrm{M})} = (\lambda^\alpha,\ \bar{\lambda}_{\dot{\alpha}})\,, \qquad \alpha = \begin{pmatrix}\vartheta_\alpha \\ \bar{\vartheta}^{\dot{\alpha}}\end{pmatrix}, \qquad \bar{\alpha} = (\vartheta^\alpha,\ \bar{\vartheta}_{\dot{\alpha}}) \\ &\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0\end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -i & 0 \\ 0 & i\end{pmatrix}, \qquad \text{etc.}\,, \end{aligned} \tag{C.14}$$

so that

$$\begin{aligned} &\bar{\lambda}^{(\mathrm{M})}\lambda^{(\mathrm{M})} = \lambda\lambda + \bar{\lambda}\bar{\lambda}\,, && \bar{\lambda}^{(\mathrm{M})}\mathrm{i}\gamma_5\lambda^{(\mathrm{M})} = \lambda\lambda - \bar{\lambda}\bar{\lambda}\,, \\ &\bar{\lambda}^{(\mathrm{M})}\gamma^\mu\lambda^{(\mathrm{M})} = \lambda\sigma^\mu\bar{\lambda} + \bar{\lambda}\bar{\sigma}^\mu\lambda\,, && \frac{1}{2}\bar{\lambda}^{(\mathrm{M})}\gamma^{\mu\nu}\alpha = \lambda\sigma^{\mu\nu}\vartheta + \bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\vartheta}\,, \end{aligned} \tag{C.15}$$

and so on, where the l.h.s. are in the four-component Majorana basis and the r.h.s. are in the two-component Weyl basis. Additionally, we need the chiral projectors

$$\mathrm{P}^\pm = \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_5)\,, \qquad \bar{\lambda}^{(\mathrm{M})}\mathrm{P}^+\lambda^{(\mathrm{M})} = \lambda\lambda\,, \quad \bar{\lambda}^{(\mathrm{M})}\mathrm{P}^-\lambda^{(\mathrm{M})} = \bar{\lambda}\bar{\lambda}\,. \tag{C.16}$$

This leads to the Lagrangian in Majorana notation (5.12) (leaving the superscript $^{(\mathrm{M})}$ implicit from now on) and to the penultimate component

$$\begin{aligned} \mathring{\Delta} = \bar{\alpha}\Big\{&-D\gamma_5\lambda - \tfrac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\lambda + 2\epsilon_{IJK}f^{abc}\big[\mathrm{P}^+\psi^a_I\phi^b_J\phi^c_K + \mathrm{P}^-\psi^a_I\phi^{b\dagger}_J\phi^{c\dagger}_K\big] + 2\mathrm{i}f^{abc}\gamma_5\phi^{a\dagger}_I\lambda^b\phi^c_I \\ &+ \mathrm{i}\sqrt{2}\big[\gamma^\mu\mathrm{P}^-\psi^a_I\mathrm{D}_\mu\phi^a_I + \gamma^\mu\mathrm{P}^+\psi^a_I\mathrm{D}_\mu\phi^{a\dagger}_I\big] - \sqrt{2}\big[\mathrm{P}^+\psi^a_I F^{a\dagger}_I + \mathrm{P}^-\psi^a_I F^a_I\big]\Big\}\,. \end{aligned} \tag{C.17}$$

In our conventions, we can state the following hermiticity properties for Majorana spinors χ, ξ:

$$\begin{aligned} &\bar{\chi}\xi = \bar{\xi}\chi\,, \qquad \bar{\chi}\gamma^\mu\xi = -\bar{\xi}\gamma^\mu\chi\,, \qquad \bar{\chi}\gamma_5\xi = \bar{\xi}\gamma_5\chi\,, \qquad \bar{\chi}\gamma^\mu\gamma_5\xi = \bar{\xi}\gamma^\mu\gamma_5\chi\,, \\ &\bar{\chi}\gamma^{\mu\nu}\xi = -\bar{\xi}\gamma^{\mu\nu}\chi\,, \qquad \bar{\chi}\gamma^{\mu\nu}\gamma_5\xi = -\bar{\xi}\gamma^{\mu\nu}\gamma_5\chi\,, \qquad \bar{\chi}\gamma^{\rho\lambda}\gamma_\mu\xi = \bar{\xi}\gamma_\mu\gamma^{\rho\lambda}\chi\,. \end{aligned} \tag{C.18}$$

Consistency checks: In order to cross-check the expressions above (in the

Majorana basis), one can perform three consistency checks. Firstly, by the superfield structure, the penultimate component $\mathring{\Delta}$ has to generate the Lagrangian via its supervariation (up to total derivatives)

$$\tfrac{1}{4}\delta\mathring{\Delta}\big|_{\bar{\alpha}\alpha} = g^2\mathcal{L}\,, \tag{C.19}$$

with $\bar{\alpha}(\ldots)\alpha\big|_{\bar{\alpha}\alpha} = -\frac{1}{2}\mathrm{tr}(\ldots)$. This requires the Fierz identity for Majorana spinors

$$4\xi\bar{\chi} = -(\bar{\chi}\xi) + \gamma_\mu(\bar{\chi}\gamma^\mu\xi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu}\xi) + \gamma_5\gamma_\mu(\bar{\chi}\gamma_5\gamma^\mu\xi) + \gamma_5(\bar{\chi}\gamma_5\xi)\,. \tag{C.20}$$

A second check is to verify that the Lagrangian transforms into a divergence

$$\delta\mathcal{L} = \text{divergence}\,. \tag{C.21}$$

Thirdly, the supervatiations have to generate the SUSY-algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}P_\mu = -2\mathrm{i}(\gamma^\mu)_{\alpha\beta}\partial_\mu\,, \tag{C.22}$$

up to a gauge transformation (since we have already chosen the WZ gauge). With the supercharges $\delta(...) = \bar{\alpha}_\alpha Q_\alpha(...)$ this can be checked by computing the commutator of two supervariations

$$[\delta^{(1)}, \delta^{(2)}] = [\bar{\alpha}_{1\alpha}Q_\alpha, \bar{Q}_\beta\alpha_{2\beta}] = \bar{\alpha}_{1\alpha}\{Q_\alpha, \bar{Q}_\beta\}\alpha_{2\beta}\,, \tag{C.23}$$

acting on each field. One obtains the SUSY-algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\mathrm{i}(\gamma^\mu)_{\alpha\beta}\partial_\mu - [\omega, \cdot]_{\alpha\beta} + G_{\alpha\beta}(A)\,, \tag{C.24}$$

where $\omega = 2\mathrm{i}\not{A}$, $[\omega, \cdot]^a = f^{abc}\omega^b(\cdot)^c$ and $G_{\alpha\beta}$ is a gauge transformation $A_\mu \to A_\mu + \partial_\mu\omega$ as required in the WZ gauge.

Stripping-off the susy parameter: Lastly, we stripped-off the SUSY parameter by setting $\delta \equiv \delta_\alpha\alpha_\alpha$ and $\mathring{\Delta} \equiv \bar{\alpha}_\alpha\mathring{\Delta}_\alpha$, resulting in the supervariations (5.13) and the penultimate component (5.14), up to an overall normalization. When doing so, one has to take into account that the fermionic supervariations gain an extra minus sign, since

$$\bar{\chi}\delta\lambda = \bar{\chi}_\beta M_{\beta\alpha}\alpha_\alpha = \bar{\chi}_\beta\delta_\alpha\alpha_\alpha\lambda_\beta = -\bar{\chi}_\beta\delta_\alpha\lambda_\beta\alpha_\alpha \quad \Rightarrow \quad \delta_\alpha\lambda_\beta = -M_{\beta\alpha}\,, \tag{C.25}$$

with some arbitrary spinor $\bar{\chi}$.

Appendix D

Construction of the $\mathcal{N}=4$ coupling flow operator

Note: *This appendix is adopted from the author's published work [31].*

In this appendix we fill in the technical details for getting from (5.38) to (5.39). The method is analogous to the $\mathcal{N}=1$ case, but there is more structure coming from R-symmetry and the additional fields. We start with

$$\begin{aligned}\delta_\alpha X[\tilde{\mathscr{A}}] &= -\mathrm{i}\int \mathrm{d}^4x\,\left(\tilde{\bar{\psi}}_4\gamma_\mu\tfrac{\delta}{\delta\tilde{A}_\mu}+\tilde{\bar{\psi}}_J(c^i)^J{}_4\tfrac{\delta}{\delta\tilde{\varphi}_i}\right)_\alpha X[\tilde{\mathscr{A}}]\\ &= -\mathrm{i}\int \mathrm{d}^4x\,\left(\tilde{\bar{\psi}}_A(\widehat{\mathscr{C}}_\Sigma)^A{}_4\tfrac{\delta}{\delta\tilde{\mathscr{A}}_\Sigma}\right)_\alpha X[\tilde{\mathscr{A}}]\,,\end{aligned}\tag{D.1}$$

where we introduced the object

$$(\widehat{\mathscr{C}}_\Sigma)^A{}_4 = \begin{cases}\delta^A{}_4\gamma_\mu & \text{for}\quad \Sigma=\mu=0,1,2,3\\ (c^i)^A{}_4 & \text{for}\quad \Sigma=3+i=4,5,\dots,9\end{cases}\,,\tag{D.2}$$

with matrix-valued entries. It is defined via

$$\delta_\alpha^{(4)}\tilde{\mathscr{A}}_\Sigma = -\mathrm{i}(\tilde{\bar{\psi}}_A(\widehat{\mathscr{C}}_\Sigma)^A{}_4)_\alpha\,,\tag{D.3}$$

where the (4) indicates that we have singled out one of the four supersymmetries (the 'fourth' one). Further, we have

$$sX[\tilde{\mathscr{A}}] = \sqrt{g}\int \mathrm{d}^4x\;\tilde{\mathscr{D}}_\Gamma\tilde{c}\tfrac{\delta}{\delta\tilde{\mathscr{A}}_\Gamma}X[\tilde{\mathscr{A}}]\,,\tag{D.4}$$

as well as the gaugino and ghost propagators

$$\tilde{\psi}^A(x)\tilde{\bar{\psi}}_B(y) = -\tilde{S}^A{}_B(x,y;\tilde{\mathscr{A}})\,,\qquad \tilde{\not{\mathscr{D}}}^A{}_C\,\tilde{S}^C{}_B(x,y;\tilde{\mathscr{A}}) = \delta^A{}_B\delta(x-y)\,,\tag{D.5}$$

and

$$\mathrm{i}\tilde{c}(x)\tilde{\bar{c}}(y) = \tilde{G}(x,y;\tilde{\mathscr{A}})\,,\qquad \tfrac{\partial\mathcal{G}(\tilde{\mathscr{A}})}{\partial\tilde{\mathscr{A}}_\Gamma}\tilde{\mathscr{D}}_\Gamma\,\tilde{G}(x,y;\tilde{\mathscr{A}}) = \delta(x-y)\,,\tag{D.6}$$

respectively. The rescaled coupling flow operator then reads

$$\overleftarrow{\tilde{R}}_g\,[\tilde{\mathscr{A}}] = \overleftarrow{\tfrac{\delta}{\delta\tilde{\mathscr{A}}_\Gamma}}\,\tilde{P}_\Gamma{}^\Sigma\tilde{R}_\Sigma + \overleftarrow{\tfrac{\delta}{\delta\tilde{\mathscr{A}}_\Gamma}}\,\tilde{\mathscr{D}}_\Gamma\tilde{G}\,\mathcal{G}(\tilde{\mathscr{A}})\,,\tag{D.7}$$

where we use the covariant projector

$$\widetilde{P}_{\Gamma}{}^{\Sigma} = \delta_{\Gamma}{}^{\Sigma} - \widetilde{\mathscr{D}}_{\Gamma}\widetilde{G}\frac{\partial\mathcal{G}(\widetilde{\mathscr{A}})}{\partial\widetilde{\mathscr{A}}_{\Sigma}} \,, \tag{D.8}$$

and introduced

$$\widetilde{R}_{\Sigma} = -\tfrac{1}{4}\mathrm{tr}\Big\{\big[\tfrac{1}{2}\widetilde{F}_{\mu\nu}\gamma^{\mu\nu}\widetilde{S}^{4}{}_{C} + (\widetilde{\Phi}^{4}{}_{A})^{\dagger}\widetilde{\mathscr{D}}^{A}{}_{B}\widetilde{S}^{B}{}_{C} - \tfrac{1}{2}(\widetilde{\Phi}^{4}{}_{A})^{\dagger}\widetilde{\Phi}^{A}{}_{B}\times\widetilde{S}^{B}{}_{C}\big](\widehat{\mathscr{C}}_{\Sigma})^{C}{}_{4}\Big\} \,. \tag{D.9}$$

The original (unrescaled) coupling flow operator is given by (c.f. (4.44))

$$R_{g}[\mathscr{A}] = \tfrac{1}{g}\left(\tilde{R}_{g}[\widetilde{\mathscr{A}}] - E\right) \qquad \text{with} \qquad E = \widetilde{\mathscr{A}}_{\Gamma}\,\frac{\delta}{\delta\widetilde{\mathscr{A}}_{\Gamma}} \,. \tag{D.10}$$

To isolate the Euler operator E, we use the identities

$$\gamma^{\rho\lambda}\widetilde{F}_{\rho\lambda} = 2\widetilde{\not{D}}\widetilde{\not{A}} + 2\partial\cdot\widetilde{A} - \widetilde{\not{A}}\times\widetilde{\not{A}} \,, \tag{D.11}$$

$$\widetilde{\not{D}}\widetilde{S}^{4}{}_{C} = \delta^{4}{}_{C} - \widetilde{\Phi}^{4}{}_{B}\times\widetilde{S}^{B}{}_{C} \,, \tag{D.12}$$

which (next to other contributions) generate the $\widetilde{A}_{\mu}\frac{\delta}{\delta\widetilde{A}_{\mu}}$ part of E. Further we use $\widetilde{\mathscr{D}}^{A}{}_{B}\,\widetilde{S}^{B}{}_{C} = \delta^{A}{}_{C}$ in the second term of (D.9) and

$$(\widetilde{\Phi}^{4}{}_{A})^{\dagger}(\widehat{\mathscr{C}}_{\Sigma})^{A}{}_{4} = \left\{\begin{array}{cl} 0 & \text{for} \quad \Sigma=\mu \\ -\mathbb{1}_{4}\widetilde{\varphi}_{I} - \gamma_{5}\widetilde{\varphi}_{I+3} & \text{for} \quad \Sigma = 3+I \\ +\gamma_{5}\widetilde{\varphi}_{I} - \mathbb{1}_{4}\widetilde{\varphi}_{I+3} & \text{for} \quad \Sigma = 6+I \end{array}\right. \,. \tag{D.13}$$

With $\mathrm{tr}\,\gamma_{5}=0$, this gives the second part of the Euler operator. From there, one can modify the expression such that

$$\widetilde{R}_{\Sigma}{=}\widetilde{\mathscr{A}}_{\Sigma}{-}\tfrac{1}{4}\mathrm{tr}\Big\{(\mathscr{C}_{\Sigma})^{4}{}_{A}\big[\tfrac{1}{2}\widetilde{S}^{A}{}_{4}(2\partial\cdot\widetilde{A} - \widetilde{\not{A}}\times\widetilde{\not{A}}){-}\widetilde{S}^{A}{}_{B}\widetilde{\Phi}^{B}{}_{4}\times\widetilde{\not{A}}{-}\tfrac{1}{2}\widetilde{S}^{A}{}_{B}\widetilde{\Phi}^{B}{}_{C}\times(\widetilde{\Phi}^{C}{}_{4})^{\dagger}\big]\Big\} \,, \tag{D.14}$$

where we flipped the order of the quantities in the trace for a more natural implicit color structure. To do so, we have used that $\widetilde{R}_{\Sigma}$ is real and identities such as

$$\begin{gathered} \bar{\psi}{=}\psi^{\dagger}\,\gamma_{0}\,, \quad (\gamma_{0})^{2}{=}\mathbb{1}_{4}\,, \quad (\widetilde{S}^{A}{}_{B})^{\dagger}{=}\gamma_{0}\,\widetilde{S}^{B}{}_{A}\,\gamma_{0}\,, \quad (\mathscr{C}_{\Sigma})^{4}{}_{A}{:=}\gamma_{0}\,((\widehat{\mathscr{C}}_{\Sigma})^{A}{}_{4})^{\dagger}\,\gamma_{0}\,, \\ \gamma_{\mu}^{\dagger} = \gamma_{0}\,\gamma_{\mu}\,\gamma_{0}\,, \qquad \gamma_{5}^{\dagger} = \gamma_{0}\,\gamma_{5}\gamma_{0} = -\gamma_{5} \,, \qquad \gamma_{0}\,\widetilde{\Phi}^{A}{}_{B}\,\gamma_{0} = (\widetilde{\Phi}^{B}{}_{A})^{\dagger} \,. \end{gathered} \tag{D.15}$$

We have also introduced the unhatted

$$(\mathscr{C}_{\Sigma})^{4}{}_{A} := \left\{\begin{array}{ll} \delta^{4}{}_{A}\gamma_{\mu} & \text{for} \quad \Sigma=\mu=0,1,2,3 \\ (c^{i})^{4}{}_{A} & \text{for} \quad \Sigma = 3+i = 4,5,...,9 \end{array}\right. \,. \tag{D.16}$$

Since $\widetilde{R}_\Gamma = \tilde{\mathscr{A}}_\Gamma + \ldots$, the Euler operator drops out in the rescaled coupling flow operator. After inserting $\tilde{\mathscr{A}} = g\mathscr{A}$, we arrive at

$$\overleftarrow{R}_g\,[\mathscr{A}] = -\tfrac{1}{4}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\big[\tfrac{1}{2}S^A{}_4(\tfrac{2}{g}\partial\cdot A - \not{A}\times\not{A}) - S^A{}_B\Phi^B{}_4\times\not{A} \\ -\tfrac{1}{2}S^A{}_B\Phi^B{}_C\times(\Phi^C{}_4)^\dagger\big]\Big\}\,, \tag{D.17}$$

for any linear gauge. With the rescaled coupling, using

$$\not{\mathscr{D}}^A{}_B = \not{D}\delta^A{}_B + g\Phi^A{}_B\times\,, \qquad \text{with} \qquad \mathrm{D}_\mu = \partial_\mu + gA_\mu\times\,\,, \tag{D.18}$$

and $\not{\mathscr{D}}^A{}_C S^C{}_B = \delta^A{}_B$, the fermion propagators can be expanded perturbatively:

$$S^A{}_B = S_0\delta^A{}_B - gS_0\not{\mathscr{A}}^A{}_C S^C{}_B = \sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_B S_0\,, \tag{D.19}$$

with $S_0 = \not{\partial}^{\,-1} = -\not{\partial}C$ and

$$\not{\mathscr{A}}^A{}_B = \not{A}\delta^A{}_B + \Phi^A{}_B\,. \tag{D.20}$$

In particular $S^A{}_4|_{g=0} = 0$, so that the coupling flow operator R_g contains no term of order $\frac{1}{g}$. We use the same procedure to get rid of the $S_0\frac{2}{g}\partial\cdot A|_{g=0}$ contribution as in the $\mathcal{N} = 1$ case in Section 4.1 with

$$2S_0\partial\cdot A = -2\not{A}^{\mathrm{L}} = \not{A}^* - \not{A} \tag{D.21}$$

to rewrite the first term:

$$\begin{aligned}
&-\tfrac{1}{4}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\{(\mathscr{C}_\Sigma)^4{}_A[\tfrac{1}{2}S^A{}_4(\tfrac{2}{g}\partial\cdot A - \not{A}\times\not{A})]\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\Big[\sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4 S_0(\tfrac{2}{g}\partial\cdot A - \not{A}\times\not{A})\Big]\Big\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\Big[\tfrac{1}{g}\sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4 S_0(\not{A}^* - \not{A}) - \sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l}{}_{A4}S_0\not{A}\times\not{A}\Big]\Big\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\Big[\tfrac{1}{g}\sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4 S_0(\not{A}^* - \not{A})\\
&\qquad\qquad - \sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_B S_0(\not{\mathscr{A}}^B{}_4 - \Phi^B{}_4)\times\not{A}\Big]\Big\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\Big[\tfrac{1}{g}\sum_{l=0}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4 S_0(\not{A}^* - \not{A})\\
&\qquad\qquad + \tfrac{1}{g}\sum_{l=1}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4\times\not{A} + S^A{}_B\Phi^B{}_4\times\not{A}\Big]\Big\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A\Big[\tfrac{1}{g}\delta^A{}_4 S_0(\not{A}^* - \not{A}) + \tfrac{1}{g}\sum_{l=1}^{\infty}(-gS_0\not{\mathscr{A}})^{l\,A}{}_4\times\not{A}^* + S^A{}_B\Phi^B{}_4\times\not{A}\Big]\Big\}\\
&= -\tfrac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A S^A{}_B\Big[-\not{\mathscr{A}}^B{}_4\times\not{A}^* + \Phi^B{}_4\times\not{A}\Big]\Big\} - \tfrac{1}{g}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\nu A^{\mathrm{L}}_\nu\\
&= -\tfrac{1}{4}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A S^A{}_B\Big[-\tfrac{1}{2}\not{\mathscr{A}}^B{}_4\times\not{A}^* + \tfrac{1}{2}\Phi^B{}_4\times\not{A}\Big]\Big\} + \frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,\Pi_\Gamma{}^\Sigma\mathscr{A}_\Sigma G\frac{\partial\mathcal{G}(\mathscr{A})}{\partial A_\nu}A^{\mathrm{L}}_\nu\,,
\end{aligned} \tag{D.22}$$

containing no term of order $1/g$. Finally, we arrive at

$$\overleftarrow{R}_g\,[\mathscr{A}] = \frac{1}{8}\,\frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,P_\Gamma{}^\Sigma\,\mathrm{tr}\Big\{(\mathscr{C}_\Sigma)^4{}_A S^A{}_B\big[\not{\mathscr{A}}^B{}_4 \times \not{A}^* + \Phi^B{}_4 \times \not{A} + \Phi^B{}_C \times \Phi^{\dagger C}{}_4\big]\Big\}$$
$$+ \frac{\overleftarrow{\delta}}{\delta\mathscr{A}_\Gamma}\,\Pi_\Gamma{}^\Sigma \mathscr{A}_\Sigma G\frac{\partial\mathcal{G}(\mathscr{A})}{\partial A_\nu}A^{\mathrm{L}}_\nu\,, \tag{D.23}$$

which after defining

$$\not{\mathscr{A}}^{*A}{}_B = \not{A}^*\delta^A{}_B + (\Phi^A{}_B)^\dagger\,, \tag{D.24}$$

takes the form (5.39).

Appendix E

Checks for the $\mathcal{N}=4$ coupling flow operator

Note: *This appendix is adopted from the author's published work [31].*

In this appendix, we present a direct proof that the coupling flow operator in the Landau gauge (5.62) satisfies the three infinitesimal conditions (2.41) and the gauge condition $R_g\mathcal{G}(\mathscr{A})=0$. The determinant matching condition follows from the other two conditions and the defining relation (1.13). The gauge condition follows automatically from the form of the covariant projector (5.45). Thus, we have only left to show the infinitesimal free action condition

$$(\partial_g + R_g)S_g^{\mathrm{b}}[\mathscr{A}] = 0\,. \tag{E.1}$$

The basic procedure of the proof is equivalent to the one in A.3 of [24] for $\mathcal{N} = 1$ SYM in $D = 3, 4, 6, 10$, but we have to take into account subtleties coming from the additional degrees of freedom in the $\mathcal{N} = 4$ case. We represent the bosonic action compactly as

$$S_g^{\mathrm{b}}[\mathscr{A}] = \int \mathrm{d}^4x\,\{-\tfrac{1}{4}\mathcal{F}^{\Sigma\Theta}\mathcal{F}_{\Sigma\Theta}\}\,, \tag{E.2}$$

with

$$\begin{aligned} &\mathcal{F}_{\Sigma\Theta} = \partial_\Sigma\mathscr{A}_\Theta - \partial_\Theta\mathscr{A}_\Sigma + g\mathscr{A}_\Sigma\times\mathscr{A}_\Theta\,, \\ &\partial_{3+i} = 0\,, \qquad \mathscr{A}_\mu = A_\mu\,, \qquad \mathscr{A}_{3+i} = \varphi_i\,. \end{aligned} \tag{E.3}$$

It is straightforward to show that

$$\partial_g S_g^{\mathrm{b}} = -\tfrac{1}{2}\mathcal{F}^{\Sigma\Theta}\mathscr{A}_\Sigma\times\mathscr{A}_\Theta \quad\text{and}\quad \frac{\delta S_g^{\mathrm{b}}}{\mathscr{A}_\Sigma} = \mathscr{D}_\Theta\mathcal{F}^{\Theta\Sigma}\,, \tag{E.4}$$

with implicit integration. We first show the statement (E.1) for the particular choice of the coupling flow operator (5.57) and afterward generalize the result to the full Lie algebra $\mathfrak{su}(4)$. Concretely, we first prove that

$$(\partial_g + R_g)S_g^{\mathrm{b}}[\mathscr{A}] = -\tfrac{1}{2}\mathcal{F}^{\Sigma\Theta}\mathscr{A}_\Sigma\times\mathscr{A}_\Theta + \tfrac{1}{8}\mathscr{D}_\Theta\mathcal{F}^{\Theta\Sigma}\,\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_B S^B{}_C \mathscr{A}^C{}_D \times \mathscr{A}^{*D}{}_4\big\} \tag{E.5}$$

vanishes. For this, we require the identities

$$\tfrac{1}{4}\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_\Gamma)^C{}_D(\bar{\mathscr{C}}_\Psi)^D{}_4\big\} = \eta_{\Sigma\Psi}\eta_{\Theta\Gamma} - \eta_{\Sigma\Gamma}\eta_{\Theta\Psi} + \eta_{\Sigma\Theta}\eta_{\Gamma\Psi}\,, \tag{E.6}$$

$$(\mathscr{C}^\Gamma)^A{}_B\mathscr{D}_\Gamma\, S^B{}_C = \not{\mathscr{D}}^A{}_B\, S^B{}_C = \delta^A{}_C\,, \tag{E.7}$$

$$(\mathscr{C}_{[\Sigma})^A{}_B(\bar{\mathscr{C}}_{\Theta]})^B{}_C(\mathscr{C}_\Gamma)^C{}_D = -2(\mathscr{C}_{[\Sigma})^A{}_D\,\eta_{\Theta]\Gamma} + (\mathscr{C}_{[\Sigma})^A{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_{\Gamma]})^C{}_D\,, \tag{E.8}$$

analogous to the ones used in [24]. Here we have introduced a 'conjugate' $\bar{\mathscr{C}}$ (in the Landau gauge), so that

$$\begin{aligned}
&\mathscr{C}_\mu = \mathbb{1}_4\gamma_\mu\,, \qquad \mathscr{C}_{3+i} = 2[(t^i)^*\mathrm{P}^+ - t^i\mathrm{P}^-]\,, \quad \mathscr{A\!\!/}^A{}_B = \mathscr{A}^\Gamma(\mathscr{C}_\Gamma)^A{}_B = A\!\!\!/ + \Phi^A{}_B \\
&\bar{\mathscr{C}}_\mu = \mathbb{1}_4\gamma_\mu\,, \qquad \bar{\mathscr{C}}_{3+i} = 2[t^i\mathrm{P}^+ - (t^i)^*\mathrm{P}^-]\,, \quad \mathscr{A\!\!/}^{*A}{}_B = \mathscr{A}^\Gamma(\bar{\mathscr{C}}_\Gamma)^A{}_B = A\!\!\!/ + (\Phi^A{}_B)^\dagger\,,
\end{aligned} \tag{E.9}$$

with the Clebsch-Gordon coefficients t^i_{AB} as matrices in R-space. It is important to note that (E.6) is only valid up to terms that vanish when contracted with fields in the adjoint representation of the gauge group due to the Jacobi identity in color space. We explicitly check (E.6) at the end of this appendix. The identity (E.8) follows from the analogous identity for the D=10 gamma matrices

$$\Gamma_\mu = \mathbb{1}_8 \otimes \gamma_\mu \quad \text{and} \quad \Gamma_{3+i} = 2\begin{pmatrix} 0 & t^i \\ (t^i)^* & 0 \end{pmatrix} \otimes (\mathrm{P}^+ - \mathrm{P}^-)\,, \tag{E.10}$$

as well as the anti-commutation relation for the Clebsch-Gordon matrices

$$\{t^i, (t^j)^*\} = -\tfrac{1}{2}\delta^{ij}\mathbb{1}_4\,. \tag{E.11}$$

This allows one to rewrite the first term in (E.5) as

$$\begin{aligned}
-\tfrac{1}{2}\mathcal{F}^{\Sigma\Theta}\mathscr{A}_\Sigma\times\mathscr{A}_\Theta &\overset{(E.6)}{=} \tfrac{1}{16}\mathcal{F}^{\Sigma\Theta}\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_\Gamma)^C{}_D(\bar{\mathscr{C}}_\Psi)^D{}_4\big\}\mathscr{A}^\Gamma\times\mathscr{A}^\Psi \\
&\overset{(E.7)}{=} \tfrac{1}{16}\mathcal{F}^{\Sigma\Theta}\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_\Gamma)^C{}_D\mathscr{D}^\Gamma\, S^D{}_E\mathscr{A\!\!/}^E{}_F\times\mathscr{A\!\!/}^{*F}{}_4\big\} \\
&\overset{\text{ibp}}{=} -\tfrac{1}{16}\mathscr{D}^\Gamma\mathcal{F}^{\Sigma\Theta}\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_\Gamma)^C{}_D S^D{}_E\mathscr{A\!\!/}^E{}_F\times\mathscr{A\!\!/}^{*F}{}_4\big\} \\
&\overset{(E.8)}{=} -\tfrac{1}{16}\mathscr{D}^\Gamma\mathcal{F}^{\Sigma\Theta}\mathrm{tr}\big\{\big[-2(\mathscr{C}_\Sigma)^4{}_D\,\eta_{\Theta\Gamma}+(\mathscr{C}_{[\Sigma})^4{}_B(\bar{\mathscr{C}}_\Theta)^B{}_C(\mathscr{C}_{\Gamma]})^C{}_D\big]S^D{}_E\mathscr{A\!\!/}^E{}_F\times\mathscr{A\!\!/}^{*F}{}_4\big\} \\
&= -\tfrac{1}{8}\mathscr{D}_\Theta\mathcal{F}^{\Theta\Sigma}\mathrm{tr}\big\{(\mathscr{C}_\Sigma)^4{}_D S^D{}_E\mathscr{A\!\!/}^E{}_F\times\mathscr{A\!\!/}^{*F}{}_4\big\}\,,
\end{aligned} \tag{E.12}$$

where in the last step the Bianchi identity $\mathscr{D}^{[\Gamma}\mathcal{F}^{\Sigma\Theta]} = 0$ is required. This concludes the proof for the special case $L = \mathrm{diag}(-1,-1,-1,+3)$ (and permutations thereof). To reach the full Lie algebra we make use of the fact that we can superimpose coupling flow operators with weight one, giving the Cartan subalgebra and that $S^{\mathrm{b}}_g[\mathscr{A}]$ is invariant under R-symmetry transformations $\mathscr{A} \to \mathscr{A}'$. From

$$0 = (\partial_g + R_g[\mathscr{A}'])S^{\mathrm{b}}_g[\mathscr{A}'] = (\partial_g + R_g[\mathscr{A}'])S^{\mathrm{b}}_g[\mathscr{A}]\,, \tag{E.13}$$

we observe the transformed $R_g[\mathscr{A}']$ also satisfies the infinitesimal free action condition, reaching all $L \in \mathfrak{su}(4)$.

We only have left to prove (E.6), which we do by explicitly checking the various possibilities of the open indices. The easiest case is the one with only gamma matrices

$$\begin{aligned}
\tfrac{1}{4}\mathrm{tr}\big\{(\mathscr{C}_\mu)^4{}_B(\bar{\mathscr{C}}_\nu)^B{}_C(\mathscr{C}_\rho)^C{}_D(\bar{\mathscr{C}}_\sigma)^D{}_4\big\} &= \tfrac{1}{4}\mathrm{tr}\big\{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\big\} \\
&= \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\nu}\eta_{\rho\sigma}\,.
\end{aligned} \tag{E.14}$$

Next, we consider the case when there are three gamma matrices (modulo chiral projectors) in the trace, i.e. one of the four indices in the range 4 to 9 and the three others in the range 0 to 3. In that case, the trace vanishes since any trace over an odd number of gamma matrices vanishes and the r.h.s. of (E.6) also clearly vanishes because in each term there is a Kronecker delta that is zero. The next case is the one where two indices are in the range 0 to 3 and the other two indices are in the range 4 to 9. We have to distinguish three arrangements of indices

$$\begin{aligned}\tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+i})^4{}_B(\bar{\mathscr{C}}_{3+j})^B{}_C(\mathscr{C}_\mu)^C{}_D(\bar{\mathscr{C}}_\nu)^D{}_4\} &= \tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+i})^4{}_B(\bar{\mathscr{C}}_{3+j})^B{}_4\gamma_\mu\gamma_\nu\}\\ &= (t^i)_{4J}(t^j)^{J4}\,\mathrm{tr}\{\mathrm{P}^+\gamma_\mu\gamma_\nu\} + (t^i)^{4J}(t^j)_{J4}\,\mathrm{tr}\{\mathrm{P}^-\gamma_\mu\gamma_\nu\}\\ &= -2[(t^i)_{4J}(t^j)^{J4}\eta_{\mu\nu} \;+\; \text{c.c.}] = \delta_{ij}\eta_{\mu\nu}\,,\end{aligned} \tag{E.15}$$

$$\begin{aligned}&\tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+i})^4{}_B(\bar{\mathscr{C}}_\mu)^B{}_C(\mathscr{C}_{3+j})^C{}_D(\bar{\mathscr{C}}_\nu)^D{}_4\}\\ &\qquad = \mathrm{tr}\{[(t^i)_{4J}\mathrm{P}^+ - (t^i)^{4J}\mathrm{P}^-]\gamma_\mu[(t^j)_{J4}\mathrm{P}^+ - (t^j)^{J4}\mathrm{P}^-]\gamma_\nu\}\\ &\qquad = -(t^i)_{4J}(t^j)^{J4}\mathrm{tr}\{\mathrm{P}^+\gamma_\mu\gamma_\nu\} - (t^i)^{4J}(t^j)_{J4}\mathrm{tr}\{\mathrm{P}^-\gamma_\mu\gamma_\nu\}\\ &\qquad = 2(t^i)_{4J}(t^j)^{J4}\eta_{\mu\nu} \;+\; \text{c.c.} = -\delta_{ij}\eta_{\mu\nu}\,,\end{aligned} \tag{E.16}$$

$$\begin{aligned}&\tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+i})^4{}_B(\bar{\mathscr{C}}_\mu)^B{}_C(\mathscr{C}_\nu)^C{}_D(\bar{\mathscr{C}}_{3+j})^D{}_4\}\\ &\qquad = \mathrm{tr}\{[(t^i)_{4J}\mathrm{P}^+ - (t^i)^{4J}\mathrm{P}^-]\gamma_\mu\gamma_\nu[(t^j)^{J4}\mathrm{P}^+ - (t^j)_{J4}\mathrm{P}^-]\}\\ &\qquad = (t^i)_{4J}(t^j)^{J4}\,\mathrm{tr}\{\mathrm{P}^+\gamma_\mu\gamma_\nu\} + (t^i)^{4J}(t^j)_{J4}\,\mathrm{tr}\{\mathrm{P}^-\gamma_\mu\gamma_\nu\}\\ &\qquad = -2[(t^i)_{4J}(t^j)^{J4}\eta_{\mu\nu} \;+\; \text{c.c.}] = \delta_{ij}\eta_{\mu\nu}\,,\end{aligned} \tag{E.17}$$

with all the other index configurations related to the three above by the cyclicity of the trace. The trace with only one gamma matrix vanishes due to the same reason as for three gamma matrices. We are left with the case

$$\begin{aligned}&\tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+i})^4{}_B(\bar{\mathscr{C}}_{3+j})^B{}_C(\mathscr{C}_{3+k})^C{}_D(\bar{\mathscr{C}}_{3+l})^D{}_4\}\\ &\quad = 4(t^i)_{4I}(t^j)^{IC}(t^k)_{CK}(t^l)^{K4}\,\mathrm{tr}\,\mathrm{P}^+ + 4(t^i)^{4I}(t^j)_{IC}(t^k)^{CK}(t^l)_{K4}\,\mathrm{tr}\,\mathrm{P}^-\\ &\quad = 8\,(t^i)_{4I}(t^j)^{IC}(t^k)_{CK}(t^l)^{K4} \;+\; \text{c.c.}\\ &\quad = 8\,[(t^i)_{4I}(t^j)^{I4}(t^k)_{4K}(t^l)^{K4} + (t^i)_{4I}(t^j)^{IJ}(t^k)_{JK}(t^l)^{K4}] \;+\; \text{c.c.}\end{aligned} \tag{E.18}$$

The last expression can be evaluated with the explicit form of the Clebsch-Gordon coefficients (5.29) and the identity

$$\epsilon_{IJM}\epsilon^{MKL} = \delta_I{}^K\delta_J{}^L - \delta_I{}^L\delta_J{}^K\,. \tag{E.19}$$

We do not quite find the desired result, because we obtain additional terms when two of the indices i, j, k, l are in the range 1 to 3 and the other two are in the range 4 to 6. For example

$$\tfrac{1}{4}\mathrm{tr}\{(\mathscr{C}_{3+I})^4{}_B(\bar{\mathscr{C}}_{6+J})^B{}_C(\mathscr{C}_{3+K})^C{}_D(\bar{\mathscr{C}}_{6+L})^D{}_4\} = \delta_{IL}\delta_{JK} - \delta_{IK}\delta_{JL} - \delta_{IJ}\delta_{KL}\,, \tag{E.20}$$

where only the second term on the r.h.s. would appear in the r.h.s. of (E.6). However, we contract (E.6) with the φ's in the adjoint representation of the gauge group. Fortunately, the additional terms are proportional to

$$(\varphi_I\times\varphi_J)\,(\varphi_{I+3}\times\varphi_{J+3})+(\varphi_I\times\varphi_{J+3})\,(\varphi_J\times\varphi_{I+3})+(\varphi_I\times\varphi_{I+3})\,(\varphi_{J+3}\times\varphi_J)=0\,, \tag{E.21}$$

and thus vanish by the Jacobi identity in color space.

Bibliography

[1] Muon g-2 collaboration, *Measurement of the positive muon anomalous magnetic moment to 0.46 ppm*, *Phys. Rev. Lett.* **126** (2021) 141801.

[2] Yu. A. Golfand and E. P. Likhtman, *Extension of the algebra of Poincaré group generators and violation of P invariance*, *JETP Lett.* **13** (1971) 323.

[3] M. F. Sohnius, *Introducing supersymmetry*, *Phys. Rep.* **128** (1985) 39.

[4] M. Drees, *An introduction to supersymmetry*, Lectures given at Seoul summer symposium on field theory (1996) [arXiv:9611409 [hep-ph]].

[5] S. P. Martin, *A supersymmetry primer*, *ASDHEP* **Perspectives on Supersymmetry** (1998) 1 [arXiv:9709356 [hep-ph]].

[6] H. Nicolai, *On a new characterization of scalar supersymmetric theories*, *Phys. Lett. B* **89** (1980) 341.

[7] H. Nicolai, *Supersymmetry and functional integration measures*, *Nucl. Phys. B* **176** (1980) 419.

[8] H. Nicolai, *Supersymmetry without anticommuting variables*, Contribution to: *Europhys. Study Conference Unif. Fund. Int.* (1980) 689.

[9] R. Flume and O. Lechtenfeld, *On the stochastic structure of globally supersymmetric field theories*, *Phys. Lett. B* **135** (1984) 91.

[10] K. Dietz and O. Lechtenfeld, *Nicolai maps and stochastic observables from a coupling constant flow*, *Nucl. Phys. B* **255** (1985) 149.

[11] O. Lechtenfeld, *Construction of the Nicolai mapping in supersymmetric field theories*, Ph.D. Thesis, Bonn University (1984), internal report *BONN-IR-84-42*, ISSN-0172-8741.

[12] K. Dietz and O. Lechtenfeld, *Ghost-free quantisation of non-Abelian gauge theories via the Nicolai transformation of their supersymmetric extensions*, *Nucl. Phys. B* **259** (1985) 397.

[13] O. Lechtenfeld, *Stochastic variables in ten dimensions?*, *Nucl. Phys. B* **274** (1986) 633.

[14] V. de Alfaro, S. Fubini, G. Furlan and G. Veneziano, *Stochastic identities in supersymmetric theories*, *Phys. Lett. B* **142** (1984) 389.

[15] V. de Alfaro, S. Fubini, G. Furlan and G. Veneziano, *Stochastic identities in quantum theory*, *Nucl. Phys. B* **255** (1985) 1.

[16] V. de Alfaro, S. Fubini, G. Furlan and G. Veneziano, *Nicolai mapping and stochastic identities in supersymmetric field theories*, *Phys. Rept.* **137** (1986) 55.

[17] R. Floreani, J.P. Leroy, J. Michel and G.C. Rossi, *A perturbative study of the Nicolai mapping*, *Phys. Lett. B* **158** (1985) 47.

[18] V. de Alfaro, S. Fubini and G. Furlan, *Stochastic identities in the light cone gauge*, *Phys. Lett. B* **163** (1985) 176.

[19] M. Bochicchio and A. Pilloni, *Gauge theories in anti-selfdual variables*, *JHEP* **09** (2013) 039 [arXiv:1304.4949 [hep-th]].

[20] H. Ezawa and J. Klauder, *Fermions without fermions: the Nicolai map revisited*, *Prog. Theor. Phys.* **74** (1985) 904.

[21] H. Nicolai, *Supersymmetric functional integration measures*, lectures delivered at the NATO Advanced Study Institute on Supersymmetry, Bonn, Germany, 20–31 Aug 1984, pp.393–420, eds. K. Dietz et. al., *Plenum Press* (1984).

[22] S. Ananth, H. Nicolai, C. Pandey and S. Pant, *Supersymmetric Yang–Mills theories: not quite the usual perspective*, *J. Phys. A* **53** (2020) 174001 [arXiv:2001.02768 [hep-th]].

[23] H. Nicolai and J. Plefka, *$\mathcal{N}=4$ super-Yang–Mills correlators without anticommuting variables*, *Phys. Rev. D* **101** (2020) 125013 [arXiv:2003.14325 [hep-th]].

[24] S. Ananth, O. Lechtenfeld, H. Malcha, H. Nicolai, C. Pandey and S. Pant, *Perturbative linearization of supersymmetric Yang–Mills theory*, *JHEP* **10** (2020) 199 [arXiv:2005.12324 [hep-th]].

[25] L. Brink, J. H. Schwarz and J. Scherk, *Supersymmetric Yang–Mills theories*, *Nucl. Phys. B* **121** (1977) 77.

[26] S. Ananth, H. Malcha, C. Pandey and S. Pant, *Supersymmetric Yang–Mills theory in $D=6$ without anti-commuting variables*, *Phys. Rev. D* **103** (2021) 025010 [arXiv:2006.02457 [hep-th]].

[27] O. Lechtenfeld and M. Rupprecht, *Universal form of the Nicolai map*, *Phys. Rev. D* **104** (2021) L021701 [arXiv:2104.00012 [hep-th]].

[28] H. Malcha and H. Nicolai, *Perturbative linearization of super-Yang–Mills theories in general gauges*, *JHEP* **06** (2021) 001 [arXiv:2104.06017 [hep-th]].

[29] O. Lechtenfeld and M. Rupprecht, *Construction method for the Nicolai map in supersymmetric Yang–Mills theories*, *Phys. Lett. B* **819** (2021) 136413 [arXiv:2104.09654 [hep-th]].

[30] O. Lechtenfeld and H. Nicolai, *A perturbative expansion scheme for supermembrane and matrix theory*, *JHEP* **02** (2022) 114 [arXiv:2109.00346 [hep-th]].

[31] M. Rupprecht, *The coupling flow of $\mathcal{N}=4$ super Yang–Mills theory*, *JHEP* **04** (2022) 004 [arXiv:2111.13223 [hep-th]].

[32] H. Malcha, *Two loop ghost free quantisation of Wilson loops in $\mathcal{N}=4$ supersymmetric Yang–Mills*, *Phys. Lett. B* **833** (2022) 137377 [arXiv:2206.02919 [hep-th]].

[33] O. Lechtenfeld and M. Rupprecht, *Is the Nicolai map unique?*, *JHEP* **09** (2022) 139 [arXiv:2207.09471 [hep-th]].

[34] O. Lechtenfeld, *Supersymmetric large-order perturbation with the Nicolai map*, *Phys. Lett. B* **835** (2022) 137507 [arXiv:2208.06420 [hep-th]].

[35] O. Lechtenfeld and M. Rupprecht, *An improved Nicolai map for super Yang–Mills theory*, *Phys. Lett. B* **838** (2023) 137681 [arXiv:2211.07660 [hep-th]].

[36] H. Nicolai, *Supersymmetry and spin systems*, *J. Phys. A* **9** (1976) 1497.

[37] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, *Phys. Rep.* **251** (1995) 267 [arXiv:hep-th/9405029].

[38] E. Witten, *Dynamical breaking of supersymmetry*, *Nucl. Phys. B* **188** (1981) 513.

[39] J.W. van Holten, *Instantons in supersymmetric quantum mechanics*, Erice Lectures 1981, pp.505–513, *Springer, Boston, MA* (1983).

[40] P. Salomonson and J.W. van Holten, *Fermionic coordinates and supersymmetry in quantum mechanics*, *Nucl. Phys. B* **196** (1982) 509.

[41] F. Cooper and B. Freedman, *Aspects of supersymmetric quantum mechanics*, *Ann. Phys.* **146** (1983) 262.

[42] B. Zumino, *Supersymmetry and the vacuum*, *Nucl. Phys. B* **89** (1975) 535.

[43] S. Coleman, *Aspects of symmetry, Chapter 7*, pp.265–350, *Cambridge University Press* (1985).

[44] S. Cecotti and L. Girardello, *Functional measure, topology and dynamical supersymmetry breaking*, *Phys. Lett. B* **110** (1982) 39.

[45] Joshua Ellis, *TikZ-Feynman: Feynman diagrams with TikZ*, [arXiv:1601.05437 [hep-ph]].

[46] J. Wess and J. Bagger, *Supersymmetry and supergravity*, *Princeton University Press, Princeton, NJ* (1992).

[47] S. Kovacs, *$\mathcal{N}=4$ supersymmetric Yang–Mills theory and the AdS / SCFT correspondence*, Ph.D. Thesis, University of Rome Tor Vergata (1999) [arXiv:9908171 [hep-th]].

[48] G. Dall'Agata and M. Zagermann, *Simple supergravity*, Book chapter for the upcoming 'Handbook of Quantum Gravity' (2023) [arXiv:2212.10044 [hep-th]].

[49] S. Cecotti and L. Girardello, *Stochastic and parastochastic aspects of supersymmetric functional measures: A new non-perturbative approach to supersymmetry*, *Ann. Phys.* **145** (1983) 81.

[50] G. Parisi and N. Sourlas, *Supersymmetric field theories and stochastic differential equations*, *Nucl. Phys. B* **206** (1982) 321.

[51] H. Nicolai, *On the functional integration measure of supersymmetric Yang-Mills theories*, *Phys. Lett. B* **117** (1982) 408

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List of Publications

1. O. Lechtenfeld and **M. Rupprecht**,
An improved Nicolai map for super Yang–Mills theory,
Phys. Lett. B **838** (2023) 137681 [arXiv:2211.07660 [hep-th]].

2. O. Lechtenfeld and **M. Rupprecht**,
Is the Nicolai map unique?,
JHEP **09** (2022) 139 [arXiv:2207.09471 [hep-th]].

3. **M. Rupprecht**,
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JHEP **04** (2022) 004 [arXiv:2111.13223 [hep-th]].

4. O. Lechtenfeld and **M. Rupprecht**,
Construction method for the Nicolai map in supersymmetric Yang–Mills theories,
Phys. Lett. B **819** (2021) 136413 [arXiv:2104.09654 [hep-th]].

5. O. Lechtenfeld and **M. Rupprecht**,
Universal form of the Nicolai map,
Phys. Rev. D **104** (2021) L021701 [arXiv:2104.00012 [hep-th]].

6. A. Erschfeld, S. Flörchinger and **M. Rupprecht**,
General relativistic nonideal fluid equations for dark matter from a truncated cumulant expansion, *Phys. Rev. D* **102** (2020) 063520 [arXiv:2005.12923 [hep-ph]].