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Stationary states to a free boundary transmission problem for an electrostatically actuated plate

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Abstract. A two-dimensional free boundary transmission problem arising in the modeling of an electrostatically actuated plate is considered and a representation formula for the derivative of the associated electrostatic energy with respect to the deflection of the plate is derived. The latter paves the way for the construction of energy minimizers and also provides the Euler–Lagrange equation satisfied by these minimizers. A by-product is the monotonicity of the electrostatic energy with respect to the deflection.

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1. Introduction

We consider a model for a microelectromechanical system (MEMS) featuring an elastic, electrostatically actuated plate with positive thickness as introduced in [3]. More precisely, given a finite interval $D := (-L, L)$ with $L > 0$, let the function $u \in C(\bar{D}, [-H, \infty))$ with $u(\pm L) = 0$ measure the deflection from rest of the lower part of an elastic plate with thickness $d > 0$, clamped at its boundaries and suspended above a fixed ground plate, the latter being located at $z = -H$ with $H > 0$ and represented by $D \times \{-H\}$ (see Fig. 1). The deflected elastic plate is then

$$\Omega_2(u) := \{(x, z) \in D \times \mathbb{R} \,:\, u(x) < z < u(x) + d\} \ ,$$

while the region between the ground plate and the deflected elastic plate is

$$\Omega_1(u) := \{(x, z) \in D \times \mathbb{R} \,:\, -H < z < u(x)\} \ .$$

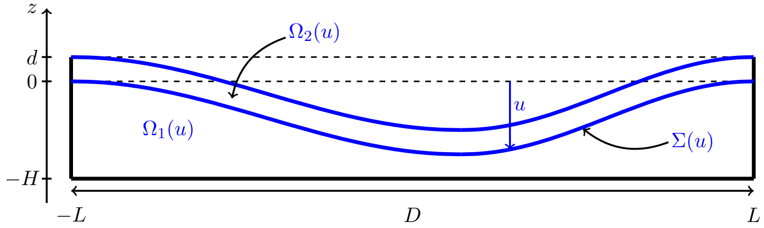

FIGURE 1. Geometry of $\Omega(u)$ for a state $u \in \mathcal{S}$ with empty coincidence set

The two regions are separated by the interface

$$\Sigma(u) := \{(x,z) \in D \times \mathbb{R} \,:\, z = u(x) > -H\} ,$$

and the subdomain of $D \times (-H, \infty)$ spanned by the MEMS device is

$$\Omega(u) := \{(x,z) \in D \times \mathbb{R} \,:\, -H < z < u(x) + d\} = \Omega_1(u) \cup \Omega_2(u) \cup \Sigma(u) .$$

The deflection of the plate being triggered by electrostatic actuation, the total energy of the device is

$$E(u) := E_{\mathrm{m}}(u) + E_{\mathrm{e}}(u) \tag{1.1a}$$

with mechanical energy $E_{\mathrm{m}}(u)$ and electrostatic energy $E_{\mathrm{e}}(u)$. The former is given by

$$E_{\mathrm{m}}(u) := \frac{\beta}{2}\|\partial_x^2 u\|_{L_2(D)}^2 + \left(\frac{\tau}{2} + \frac{a}{4}\|\partial_x u\|_{L_2(D)}^2\right) \|\partial_x u\|_{L_2(D)}^2 \tag{1.1b}$$

with $\beta > 0$ and $a, \tau \geq 0$, taking into account bending and external stretching effects of the elastic plate. The electrostatic energy

$$E_{\mathrm{e}}(u) := -\frac{1}{2}\int_{\Omega(u)} \sigma |\nabla \psi_u|^2 \,\mathrm{d}(x,z) \tag{1.1c}$$

involves the electrostatic potential ψ_u in the domain $\Omega(u)$ with ψ_u being the solution to the transmission problem

$$\mathrm{div}(\sigma \nabla \psi_u) = 0 \quad \text{in} \quad \Omega(u) , \tag{1.2a}$$

$$[\![\psi_u]\!] = [\![\sigma \nabla \psi_u]\!] \cdot \mathbf{n}_{\Sigma(u)} = 0 \quad \text{on} \quad \Sigma(u) , \tag{1.2b}$$

$$\psi_u = h_u \quad \text{on} \quad \partial\Omega(u) , \tag{1.2c}$$

where $[\![\cdot]\!]$ denotes the (possible) jump across the interface $\Sigma(u)$; that is,

$$[\![f]\!](x, u(x)) := f|_{\Omega_1(u)}(x, u(x)) - f|_{\Omega_2(u)}(x, u(x)) , \qquad x \in D ,$$

whenever meaningful for a function $f : \Omega(u) \to \mathbb{R}$. Moreover,

$$\sigma := \sigma_1 \mathbf{1}_{\Omega_1(u)} + \sigma_2 \mathbf{1}_{\Omega_2(u)} \tag{1.3}$$

involves the material dependent constant permittivities $\sigma_2, \sigma_1 > 0$. The unit normal vector field to $\Sigma(u)$ (pointing into $\Omega_2(u)$) is

$$\mathbf{n}_{\Sigma(u)} := \frac{(-\partial_x u, 1)}{\sqrt{1 + (\partial_x u)^2}} .$$

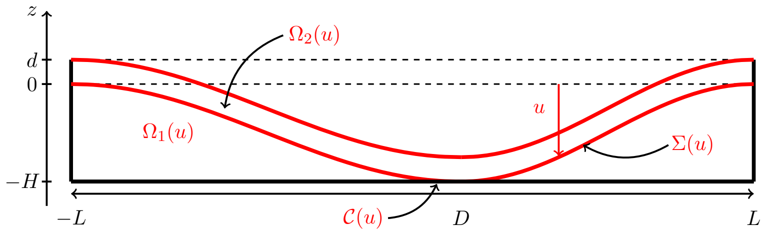

FIGURE 2. Geometry of $\Omega(u)$ for a state $u \in \bar{\mathcal{S}}$ with non-empty coincidence set

As for the boundary values in (1.2c) we assume the particular form

$$h_u(x,z) := \zeta(z - u(x) + 1)\,, \qquad (x,z) \in \bar{D} \times [-H,\infty)\,, \tag{1.4a}$$

where

$$\zeta \in C^2(\mathbb{R})\,, \qquad \zeta|_{(-\infty,1]} \equiv 0\,, \qquad \zeta|_{[1+d,\infty)} \equiv V\,, \tag{1.4b}$$

with $V > 0$. For instance, $\zeta(r) := V \min\{1, (r-1)^m/d^m\}$ for $r > 1$ and $m > 2$ and $\zeta \equiv 0$ on $(-\infty, 1]$ is a possible choice. Note that

$$h_u(x,-H) = 0\,, \quad h_u(x, u(x) + d) = V\,, \qquad x \in D\,;$$

that is, the ground plate and the top of the elastic plate are kept at different constant potentials. Let us emphasize that we explicitly allow that the elastic plate touches upon the ground plate when u reaches the value $-H$ somewhere, a situation corresponding to a non-empty *coincidence set*

$$\mathcal{C}(u) := \{x \in D \,:\, u(x) = -H\}\,, \tag{1.5}$$

as depicted in Fig. 2. In this case, the region $\Omega_1(u)$ is not connected and its boundary features cusps, so that its connected components are not Lipschitz domains.

In this research we shall be interested in minimizers of the total energy E which correspond to stationary states of the MEMS device. More precisely, we shall show the existence of minimizers and derive the corresponding Euler–Lagrange equation they satisfy, which, due to the nature of the problem, is a variational inequality. Obviously, the main difficulty in this regard is related to the electrostatic energy E_e and the associated transmission problem (1.2) for the electrostatic potential. The latter was investigated in [5] for deflections belonging to the set

$$\bar{\mathcal{S}} := \{u \in H^2(D) \cap H_0^1(D) \,:\, u \ge -H \text{ in } D \text{ and } \pm [\![\sigma]\!] \partial_x u(\pm L) \le 0\}$$

with $[\![\sigma]\!] = \sigma_1 - \sigma_2$. More precisely, the following result is shown in [5].

Theorem 1.1. *[5, Theorem 1.1] Suppose (1.4).*

(a) For each $u \in \bar{\mathcal{S}}$, there is a unique variational solution $\psi_u \in h_u + H_0^1(\Omega(u))$ to (1.2). Moreover,

$$\psi_{u,1} := \psi_u|_{\Omega_1(u)} \in H^2(\Omega_1(u)) \text{ and } \psi_{u,2} := \psi_u|_{\Omega_2(u)} \in H^2(\Omega_2(u)),$$

and ψ_u is a strong solution to the transmission problem (1.2).

(b) Given $\kappa > 0$, there is $c(\kappa) > 0$ such that ψ_u satisfies

$$\|\psi_u\|_{H^1(\Omega(u))} + \|\psi_{u,1}\|_{H^2(\Omega_1(u))} + \|\psi_{u,2}\|_{H^2(\Omega_2(u))} \leq c(\kappa)$$

for every $u \in \bar{\mathcal{S}}$ with $\|u\|_{H^2(D)} \leq \kappa$.

The H^2-regularity of the electrostatic potential ψ_u provided by Theorem 1.1 is then the basis for deriving the existence of minimizers of the total energy E. We shall look for minimizers with clamped boundary conditions; that is, minimizers in the closed convex subset

$$\bar{\mathcal{S}}_0 := \{u \in H^2(D) \cap H_0^1(D) \,:\, u \geq -H \text{ in } D \text{ and } \partial_x u(\pm L) = 0\}$$

of $H^2(D)$. We denote by $\partial \mathbb{I}_{\bar{\mathcal{S}}_0}$ the subdifferential of the indicator function $\mathbb{I}_{\bar{\mathcal{S}}_0}$. Our main result then reads:

Theorem 1.2. *Assume $a > 0$ or $[\![\sigma]\!] < 0$, and let (1.4) be satisfied. Then, the total energy E has at least one minimizer in $\bar{\mathcal{S}}_0$. Moreover, any minimizer $u_* \in \bar{\mathcal{S}}_0$ of E in $\bar{\mathcal{S}}_0$ with*

$$E(u_*) = \min_{\bar{\mathcal{S}}_0} E \tag{1.6}$$

is an H^2-weak solution to the variational inequality

$$\beta \partial_x^4 u_* - (\tau + a\|\partial_x u_*\|_{L_2(D)}^2)\partial_x^2 u_* + \partial \mathbb{I}_{\bar{\mathcal{S}}_0}(u_*) \ni -g(u_*) \quad \text{in} \quad D\,; \tag{1.7}$$

that is,

$$\int_D \Big\{\beta \partial_x^2 u_*\, \partial_x^2 (w - u_*) + \big[\tau + a\|\partial_x u_*\|_{L_2(D)}^2\big] \partial_x u_*\, \partial_x (w - u_*)\Big\}\, \mathrm{d}x \geq - \int_D g(u_*)(w - u_*)\, \mathrm{d}x$$

for all $w \in \bar{\mathcal{S}}_0$. For $u \in \bar{\mathcal{S}}_0$, the function $g(u) \in L_2(D)$ is given by

$$\begin{aligned} g(u) := & -\frac{[\![\sigma]\!]}{2(1 + (\partial_x u(x))^2)} \big(\partial_x \psi_{u,2} + \partial_x u \partial_z \psi_{u,2}\big)^2 (x, u(x)) \\ & - \frac{[\![\sigma]\!]\sigma_2}{2\sigma_1 (1 + (\partial_x u(x))^2)} \big(\partial_x u \partial_x \psi_{u,2} - \partial_z \psi_{u,2}\big)^2 (x, u(x)) \\ & + \frac{\sigma_2}{2} \big|\nabla \psi_{u,2}(x, u(x) + d)\big|^2 . \end{aligned} \tag{1.8}$$

Finally, if $[\![\sigma]\!] < 0$, then $u_* \leq 0$ in D.

Even though the total energy E consists of two competing terms with different signs, it is not difficult to see that it is H^2-coercive if $a > 0$ in (1.1b), see [4], and the existence of a minimizer for E in $\bar{\mathcal{S}}_0$ follows directly. When $a = 0$, the coercivity of E is no longer obvious without additional assumptions. In fact, we are not able to prove directly that E is bounded below and thus have to proceed differently. In this case, the coercivity of the functional can be enforced by adding a penalty term which vanishes when u is bounded, an idea already used in [2]. The minimizers of the penalized energy functional on $\bar{\mathcal{S}}_0$ then satisfy the Euler–Lagrange equation (1.7) with an additional term. The assumption $[\![\sigma]\!] < 0$ now guarantees that $g(u) \geq 0$ in D according to (1.8) which, in turn, yields an a priori bound on the minimizers by a comparison argument. This then implies that the minimizers of the penalized energy actually minimize the total energy E. It is worth emphasizing that the non-negative sign of $g(u)$ — read off from the explicit formula (1.8) when $[\![\sigma]\!] < 0$ — is essential for this approach.

The main motivation of this research is thus the derivation of an explicit formula for the electrostatic force $g(u)$ as the (directional) derivative of the electrostatic energy $E_{\mathrm{e}}(u)$. By definition of $E_{\mathrm{e}}(u)$, such a computation corresponds to that of a shape derivative and thus follows the guidelines of classical results [1,6,7]. In fact, a computation in the same spirit is performed in [4] for a related MEMS model but with a flat transmission interface. As we shall see in Sect. 2, the non-flat transmission interface $\Sigma(u)$ in (1.2b) leads to additional terms in the electrostatic force, making the computation of the latter noticeably more involved. We first establish in Sect. 2 differentiability properties of the electrostatic potential ψ_u with respect to u which then ensure the Fréchet differentiability of the electrostatic energy E_{e} on $\mathcal{S}_0$. The subsequent identification of $g(u)$ as the (directional) derivative of the electrostatic energy $E_{\mathrm{e}}(u)$ is the main contribution of Sect. 2. It is worth already pointing out here that the derivation of the explicit formula (1.8) of $g(u)$ does not require the explicit computation of the derivative of the electrostatic potential ψ_u with respect to u. Once the formula (1.8) is established, the existence of minimizers of E in $\bar{\mathcal{S}}_0$ follows along the lines of [2] as described above.

As already pointed out, the electrostatic force $g(u)$ has a sign if one assumes that $[\![\sigma]\!] < 0$; that is, if $\sigma_2 > \sigma_1$. For instance, this is a natural assumption when the region between the two plates is vacuumed or filled with air. We also point out that this assumption implies the monotonicity of the electrostatic energy E_{e} as stated explicitly in Corollary 2.7.

Remark 1.3. The total energy E can also be minimized in $\bar{\mathcal{S}}$ leading then to weak solutions to (1.7) with $\mathbb{I}_{\bar{\mathcal{S}}}$ instead of $\mathbb{I}_{\bar{\mathcal{S}}_0}$ which satisfy pinned boundary conditions $u(\pm L) = \partial_x^2 u(\pm L) = 0$ instead of the clamped boundary conditions involved in $\bar{\mathcal{S}}_0$. In this case, however, one has to be slightly more careful when computing the shape derivative of the electrostatic energy E_{e} due to the different constraints on the boundary.

2. Shape derivative of the electrostatic energy

The heart of the proof of Theorem 1.2 is the differentiability of the electrostatic energy E_{e} and, in particular, the identification of $g(u)$ as its derivative at $u \in \bar{\mathcal{S}}_0$. On a formal level, this derivative is computed in [3] (in a three-dimensional setting). Here we provide a rigorous proof. Actually, we shall show that the electrostatic energy E_{e} is Fréchet differentiable on

$$\mathcal{S}_0 := \{u \in H^2(D) \cap H_0^1(D) \,:\, u > -H \text{ in } D \ \text{ and } \ \partial_x u(\pm L) = 0\},$$

i.e., for points with empty coincidence set, while it admits a directional derivative at $u \in \bar{\mathcal{S}}_0$ in the directions $-u + \mathcal{S}_0$. Here and in the following, $\mathcal{S}_0$ and $\bar{\mathcal{S}}_0$ are endowed with the $H^2(D)$-topology. The precise result reads as follows:

Theorem 2.1. *Assume (1.4). The electrostatic energy $E_{\mathrm{e}} : \mathcal{S}_0 \to \mathbb{R}$ is continuously Fréchet differentiable with*

$$\partial_u E_{\mathrm{e}}(u)[\vartheta] = \int_D g(u)(x)\, \vartheta(x)\, \mathrm{d}x$$

for $u \in \mathcal{S}_0$ and $\vartheta \in H^2(D) \cap H_0^1(D)$, where $g(u)$ is defined in (1.8). Moreover, if $u \in \bar{\mathcal{S}}_0$ and $w \in \mathcal{S}_0$, then

$$\lim_{t \to 0^+} \frac{1}{t}\big(E_{\mathrm{e}}(u + t(w-u)) - E_{\mathrm{e}}(u)\big) = \int_D g(u)(x)\,(w-u)(x)\, \mathrm{d}x\,.$$

The function $g : \bar{\mathcal{S}}_0 \to L_p(D)$ is continuous for each $p \in [1, \infty)$.

The proof of Theorem 2.1 follows from Proposition 2.5 and Corollary 2.7 below. We will need the following result which is contained in [5].

Proposition 2.2. [5, Theorem 1.3, Proposition 3.3] *Assume (1.4). Let $u \in \bar{\mathcal{S}}_0$ and consider a bounded sequence $(u_n)_{n \geq 1}$ in $\bar{\mathcal{S}}_0$ such that*

$$\lim_{n \to \infty} \|u_n - u\|_{H^1(D)} = 0\,.$$

Then, for any $p \in [1, \infty)$,

$$\lim_{n \to \infty} \big\|\nabla \psi_{u_n,2}(\cdot, u_n) - \nabla \psi_{u,2}(\cdot, u)\big\|_{L_p(D,\mathbb{R}^2)} = 0\,, \tag{2.1a}$$

$$\lim_{n \to \infty} \big\|\nabla \psi_{u_n,2}(\cdot, u_n + d) - \nabla \psi_{u,2}(\cdot, u + d)\big\|_{L_p(D,\mathbb{R}^2)} = 0\,. \tag{2.1b}$$

Moreover,

$$\lim_{n \to \infty} E_{\mathrm{e}}(u_n) = E_{\mathrm{e}}(u)\,. \tag{2.2}$$

Finally, setting

$$M := d + \max\big\{\|u\|_{L_\infty(D)}\,,\, \sup_{n \geq 1}\{\|u_n\|_{L_\infty(D)}\}\big\}\,,$$

one has

$$\lim_{n \to \infty} \|(\psi_{u_n} - h_{u_n}) - (\psi_u - h_u)\|_{H_0^1(D \times (-H, M))} = 0\,. \tag{2.3}$$

The first step of the proof of Theorem 2.1 is to show that the electrostatic energy E_{e} is Fréchet differentiable on $\mathcal{S}_0$. The next lemma is adapted from [1, Theorem 5.3.2], see also [4, Lemma 4.1]. We include the proof for the reader's ease.

Lemma 2.3. *Assume (1.4). Let $u \in \mathcal{S}_0$ be fixed and define, for $v \in \mathcal{S}_0$, the transformation*

$$\Theta_{u,v} = (\Theta_{u,v,1}, \Theta_{u,v,2}) : \Omega(u) \to \Omega(v)$$

by

$$\Theta_{u,v,1}(x,z) := \left(x, z + \frac{v(x)-u(x)}{H+u(x)}(z+H)\right), \qquad (x,z) \in \Omega_1(u), \tag{2.4a}$$

$$\Theta_{u,v,2}(x,z) := (x, z + v(x) - u(x)), \qquad (x,z) \in \Omega_2(u). \tag{2.4b}$$

Then there exists a neighborhood $\mathcal{U}$ of u in $\mathcal{S}_0$ such that the mapping

$$\mathcal{U} \to H_0^1(\Omega(u)), \quad v \mapsto \xi_v := \big(\psi_v - h_v\big) \circ \Theta_{u,v}$$

is continuously differentiable, recalling that $\mathcal{S}_0$ and thus also $\mathcal{U}$ are endowed with the $H^2(D)$-topology.

Remark 2.4. Lemma 2.3 is only an intermediate step in the computation of the Fréchet derivative of the electrostatic energy E_{e}. As we shall see later in the proof of Proposition 2.5, the computation does not require an explicit formula for the derivative of $v \mapsto \xi_v$. Moreover, we do not strive for optimal assumptions (e.g., the topology of $\mathcal{S}_0$ can be weakened, as long as it is stronger than that of $W_\infty^1(D)$).

Proof of Lemma 2.3. The differentiability property relies on a classical approach: we shall first identify a suitable C^1-function

$$\mathcal{F} : \mathcal{S}_0 \times H_0^1(\Omega(u)) \to H^{-1}(\Omega(u))$$

which vanishes at (v, ξ_v) whenever $v \in \mathcal{S}_0$. We then show that the implicit function theorem applies to $\mathcal{F}$ near (u, ξ_u).

To this end, set $\chi_v := \psi_v - h_v$ for $v \in \mathcal{S}_0$. Owing to Theorem 1.1, the function χ_v belongs to $H_0^1(\Omega(v))$ and satisfies the integral identity

$$\int_{\Omega(v)} \sigma \nabla \chi_v \cdot \nabla \theta \,\mathrm{d}(\bar{x},\bar{z}) = -\int_{\Omega(v)} \sigma \nabla h_v \cdot \nabla \theta \,\mathrm{d}(\bar{x},\bar{z}), \quad \theta \in H_0^1(\Omega(v)), \tag{2.5}$$

which we next shall write as integrals over $\Omega(u)$. To this end, we first note that, due to $\Theta_{u,u} = \mathrm{id}$,

$$\xi_u = \chi_u, \qquad \nabla \xi_v = D\Theta_{u,v}^T \nabla \chi_v \circ \Theta_{u,v}, \tag{2.6}$$

where

$$D\Theta_{u,v,1}(x,z) = \begin{pmatrix} 1 & 0 \\ (z+H)\partial_x\left(\dfrac{v-u}{H+u}\right)(x) & \dfrac{H+v(x)}{H+u(x)} \end{pmatrix}, \quad (x,z) \in \Omega_1(u),$$

and

$$D\Theta_{u,v,2}(x,z) = \begin{pmatrix} 1 & 0 \\ \partial_x(v-u)(x) & 1 \end{pmatrix}, \qquad (x,z) \in \Omega_2(u).$$

For $\phi \in H_0^1(\Omega(u))$ we set

$$\phi_v := \phi \circ \Theta_{u,v}^{-1} \in H_0^1(\Omega(v))$$

and note that

$$\nabla \phi_v = \big((D\Theta_{u,v}^T)^{-1} \nabla \phi\big) \circ \Theta_{u,v}^{-1}.$$

Performing the change of variables $(\bar{x},\bar{z})=\Theta_{u,v}(x,z)$ in (2.5) with $\theta=\phi_v$ and using (1.3) give

$$\begin{aligned}\int_{\Omega(u)} \sigma\, J_v(D\Theta_{u,v})^{-1}(D\Theta_{u,v}^T)^{-1}\nabla\xi_v\cdot\nabla\phi\,\mathrm{d}(x,z)\\ = -\int_{\Omega(u)} \sigma\, J_v\,(D\Theta_{u,v})^{-1}\nabla h_v\circ\Theta_{u,v}\cdot\nabla\phi\,\mathrm{d}(x,z)\,,\end{aligned}\tag{2.7}$$

where the Jacobian $J_v:=|\det(D\Theta_{u,v})|$ is given by

$$J_{v,1}=\frac{H+v}{H+u}\quad\text{in}\quad\Omega_1(u)\,,\qquad J_{v,2}=1\quad\text{in}\quad\Omega_2(u)\,.\tag{2.8}$$

Introducing the notations

$$A(v):=\sigma\,J_v\,(D\Theta_{u,v})^{-1}(D\Theta_{u,v}^T)^{-1}$$

and

$$B(v):=\mathrm{div}\big(\sigma\,J_v\,(D\Theta_{u,v})^{-1}\nabla h_v\circ\Theta_{u,v}\big)\,,$$

we define the function

$$\mathcal{F}:\mathcal{S}_0\times H_0^1(\Omega(u))\to H^{-1}(\Omega(u))\,,\quad (v,\xi)\mapsto -\mathrm{div}\big(A(v)\nabla\xi\big)-B(v)$$

and observe that (2.7) is equivalent to

$$\mathcal{F}(v,\xi_v)=0\,,\quad v\in\mathcal{S}_0\,.\tag{2.9}$$

We then shall use the implicit function theorem to show that ξ_v depends smoothly on v. For that purpose, let us first show that $\mathcal{F}$ is Fréchet differentiable in $\mathcal{S}_0\times H_0^1(\Omega(u))$. Indeed, by (1.4), it is readily checked that

$$\nabla h_v\circ\Theta_{u,v}(x,z)=\mathbf{1}_{\Omega_2(u)}\zeta'\big(z-u(x)+1\big)\begin{pmatrix}-\partial_x v(x)\\ 1\end{pmatrix}$$

is linear in v, so that its Fréchet derivative with respect to v is

$$\partial_v\big(\nabla h_v\circ\Theta_{u,v}\big)[\vartheta](x,z)=\mathbf{1}_{\Omega_2(u)}\zeta'\big(z-u(x)+1\big)\begin{pmatrix}-\partial_x\vartheta(x)\\ 0\end{pmatrix}\tag{2.10}$$

for $\vartheta\in H^2(D)\cap H_0^1(D)$. Thus,

$$\big[v\mapsto\nabla h_v\circ\Theta_{u,v}\big]\in C^1\big(\mathcal{S}_0,L_2(\Omega(u),\mathbb{R}^2)\big)\,.$$

Moreover, $v\mapsto J_v$ and $v\mapsto(D\Theta_{u,v})^{-1}$ are continuously differentiable from $\mathcal{S}_0$ to $L_\infty(\Omega(u))$ and $L_\infty(\Omega(u),\mathbb{R}^{2\times 2})$, respectively, and we conclude that

$$v\mapsto\sigma\,J_v\,(D\Theta_{u,v})^{-1}\nabla h_v\circ\Theta_{u,v}$$

is continuously differentiable from $\mathcal{S}_0$ to $L_2(\Omega(u),\mathbb{R}^2)$. Hence

$$B\in C^1(\mathcal{S}_0,H^{-1}(\Omega(u))).$$

The C^1-smoothness of $(v,\xi)\mapsto\mathrm{div}(A(v)\nabla\xi)$ is proven as in [1, Theorem 5.3.2] and we have thus established that

$$\mathcal{F}\in C^1\big(\mathcal{S}_0\times H_0^1(\Omega(u)),H^{-1}(\Omega(u))\big)\,.$$

The Lax–Milgram theorem and the open mapping theorem imply that the mapping

$$\omega \mapsto \partial_\xi \mathcal{F}(u,\xi_u)[\omega] = -\mathrm{div}(\sigma\nabla\omega)$$

is an isomorphism from $H_0^1(\Omega(u))$ to $H^{-1}(\Omega(u))$. Consequently, by the implicit function theorem there is a neighborhood $\mathcal{W}$ of (u,ξ_u) in $\mathcal{S}_0 \times H_0^1(\Omega(u))$, a neighborhood $\mathcal{U}$ of u in $\mathcal{S}_0$, and a function $\Xi \in C^1(\mathcal{U}, H_0^1(\Omega(u)))$ with $\Xi(u) = \xi_u$ such that

$$\Big((v,\xi)\in\mathcal{W} \quad \text{with} \quad \mathcal{F}(v,\xi)=0\Big) \iff \Big(v\in\mathcal{U} \quad \text{and} \quad \xi = \Xi(v)\Big).$$

By (2.3), we may assume that $(v,\xi_v) \in \mathcal{W}$ for $v\in\mathcal{U}$. Hence, $\xi_v = \Xi(v)$ for $v\in\mathcal{U}$ and the proof is complete. $\square$

We next compute the Fréchet derivative of the electrostatic energy on $\mathcal{S}_0$ and thereby provide a proof of the first part of Theorem 2.1. This computation follows the classical approach developed in [1,6,7] for shape derivatives and is performed in a similar way in [4] for a geometry with a flat interface instead of $\Sigma(u)$. It is worth pointing out that the non-flat transmission interface considered herein leads to additional terms. The concise formula (1.8) that we derive for the derivative $g(u)$ of $E_\mathrm{e}(u)$ reveals the importance of these contributions to the electrostatic force, as it involves terms that counteract the contributions from the top of the elastic plate when $[\![\sigma]\!] > 0$. As the identification of these additional terms and the derivation of the concise formula (1.8) do not seem to be straightforward, we give a detailed proof (see also Remark 2.6 below).

Proposition 2.5. *Assume (1.4). The electrostatic energy $E_\mathrm{e} : \mathcal{S}_0 \to \mathbb{R}$ is continuously Fréchet differentiable with*

$$\partial_u E_\mathrm{e}(u)[\vartheta] = \int_D g(u)(x)\,\vartheta(x)\,\mathrm{d}x$$

for $u\in\mathcal{S}_0$ and $\vartheta \in H^2(D)\cap H_0^1(D)$, where $g(u)$ is defined in (1.8).

Proof. The proof is quite technical and basically includes three steps. As a starting point, we shall use Lemma 2.3 which guarantees the differentiability of E_e and yields an abstract formula for its derivative, see (2.11) below. Computing then this derivative explicitly, we first derive in (2.20) an expression involving only the trace of the gradient of ψ_u on the top of the elastic plate and the jumps of σ and the partial derivatives of ψ_u on the interface. Finally, we write the interface integrals in terms of $\psi_{u,2}$ only and thus obtain the desired formula (1.8).

To be more precise, we fix $u\in\mathcal{S}_0$ and use the notation introduced in Lemma 2.3. Recall that, according to Lemma 2.3, there is a neighborhood $\mathcal{U}$ of u in $\mathcal{S}_0$ such that the mapping

$$v \mapsto \xi_v = \big(\psi_v - h_v\big)\circ\Theta_{u,v}$$

belongs to $C^1(\mathcal{U}, H_0^1(\Omega(u)))$, the transformation $\Theta_{u,v} : \Omega(u) \to \Omega(v)$ being defined in (2.4). Now, for $v\in\mathcal{U}$, we use (2.6), the relation $\chi_v = \psi_v - h_v$, and the change of variable $(\bar{x},\bar{z}) = \Theta_{u,v}(x,z)$ in the integral defining $E_{\rm e}(v)$ to obtain

$$E_{\rm e}(v) = -\frac{1}{2}\int_{\Omega(v)} \sigma|\nabla\psi_v|^2\,{\rm d}(\bar{x},\bar{z}) = -\frac{1}{2}\int_{\Omega(u)} \sigma|j(v)|^2 J_v\,{\rm d}(x,z)\,,$$

where

$$j(v) := (D\Theta_{u,v}^T)^{-1}\nabla\xi_v + \nabla h_v\circ\Theta_{u,v}\,.$$

Owing to the differentiability of $v\mapsto\xi_v$ in $\mathcal{U}$, we deduce that the Fréchet derivative of $E_{\rm e}$ at u applied to some $\vartheta\in H^2(D)\cap H_0^1(D)$ is given by

$$\begin{aligned}\partial_u E_{\rm e}(u)[\vartheta] = \partial_v E_{\rm e}(v)[\vartheta]\big|_{v=u} = &-\int_{\Omega(u)} \sigma j(u)\cdot(\partial_v j(v))[\vartheta]\big|_{v=u}\,J_u\,{\rm d}(x,z)\\ &-\frac{1}{2}\int_{\Omega(u)} \sigma|j(u)|^2\,(\partial_v J_v)[\vartheta]\big|_{v=u}\,{\rm d}(x,z)\,.\end{aligned}$$

Taking the identity $j(u) = \nabla\chi_u + \nabla h_u = \nabla\psi_u$ into account, we infer from (2.8) that

$$\begin{aligned}\partial_u E_{\rm e}(u)[\vartheta] = &-\int_{\Omega(u)} \sigma\nabla\psi_u\cdot\big(\partial_v j(v)[\vartheta]\big|_{v=u}\big)\,{\rm d}(x,z)\\ &-\frac{1}{2}\int_{\Omega_1(u)} \sigma_1|\nabla\psi_{u,1}|^2\,\frac{\vartheta}{H+u}\,{\rm d}(x,z)\,.\end{aligned}\tag{2.11}$$

We next use that $\Theta_{u,u}$ is the identity on $\Omega(u)$ and that $\xi_u = \chi_u$ to compute from the definition of $j(v)$ that

$$\begin{aligned}\partial_v j(v)[\vartheta]\big|_{v=u} = &-\partial_v(D\Theta_{u,v}^T)[\vartheta]\big|_{v=u}\nabla\chi_u + \partial_v(\nabla\xi_v)[\vartheta]\big|_{v=u}\\ &+\partial_v(\nabla h_v\circ\Theta_{u,v})[\vartheta]\big|_{v=u}\,.\end{aligned}\tag{2.12}$$

Now, $\chi_{u,1} = \psi_{u,1}$ in $\Omega_1(u)$ due to (1.4), so that

$$-\partial_v(D\Theta_{u,v}^T)[\vartheta]\big|_{v=u}\nabla\chi_u = -\partial_z\psi_u\nabla\left(\frac{\vartheta(z+H)}{H+u}\right)\quad \text{in}\ \ \Omega_1(u)\,,\tag{2.13}$$

while

$$-\partial_v(D\Theta_{u,v}^T)[\vartheta]\big|_{v=u}\nabla\chi_u = -\begin{pmatrix}\partial_z\chi_u\partial_x\vartheta\\ 0\end{pmatrix}\quad \text{in}\ \ \Omega_2(u)\,.\tag{2.14}$$

Also note that

$$\partial_v(\nabla\xi_v)[\vartheta]\big|_{v=u} = \nabla\big(\partial_v\xi_v[\vartheta]\big|_{v=u}\big)\quad \text{in}\ \ \Omega(u)\,.\tag{2.15}$$

Consequently, gathering (2.11)–(2.15) and recalling (2.10) lead us to

$$\partial_u E_{\rm e}(u)[\vartheta] = I_0(u)[\vartheta] + I_1(u)[\vartheta] + I_2(u)[\vartheta]\,,\tag{2.16}$$

where

$$I_0(u)[\vartheta] := -\int_{\Omega(u)} \sigma\,\nabla\psi_u\cdot\nabla\big(\partial_v\xi_v[\vartheta]\big|_{v=u}\big)\,\mathrm{d}(x,z)\,,$$

$$\begin{aligned}I_1(u)[\vartheta] :=& \int_{\Omega_1(u)} \sigma_1\,\partial_z\psi_{u,1}\,\nabla\psi_{u,1}\cdot\nabla\left(\frac{\vartheta(z+H)}{H+u}\right)\mathrm{d}(x,z)\\ &-\frac{1}{2}\int_{\Omega_1(u)} \sigma_1\,|\nabla\psi_{u,1}|^2\,\frac{\vartheta}{H+u}\,\mathrm{d}(x,z)\,,\end{aligned}$$

and

$$\begin{aligned}I_2(u)[\vartheta] :=& \int_{\Omega_2(u)} \sigma_2\,\partial_x\psi_{u,2}\,\zeta'\big(z-u+1\big)\,\partial_x\vartheta\,\mathrm{d}(x,z)\\ &+\int_{\Omega_2(u)} \sigma_2\,\partial_x\psi_{u,2}\,\partial_z\chi_{u,2}\,\partial_x\vartheta\,\mathrm{d}(x,z)\,.\end{aligned}$$

We are left with simplifying these three integrals and begin with $I_0(u)[\vartheta]$. We use Gauß' theorem and (1.2a) to get

$$\begin{aligned}I_0(u)[\vartheta] =& -\int_{\partial\Omega(u)} \big(\partial_v\xi_v[\vartheta]\big|_{v=u}\big)\sigma\nabla\psi_u\cdot\mathbf{n}_{\partial\Omega(u)}\,\mathrm{d}S\\ &-\int_{\Sigma(u)} [\![\partial_v\xi_v[\vartheta]\big|_{v=u}\sigma\nabla\psi_u]\!]\cdot\mathbf{n}_{\Sigma(u)}\,\mathrm{d}S\,.\end{aligned}$$

Now, recall that $\partial_v\xi_v[\vartheta]\big|_{v=u}$ belongs to $H_0^1(\Omega(u))$ according to Lemma 2.3. On the one hand, this entails that $\partial_v\xi_v[\vartheta]\big|_{v=u}$ vanishes on $\partial\Omega(u)$, so that the first integral on the right-hand side of the above identity is zero. On the other hand, the H^1-regularity of $\partial_v\xi_v[\vartheta]\big|_{v=u}$ also implies that $[\![\partial_v\xi_v[\vartheta]\big|_{v=u}]\!]=0$ on $\Sigma(u)$, so that

$$[\![\partial_v\xi_v[\vartheta]\big|_{v=u}\sigma\nabla\psi_u]\!]\cdot\mathbf{n}_{\partial\Sigma(u)} = \partial_v\xi_v[\vartheta]\big|_{v=u}\,[\![\sigma\nabla\psi_u]\!]\cdot\mathbf{n}_{\Sigma(u)} = 0 \quad\text{on}\quad \Sigma(u)$$

due to (1.2b). Therefore,

$$I_0(u)[\vartheta] = 0\,. \tag{2.17}$$

We next deal with $I_1(u)[\vartheta]$. Since $\sigma_1\Delta\psi_{u,1} = \mathrm{div}(\sigma\nabla\psi_u) = 0$ in $\Omega_1(u)$ by (1.2a), it follows from Gauß' theorem that

$$\begin{aligned}I_1(u)[\vartheta] =& \int_{\Omega_1(u)} \sigma_1\,\partial_z\psi_{u,1}\,\mathrm{div}\left(\left(\frac{\vartheta(z+H)}{H+u}\right)\nabla\psi_{u,1}\right)\mathrm{d}(x,z)\\ &-\frac{1}{2}\int_{\Omega_1(u)} \sigma_1\,|\nabla\psi_{u,1}|^2\,\frac{\vartheta}{H+u}\,\mathrm{d}(x,z)\\ =& \int_{\partial\Omega_1(u)} \sigma_1\,\frac{\vartheta(z+H)}{H+u}\partial_z\psi_{u,1}\nabla\psi_{u,1}\cdot\mathbf{n}_{\partial\Omega_1(u)}\,\mathrm{d}S\\ &-\int_{\Omega_1(u)} \sigma_1\,\nabla\psi_{u,1}\cdot\nabla\left(\partial_z\psi_{u,1}\right)\frac{\vartheta(z+H)}{H+u}\,\mathrm{d}(x,z)\\ &-\frac{1}{2}\int_{\Omega_1(u)} \sigma_1\,|\nabla\psi_{u,1}|^2\,\frac{\vartheta}{H+u}\,\mathrm{d}(x,z)\,.\end{aligned}$$

Recalling that $\vartheta \in H_0^1(D)$ and noticing that

$$\nabla\psi_{u,1} \cdot \nabla\left(\partial_z\psi_{u,1}\right) = \partial_z\big(|\nabla\psi_{u,1}|^2\big)/2\,,$$

we further obtain

$$\begin{aligned}I_1(u)[\vartheta] = &\int_D \sigma_1\,\partial_z\psi_{u,1}(x,u(x))\big(-\partial_x u\partial_x\psi_{u,1} + \partial_z\psi_{u,1}\big)(x,u(x))\vartheta(x)\,\mathrm{d}x\\ &- \frac{1}{2}\int_D \sigma_1\,|\nabla\psi_{u,1}(x,u(x))|^2\vartheta(x)\,\mathrm{d}x\,.\end{aligned}$$

Hence,

$$\begin{aligned}I_1(u)[\vartheta] = &-\frac{1}{2}\int_D \sigma_1\,\left(|\partial_x\psi_{u,1}|^2 - |\partial_z\psi_{u,1}|^2\right)(x,u(x))\vartheta(x)\,\mathrm{d}x\\ &- \int_D \sigma_1\,\partial_x u(x)\big(\partial_x\psi_{u,1}\partial_z\psi_{u,1}\big)(x,u(x))\vartheta(x)\,\mathrm{d}x\,.\end{aligned} \tag{2.18}$$

Finally, using (1.4a), $\chi_u = \psi_u - h_u$, and $\vartheta \in H_0^1(D)$, it follows from Green's formula that

$$\begin{aligned}I_2(u)[\vartheta] &= \int_{\Omega_2(u)} \sigma_2\,\partial_x\psi_{u,2}\partial_z\psi_{u,2}\partial_x\vartheta\,\mathrm{d}(x,z)\\ &= -\int_D \sigma_2\,\big(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\big)(x,u(x)+d)\partial_x u(x)\,\mathrm{d}x\\ &\quad + \int_D \sigma_2\,\big(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\big)(x,u(x))\partial_x u(x)\,\mathrm{d}x\\ &\quad - \int_{\Omega_2(u)} \sigma_2\,\partial_x\big(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\big)\vartheta\,\mathrm{d}(x,z)\,.\end{aligned}$$

Owing to (1.2a), we have $\sigma_2\partial_x^2\psi_{u,2} = -\sigma_2\partial_z^2\psi_{u,2}$ in $\Omega_2(u)$ from which we deduce that

$$\begin{aligned}&\int_{\Omega_2(u)} \sigma_2\,\partial_x\big(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\big)\vartheta\,\mathrm{d}(x,z)\\ &\quad= \int_{\Omega_2(u)} \sigma_2\,\big(\partial_x^2\psi_{u,2}\partial_z\psi_{u,2} + \partial_x\psi_{u,2}\partial_x\partial_z\psi_{u,2}\big)\vartheta\,\mathrm{d}(x,z)\\ &\quad= \int_{\Omega_2(u)} \sigma_2\,\big(-\partial_z\psi_{u,2}\partial_z^2\psi_{u,2} + \partial_x\psi_{u,2}\partial_x\partial_z\psi_{u,2}\big)\vartheta\,\mathrm{d}(x,z)\\ &\quad= \frac{1}{2}\int_{\Omega_2(u)} \sigma_2\,\partial_z\big(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\big)\vartheta\,\mathrm{d}(x,z)\\ &\quad= \frac{1}{2}\int_D \sigma_2\,\big(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\big)(x,u(x)+d)\vartheta(x)\,\mathrm{d}x\\ &\qquad - \frac{1}{2}\int_D \sigma_2\,\big(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\big)(x,u(x))\vartheta(x)\,\mathrm{d}x\,.\end{aligned}$$

Consequently,

$$\begin{aligned} I_2(u)[\vartheta] = & -\int_D \sigma_2 \left(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\right)(x,u(x)+d)\partial_x u(x)\,\mathrm{d}x \\ & + \int_D \sigma_2 \left(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\right)(x,u(x))\partial_x u(x)\,\mathrm{d}x \\ & - \frac{1}{2}\int_D \sigma_2 \left(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\right)(x,u(x)+d)\vartheta(x)\,\mathrm{d}x \\ & + \frac{1}{2}\int_D \sigma_2 \left(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\right)(x,u(x))\vartheta(x)\,\mathrm{d}x\,. \end{aligned}$$

We finally note that

$$\partial_x\psi_{u,2}(x,u(x)+d)) = -\partial_x u(x)\partial_z\psi_{u,2}(x,u(x)+d)\,,$$

since $\psi_{u,2}(x,u(x)+d) = V$ owing to (1.2c) and (1.4b). This identity allows us to simplify further the formula for $I_2(u)[\vartheta]$, so that we end up with

$$\begin{aligned} I_2(u)[\vartheta] = & \frac{1}{2}\int_D \sigma_2\,|\nabla\psi_{u,2}(x,u(x)+d)|^2\,\mathrm{d}x \\ & + \int_D \sigma_2 \left(\partial_x\psi_{u,2}\partial_z\psi_{u,2}\right)(x,u(x))\partial_x u(x)\,\mathrm{d}x \\ & + \frac{1}{2}\int_D \sigma_2 \left(|\partial_x\psi_{u,2}|^2 - |\partial_z\psi_{u,2}|^2\right)(x,u(x))\vartheta(x)\,\mathrm{d}x\,. \end{aligned} \tag{2.19}$$

Collecting (2.16), (2.17), (2.18), and (2.19) gives

$$\begin{aligned} \partial_u E_{\mathrm{e}}(u)[\vartheta] = & -\frac{1}{2}\int_D [\![\sigma(\partial_x\psi_u)^2 - \sigma(\partial_z\psi_u)^2]\!](x,u(x))\,\vartheta(x)\,\mathrm{d}x \\ & - \int_D \partial_x u(x)\,[\![\sigma\partial_x\psi_u\partial_z\psi_u]\!](x,u(x))\,\vartheta(x)\,\mathrm{d}x \\ & + \frac{1}{2}\int_D \sigma_2\,\left|\nabla\psi_{u,2}(x,u(x)+d)\right|^2\vartheta(x)\,\mathrm{d}x\,. \end{aligned} \tag{2.20}$$

Finally, we shall write (2.20) only in terms of $\psi_{u,2}$. To this end, we set

$$F_u := \partial_x\psi_u + \partial_x u\partial_z\psi_u\,, \qquad G_u := -\partial_x u\partial_x\psi_u + \partial_z\psi_u\,, \tag{2.21}$$

and observe that differentiating the transmission condition $[\![\psi_u]\!] = 0$ on $\Sigma(u)$, along with the second transmission condition in (1.2b), ensures that

$$[\![F_u]\!] = [\![\sigma G_u]\!] = 0 \quad \text{on} \quad \Sigma(u)\,.$$

These properties in turn imply that

$$[\![\sigma F_u^2]\!] = [\![\sigma]\!]F_{u,2}^2\,, \quad [\![\sigma F_u G_u]\!] = 0\,, \qquad [\![\sigma G_u^2]\!] = [\![\frac{1}{\sigma}]\!]\sigma_2^2 G_{u,2}^2 \quad \text{on} \quad \Sigma(u)\,. \tag{2.22}$$

Guided by (2.22), we next express the jump terms in (2.20) using F_u and G_u. Since

$$\left[1+(\partial_x u)^2\right]\partial_x\psi_u = F_u - G_u\partial_x u \quad \text{and} \quad \left[1+(\partial_x u)^2\right]\partial_z\psi_u = F_u\partial_x u + G_u\,,$$

we compute

$$\begin{aligned}&\left[1+(\partial_x u)^2\right]^2\left[(\partial_x\psi_u)^2-(\partial_z\psi_u)^2+2\partial_x u\partial_x\psi_u\partial_z\psi_u\right]\\&\quad=(F_u-G_u\partial_x u)^2-(F_u\partial_x u+G_u)^2+2\partial_x u(F_u-G_u\partial_x u)(F_u\partial_x u+G_u)\\&\quad=\left[1+(\partial_x u)^2\right]\left(F_u^2-2F_uG_u\partial_x u-G_u^2\right)\,.\end{aligned}$$

Therefore, by (2.22),

$$\begin{aligned}&\left[1+(\partial_x u)^2\right][\![\sigma(\partial_x\psi_u)^2-\sigma(\partial_z\psi_u)^2+2\sigma\partial_x u\partial_x\psi_u\partial_z\psi_u]\!]\\&\quad=[\![\sigma F_u^2-2\sigma F_uG_u\partial_x u-\sigma G_u^2]\!]=[\![\sigma]\!]F_{u,2}^2-[\![\frac{1}{\sigma}]\!]\sigma_2^2G_{u,2}^2\\&\quad=[\![\sigma]\!]F_{u,2}^2+[\![\sigma]\!]\frac{\sigma_2}{\sigma_1}G_{u,2}^2\,.\end{aligned}$$

Consequently, plugging this formula into (2.20) and recalling (2.21) yield

$$\begin{aligned}&\partial_u E_{\mathrm{e}}(u)[\vartheta]\\&\quad=-\frac{[\![\sigma]\!]}{2}\int_D\frac{1}{1+(\partial_x u(x))^2}\big(\partial_x\psi_{u,2}+\partial_x u(x)\partial_z\psi_{u,2}\big)^2(x,u(x))\,\vartheta(x)\,\mathrm{d}x\\&\qquad-\frac{[\![\sigma]\!]\sigma_2}{2\sigma_1}\int_D\frac{1}{1+(\partial_x u(x))^2}\big(\partial_x u(x)\partial_x\psi_{u,2}-\partial_z\psi_{u,2}\big)^2(x,u(x))\,\vartheta(x)\,\mathrm{d}x\\&\qquad+\frac{1}{2}\int_D\sigma_2\left|\nabla\psi_{u,2}(x,u(x)+d)\right|^2\vartheta(x)\,\mathrm{d}x\,;\end{aligned}$$

that is,

$$\partial_u E_{\mathrm{e}}(u)[\vartheta]=\int_D g(u)(x)\,\vartheta(x)\,\mathrm{d}x$$

for $u\in\mathcal{S}_0$ and $\vartheta\in H^2(D)\cap H_0^1(D)$ with $g(u)$ being defined in (1.8). It then readily follows from (2.1) that

$$\partial_u E_{\mathrm{e}}:\mathcal{S}_0\to\mathcal{L}\big(H^2(D)\cap H_0^1(D),\mathbb{R}\big)$$

is continuous. □

Remark 2.6. Compared to the proof of [4, Proposition 4.2], the main difference in the proof of Proposition 2.5 is the term $I_1(u)$ stemming from the non-flatness of the interface $\Sigma(u)$. Additionally, even though the terms $I_0(u)$ and $I_2(u)$ already appear in the flat geometry considered in [4, Proposition 4.2], they give herein different contributions to $g(u)$ due to the specific choice (1.4) of the boundary values (1.2c).

The final step of the proof of Theorem 2.1 is to show that the electrostatic energy E_{e} admits directional derivatives in the directions $-u+\mathcal{S}_0$.

Corollary 2.7. *Assume (1.4). Let $u_0\in\bar{\mathcal{S}}_0$ and $u_1\in\mathcal{S}_0$. Then*

$$\lim_{t\to 0^+}\frac{1}{t}\left[E_{\mathrm{e}}(u_0+t(u_1-u_0))-E_{\mathrm{e}}(u_0)\right]=\int_D g(u_0)(x)\,(u_1-u_0)(x)\,\mathrm{d}x\,.$$

Moreover, the function $g:\bar{\mathcal{S}}_0\to L_p(D)$ is continuous for each $p\in[1,\infty)$.

Proof. The stated continuity of g is a straightforward consequence of (2.1). Next, given $u_0 \in \bar{\mathcal{S}}_0$ and $u_1 \in \mathcal{S}_0$, we set

$$u_s := u_0 + s(u_1 - u_0) = (1-s)u_0 + su_1 \in \mathcal{S}_0\,, \qquad s \in (0,1]\,.$$

Since $u_s \in \mathcal{S}_0$ for $s \in (0,1]$, we deduce from Proposition 2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{e}}(u_s) = \int_D g(u_s)(x)\,(u_1 - u_0)(x)\,\mathrm{d}x\,, \qquad s \in (0,1]\,. \tag{2.23}$$

Therefore, letting $s \to 0$, the continuity of g entails

$$\lim_{s\to 0^+} \frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{e}}(u_s) = \int_D g(u_0)(x)\,(u_1 - u_0)(x)\,\mathrm{d}x\,. \tag{2.24}$$

Now (2.2) guarantees that $E_{\mathrm{e}}(u_s) \to E_{\mathrm{e}}(u_0)$ as $s \to 0$, so that

$$E_{\mathrm{e}}(u_t) - E_{\mathrm{e}}(u_0) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{e}}(u_s)\,\mathrm{d}s\,, \quad t \in (0,1]\,, \tag{2.25}$$

and we conclude from (2.24) that

$$\begin{aligned}\lim_{t\to 0^+} \frac{1}{t}\big(E_{\mathrm{e}}(u_t) - E_{\mathrm{e}}(u_0)\big) &= \lim_{t\to 0^+} \frac{1}{t} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{e}}(u_s)\,\mathrm{d}s \\ &= \int_D g(u_0)(x)\,(u_1 - u_0)(x)\,\mathrm{d}x\end{aligned}$$

as claimed. □

If $[\![\sigma]\!] < 0$, then an obvious consequence of (1.8) is that g is non-negative on $\bar{\mathcal{S}}_0$. This yields the monotonicity of the electrostatic energy E_{e}.

Corollary 2.8. *Assume $[\![\sigma]\!] < 0$ and (1.4). If $u_0 \in \bar{\mathcal{S}}_0$ and $u_1 \in \mathcal{S}_0$ are such that $u_0 \le u_1$ in D, then $E_{\mathrm{e}}(u_0) \le E_{\mathrm{e}}(u_1)$.*

Proof. The assumption $[\![\sigma]\!] < 0$ implies that $g(u_s) \ge 0$ for $s \in (0,1]$ according to (1.8), where $u_s = (1-s)u_0 + su_1$ as in the proof of Corollary 2.7. Hence, (2.23) and (2.25) with $t = 1$ imply the assertion. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 now follows from Theorem 2.1 as in [2]. Indeed, Theorem 2.1 guarantees that any minimizer of the total energy E on $\bar{\mathcal{S}}_0$ satisfies the Euler–Lagrange equation (1.7). In case that $a > 0$, the total energy E is coercive and thus the existence of a minimizer of E in $\bar{\mathcal{S}}_0$ can be shown as in [2, Section 7]. In the more complex case $a = 0$, the total energy E need not be coercive. But, as pointed out in the introduction, one may enforce its coercivity by adding a penalizing term and proceed along the lines of [2, Section 6], recalling that the assumption $[\![\sigma]\!] < 0$ guarantees that $g(u) \ge 0$ in D, which is essential in this case (see, in particular, [2, Equation (6.4)]).

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