Hyperkähler structure of bow varieties

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Kurzzusammenfassung

In dieser Dissertation untersuchen wir Cherkis Bogenvarietäten und deren Beschreibung als lineare Flüsse auf der Jacobischen Varietät einer bestimmten Spektralkurve. Wir beschreiben explizit die Bogenvarietät eines deformierten Instanton-Modulraums über der Taub-NUT-Mannigfaltigkeit, das heißt wir beschreiben die Bogenvarietät, die aus einem Pfeil und einem Intervall mit r λ-Punkte besteht, und finden eine spektrale Darstellung in Form von Bedingungen an spezielle Divisoren. Wir finden eine asymptotische Metrik für diese Bogenvarietät, indem wir mittels Methoden aus der Twistortheorie einen Modellraum konstruieren und zeigen, dass die zugehörige Metrik asymptotisch nah an der eigentlichen Metrik der Bogenvarietät liegt.

Abstract

In this thesis we study Cherkis bow varieties and its description in terms of linear flows on the Jacobian variety of certain spectral curve. We describe explicitly the bow variety of a deformed instanton moduli space over Taub-NUT, i.e. the bow variety consisting of one arrow and interval with r λ-points, and find a spectral description in terms of conditions on certain divisors. We find an asymptotic metric for the bow variety by constructing a model space using twistor methods and showing that the corresponding metric is asymptotically close to the one of the bow variety.

Keywords

Hyperkähler geometry, bow varieties, Nahm equations, moduli space, instanons, asymptotic metric

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Contents

1 Introduction

Sergey Cherkis introduced a method of constructing complex varieties, the so called *bow varieties*, in [Che09], [Che10] and [Che11]. These are moduli spaces of solutions to Nahm equations with certain boundary conditions which carry a hyperkähler structure. The idea of the construction of bow varieties is the same as for quiver varieties. In fact, a bow can be seen as a generalization of quiver in the sense that all vertices of a quiver are replaced by intervals of certain length. If the length of all those intervals goes to zero, we obtain the original quiver back.

The bow variety can be obtained from a certain bow by choosing a representation, i.e. defining some data. A hermitian vector bundle is associated to every interval contained in the bow. These bundles carry Nahm data, meaning there are $\mathfrak{u}(k)$-valued functions T_i defined on every hermitian bundle which satisfy Nahm equations

$$\frac{\mathrm{d}}{\mathrm{d}s}T_i(s) - [T_0(s), T_i(s)] = \frac{1}{2}\epsilon_{ijk}[T_j(s), T_k(s)],$$

for $i, j, k \in \{1, 2, 3\}$ and ϵ_{ijk} being the Levi-Civita symbol. These functions may have some rank 1 jumps on the interval. The arrows between the intervals yield data in form of homomorphisms between the fibres over the boundary points of the interval. This space of data, denoted by $\mathrm{Dat}(\mathcal{R})$, carries a natural action of a gauge group $\mathcal{G}$ given by $U(n)$-valued functions. The bow variety is then given as the hyperkähler quotient of the space of data with the gauge group action

$$\mathcal{M} := \mathrm{Dat}(\mathcal{R})/\!\!/\!\!/\mathcal{G}.$$

It is well known that solutions to Nahm equations on the line can be used to describe monopole spaces. This was done by Nahm [Nah81] and has been put into a larger picture by Hitchin [Hit83].

In [Che11] Cherkis identified some basic bow varieties with ALF-spaces and the varieties of some more complicated representations with moduli spaces of instantons on these ALF-spaces.

There is another correspondence which is necessary to appropriately describe bow varieties. This correspondence is between bow varieties and spectral data. A spectral curve is an algebraic curve S embedded in the total space $|\mathcal{O}_{\mathbb{CP}^1}(2)|$ of the bundle $\mathcal{O}(2)$ over the complex projective space $\mathbb{CP}^1$. It is given by the vanishing of the characteristic polynomial of some matrix valued polynomial on $\mathbb{CP}^1$:

$$S = \{(\eta, \zeta) \in |\mathcal{O}(2)| : \det(\eta\mathrm{Id} - A(\zeta)) = 0\}.$$

Nahm equations define such a spectral curve in a natural way. They can be written as a Lax pair equation

$$\frac{\mathrm{d}}{\mathrm{d}s}A(s,\zeta) = [A(s,\zeta), B(s,\zeta)]$$

where

$$\begin{aligned} A(s,\zeta) &= (T_2(s) + iT_3(s)) + 2iT_1(s)\zeta + (T_2(s) - iT_3(s))\zeta^2, \\ B(\zeta,s) &= iT_1(s) + \zeta(T_2(s) - iT_3(s)). \end{aligned}$$

This polynomial in ζ now defines a spectral curve which is independent of the value of the parameter s. This is because Nahm equations are isospectral.

For a smooth spectral curve S there is an important result of Beauville [Bea90]: The quotient by $PGL_k(\mathbb{C})$ of the variety of matrix valued polynomials in ζ whose characteristic polynomial $P(\eta,\zeta)$ is fixed is in 1-1 correspondence with the Jacobian variety $\mathrm{Jac}^{g-1}(S)\backslash\Theta$ of the spectral curve S in $|\mathcal{O}(2)|$ cut out by the equation

$$P(\eta,\zeta) := \det(\eta\mathrm{Id} - A(\zeta)) = 0$$

without the theta divisor Θ.

The Jacobian variety $\mathrm{Jac}^{g-1}(S)$ of the spectral curve S is a variety consisting of equivalence classes of line bundles on S with a fixed degree d and the canonical theta divisor Θ is the set of classes of line bundles which have a non trivial global section.

Given this correspondence, it can be shown (see [Hit83]) that solutions to Nahm equations, i.e. solutions to Lax pair equations, are in 1-1 correspondence with linear flows on the Jacobian $\mathrm{Jac}^{g-1}(S)\backslash\Theta$, i.e. they belong to a linear flow of line bundles $L \mapsto L \otimes L^s$ where L^s is of degree zero.

In this work we will explicitly describe the bow variety which arises from the simple quiver consisting of a single point together with a single arrow leading from the point to itself. The complexity will lie in the fact that we allow the solutions to Nahm equations to have several rank 1 jumps on the interval. Moreover, we will describe this bow variety in terms of spectral data, which will need some more methods to translate all the boundary conditions into the language of spectral curves.

In addition to that, we are interested in the instanton moduli space that arises from this bow variety. There are some examples of instanton moduli spaces that look similar to our case, but are more restrictively (see [Nak99], [Tak15], [Nek98] and also [CLHS21a] and [CLHS21b]). [CH19] and [CH21] give a description of how to relate bows and bow varieties with instantons and how to define a classification for them. Further, there is an interpretation of bow varieties in terms of Coulomb branches of the underlying quivers in [NT17], which also fits into the construction in [CH21] describing the same manifold as a Higgs branch.

Bielawski [Bie99] already gave a construction of an asymptotic metric for $SU(n)$ monopole spaces by building a model space using twistor methods. Therefore, we will construct an asymptotic metric for the bow variety in a similar way.

This work is structured the following way:

In chapter 2 we will recall some basic concepts and introduce notations for later purposes. These are all well known. We will describe the construction of hyperkähler quotients according to [HKLR87], because bow varieties are given as those quotients. Further, we will repeat some theory about sheaves, line bundles and divisors, because this will be needed when describing the spectral picture of bow varieties. Lastly, we will recall briefly the definition of spectral curves and their Jacobian varieties and, thus, we will already give an example for the spectral curve corresponding to Nahm equations of rank 2.

Chapter 3 will focus on the introduction of the object of main interest for this work. We will define bow diagrams and explain how bow varieties arise from them. By doing so, we will mostly follow the description in [Che11]. We will explain how bow data can be defined over a bow diagram and give a description of the gauge group action. Eventually, we will explicitly calculate the moment maps and their conditions and describe the bow variety as a hyperkähler quotient.

This chapter further explains Nahm equations. This is due to the fact that the essential point in writing down a bow variety is solving Nahm equations. We will show how Nahm equations are related to Yang-Mills equations and recall some known solutions for rank one and two. Furthermore, we will introduce some different forms of expressing Nahm equations, which will be used later in this work. One is the complex form, where two $\mathfrak{u}(k)$-valued functions each are conbined to $\mathfrak{gl}_k(\mathbb{C})$-valued functions and we obtain a complex and a real equation. The other one are the so called Basu-Harvey-Terashima equations. They are the lift of Nahm equations to their double cover. This form is inspired by the work in [Bie15]. At the end of the chapter we will use the Basu-Harvey-Terashima equations to illustrate our result that higher order representations of the bow diagrams in [Che11], which lead to A_k-ALF spaces, do not contain any more information. In fact, if we choose a representation of rank n instead of rank 1, we obtain the n-th product of the corresponding ALF space.

In chapter 4 we will describe the spectral picture of bow varieties. Therefore, we will state the Beauville correspondence [Bea90] and give a sketch of the proof. Then we will show, that we can assign linear flows on the Jacobian of the spectral curve which do not lie in the theta divisor to solutions of Nahm equations. Further, we have an additional real structure on spectral curves that arises from Nahm data, which will be explained. At the end of the chapter we will discuss how the rank 1 jump of solutions to Nahm equations translate into a condition on the spectral curve. In fact, we will find, following [HM89], that it corresponds to conditions on certain divisors on the involved spectral curves.

Chapter 5 finally yields the main results of this work. An explicit example of a bow variety will be discussed, namely a one given by a single interval together with a single arrow, which has a representation in terms of rank k Nahm equations which have r rank 1 jumps on the interval. We will describe this bow diagram with all the boundary conditions that arise in this case and translate it to the spectral curve to give a full description of that viewpoint.

Further, we recall the method of [Bie99] to construct a model space for the asymptotic metric and adapt it to our variety. Eventually, we will use an ansatz with the Gibbons-Manton metric and show that it is asymptotically close to the metric on the bow variety. There is also an article about this bow variety by Roger Bielawski, Sergey Cherkis and the author of this thesis [BBC22]

2 Preliminaries

2.1 Hyperkähler quotients

In this section we want to recall some basic properties about hyperkähler manifolds and discuss a method for constructing new hyperkähler manifolds the so called hyperkähler quotient invented by Hitchin et al [HKLR87]. This construction will be of great importance as all bow varieties will arise as such quotients. Finally, we will give an explicit example of how the Taub-NUT space can be seen as a hyperkähler quotient. This example was studied together with many others in [GRG97], but nonetheless it is worth studying it here, because on the one hand it is a quite simple example to demonstrate how the quotient construction works and on the other hand it will be the basis of an important class bow varieties, namely the ones that describe instantons on the Taub-NUT space.

2.1.1 Hyperkähler manifolds

The first thing we need to do is to define Kähler and especially hyperkähler manifolds and discuss some of their properties.

Definition 2.1. Let (M, J) be a complex manifold of real dimension $2n$ and let g be a hermitian metric on it. M is called a *Kähler manifold*, if the associated fundamental 2-form of ω of g (i.e. $\omega(-,-) = g(-, J-)$) is closed. ω is called the *Kähler form*.

Alternatively, one could define a Kähler manifold as a symplectic manifold with an integrable almost complex structure that is compatible with the symplectic form.

Proposition 2.2. *(M, J, ω) is a Kähler manifold if and only if the fundamental form ω has locally a potential K (called the* Kähler potential*) such that*

$$\omega = i\partial\bar{\partial}K.$$

Definition 2.3. A *hyperkähler manifold* is the tuple (M, g, I, J, K) where M is a real $4n$ dimensional manifold, g is a Riemannian metric and I, J and K are complex structures such that all three triples (M, g, I), (M, g, J) and (M, g, K) are Kähler manifolds and the three complex structures satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}.$$

Now, we obtain three Kähler forms named after the respective complex strucures ω_I, ω_J and ω_K and also three Kähler potentials K_I, K_J and K_K.

As all hyperkähler manifolds must be of dimension $4n$ the first example is the quaternionic space $\mathbb{H}$ itself. We can identify $\mathbb{H}$ with $\mathbb{R}^4$ with the standard metric and find local (matrix) representations for the three complex structures, e.g. the following

$$I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2.1.2 Moment maps

A basic ingredient for defining hyperkähler quotients is the so called moment map. This is a concept which does not require hyperkähler manifolds but only symplectic manifolds.

So, let (M, ω) be a symplectic manifold and G be a Lie group acting smoothly and by symplectomorphisms on M, i.e. the symplectic form ω is preserved by the action $\sigma : G \to \mathrm{Diff}(M)$:

$$\sigma(g)^* \omega = \omega, \quad \forall g \in G.$$

Now, let $X \in \mathfrak{g}$ be an element of the Lie algebra of G. Then $\exp(tX)$ is a smooth path in the Lie group G and the action of G on M gives us the fundamental vector field $X^\#$ corresponding to X defined by

$$X^\#\big|_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sigma(\exp(tX)) \cdot p$$

describing the infinitesimal action of X. As G acts via symplectomorphisms, we have $\mathcal{L}_{X^\#}\omega = 0$ and therefore Cartan's formula yields

$$0 = \mathrm{d}(\iota_{X^\#}\omega) + \iota_{X^\#}\mathrm{d}\omega = \mathrm{d}(\iota_{X^\#}\omega).$$

Here $\iota_{X^\#}\omega$ is the contraction of ω with $X^\#$, i.e. $\iota_{X^\#}\omega(-) = \omega(X^\#, -)$. The second term above vanishes because as a symplectic form ω is closed and thus $\iota_{X^\#}\omega$ is a closed 1-form. If $\iota_{X^\#}\omega$ is also exact for every $X \in \mathfrak{g}$ then we obtain a smooth function μ^X on M such that $\mathrm{d}\mu^X = \iota_{X^\#}\omega$. This function is called a *Hamiltonian function* and it is unique up to an additive constant. Such a set of Hamiltonian functions induces a map $\mu : M \to \mathfrak{g}^*$ from the manifold in the dual of the Lie algebra by

$$p \mapsto \mu_p(X\big|_p)$$

where we define

$$\mu_p(X\big|_p) := \mu^X(p).$$

μ is called a *moment map*, if it is G-equivariant in terms of the coadjoint action of G on $\mathfrak{g}^*$, i.e. $\mathrm{Ad}(g^{-1})^*\mu_p = \mu_{\sigma(g)\cdot p}$. This is always the case, if G is semi-simple or compact [HKLR87].

If a moment map exists (i.e. if $\iota_{X^\#}\omega$ is exact for all $X \in \mathfrak{g}$ and μ is G-equivariant), then the action of G on M is called *Hamiltonian*.

2.1.3 Hyperkähler quotients

It is well known that for a compact Lie group G acting smoothly and freely on a manifold M the quotient M/G is again a manifold such that the projection $\pi : M \to M/G$ is a smooth submersion. In this section we define a special case of quotient, namely the hyperkähler quotient, and show under which circumstances they inherit properties like being a manifold or being hyperkähler from the original space.

So let us consider a hyperkähler manifold (M, g, I, J, K) and a compact Lie group G acting smoothly, freely and by isometries on M and suppose the action to be tri-hamiltonian. As Kähler manifolds are also symplectic manifolds this means that there exists a moment map for each of the three Kähler manifolds (M, g, I), (M, g, J) and (M, g, K) which we denote by μ_I, μ_J and μ_K. We can write this in a single moment map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3.$$

Let us suppose that $c = (c_I, c_J, c_K)$ is a regular value of μ. Then we define the *hyperkähler quotient* as the following quotient:

$$M/\!/\!/G := \mu^{-1}(c)/G = \left(\mu_I^{-1}(c_I) \cap \mu_J^{-1}(c_J) \cap \mu_K^{-1}(c_K)\right)/G.$$

We have the following theorem from Hitchin et. al. [HKLR87]:

Theorem 2.4. *Let (M, g, I, J, K) be a hyperkähler manifold of real dimension $4n$ and G a compact Lie group acting on M as above. Then the hyperkähler quotient $M/\!/\!/G$ is again a manifold of dimension $\dim(M) - 4\dim(G)$ and it carries a hyperkähler structure.*

The idea of the proof is the following. For convenience we choose the level set $c = 0$. We start with a Kähler manifold (M, g, I). We have the $\dim(M) - \dim(G)$ dimensional submanifold $\mu^{-1}(0)$ of M. Since we chose μ to be equivariant, G acts freely on $\mu^{-1}(0)$ and we can write down the quotient manifold $\mu^{-1}(0)/G$ of dimension $\dim(M) - 2\dim(G)$. We have a natural 2-form ϱ on the quotient given by the symplectic form on $\mu^{-1}(0)$. Explicitly, if $\tilde{X}_1, \tilde{X}_2 \in \Gamma(T\mu^{-1}(0))$ and $X_1, X_2 \in \Gamma(T(\mu^{-1}(0)/G))$ such that the projection sends $\tilde{X}_i$ to X_i, the 2-form is given by $\varrho(X_1, X_2) = \omega(\tilde{X}_1, \tilde{X}_2)$. One can show (cf. [HKLR87]) that this defines a symplectic form on the quotient.

To show that the quotient is a Kähler manifold we need to show that the induced metric on the quotient is again Kähler. The idea for this is to split the tangent bundle of $\mu^{-1}(0)$ into a horizontal and a vertical subbundle. Hitchin shows [HKLR87] that the Levi-Civita connection of the quotient metric on $\mu^{-1}(0)/G$ lifts to a connection on the horizontal subbundle of $T\mu^{-1}(0)$, which is the orthogonal projection of the Levi-Civita connection on $T\mu^{-1}(0)$. The orthogonal subbundle H is a complex vector bundle, which can be seen by showing explicitly that its orthogonal complement in TM is complex. Therefore, the complex structure I commutes with the Levi-Civita connection and with the orthogonal projection (as g is a Kähler metric and H is a complex vector bundle) and therefore it commutes with the Levi-Civita connection on the quotient, what turns the quotient into a Kähler manifold.

This proof extends to hyperkähler quotients the following way: We define the complex function $\mu_+ = \mu_2 + i\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}$, such that $\mathrm{d}\mu_+^X(Y) = \omega_2(X,Y) + i\omega_3(X,Y)$ satisfies $\mathrm{d}\mu_+^X(IY) = i\mathrm{d}\mu_+^X(Y)$ for all vector fields Y. μ_+^X is a holomorphic function and hence $\mu_+^{-1}(0)$ is a complex submanifold of M with respect to the complex structure I and the induced metric is a Kähler metric. G acts on $\mu_+^{-1}(0)$ preserving the Kähler form and the moment map of that action is the restriction of the moment map μ_1 to $\mu_+^{-1}(0)$. This means by the argument above for Kähler quotients that the quotient metric on $(\mu_+^{-1}(0) \cap \mu_1^{-1}(0))/G$ is again a Kähler with respect to the complex structure I. By repeating this procedure for the complex structures J and K, we obtain a Kähler structure with respect to J and K on the quotient and therefore a hyperkähler structure. The space $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ is the intersection of the level sets of three moment maps, each of them giving $\dim G$ independent constraints (as the complex structures are linearly independent). Thus, the dimension of the hyperkähler quotient $M/\!\!/\!\!/G$ is $\dim(M) - 3\dim(G) - \dim(G) = \dim(M) - 4\dim(G)$.

2.1.4 Example: The Taub-NUT space

To complete this section we want to consider an explicit example to see how the hyperkähler quotient construction works. For this purpose the Taub-NUT space is an easy example and it is an important basis for describing bow varieties. This example together with many others was discussed in [GRG97] and we follow the calculation there.

We start with the product manifold

$$M = \mathbb{H} \times \mathbb{H}.$$

There are different ways to express quaternions. We choose coordinates $(y, \mathbf{y}, \psi, \mathbf{r})$ on M, such that for each $(q, w) \in \mathbb{H} \times \mathbb{H}$ we have $w = (y + \mathbf{y})$ and $q = (\psi, \mathbf{r})$ with $\mathbf{r} = qi\bar{q}$, $\psi \in [0, 2\pi]$ and $\mathbf{y} = \frac{1}{2}(w - \bar{w})$ with $y \in \mathbb{R}$. This choice of coordinates needs some explanation.

While w is the standard representation of a quaternion (y is the real part whereas $\mathbf{y}$ is the 3-dimensional imaginary part) the choice of coordinates q comes from the fact that we have a $U(1)$ action $q \mapsto qe^{it}$, $t \in (0, 2\pi]$ of right multiplications on $\mathbb{H}$ which is free away from the origin and preserves the hyperkähler structure $-\frac{1}{2}\mathrm{d}q \wedge \mathrm{d}\bar{q}$ and hence induces a moment map $\mu = \frac{1}{2}qi\bar{q}$ whose orbits are $\mathbb{R}^3$. This moment map induces a Riemannian submersion $\mathbb{R}^4\backslash\{0\} \to \mathbb{R}^3\backslash\{0\}$ whose fibres are circles S^1. The choice of coordinates $q = (\psi, \mathbf{r})$ is adapted to that submersion. In these coordinates the flat metric $\mathrm{d}q\mathrm{d}\bar{q}$ on $\mathbb{H}$ reads $\frac{1}{4}\left(\frac{1}{r}\mathrm{d}\mathbf{r}^2 + r(\mathrm{d}\psi + \langle\omega, \mathrm{d}\mathbf{r}\rangle)^2\right)$. Hence the flat product metric on $\mathbb{H} \times \mathbb{H}$ is

$$\mathrm{d}s^2 = \frac{1}{4}\left(\frac{1}{r}\mathrm{d}\mathbf{r}^2 + r(\mathrm{d}\psi + \langle\omega, \mathrm{d}\mathbf{r}\rangle)^2\right) + \mathrm{d}y^2 + \mathrm{d}\mathbf{y}^2.$$

Here, $r = ||\mathbf{r}||$ is the standard norm on $\mathbb{R}^3$, $\langle -, - \rangle$ is the standard inner product on $\mathbb{R}^3$ and $\mathrm{rot}(\omega) = \frac{1}{r}$ is the rotation operator on $\mathbb{R}^3$.

We now want to construct the Taub-NUT space as a hyperkähler quotient from this. The hyperkähler structure of $\mathbb{H}$ is invariant under right multiplications as above and also under real translations. Therefore, we have the following action on M. Let $\lambda \in \mathbb{R}$ be fixed. We have an $\mathbb{R}$-action on $\mathbb{H} \times \mathbb{H}$ given by

$$(q, w) \mapsto (qe^{it}, w - \lambda t).$$

The moment map for this action is

$$\mu = \frac{1}{2}qi\bar{q} + \frac{\lambda}{2}(w - \bar{w}) = \frac{1}{2}\mathbf{r} + \lambda\mathbf{y}.$$

The level set $\mu^{-1}(0) \in M$ is therefore defined by the equation $\mathbf{y} = -\frac{1}{2\lambda}\mathbf{r}$. Further, writing the action in terms of the coordinates $(y, \mathbf{y}, \psi, \mathbf{r})$ on M we have $(\psi, y) \mapsto (\psi + 2t, y - \lambda t)$, which leaves $\tau := \psi + \frac{2y}{\lambda}$ invariant. Chosing τ as a new coordinate we can write the induced metric on $\mu^{-1}(0)$ as

$$\mathrm{d}s^2 = \frac{1}{4}\left(\frac{1}{r}\mathrm{d}\mathbf{r}^2 + r(\mathrm{d}\tau - \frac{2}{\lambda}\mathrm{d}y + \langle\omega, \mathrm{d}\mathbf{r}\rangle)^2\right) + \mathrm{d}y^2 + \frac{1}{2\lambda}\mathrm{d}\mathbf{r}^2.$$

We obtain the hyperkähler quotient now by taking the quotient of $\mu^{-1}(0)$ with the $\mathbb{R}$-action. As the $\mathbb{R}$-action is now only a translation in y, we obtain the quotient metric on $\mu^{-1}(0)/\mathbb{R}$ by the projection of the metric orthogonal to $\frac{\partial}{\partial y}$ which is

$$\mathrm{d}s^2 = \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)\mathrm{d}\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1}(\mathrm{d}\tau + \langle\omega, \mathrm{d}\mathbf{r}\rangle)^2,$$

which is the standard metric on the Taub-NUT space.

2.2 Sheaves and line bundles

2.2.1 Sheaves and sheaf cohomology

In this section we repeat the definitions of sheaves and give a cohomology theory for them. All of these constructions can be found in [GH78] and [Har77].

Definition 2.5. Let $(X, \mathcal{T})$ be a topological space. A *presheaf of (abelian) groups* on X is a collection denoted by $\mathcal{F}$ of the following data:

- an abelian group $\mathcal{F}(U)$ for every open subset $U \in \mathcal{T}$,
- a group homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subseteq U$ of open subsets.

Those *restriction homomorphisms* must satisfy the following conditions:

1. $\mathcal{F}_{UU} = \mathrm{Id}$,
2. $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every inclusion of open sets $W \subseteq V \subseteq U$.

The group $\mathcal{F}(U)$ is the space of sections on U, i.e. an element $s \in \mathcal{F}(U)$ is a local section, whereas an element $s \in \mathcal{F}(X)$ is a global section. We introduce the following notation of the restriction map: Let $V \subseteq U$ be open sets and let $s \in \mathcal{F}(U)$. We set $s|_V = \rho_{UV}(s)$.

Definition 2.6. A *sheaf* on X is a presheaf on X satisfying the following conditions:

1. *Locality:* Let $U, V \in \mathcal{T}$ be open sets and let $s \in \mathcal{F}(U \cup V)$ be a local section. If
$$s|_U = 0 = s|_V$$
then $s = 0$.
2. *Gluing property:* Let $U, V \in \mathcal{T}$ be open sets and let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ be local sections. If
$$s|_{U \cap V} = t|_{U \cap V}$$
then there is a local section $m \in \mathcal{F}(U \cup V)$ such that
$$m|_U = s, \quad m|_V = t.$$

There are many examples of sheaves. Here we want to give the ones used later in this work. Let M be a Riemann surface or more generally a complex manifold. Then we have the following sheaves

- $\mathbb{Z}$ is the sheaf is locally constant functions with values in $\mathbb{Z}$,
- $\mathbb{C}$ is the sheaf is locally constant functions with values in $\mathbb{C}$,
- $\mathcal{O}$ also called the structure sheaf of a complex manifold is the sheaf of holomorphic functions,
- $\mathcal{O}^*$ is the sheaf of nowhere vanishing holomorphic functions,
- $\mathcal{O}(L)$ is the sheaf of section in the holomorphic line bundle L over M.

Definition 2.7. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ and $\mathcal{G}$ be (pre-)sheaves on X. A *morphism* between (pre-)sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms between (abelian) groups $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \in \mathcal{T}$ such that the following diagram commutes for all inclusions of open set $V \subseteq U$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle \rho_{UV}} & & \big\downarrow{\scriptstyle \tilde{\rho}_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

This enables us to define kernels and cokernels of sheaf morphisms in order to write down a cohomology theory. If we have a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ we define the kernel of φ to be the collection $\ker \varphi_U = \ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ for every open set $U \in \mathcal{T}$. Together with the restriction map inherited from the sheaf $\mathcal{F}$ we see that $\ker \varphi$ defined this way is again a sheaf.

Defining the cokernel in the same way defines in general not a sheaf but only a presheaf. The standard example for this is the morphism $\exp : \mathcal{O} \to \mathcal{O}^*$ on $\mathbb{C}\backslash\{0\}$ given by $f \mapsto \exp(2\pi i f)$. The section $f(z) = z \in \mathcal{O}^*(\mathbb{C}\backslash\{0\})$ is not in the image of exp as log is not defined on $\mathbb{C}\backslash\{0\}$, but its restriction to any open contractible set $U \in \mathbb{C}\backslash\{0\}$ is. This means, z is a nonzero element of $\operatorname{coker}(\exp)(\mathbb{C}\backslash\{0\})$ but its restriction to any contractible open set is locally the zero section. This contradicts the gluing property of sheaves and thus the cokernel defines only a presheaf. There is a general method of turning a presheaf into a sheaf called sheafification (see [Har77]). For the special case of the cokernel this works as follows. Let $\{U_i\}$ be an open cover of U and let $s_i \in \mathcal{G}(U_i)$. We define the cokernel of a homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ to be the collection $(U_i, s_i)_{i \in I}$ such that

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \varphi_{U_i \cap U_j}(\mathcal{F}(U_i \cap U_j)).$$

If we have two such collection (U_i, s_i) and $(\tilde{U}_i, \tilde{s}_i)$, we identify them, if for every $p \in U_i \cap \tilde{U}_j$ there exists an open neighbourhood V of p with $V \subseteq U_i \cap \tilde{U}_j$ such that

$$s_i|_V - \tilde{s}_j|_V \in \varphi_V(\mathcal{F}(V)).$$

Definition 2.8. A short sequence of sheaves

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

is called *exact*, if $\mathcal{E} = \ker \beta$ and $\mathcal{G} = \operatorname{coker}(\alpha)$. We call $\mathcal{G}$ the *quotient sheaf* $\mathcal{G} := \mathcal{F}/\mathcal{E}$ of $\mathcal{F}$ by $\mathcal{E}$ and the sequence is called the *cokernel sequence*.

A basic example for a short exact sequence is the *exponential sequence* on a complex manifold, which we will also need later. It is the following sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{\exp(2\pi i f)} \mathcal{O}^* \longrightarrow 1,$$

where $j : \mathbb{Z} \to \mathcal{O}$ in the inclusion map. We write 1 for the last sheaf instead of 0, because $\mathcal{O}^*$ is a sheaf of multiplicative groups (multiplication is given by multiplication of holomorphic functions such that the identity is the neutral element).

More generally, we can also define exactness for long sequences of sheaves.

Definition 2.9. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}_n$ be sheaves on X and $\varphi_n : \mathcal{F}_n \to \mathcal{F}_{n+1}$ be sheaf morphisms for all $n \in \mathbb{N}$. The long sequence of sheaves

$$\ldots \longrightarrow \mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{F}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{F}_{n+2} \longrightarrow \ldots$$

is called *exact*, if $\varphi_{n+1} \circ \varphi_n = 0$ and the short sequence

$$0 \longrightarrow \ker(\varphi_n) \xrightarrow{j} \mathcal{F}_n \xrightarrow{\varphi_n} \ker(\varphi_{n+1}) \longrightarrow 0$$

is exact for all $n \in \mathbb{N}$.

This now gives us all we need to define a cohomology theory for sheaves, which we call the *Čech cohomology*. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open cover for X. We define

$$\begin{aligned}
C^0(\mathcal{U}, \mathcal{F}) &= \prod_{i \in I} \mathcal{F}(U_i) \\
C^1(\mathcal{U}, \mathcal{F}) &= \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}) \\
&\vdots \\
C^p(\mathcal{U}, \mathcal{F}) &= \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p})
\end{aligned}$$

Elements $s \in C^p(\mathcal{U}, \mathcal{F})$ are called p-cochains and are given by a family $s = (s_{\mathcal{I}})_{\mathcal{I}=(i_0,\ldots,i_p) \in I^{p+1}}$ such that $s_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p})$.

We define the coboundary operator $\delta_p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta_p s)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0} \cap \ldots \cap U_{i_p}}$$

We have the following proposition:

Proposition 2.10. *The composition $\delta_{p+1} \circ \delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+2}(\mathcal{U}, \mathcal{F})$ is the zero map.*

Proof. This can easily be calculated:

$$
\begin{aligned}
(\delta_{p+1} \circ \delta_p s)_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k (\delta_p s)_{i_0,\dots,\hat{i}_k,\dots,i_{p+2}} \\
&= \sum_{k=0}^{p+2} (-1)^k \sum_{\substack{l=0 \\ l\neq k}}^{p+2} (-1)^l s_{i_0,\dots\hat{i}_k,\dots,\hat{i}_l,\dots,i_{p+2}} \\
&= \sum_{\substack{k,l=0 \\ k\neq l}}^{p+2} \Big(\underbrace{(-1)^k(-1)^{l-1}}_{k<l} + \underbrace{(-1)^k(-1)^l}_{k>l} \Big) s_{i_0,\dots\hat{i}_k,\dots,\hat{i}_l,\dots,i_{p+2}} \\
&= 0.
\end{aligned}
$$

□

A p-cochain s is called a *p-cocycle*, if $\delta_p s = 0$ and we write

$$Z(\mathcal{U}, \mathcal{F}) = \ker(\delta_p) \subset C^p(\mathcal{U}, \mathcal{F}).$$

Further, a p-cochain s is called a *p-coboundary*, if $s = \delta_{p-1} t$ for some $p-1$-cochain t and we write

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{Z^p(\mathcal{U})}{\delta_{p-1}(C^{p-1}(\mathcal{U}, \mathcal{F}))}.$$

Thus, $\check{H}^p(\mathcal{U}, \mathcal{F})$ is the p-th Čech cohomology group.

Note that this definition of cohomology depends on the choice of the cover $\mathcal{U}$. We can it make independent of the choice by taking the direct limit over refinements of open covers

$$\check{H}^p(M, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}).$$

This is well defined (see e.g. [GH78]), but not suitable for calculation. A solution to this problem is to take special covers (so called *Leray covers*) which have the property that the Čech cohomology groups with respect to that cover coincide with the direct limit. This is the following theorem.

Definition 2.11. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ be a sheaf on X. A locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ for X is called a *Leray cover*, if $\check{H}^p(U_{i_1} \cap \dots \cap U_{i_q}) = 0$ for all $p > 0$ and all $i_1, \dots, i_q \in I$.

Theorem 2.12 (Leray's theorem). *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ be a sheaf on X. If $\mathcal{U}$ is a Leray cover of X, then*

$$\check{H}^*(\mathcal{U}, \mathcal{F}) = \check{H}^*(X, \mathcal{F}).$$

A proof of Leray's theorem can be found in [GPR94].

Let us discuss an important example.

Example 2.13. *Let M be a Riemann surface and L be a holomorphic line bundle over M. Consider the space of holomorphic sections $\mathcal{O}(L)$ of L. If $\mathcal{U}$ is a locally finite open cover and $s \in \check{H}^0(\mathcal{U}, \mathcal{O}(L))$, then we have*

$$(\delta_0 s)_{ij} = s_i - s_j.$$

This vanishes if and only if these local expressions glue together to a global section. Thus, $\check{H}^0(M, \mathcal{O}(L)) = \ker \delta_0 = H^0(M, L)$ is the space of global sections of L, where $H^0(M, L)$ is the standard Dolbeault cohomology group.

To finish this section, let us introduce the notation of *locally free sheaves* and *invertible sheaves*.

Definition 2.14. Let M be a complex manifold and $\mathcal{O}$ its structure sheaf. A sheaf $\mathcal{F}$ on M is called *locally free* of rank k, if for every $p \in M$ there is an open neighbourhood U around p and an isomorphism $\phi : \mathcal{F}(U) \to \mathcal{O}(U)^{\oplus k}$. If the rank $k = 1$, the sheaf $\mathcal{F}$ is called *invertible*.

We can identify locally free sheaves with holomorphic vector bundles, as there are two inverse constructions.

First, we start with a holomorphic vector bundle E of rank k over a complex manifold M and we denote by $\mathcal{E}$ the sheaf of holomorphic sections of E. In a local trivialization we can write for every $s \in \mathcal{E}$ the local representation $s = \sum_{i=1}^{k} f_i e_i$, where the e_i are a local frame for E and the f_i are holomorphic functions. Hence, the bundle $\mathcal{E}$ is locally free.

On the other hand, let M be a complex manifold and $\mathcal{E}$ a localy free sheaf on it. This means, we have an open cover $\{U_\alpha\}$ of M and local isomorphisms $g_\alpha : \mathcal{E}(U_\alpha) \to \mathcal{O}(U_\alpha)^{\oplus k}$. On the overlap $U_\alpha \cap U_\beta$ we obtain an isomorphism

$$g_{\alpha\beta} = g_\alpha|_{U_\alpha \cap U_\beta} \circ g_\beta^{-1}|_{\mathcal{O}(U_\alpha \cap U_\beta)^{\oplus k}} : \mathcal{O}(U_\alpha \cap U_\beta)^{\oplus k} \to \mathcal{O}(U_\alpha \cap U_\beta)^{\oplus k},$$

which is given by an invertible $k \times k$-matrix, whose entries are holomorphic functions. These matrices $g_{\alpha\beta}$ satisfy the cocycle condition and are therefore transition functions, which define a holomorphic vector bundle E.

As these constructions are inverse to each other, we have a 1-1 correspondence between locally free sheaves and holomorphic vector bundles and hence a correspondence between invertible sheaves and holomorphic line bundles. In this work we will use both expressions depending on the context.

2.2.2 Divisors and line bundles

Another important tool from algebraic geometry are divisors. These are roughly speaking formal linear combinations of analytic hypersurfaces in a complex manifold.

With our knowledge from the previous section we can relate them to the zeros and poles of meromorphic functions and we can associate line bundles to divisors. This will be important for this work, because from the spectral data corresponding to bow varieties we can get some obstructions on several divisors occurring in the theory and helping us to describe bow varieties. In this section we follow the argumentation in [GH78].

We start with defining the group of so called *Weil divisors* $\mathrm{Div}(M)$. These are more or less linear combinations of some hypersurfaces, so we need to discus them first.

Definition 2.15. Let M be an n-dimensional complex manifold. An *analytic hypersurface* $V \subset M$ is an $n-1$ dimensional subvariety locally given as the zero set of a single holomorphic function on M, i.e. V is an analytic hypersurface, if for every point $p \in V$ there exists a neighbourhood U around p such that $V \cap U = \{f^{-1}(0)\}$. The function f is called a *local defining function* for V and it divides every holomorphic function g defined around p and vanishing on $V \cap U$.

Definition 2.16. An analytic hypersurface $V \subset M$ is called *irreducible*, if it can not be written as the union of analytic hypersurfaces $V \neq V_1 \cup V_2$ for $V_1, V_2 \subset M$ with $V_1, V_2 \neq V$. V is *irreducible at a point* $p \in V$, if there is a small neighbourhood around p such that $V \cap U$ is irreducible.

An analytic hypersurface being irreducible means that the local defining function can not be factorized into holomorphic functions f_1 and f_2 having their zeros on V. Locally in a neighbourhood around a point $p \in V$ an analytic hypersurface V can uniquely be written as the finite union

$$V = V_1 \cup ... \cup V_k$$

of irreducible analytic hypersurfaces at the point p. We can now define Weil divisors as follows.

Definition 2.17. A *Weil divisor* D is the locally finite formal linear combination of irreducible analytic hypersurfaces of M:

$$D = \sum_i a_i V_i.$$

The set of Weil divisors on M is an additive group denoted by $\mathrm{Div}(M)$.

Locally finite means that every point $p \in M$ has a neighbourhood that intersects with only finitely many of the V_i. If M is compact, then the sum is finite. We can identify every analytic hypersurface V on M with the divisor $D = \sum_i V_i$ where the V_i are the unique irreducible components of V ($V = V_1 \cup ... \cup V_k$).

On the other hand we can define divisors in terms of sheaves of meromorphic functions. These will be the so called *Cartier divisors*.

Recall that a meromorphic function f on M is an equivalence class locally given by the quotient of holomorphic functions $f = \frac{g}{h}$. I.e. for an open cover $\{U_i\}$ f is given as

$f_i = f|_{U_i} = \frac{g_i}{h_i}$ for relatively prime holomorphic functions g_i and h_i satisfying $g_i h_j = g_j h_i$ on the overlap $U_i \cap U_j$.

We can now define the sheaves $\mathcal{M}$ of meromorphic functions on M (additive) and $\mathcal{M}^*$ of meromorphic functions not being identically zero (multiplicative). We then can define the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. A global section of this quotient sheaf is given by a locally finite open cover $\{U_i\}$ of M together with meromorphic functions $f_i \in \mathcal{M}^*(U_i)$ such that on every intersection $U_i \cap U_j$ we have

$$\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j).$$

As $\mathcal{M}^*/\mathcal{O}^*$ is a sheaf on M we can define Čech cohomology for it. As we have seen in the previous section the space $\check{H}^0(M, \mathcal{F})$ is the space of global sections of the sheaf $\mathcal{F}$.

Definition 2.18. The group $\check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*)$ is called the group of *Cartier divisors*.

In the following we will see that on complex manifolds the group of Weil divisors and the group of Cartier divisors can be identified with each other. To see this, we have to introduce the order of holomorphic and meromorphic functions and discuss the zeros and poles of them.

Definition 2.19. Let $V \subset M$ be an irreducible analytic hypersurface of a complex manifold M, let $p \in V$ and f be a local defining function for V near p. Let g be a holomorphic function on M. We define the *order of g along V near p* as the largest $m \in \mathbb{N}$ such that f^m divides g, i.e. such that $g = f^m h$ for some holomorphic function h. We write $\operatorname{ord}_{V,p}(g)$ for the order of g along V at p.

The order is in fact independent of the choice of the point p, because it is locally constant and V is connected as it was supposed to be irreducible. We clearly have

$$\operatorname{ord}_V(gh) = \operatorname{ord}_V(g) + \operatorname{ord}_V(h).$$

Thus, we can define the order also for meromorphic functions as follows: Let f be a meromorphic functions locally given as $f = \frac{g}{h}$ for some holomorphic functions g, h. Then we set $\operatorname{ord}_V(f) = \operatorname{ord}_V(g) - \operatorname{ord}_V(h)$. We say f has a *zero of order k along V* if $\operatorname{ord}_V(f) = k > 0$ and f has a *pole of order k along V* if $\operatorname{ord}_V(f) = -k < 0$. The order of a meromorphic function gives rise to a special Weil divisor.

Definition 2.20. Let f be a meromorphic function on M. Then we define the following Weil divisor

$$(f) = \sum_V \operatorname{ord}_V(f) V,$$

where the sum goes over all irreducible analytic hypersurfaces in M.

This is well defined as the sum is locally finite. Now we have the following theorem ([GH78]):

Theorem 2.21. *Let M be a complex manifold. Then we have*

$$\mathrm{Div}(M) \cong \check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*).$$

Proof. Take a Cartier divisor, i.e. a global section of $\mathcal{M}^*/\mathcal{O}^*$. The condition $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$ means that $\mathrm{ord}_V(f_i) = \mathrm{ord}_V(f_j)$. Therefore, the Weil divisor $D = \sum_V \mathrm{ord}_V(f_i)V$ is well defined, where for each V we choose i such that $V \cap U_i \neq \emptyset$.

On the other hand, let $D = \sum_i a_i V_i$ be a Weil divisor. Let $\{U_\alpha\}$ be an open cover such that only finitely many V_i intersect each of the U_α. Let each V_i have a local defining function $f_i \in \mathcal{O}(U_\alpha)$. Then $f_\alpha := \prod_i f_i^{a_i}$ is a meromorphic function on U_α. f_α is defined up to multiplication with an element of $\mathcal{O}^*(U_\alpha)$, because the local defining functions are unique up to a non vanishing factor and thus $\{(U_\alpha, f_\alpha)\}$ defines a global section of $\mathcal{M}^*/\mathcal{O}^*$ and therefore a Cartier divisor. □

As an example consider M to be a Riemann surface. Then the divisor (f) is just given as $(f) = \sum_i m_i z_i$, where the z_i are the distinct zeros and poles of f and the m_i are the corresponding multiplicities.

Now, having a description of divisors, we want to give a construction to associate a line bundle with a divisor. This is crucial describing flows on a Jacobian variety in the spectral picture of bow varieties.

On a complex manifold M we can define the so called *Picard group*, which gives the set of holomorphic line bundles on M the structure of a group.

Definition 2.22. Let M be a complex manifold. The set of holomorphic line bundles on M can be equipped with a group structure the following way: Let L and $\tilde{L}$ be line bundles on M given by their transition functions g_{ab} and $\tilde{g}_{ab}$ respectively. We define the product of the two bundles via tensor product $L\tilde{L} = L \otimes \tilde{L}$ given by the transition functions $g_{ab}\tilde{g}_{ab}$. The inverse is given by the dual bundle L^* with transition functions g_{ab}^{-1}. We call this group the *Picard group* and denote it by $\mathrm{Pic}(M)$.

We have the following observation (se e.g. [GH78]):

Lemma 2.23. $\mathrm{Pic}(M) = \check{H}^1(M, \mathcal{O}^*)$.

Proof. Let $\pi : L \to M$ be a holomorphic line bundle on a complex manifold M given by transition functions $g_{ab} = \varphi_a \circ \varphi_b^{-1} : U_a \cap U_b \to \mathbb{C}^*$ for some local trivialization $\{U_a, \varphi_a\}$ with $\varphi_a : \pi^{-1}(U_a) \to U_a \times \mathbb{C}$ of M. The transition functions are non-vanishing, holomorphic functions satisfying $g_{ab} = g_{ba}^{-1}$ and are therefore a Čech 1-cochain. We have

$$(\delta g)_{abc} = g_{ab}g_{bc}g_{ca} = \mathrm{Id}$$

and thus it is a Čech cocycle.

On the other hand we obtain other trivializations over $\{U_a\}$ and therefore other transition function for L by multiplication with non-vanishing holomorphic functions f_a such that we have $\tilde{\varphi}_a = f_a\varphi_a$ and $\tilde{g}_{ab} = \frac{f_a}{f_b}g_{ab}$. We see that any other trivialization over $\{U_a\}$ can be obtained this way and we see that the difference is a Čech coboundary $(\delta f)_a$. This means the set of (isomorphism classes of) line bundles on M is the group $H^1(M, \mathcal{O}^*)$ and the group structure coincides with the one on $\mathrm{Pic}(M)$. □

To see that there is a homomorphism associating a line bundle to a divisor on M, observe the following short exact sequence on M:

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}^* \longrightarrow 0.$$

This gives rise to a long exact sequence of cohomology groups

$$0 \longrightarrow \check{H}^0(M, \mathcal{O}^*) \longrightarrow \check{H}^0(M, \mathcal{M}^*) \longrightarrow \check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*) \longrightarrow \check{H}^1(M, \mathcal{O}^*) \longrightarrow ...$$

where $H^0(M, \mathcal{M}^*/\mathcal{O}^*) \simeq \mathrm{Div}(M)$ and $H^1(M, \mathcal{O}^*) \simeq \mathrm{Pic}(M)$. Thus, there is a natural group homomorphism sending a divisor to its associated line bundle. This map is given in the following way: Let $\{U_\alpha\}$ be a locally finite open cover of M. Let $D = \sum_i a_i V_i$ be a divisor given by the local defining functions $f_i \in \mathcal{M}^*(U_\alpha)$ (for finitely many α) and define $f_\alpha = \prod_i f_i^{a_i} \in \mathcal{M}^*(U_\alpha)$ as in the proof of theorem 2.21. Then on every intersection $U_\alpha \cap U_\beta$ the function $g_{\alpha\beta} = \frac{f\alpha}{f\beta}$ is holomorphic and nowhere vanishing. Furthermore, we have

- $g_{\alpha\alpha} = \frac{f_\alpha}{f_\alpha} = \mathrm{Id}$ on each U_α,
- $g_{\alpha\beta} g_{\beta\alpha} = \frac{f_\alpha}{f_\beta} \frac{f_\beta}{f_\alpha} = \mathrm{Id}$ on each intersection $U_\alpha \cap U_\beta$,
- $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \frac{f_\alpha}{f_\beta} \frac{f_\beta}{f_\gamma} \frac{f_\gamma}{f_\alpha} = \mathrm{Id}$ on every triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$.

Hence, there is a line bundle given by its transition function $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ defined as before. We call this bundle the *line bundle associated to* D and denote it by $[D]$.

This homomorphism $\mathrm{Div}(M) \to \mathrm{Pic}(M)$ is well defined: Choosing a different cover $\{I'_\alpha\}$ with local defining functions f'_α, we obtain

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = g_{\alpha\beta} \frac{h_\beta}{h_\alpha},$$

where $h_\alpha = \frac{f_\alpha}{f'_\alpha} \in \mathcal{O}^*(U_\alpha)$ and thus the corresponding line bundle is isomorphic to the original one.

Observe that a line bundle associated to a divisor D is trivial if and only if $D = (f)$ is the divisor of a meromorphic function. If $D = (f)$ then $f|_{U_\alpha}$ are local defining functions and on every intersection we have $g_{\alpha\beta} = \frac{f_{U_\alpha \cap u_\beta}}{f_{U_\alpha \cap u_\beta}} = \mathrm{Id}$ and hence a trivial line bundle. On the other hand, if $[D]$ is trivial and f_α are local defining functions for D, then there exist holomorphic nowhere vanishing functions $h_\alpha \in \mathcal{O}^*(U_\alpha)$ such that $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$.

This means $f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$ on every intersection $U_\alpha \cap U_\beta$ and thus the function f locally given as $f_\alpha = f|_\alpha$ is a global meromorphic function and hence $D = (f)$.

Further observe that if D and D' are two divisors with local defining functions f_i and f_i' and associated line bundles $[D]$ and $[D']$ then $D + D'$ is a divisor with local defining functions $f_i f_i'$ and the associated bundles satisfy

$$[D + D'] = [D] \otimes [D'].$$

2.3 Spectral curves and their Jacobian varieties

We wish to solve Nahm equations with certain boundary conditions in order to describe bow varieties. It is not clear in general how to do so with standard methods for ODEs. But as we will see, there is a link to algebraic geometry. In particular, there is a one to one correspondence of some matrix valued polynomials with certain properties (this will be the solutions to Nahm equations) and linear flows on the Jacobian variety of the corresponding spectral curve. This algebraic treatment will enable us to give a description of bow varieties. The correspondence itself will be discussed in later chapters. Here, we want to recall the necessary background from algebraic geometry by briefly introducing spectral curves, Jacobian varieties and their theta divisors.

2.3.1 The total space $|\mathcal{O}_{\mathbb{CP}^1}(2)|$

First, we want to discuss the total space $|\mathcal{O}_{\mathbb{CP}^1}(2)|$ of the line bundle $\mathcal{O}(2)$ over $\mathbb{CP}^1$ as a two-dimensional complex manifold, because the spectral curves we wish to construct later in this chapter turn out to be subsets of $|\mathcal{O}_{\mathbb{CP}^1}(2)|$.

We begin the construction with the complex projective space $\mathbb{CP}^1 = \mathbb{C}^2 \backslash \{0\}/_\sim$ where $z \sim z' \Leftrightarrow z' = az$ for some $a \in \mathbb{C}^*$ being the space of one-dimensional subspaces in $\mathbb{C}^2$, which can be identified with $\mathbb{C} \cup \{\infty\}$. The standard open cover consists of two open sets (U_0, U_1) where $U_i = \{[z_0, z_1] \in \mathbb{CP}^1 | z_i \neq 0\}$. We have charts $\varphi_0 : U_0 \to \mathbb{C}$, $[z_0, z_1] \mapsto \frac{z_1}{z_0} =: \zeta$ and $\varphi_1 : U_1 \to \mathbb{C}$, $[z_0, z_1] \mapsto \frac{z_0}{z_1} =: \tilde{\zeta}$. On $U_0 \cap U_1$ we have the transition function $\tilde{\zeta} = \frac{1}{\zeta}$.

We can define the structure sheaf of holomorphic functions on each of the two open sets U_i, identified with $\mathbb{C}$, and glue them together (with the gluing property of sheaves) to the structure sheaf $\mathcal{O}$ of holomorphic functions on $\mathbb{CP}^1$. Let $U \subset \mathbb{CP}^1$ be an open subset. Then, a local section $f \in \mathcal{O}(U)$ is given by a pair $f = (f_0, f_1)$ of holomorphic functions defined on $U \cap U_0$ and $U \cap U_1$ respectively satisfying $f_1(\tilde{\zeta}) = f_1(\frac{1}{\zeta}) = f_0(\zeta)$ on the overlap $U_0 \cap U_1$.

We can define other holomorphic line bundles on $\mathbb{CP}^1$ via their transition function. The first bundles, that one would imagine are the bundles $\mathcal{O}(k)$ for $k \in \mathbb{Z}$. They are given by transition function ζ^{-k}, i.e. sections are given by local expression $s_0(\zeta)$ and $s_1(\tilde{\zeta})$ on U_0 and U_1 respectively satisfying $s_1(\tilde{\zeta}) = \zeta^{-k} s_0(\zeta)$. Line bundles with negative integer $\mathcal{O}(-k)$ for $k > 0$ are therefore given as the dual bundles of the $\mathcal{O}(k)$. Observe that the section of $\mathcal{O}(1)$ is given by setting $s_0(\zeta) = \zeta$ and $s_1(\tilde{\zeta}) = 1$.

For each of the $\mathcal{O}(k)$ we can write down their total space $|\mathcal{O}(k)|$. As they are line bundles (precisely: $\pi_{\mathcal{O}(k)} : |\mathcal{O}_{\mathbb{CP}^1}(k)| \to \mathbb{CP}^1$) the total spaces are complex two-dimensional manifolds. With the given transition function we have two coordinate patches (V_0 for $\pi^{-1}(U_0)$ and V_1 for $\pi^{-1}(U_1)$) and hence they are given as

$$V_0 = \{(\zeta, \eta) \in \mathbb{C}^2\} \text{ and } V_1 = \{(\tilde{\zeta}, \tilde{\eta}) \in \mathbb{C}^2\}.$$

Observe, that $\zeta \neq 0$ in V_0 and $\tilde{\zeta} \neq 0$ in V_1 because of the definition of the U_i. We can identify both spaces with $\mathbb{C}^* \times \mathbb{C}$ and a coordinate change $(\tilde{\zeta}, \tilde{\eta}) = (\frac{1}{\zeta}, \frac{\eta}{\zeta^k})$ on the intersection $V_0 \cap V_1$.

Now, the bundle $\pi_{\mathcal{O}(2)} : |\mathcal{O}_{\mathbb{CP}^1}(2)| \to \mathbb{CP}^1$ is of special interest as it is the holomorphic tangent bundle $T\mathbb{CP}^1$ of $\mathbb{CP}^1$, as $\mathcal{O}(2)$ is the dual of the canonical bundle $K_{\mathbb{CP}^1} = \mathcal{O}(-2)$. Observe that the cotangent bundle $T^*\mathbb{CP}^1$ is trivialized by $\mathrm{d}\zeta$ on U_0 and $-\mathrm{d}\tilde{\zeta}$ on U_1 in the standard coordinates of $\mathbb{CP}^1$. Thus, we have $-\mathrm{d}\tilde{\zeta} = -\mathrm{d}\frac{1}{\zeta} = \frac{1}{\zeta^2}\mathrm{d}\zeta$. Changing the sign of the transition function gives us $T^*\mathbb{CP}^1 = \mathcal{O}(-2)$ and as the tangent bundle is the dual bundle of this we obtain $|\mathcal{O}(2)| = T\mathbb{CP}^1$.

2.3.2 The Jacobian variety and the theta divisor

The first thing we need to define is the *Jacobian variety* of a Riemann surface M. This construction will also work for a more general setting, but here we only need it for Riemann surfaces.

We have the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ j\ } \mathcal{O} \xrightarrow{\exp(2\pi i f)} \mathcal{O}^* \longrightarrow 1,$$

where j is the inclusion. This gives rise to a long exact sequence of cohomology groups

$$0 \to H^0(M,\mathbb{Z}) \to H^0(M,\mathcal{O}) \to H^0(M,\mathcal{O}^*) \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \to$$
$$\to H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{O}) \to H^2(M,\mathcal{O}^*) \to \ldots$$

We can make some observations: We have $H^0(M,\mathbb{Z}) = \mathbb{Z}$, $H^0(M,\mathcal{O}) = \mathbb{C}$ and $H^0(M,\mathcal{O}^*) = \mathbb{C}^*$, because the only holomorphic functions on compact Riemann surfaces are the constants. Since M is a compact manifold of real dimension 2, we have Poincaré duality $\mathbb{Z} \cong H_0(M,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$. We have a Leray cover for the structure sheaf $\mathcal{O}$ consisting of two open sets (take coordinate neighbourhoods homeomorphic to a disc) and therefore $H^2(M,\mathcal{O}) = 0$. This simplifies the long exact sequence to the following:

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \to \mathbb{Z} \to 0$$

Another observation is the following: Since the map $\exp : \mathbb{C} \to \mathbb{C}^*$ is surjective, we find by exactness of the above sequence that the map $\mathbb{C}^* \to H^1(M, \mathbb{Z})$ must be the 0-map and the map $H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O})$ must be injective. This means we can take the quotient and finally obtain the short exact sequence

$$0 \to \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})} \stackrel{\exp}{\to} H^1(M, \mathcal{O}^*) \stackrel{\delta}{\to} \mathbb{Z} \to 0.$$

We call the map δ, which sends an isomorphism class of holomorphic line bundles on M to an integer the degree map and the image of a line bundle under this map the degree of the line bundle. As the degree map must be a homomorphism, i.e. $\deg(L \otimes L') = \deg(L) + \deg(L')$, we find that any two isomorphism classes of line bundles with given degree d differ by an element of the kernel, a line bundle with degree 0 (see [GH78]). Thus all equivalence classes of isomorphism classes of line bundle with a fix degree are isomorphic to each other and isomorphic to the kernel of the degree map, which is the image of

$$\frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})}$$

under the exponential map (because of exactness), and this is what we call the *Jacobian variety* or just the *Jacobian* of the Riemann surface M and denote it by $\mathrm{Jac}^{g-1}(M)$, where g is the genus of M.

Now, we want to construct the theta divisor on a Jacobian. The theta divisor is defined to be the set of line bundles which have a non trivial global section, i.e. the set of sheaves

$$\Theta = \{\mathcal{F} \in \mathrm{Jac}^{g-1}(M) | \dim H^0(M, \mathcal{F}) \neq 0\}.$$

This set is special because of the following behaviour:

There is a natural way to assign a vector bundle E to a line bundle L as a direct image sheaf. This vector bundle has a a representation as a direct sum of of line bundles $\mathcal{O}(a_i)$. Now, as the line bundle $L(-1)$ meets the theta divisor, the a_i in the decomposition of E will have a jump. The rest of this paragraph will explain how this phenomenon occurs.

We start by defining the *direct image sheaf* $f_*\mathcal{F}$.

Definition 2.24. Let $f : M' \to M$ be a holomorphic map and $\mathcal{F}$ a sheaf on M'. The *direct image sheaf* on M to be the sheaf $f_*\mathcal{F}$ given by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for each open set $U \subset M$.

If L is a line bundle over M' and $\mathcal{O}(L)$ the sheaf of holomorphic sections of L, we have the property

$$f_*\mathcal{O}(L) = \mathcal{O}(E)$$

for a holomorphic vector bundle E on M with rank $m := \mathrm{rk}(E) = \deg f$ (cf. [HSW99]). There is a relation between the degree of E and the degree of L. For our case of interest, we take $M = \mathbb{CP}^1$. Then we have

$$\deg E = \deg L - \deg f + 1 - g'. \tag{2.1}$$

This is essentially the Riemann Roch theorem and some properties of the direct image sheaf [HSW99].

Recall that the famous Riemann-Roch theorem reads as follows

Theorem 2.25. *Let E be a vector bundle on a compact Riemann surface M of genus g. Then*

$$\dim \check{H}^0(M, E) - \dim \check{H}^1(M, E) = \deg(E) + \mathrm{rank}(E)(1 - g). \tag{2.2}$$

For the next step we need another famous theorem, the Birkhoff-Grothendieck theorem.

Theorem 2.26. *Let E be a holomorphic vector bundle over $\mathbb{CP}^1$. Then E can be written as a direct sum of line bundles of the form $\mathcal{O}(k)$:*

$$E \cong \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_m)$$

for some $a_i \in \mathbb{Z}$.

With this we can show the following statement:

Lemma 2.27. *Now, a vector bundle E on $\mathbb{CP}^1$ is trivial if $\deg E = 0$ and $H^0(\mathbb{CP}^1, E \otimes \mathcal{O}(-1)) = 0$.*

Proof. As we can write $E \cong \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_m)$ (Birkhoff-Grothedieck), we have

$$0 = H^0(\mathbb{CP}^1, E \otimes \mathcal{O}(-1)) = \bigoplus_i H^0(\mathbb{CP}^1, \mathcal{O}(a_i - 1))$$

and this means, $a_i \leq 0$ for all i. With

$$\deg E = \sum_i a_i = 0$$

we obtain that all $a_i = 0$, and therefore E is trivial. □

For the line bundle L the condition $\deg E = 0$ is equivalent to $\deg L = \deg f + g' - 1$ (cf. (2.1)) and (using the projection formula [Har77]) the condition $H^0(\mathbb{CP}^1, E \otimes \mathcal{O}(-1)) = 0$ is equivalent to $H^0(M', L \otimes f^*\mathcal{O}(-1)) = 0$, i.e. to the condition that the bundle $L \otimes f^*\mathcal{O}(-1)$ has no non-trivial global sections. Now,

$$\deg(L \otimes f^*\mathcal{O}(-1)) = \deg L + \deg f^*\mathcal{O}(-1) = \deg f + g' - 1 + (-1)\deg f = g' - 1$$

and therefore the line bundle $L \otimes f^*\mathcal{O}(-1) \in \mathrm{Jac}^{g'-1}(M')$ belongs to the Jacobian of M'. As it has no non-trivial global sections it does not belong to the theta divisor, but if we take a continuous family $L^t \otimes \mathcal{O}(-1)$, there might be a t_0 so that the bundle belongs to the theta divisor. In this case the a_i of the corresponding vector bundle will jump from all being zero to non zero values.

2.3.3 Spectral curves

The most important object for the algebraic part of the problem is the so called *spectral curve*. Generally speaking, a spectral curve is an algebraic curve embedded into (in our case) some complex manifold.

Let us consider the line bundle $\mathcal{O}(n)$ on $\mathbb{CP}^1$ and let M be a Riemann surface. Let $f : M \to \mathbb{CP}^1$ be a complex map of degree m.

We start with a global section η in the pullback bundle $f^*\mathcal{O}(n)$ on M, $\eta \in H^0(M, f^*\mathcal{O}(n))$. As it is a global section, it induces a map

$$H^0(f^{-1}(U), L) \to H^0(f^{-1}(U), L \otimes f^*\mathcal{O}(n)).$$

In other words, a section of L is mapped to a section on the tensor product by multiplication with η. By the definition of E via the direct image sheaf ($\mathcal{O}(L)(f^{-1}(U)) = f_*\mathcal{O}(L)(U) = \mathcal{O}(E)(U)$) we can identify

$$H^0(f^{-1}(U), L) \cong H^0(U, E)$$

and (by using again the projection formula [Har77])

$$H^0(f^{-1}(U), L \otimes f^*\mathcal{O}(n)) \cong H^0(U, E \otimes \mathcal{O}(n)).$$

Because the multiplication is globally defined, we obtain a map

$$H^0(\mathbb{CP}(1), E) \to H^0(\mathbb{CP}^1, E \otimes \mathcal{O}(n)).$$

E is a trivial bundle of degree m, thus $H^0(\mathbb{CP}^1, E) \cong \mathbb{C}^m$. I.e. we obtain a global section of $\mathbb{C}^m \otimes H^0(\mathbb{CP}^1, \mathcal{O}(n))$. Since sections of $\mathcal{O}(n)$ are polynomials in ζ of degree $\leq n$, we obtain a pair of matrix valued polynomials

$$A(\zeta) = A_0 + A_1\zeta + ... + a_n\zeta^n, \qquad \tilde{A}(\tilde{\zeta}) = \tilde{A}_0 + \tilde{A}_1\tilde{\zeta} + ... + \tilde{A}_n\tilde{\zeta}^n.$$

As we will see later, the matrix polynomial A representing Nahms equations in the Lax pair will be of degree 2, therefore it is enough for our purposes to restrict ourselves to $\mathcal{O}(2)$ here. For this to be a global section we need the polynomials to satisfy

$$\tilde{A}(\frac{1}{\zeta}) = \frac{1}{\zeta^2}A(\zeta)$$

on $U_0 \cap U_1$ assuming the standard charts on $\mathbb{CP}^1$. This implies $\tilde{A}_0 = A_2$, $\tilde{A}_1 = A_1$ and $\tilde{A}_2 = A_0$.

We now want to observe that the two characteristic polynomials of these two matrix valued functions define a global section of some line bundle on the total space $|\mathcal{O}(2)|$.

This total space is a two dimensional manifold, which we can trivialise the following way: Let $U_0 = \mathbb{C}^2$ with coordinates (ζ, η) and $U_1 = \mathbb{C}^2$ with coordinates $(\tilde{\zeta}, \tilde{\eta})$. Then we can identify $|\mathcal{O}(2)| \cong U_0 \cup U_1 / \sim$ with $(\zeta, \eta) \sim (\tilde{\zeta}, \tilde{\eta}) \Leftrightarrow \zeta \neq 0$ and $(\tilde{\zeta}, \tilde{\eta}) = (\frac{1}{\zeta}, \frac{\eta}{\zeta^2})$.

Calculating the characteristic polynomials we find

$$P_1(\tilde{\zeta},\tilde{\eta}) = \det(\tilde{\eta} Id_m - \tilde{A}(\tilde{\zeta})) = \det(\frac{\eta}{\zeta^2} Id_m - \frac{1}{\zeta^2} A(\zeta)) = \frac{1}{\zeta^{2m}} \det(\eta Id_m - A(\zeta))$$
$$= \frac{1}{\zeta^{2m}} P_0(\zeta,\eta).$$

Thus, we have a section of the bundle $\mathcal{O}_{|\mathcal{O}(2)|}(2m)$.

To understand what this means we choose some local trivialisations. If $p \in \mathbb{CP}^1$ is a regular value of f, then its preimage consists of m distinct point $p_1, ..., p_m \in M$. If we take a small neighbourhood U of p, we have locally $H^0(U,E) = \bigoplus_i H^0(U_i, L)$. As E is a trivial vector bundle of rank m, we can choose a local basis $\{\sigma_1, ..., \sigma_m\}$ of the space of sections of L over $f^{-1}(U)$, such that $\sigma_i|_{U_j} = 0$ for $i \neq j$ and identify this with a local basis $\{s_1, ..., s_m\}$ of the space of sections of E over U. By the definition we obtain an eigenvalue problem

$$A(\zeta)s_i = \eta|_{U_i}\sigma_i = \eta|_{U_i} \circ f_i^{-1}|_U(s_i). \tag{2.3}$$

Here, $\eta|_{U_i} \circ f_i^{-1}|_U$ is meant to be the following function: as U is a small neighbourhood around a regular value p of f the preimage $f^{-1}(U)$ consists of m small neighbourhoods $U_1, ..., U_m$ around the preimages $p_1, ..., p_m$ of p such that $f|_{U_i} : U_i \to f(U_i)$ is invertible. Denote by f_i^{-1} the inverse function of the restriction of f to the neighbourhood U_i. Then we obtain m functions $\eta|_{U_i} \circ f_i^{-1}|_U$ which are eigenvalues of $A(\zeta)$.

Therefore, at every point in M the value of the section η satisfies the characteristic equation $\det(\eta Id_m - A(\zeta)) = 0$. Thus, η sends the Riemann surface M to the set of points $(\zeta, \eta) \in |\mathcal{O}(2)|$ where the section given by characteristic equation is zero. This subset $\eta(M) \subset |\mathcal{O}(2)|$ is what we call the spectral curve. Observe, that all matrix valued polynomials $A(\zeta)$ with the same characteristic polynomial $P(\zeta,\eta) = \det(\eta \mathrm{Id} - A(\zeta))$ define the same spectral curve. Hence, the spectral curve depends only on the characteristic polynomial.

2.3.4 Example: The spectral curve for Nahm equations

As we have seen, we can construct a spectral curve from a given matrix valued polynomial. To finish this chapter, we want to construct the spectral curve for Nahm equations of rank 2 as an example.

So taking T_1, T_2, T_3 as $\mathfrak{u}(2)$-valued meromorphic functions satisfying Nahm equations

$$\frac{\mathrm{d}}{\mathrm{d}s} T_i(s) = \frac{1}{2}\epsilon_{ijk}[T_j(s), T_k(s)]$$

it is well known (see e.g. [Hit83]) that we can assemble the T_i into some matrix valued polynomials such that Nahm equations are equivalent to a Lax pair equation.

Thus, we define $A_0(s) = T_2(s) + iT_3(s)$, $A_1(s) = 2iT_1(s)$ and $A_2(s) = T_2(s) - iT_3(s)$ and write

$$A(\zeta, s) = A_0(s) + \zeta A_1(s) + \zeta^2 A_2(s).$$

Taking

$$B(\zeta, s) = \frac{1}{2}\frac{\mathrm{d}A}{\mathrm{d}\zeta} = \frac{1}{2}A_1(s) + \zeta A_2(s)$$

we find that Nahm equations are equivalent to a Lax pair equation

$$\frac{\mathrm{d}A}{\mathrm{d}s} = [B, A].$$

Observe that

$$\frac{\mathrm{d}A(\zeta, s)}{\mathrm{d}s} = \dot{T}_2(s) + i\dot{T}_3(s) + 2i\dot{T}_1(s)\zeta + (\dot{T}_2(s) - i\dot{T}_3(s))\zeta^2$$

and

$$[B(\zeta, s)A(\zeta, s)] = [T_3(s), T_1(s)] + 2i[T_2(s), T_3(s)]\zeta + \big([T_3(s), T_1(s)] - i[T_1(s), T_2(s)]\big)\zeta^2.$$

Comparing coefficients in ζ shows the equivalence between Lax pair equations and Nahm equations. This different form of Nahm equations will be important for chapter 4, when we translate the conditions of a bow variety into the language of spectral curves.

This also shows that the spectrum of A is independent of s [Hit83]: As the commutator is skew symmetric we have $\frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Tr} A^n = 0$ for all $n \geq 0$. Therefore, also the characteristic polynomial $\det(\eta \mathrm{Id} - A(\zeta, s))$ is in fact independent of s and therefore it makes sense to talk about a spectral curve for Nahm equations and not about a different one for every different value of the parameter s. For this reason we call the Nahm equations to be *isospectral*.

Finally, observe that

$$A_0^\dagger = -A_2, \qquad A_2^\dagger = -A_0, \qquad A_1^\dagger = A_1, \tag{2.4}$$

as the Nahm matrices T_i are $\mathfrak{u}(2)$-valued. This is the so called reality condition. Every matrix polynomial $A(\zeta)$ arising from Nahm equations this way satisfies the reality condition. More implications of this fact will be discussed in chapter 4. Therefore, we can write down the set $S = \{(\eta, \zeta) \in |\mathcal{O}(2)| : \det(\eta \mathrm{Id} - A(\zeta)) = 0\} \in |\mathcal{O}(2)|$ being an algebraic curve in the tangent bundle of $\mathbb{CP}^1$, which is the spectral curve. The defining polynomial $P(\eta, \zeta) = \det(\eta \mathrm{Id} - A(\zeta))$ is explicitly given by

$$P(\eta, \zeta) = \eta^2 + \eta(a_0\zeta^2 + a_2\zeta + a_2) + b_0\zeta^4 + b_1\zeta^3 + b_2\zeta^2 + b_3\zeta + b_4$$

with the following coefficients:

$$\begin{aligned}
a_0 =& -\operatorname{Tr} T_2 - i \operatorname{Tr} T_3 \\
a_1 =& -2i \operatorname{Tr} T_1 \\
a_2 =& -\operatorname{Tr} T_2 + i \operatorname{Tr} T_3 \\
b_0 =& \frac{1}{2}(\langle T_2, T_2\rangle - \langle T_3, T_3\rangle + 2i\langle T_2, T_3\rangle + (\operatorname{Tr} T_2 + i \operatorname{Tr} T_3)^2) \\
b_1 =& 2i(\operatorname{Tr} T_1(\operatorname{Tr} T_2 + i \operatorname{Tr} T_3) + \langle T_1, T_2\rangle + i\langle T_1, T_3\rangle) \\
b_2 =& \langle T_2, T_2\rangle + \langle T_3, T_3\rangle - 2\langle T_1, T_1\rangle + (\operatorname{Tr} T_2)^2 + (\operatorname{Tr} T_3)^2 - 2(\operatorname{Tr} T_1)^2 \\
b_3 =& 2i(\operatorname{Tr} T_1(\operatorname{Tr} T_2 - i \operatorname{Tr} T_3) + \langle T_1, T_2\rangle - i\langle T_1, T_3\rangle) \\
b_4 =& \frac{1}{2}(\langle T_2, T_2\rangle - \langle T_3, T_3\rangle - 2i\langle T_2, T_3\rangle + (\operatorname{Tr} T_2 - i \operatorname{Tr} T_3)^2).
\end{aligned} \tag{2.5}$$

Because the characteristic polynomial is independent of s, these coefficient must be as well. In particular, we find that the 3 traces $\operatorname{Tr} T_1$, $\operatorname{Tr} T_2$ and $\operatorname{Tr} T_3$ are preserved as well as the the 5 quantities

$$\begin{aligned}
\alpha_1 &= \langle T_1, T_2\rangle - \langle T_2, T_2\rangle \\
\alpha_2 &= \langle T_1, T_1\rangle - \langle T_3, T_3\rangle \\
\alpha_3 &= \langle T_1, T_2\rangle \\
\alpha_4 &= \langle T_1, T_3\rangle \\
\alpha_5 &= \langle T_2, T_3\rangle.
\end{aligned} \tag{2.6}$$

These quantities will play an important role later when we construct a solution for Nahm equations of rank 2.

2.4 Twistor spaces

The last point, we want to mention here is a theorem about twistor spaces, which we will need later in chapter 5. This basic theorem was shown by Hitchin et al. [HKLR87] and therefore, we will follow the construction of twistor spaces as it is presented there.

The sphere S^2 carries a complex structure given as the projective space $\mathbb{CP}^1$. We have complex coordinates for S^2 given as

$$(a, b, c) = \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}}\right).$$

Definition 2.28. Let M be a hyperkähler manifold of real dimension $4n$ with complex structures (I, J, K). The *twistor space* of M is the product manifold $Z = M \times S^2$ of M with the 2-sphere equipped with a complex structure $\underline{I}$, which is given on a tangent space $T_{(p,\zeta)}Z$ in the following way: Write the tangent space as the direct sum $T_{(p,\zeta)}Z = T_pM \oplus T_\zeta S^2$ and denote by I_0 the multiplication by i on $T_\zeta S^2$. Then the complex structure $\underline{I}$ is given as

$$\underline{I} = \left(\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}I + \frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}J + \frac{i(\zeta-\bar{\zeta})}{1+\zeta\bar{\zeta}}K, I_0\right)$$

First, we have to observe that the almost complex structure $\underline{I}$ is in fact a complex structure, what can be shown by the Newlander-Nirenberg theorem. There are several versions of this theorem and we use the following: If for each $(1,0)$-form θ (i.e. each complex 1-form such that $\underline{I}\theta = i\theta$) we can write the exterior derivative as $\mathrm{d}\theta = \theta_i \wedge \alpha_i$ for $(1,0)$-forms θ_i and general 1-forms α_i, then the almost complex structure $\underline{I}$ is integrable.

Let φ be a $(1,0)$-form for the complex structure I and let $\theta = \varphi + \zeta K\varphi$. Then we have

$$(1+\zeta\bar{\zeta})\underline{I}\theta = i(1+\zeta\bar{\zeta})(\varphi + \zeta K\varphi)$$

meaning, that θ is a $(1,0)$-form for $\underline{I}$. Hence, $\varphi_i + \zeta K\varphi_i$ for $i = 1,...,2n$ and $\mathrm{d}\zeta$ form a basis of the space of local $(1,0)$-forms for $\underline{I}$, if φ_i for $i = 1,...,2n$ form a basis for local $(1,0)$-forms for I. We can now calculate the exterior derivative for those base $(1,0)$-forms [HKLR87]:

$$\mathrm{d}\theta = \mathrm{d}(\varphi + \zeta K\varphi) = \mathrm{d}x^i \wedge \nabla_{\partial/\partial x^i}(\varphi + \zeta K\varphi) + \mathrm{d}\zeta \wedge K\varphi.$$

Since $\underline{I}$ is covariantly constant with respect to $\nabla_{\partial/\partial x^i}$ the term $\nabla_{\partial/\partial x^i}(\varphi + \zeta K\varphi)$ is a $(1,0)$-form for $\underline{I}$ hence, the first term is of the form $\theta_i \wedge \alpha_i$ as defined above. The second term is also of that form by definition and this means we can write every $(1,0)$-form as $\theta_i \wedge \alpha_i$ and therefore we can apply the Newlander-Nirenberg theorem. Hence, the twistor space Z is a complex manifold of dimension $\dim_{\mathbb{C}} = 2n+1$.

We can extract some holomorphic data from this construction, which allow us to get back the hyperkähler metric of M. These data are explained in detail in [HKLR87] and are given as follows:

- The projection map $\pi : Z \to \mathbb{CP}^1$ is holomorphic and each $(p, \mathbb{CP}^1)$ is a holomorphic section of this projection. Those sections are called *twistor lines*. Geometrically, the normal bundle to each twistor line is just the product $S^2 \times T_pM$, but as a holomorphic vector bundle, it is not trivial (see below).

- The normal bundle to each twistor line is given by the following trivialization: Let (U_0, U_1) be the standard coordinate charts on $\mathbb{CP}^1$ with coordinate ζ and $\tilde{\zeta} = \frac{1}{\zeta}$ on the overlap. The normal bundle is now obtained by taking $U_0 \times \mathbb{C}^{2n}$ and $U_1 \times \mathbb{C}^{2n}$ and patching them together over $U_0 \cap U_1$ by the transition function $i\zeta\mathrm{Id}_{2n}$. Hence, the normal bundle is $\mathbb{C}^{2n} \times \mathcal{O}(1)$. This can be obtained by taking the tangent space $T_pM \cong \mathbb{R}^{4n}$ together with the action of the complex structures

I, J, K assembled into $\underline{I}$ as above. This defines a complex structure for T_pM at a point $\zeta \in S^2$, hence $\underline{I}$ acts on $\mathbb{R}^{4n} \times \mathbb{C}$ such that the complex vectors over ζ build an eigenspace of $\underline{I}$ corresponding to the eigenvalue i.

- We have a holomorphic, complex symplectic form on the fibres of the projection $\pi : Z \to \mathbb{CP}^1$ given as follows. If $\omega_+ = \omega_2 + i\omega_3$ is a holomorphic 2-form of type $(2,0)$ for the complex structure I, written as $\frac{1}{2}\omega_+ = \sum_{i=1}^n \varphi_i \wedge \varphi_{n+i}$ in a local basis $\{\varphi_i\}$ of $(1,0)$-form with respect to I, then

$$\frac{1}{2}\omega = \sum_{i=1}^{n} (\varphi_i + \zeta K\varphi_i) \wedge (\varphi_{n+i} + \zeta K\varphi_{n+i})$$

is a 2-form of type $(2,0)$ for the complex structure $\underline{I}(\zeta)$ which is quadratic in ζ. Evaluating on tangent vectors yields

$$\omega = \omega_2 + i\omega_3 + 2\omega_1\zeta + (\omega_2 - i\omega_3)\zeta^2.$$

For each $\zeta \in \mathbb{CP}^1$ this defines a holomorphic, complex symplectic form on the fibres of the projection. Globally this defines a holomorphic section of the vector bundle $\Lambda^2 T_F^*(2)$ over Z, where $T_F = \ker d\pi$ is the tangent bundle along the fibres.

- We have a real structure on Z induced by the antipodal map on $\mathbb{CP}^1$ that takes the complex structure $\underline{I}$ to its conjugate $-\underline{I}$ and that is compatible with all the other holomorphic data. It is given by

$$\begin{aligned} \tau : M \times S^2 &\to M \times S^2 \\ (p, \zeta) &\mapsto \left(p, -\frac{1}{\bar{\zeta}}\right). \end{aligned}$$

From this data we can reconstruct the hyperkähler metric. We have the following theorem from [HKLR87]:

Theorem 2.29. *Let Z be a complex manifold of (complex) dimension $2n+1$ such that the following properties are satisfied:*

1. *Z is a holomorphic bundle $\pi : Z \to \mathbb{CP}^1$,*
2. *the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,*
3. *there is a holomorphic section $\omega \in \Gamma(\Lambda^2 T_F^*(2))$ defining a holomorphic symplectic form on each fiber,*

4. *Z has a real structure τ inducing the antipodal map on $\mathbb{CP}^1$ and being compatible with (i)-(iii).*

Then the parameter space of real sections (i.e. sections that are invariant under τ) with normal bundle as in 2. is a $4n$ dimensional manifold with a natural hyperkähler metric for which Z is the twistor space.

3 Bows, bow varieties and Nahm equations

This chapter introduces the main object of this work, the bow varieties. In section 3.1 we will show following [Che11] how to construct them and in section 3.2 we will introduce Nahm equations and work out some of their properties, because solving them is crucial for explicitly writing down bow varieties.

3.1 Bows and bow varieties

In this section we will define bows as generalizations of quivers and introduce bow representations giving some additional structure on the bow. Further, we will introduce the space of bow data corresponding to a bow representation. We will work out a suitable description of this space and find a gauge group action. Finally, we will define bow varieties as a hyperkähler quotient of the space of Bow data corresponding to a bow representation with the gauge group action. We will discuss the different moment map conditions that can occur in this process for different bow representations.

3.1.1 Bows and bow data

Our starting point for this section will be a quiver. In representation theory a quiver is a directed graph, i.e. a quiver consist of a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of edges together with two functions $f, g : \mathcal{E} \to \mathcal{V}$ assigning the sources and the target to an edge. So, the edges can be viewed as arrows between the vertices. For the rest of this thesis we will call such a directed edge an arrow.

We want to generalize this construction by replacing the vertices with distinct intervals. According to Cherkis [Che11] we have the following definition.

Definition 3.1. A *bow* is a collection of intervals together with a collection of arrows. Thus, it consists of:

- a collection $\mathcal{I} = \{I_i\}$ of closed distinct intervals parametrised by a parameter s, such that $I_i = [p_{i,L}, p_{i,R}]$. We denote the length of the i-th interval with l_i.

- a collection $\mathcal{E} = \{e_{ij}\}$ of arrow between the intervals. The arrow e_{ij} leads from the i-th to the j-th interval, i.e. if we denote the head of the arrow with h and the tail with t, then we have $h(e_{ij}) = p_{j,L}$ and $t(e_{ij}) = p_{i,R}$. e^{ij} defines the same arrow with opposite orientation and we write $\bar{\mathcal{E}} = \{e^{ij}\}$.

A bow is a generalization of a quiver in the sense, that it degenerates to a quiver, as we send $l_i \to 0 \ \forall i$.

We now want to define representations of bows in analogy of representations of quivers which consists of a family of vector spaces associated to the points. We want to generalize these concepts to bows. Cherkis defines bow representations as follows [Che11]:

Definition 3.2. A regular *bow representation* is a triple (Λ, E, W) consisting of

- A collection $\Lambda = \{\lambda_i^\alpha\}$ of distinct points, the so called *λ-points*, where $\lambda_i^\alpha \in I_i$. For each interval I_i we have $\alpha = 1, ..., k_i$ with $k_i \geq 0$, i.e. we split off every interval into $k_i + 1$ closed subintervals. Denote the α-th subinterval with I_i^α. Observe that the λ-points belong to two subintervals. In particular, we have $I_i^\alpha = [\lambda_i^{\alpha-1}, \lambda_i^\alpha]$ with $\lambda_i^0 := p_{i,L}$ and $I_i^{k_i+1} = [\lambda_i^{k_i}, p_{i,R}]$.
- A collection $E_i^\alpha \to I_i^\alpha$ of hermitian vector bundles over the subintervals with ranks $rk(E_i^\alpha) = R_i^\alpha$ such that on the λ-points the matching conditions

$$E_i^\alpha|_{\lambda_i^\alpha} \subset E_i^{\alpha+1}|_{\lambda_i^\alpha} \ ,$$

  if $R_i^\alpha \leq R_i^{\alpha+1}$ or vice versa, if $R_i^\alpha \geq R_i^{\alpha+1}$. We write $E = \{E_i^\alpha\}$ for the collection of hermitian bundles. Further, denote $\Lambda_0 \subset \Lambda$ the subspace of λ-points, where the rank of the bundles do not change.
- A collection $W = \{W_\lambda | \lambda \in \Lambda_0\}$ of one dimensional hermitian spaces.

This definition gives a suitable generalization of the vector space part in the representation of quivers.

We can now define data on such a bow representation. In the same manner as for a quiver representation, where data can be defined as homomorphisms between the vector spaces over the vertices along the arrows (i.e. a homomorphism is assigned to each arrow), we have three different kinds of data for a bow representation (as we consider vector bundles over intervals here instead of vector spaces over vertices, we have more than one kind of data to define)[Che11]:

- The *bifundamental data* is the analogy of the data on the arrows of a quiver representation. We assign a homomorphism to every arrow $e_{ij} \in \mathcal{E}$ and second one to the reverse arrow $e^{ij} \in \bar{\mathcal{E}}$. Thus, the space of bifundamental data is

$$\mathcal{B} \oplus \bar{\mathcal{B}} := \bigoplus_{e \in \mathcal{E}} \left(\mathrm{Hom}\left(E|_{t(e)}, E|_{h(e)}\right) \oplus \mathrm{Hom}\left(E|_{h(e)}, E|_{t(e)}\right)\right).$$

- The *fundamental data* correspond to the hermitian spaces W. For every $\lambda \in \Lambda_0$ we have homomorphisms between W_λ and the corresponding fiber $E|_\lambda$. We define the space of fundamental data as

$$\mathcal{F}_{\text{in}} \oplus \mathcal{F}_{\text{out}} := \bigoplus_{\lambda \in \Lambda_0} \left(\mathrm{Hom}\left(W_\lambda, E|_\lambda\right) \oplus \mathrm{Hom}\left(E|_\lambda, W_\lambda\right)\right).$$

- Finally, the *Nahm data* define the behaviour on the intervals itself. They are essentially given by four sections $T_0, ..., T_3$ in the endomorphism bundle $\mathrm{End}(E)$ on every hermitian bundle over each interval. Precisely, for an hermitian bundle $E \to I$ we have the space $\mathrm{Con}(E)$ of connections on E of the form $\nabla = \frac{d}{ds} - iT_0$ where T_0 is a section in $\Gamma(\mathrm{End}(E))$. Then, we define the space of Nahm data as follows:
$$\mathcal{N} := \bigoplus_{i,\alpha} \left(\mathrm{Con}(E_i^\alpha) \oplus \mathrm{End}(E_i^\alpha) \otimes \mathbb{R}^3\right).$$

Definition 3.3. The space of bow data (of a chosen bow representation $\mathcal{R}$) is the space $\mathrm{Dat}(\mathcal{R})$ given by
$$\mathrm{Dat}(\mathcal{R}) = \mathcal{F}_{\mathrm{in}} \oplus \mathcal{F}_{\mathrm{out}} \oplus \mathcal{B} \oplus \bar{\mathcal{B}} \oplus \mathcal{N}.$$
Data sets are elements $(I, J, B, A, (T_0, T_1, T_2, T_3)) \in \mathrm{Dat}(\mathcal{R})$.

There is a 2-sphere of complex structures on $\mathrm{Dat}(\mathcal{R})$ coming from the identification of the fundamental data, the bifundamental data and the Nahm data with quaterions, i.e. these spaces carry a natural hyperkähler structure. This structure shall be explained now. Let us choose a 2 dimensional representation $\{e_i\}$, $i = 1, 2, 3$ of the quaternionic units with representation space S. In terms of Pauli matrices we have

$$e_1 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad e_2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad e_3 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If not stated otherwise, we will use this representation from now on. It may look a bit inconvenient, but it will be very useful later, when we introduce a complex form of Nahm matrices. With this we can write the data as follows [Che11]:

- For the fundamental data we write
$$Q := \bigoplus_{\lambda \in \Lambda_0} Q_\lambda : W \to S \otimes E(\Lambda_0)$$
where we define
$$Q_\lambda = \begin{pmatrix} J_\lambda^\dagger \\ I_\lambda \end{pmatrix} : W_\lambda \to S \otimes E|_\lambda.$$

- For the bifundamental data we have
$$B^- := \oplus_{e \in \mathcal{E}} B_e^- : E_L \to S \otimes E_R, \qquad B^+ := \oplus_{e \in \mathcal{E}} B_e^+ : E_R \to S \otimes E_L$$
with
$$B_e^- = \begin{pmatrix} B_e^\dagger \\ -A_e \end{pmatrix} : E|_{t(e)} \to S \otimes E|_{h(e)}, \qquad B_e^+ = \begin{pmatrix} A_e^\dagger \\ B_e \end{pmatrix} : E|_{h(e)} \to S \otimes E|_{t(e)}.$$

- For the Nahm data we write

$$\mathbf{T} := 1 \otimes T_0 + e_1 \otimes T_1 + e_2 \otimes T_2 + e_3 \otimes T_3$$

together with its quaternionic conjugate

$$\mathbf{T}^* = 1 \otimes T_0 - e_1 \otimes T_1 - e_2 \otimes T_2 - e_3 \otimes T_3.$$

The action of the quaternionic units is now via left multiplication

$$e_j : (dQ, dB^\pm, d\mathbf{T}) \mapsto \left((e_j \otimes 1)dQ, (e_j \otimes 1)dB^\pm, (e_j \otimes 1)d\mathbf{T}\right).$$

As a hyperkähler space $\mathrm{Dat}(\mathcal{R})$ has a metric (the direct product metric of the flat hyperkähler metric on the 3 parts of the data) given by

$$\mathrm{d}s^2 = \mathrm{Tr}\,\mathrm{d}Q^\dagger \mathrm{d}Q + \mathrm{Tr}\,\mathrm{d}(B^\pm)^\dagger \mathrm{d}B^\pm + \frac{1}{2}\int \mathrm{Tr}_S\,\mathrm{Tr}\,\mathrm{d}\mathbf{T}^* \mathrm{d}\mathbf{T}\mathrm{d}s, \tag{3.1}$$

which is compatible with the hyperkähler structure. The action of the hyperkähler structure gives rise to 3 Kähler forms $\omega_j(\cdot,\cdot) = g(\cdot, e_j \cdot)$ which are of the form

$$\begin{aligned}\omega :=& e_1 \otimes \omega_1 + e_2 \otimes \omega_2 + e_3 \otimes \omega_3 \\ =& \mathrm{Im}\left(\mathrm{Tr}\,\mathrm{d}Q \wedge \mathrm{d}Q^\dagger + \mathrm{Tr}\,\mathrm{d}B^\pm \wedge \mathrm{d}(B^\pm)^\dagger + \frac{1}{2}\int \mathrm{d}\mathbf{T} \wedge \mathrm{d}\mathbf{T}^* \mathrm{d}s\right).\end{aligned} \tag{3.2}$$

3.1.2 Gauge group, moment maps and bow varieties

On $\mathrm{Dat}(\mathcal{R})$ there is a natural gauge group action preserving the metric. Let $\mathcal{G}$ be the group of gauge transformations on the collection E of hermitian bundles over I, i.e. a gauge transformation $g \in \mathcal{G}$ is a smooth $U(n)$-valued function, where $n = \mathrm{rk}(E_i^\alpha)$ which may vary on different (sub)-intervals. If $\mathrm{rk}(E_i^\alpha) < \mathrm{rk}(E_i^{\alpha+1})$ we have

$$g(\lambda_+) = \begin{pmatrix} 1 & 0 \\ 0 & g(\lambda_-) \end{pmatrix}$$

where λ_- and λ_+ denote the limits from below and from above (analogously for $\mathrm{rk}(E_i^\alpha) > \mathrm{rk}(E_i^{\alpha+1})$). This gives rise to a gauge group action on the data (cf. [Kro04],[Che11]). It is given by (for simplicity we oppress the indices labelling the intervals and subintervals):

$$g(s) : \begin{pmatrix} T_0(s) \\ T_i(s) \\ B \\ A \\ I \\ J \end{pmatrix} \mapsto \begin{pmatrix} g(s)T_0(s)g^{-1}(s) + \frac{\mathrm{d}g(s)}{\mathrm{d}s}g^{-1}(s) \\ g(s)T_i(s)g^{-1}(s) \\ g(p_L)Bg^{-1}(p_R) \\ g(p_R)Ag^{-1}(p_L) \\ g(\lambda)I \\ Jg^{-1}(\lambda) \end{pmatrix}. \tag{3.3}$$

In terms of the quaternionic notation of the data this is of the form

$$g(s):\begin{pmatrix}\mathbf{T}(s)\\ B^+\\ B^-\\ Q\end{pmatrix}\mapsto\begin{pmatrix}g(s)\mathbf{T}(s)g^{-1}(s)+\frac{\mathrm{d}g(s)}{\mathrm{d}s}g^{-1}(s)\\ g(p_L)B^+g^{-1}(p_R)\\ g(p_R)B^-g^{-1}(p_L)\\ g(\lambda)Q\end{pmatrix}.$$

In this notation we observe immediately that the metric and the hyperkähler structure is preserved under this action. Therefore, we are able to perform a hyperkähler reduction [HKLR87].

Definition 3.4. [Che09],[Che11]: A *bow variety* is the hyperkähler quotient of the space $\mathrm{Dat}(\mathcal{R})$ of data of a representation of a bow with the gauge group $\mathcal{G}$ acting on $\mathrm{Dat}(\mathcal{R})$ as above.

$$\mathcal{M}=\mathrm{Dat}(\mathcal{R})/\!/\!/\mathcal{G}. \tag{3.4}$$

This moduli space is again hyperkähler as it is a hyperkähler quotient.

As a bow variety is hyperkähler, we have again a 2-sphere of complex structures and $\mathcal{M}$ can be viewed as a complex variety in each of them. There is a natural action of the complexified gauge group $\mathcal{G}^{\mathbb{C}}$ on $\mathrm{Dat}(\mathcal{R})$, which will be explained in detail later. As complex varieties $\mathcal{M}$ is isomorphic to the complex symplectic quotient of $\mathrm{Dat}(\mathcal{R})$ by the complexified gauge group $\mathcal{G}^{\mathbb{C}}$ (cf. [Kro04]):

$$\mathcal{M}=\mathrm{Dat}(\mathcal{R})/\!/\!/\mathcal{G}=\mathrm{Dat}(\mathcal{R})/\!/\mathcal{G}^{\mathbb{C}}.$$

We now want to perform the hyperkähler reduction to obtain an explicit description of such a bow variety. As bows can become very complicated we want to restrict ourselves to the most simple case which is good enough to obtain all kinds of boundary conditions that can occur.

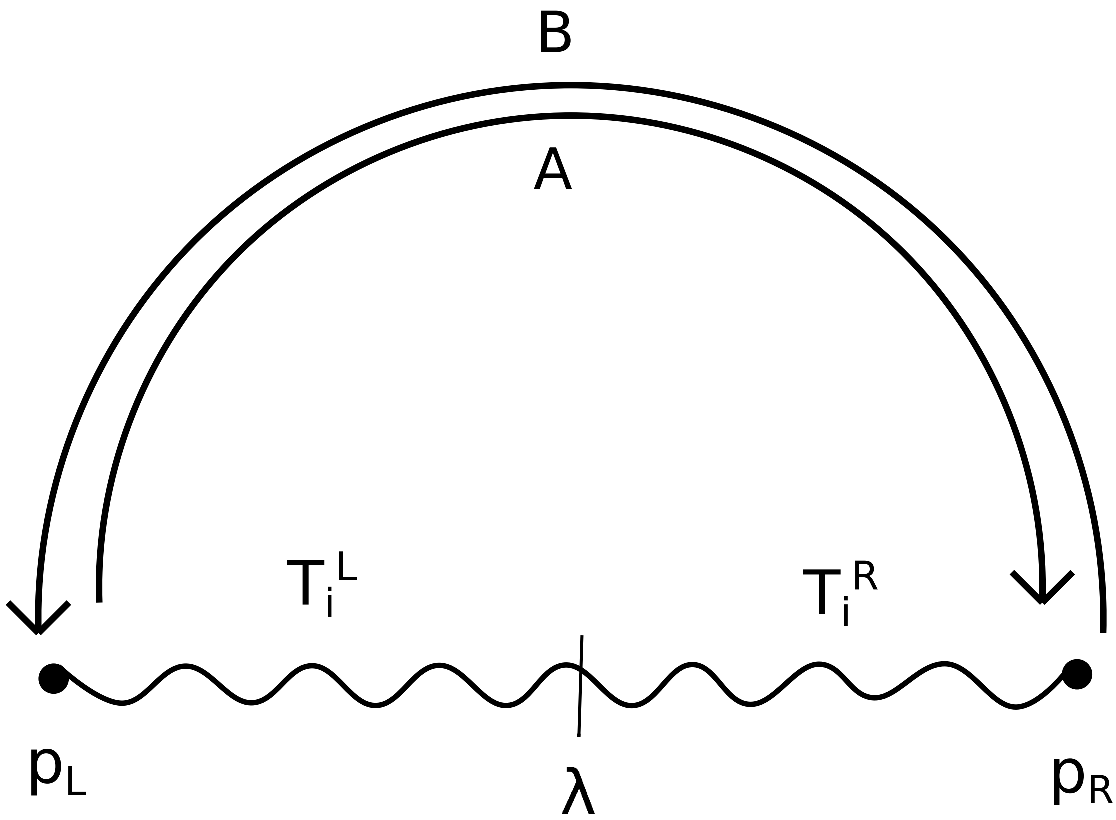

Figure 3.1: Bow diagram for the computation of the boundary conditions

This is a bow consisting of one interval $[p_L, p_R]$ together with a single arrow from the right end to the left end of the interval and one arrow in the reverse direction. We choose a representation with one λ-point at $s = \lambda$ and bundles $E_L \to [p_L, \lambda]$ and $E_R \to [\lambda, p_R]$ both of the same rank n. We can add more intervals and more λ-points and the form of boundary conditions at the special points will remain the same. The only case which is not contained here is the one of more than one arrow starting at or leading to an interval. Conditions for that case are not difficult to obtain, but as this work will mostly not consider those cases, it will not be shown here. This special representation will not lead to an interesting bow variety as shown in [Tak15], but it is sufficient to obtain all possible boundary conditions.

Let us denote by $\mathcal{G}_0$ the subgroup of $\mathcal{G}$ that consists of all gauge transformations which are equal to the identity at the special points $s = p_L$, $s = \lambda$ and $s = p_R$. Then we have

$$\mathcal{G}/\mathcal{G}_0 = G_{p_L} \times G_\lambda \times G_{p_R} = U(n) \times U(n) \times U(n)$$

and therefore we can do the hyperkähler reduction in two steps, first by $\mathcal{G}_0$ and then by $\mathcal{G}/\mathcal{G}_0$ (cf. [Che09]).

As the gauge group acts on the three parts of the data separately, we will obtain three moment maps related to each part of the data. We denote them by $\mu_{\text{Nahm}} : \mathcal{N} \to \mathfrak{g}^* \otimes \mathbb{R}^3$ and analogously for the fundamental and the bifundamental part, where $\mu_{\text{Nahm}} = e_1 \otimes \mu_{1,\text{Nahm}} + e_2 \otimes \mu_{2,\text{Nahm}} + e_3 \otimes \mu_{3,\text{Nahm}}$ in the quaternion representation.

Performing the first step we only need the Kähler form on the Nahm data which is

$$\begin{aligned}
\omega_{\text{Nahm}} =& \operatorname{Im}\left(\frac{1}{2}\int_{p_L}^{p_R} \operatorname{Tr} \mathrm{d}\mathbf{T} \wedge \mathrm{d}\mathbf{T}^\dagger \mathrm{d}s\right) \\
=& \operatorname{Im}\left(\frac{1}{2}\int_{p_L}^{p_R} \operatorname{Tr}\begin{pmatrix} \mathrm{d}T_0 - i\mathrm{d}T_1 & -\mathrm{d}T_2 + i\mathrm{d}T_3 \\ \mathrm{d}T_2 + i\mathrm{d}T_3 & \mathrm{d}T_0 + i\mathrm{d}T_1 \end{pmatrix} \wedge \begin{pmatrix} \mathrm{d}T_0 + i\mathrm{d}T_1 & \mathrm{d}T_2 - i\mathrm{d}T_3 \\ -\mathrm{d}T_2 - i\mathrm{d}T_3 & \mathrm{d}T_0 - i\mathrm{d}T_1 \end{pmatrix} \mathrm{d}s\right) \\
=& \int_{p_L}^{p_R} \Big(\operatorname{Tr}(\mathrm{d}T_1 \wedge \mathrm{d}T_0 + \mathrm{d}T_3 \wedge \mathrm{d}T_2)e_1 + \operatorname{Tr}(\mathrm{d}T_2 \wedge \mathrm{d}T_0 + \mathrm{d}T_1 \wedge \mathrm{d}T_3)e_2 \\
& \quad + \operatorname{Tr}(\mathrm{d}T_3 \wedge \mathrm{d}T_0 + \mathrm{d}T_2 \wedge \mathrm{d}T_1)e_3 \Big) \mathrm{d}s.
\end{aligned}$$

Let $\mathfrak{g}_0$ be the Lie algebra of $\mathcal{G}_0$, which means elements $\rho \in \mathfrak{g}_0$ satisfy $\rho(p_L) = \rho(\lambda) = \rho(p_R) = 0$. The fundamental vector field of the $\mathcal{G}_0$ action on the Nahm data is then given by

$$\begin{aligned}
X^\#_{\rho(s)} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp\big(-t\rho(s)\big)\cdot\big(T_0(s), T_1(s), T_2(s), T_3(s)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \begin{pmatrix} \exp\big(-t\rho(s)\big) T_0(s) \exp\big(t\rho(s)\big) + \Big(\frac{\mathrm{d}}{\mathrm{d}s}\exp\big(-t\rho(s)\big)\Big)\exp\big(t\rho(s)\big) \\ \exp\big(-t\rho(s)\big) T_i(s) \exp\big(t\rho(s)\big) \end{pmatrix} \\
&= \begin{pmatrix} [T_0(s), \rho(s)] - \frac{\mathrm{d}\rho(s)}{\mathrm{d}s} \\ [T_i(s), \rho(s)] \end{pmatrix}.
\end{aligned}$$

For the moment map we obtain (we only consider the Kähler form ω_1 here, the other two are completely analogous, I is the interval $[p_L, p_R]$)

$$\iota_{X^\#_{\rho(s)}}\omega_{1,\text{Nahm}} = \int_I \operatorname{Tr}\Big(\rho(s)\cdot \mathrm{d}\left([T_0(s), T_1(s)] + [T_2(s), T_3(s)]\right)\Big)\mathrm{d}s + \int_I \operatorname{Tr}\left(\frac{\mathrm{d}\rho(s)}{\mathrm{d}s}\mathrm{d}T_1(s)\right)\mathrm{d}s.$$

We can solve the second integral using integration by parts. The constant term vanishes, as $\rho(p_L) = \rho(p_R) = 0$. We obtain

$$\begin{aligned}
\iota_{X^\#_{\rho(s)}}\omega_{1,\text{Nahm}} =& \int_I \operatorname{Tr}\Big(\rho(s)\cdot \mathrm{d}\left([T_0(s), T_1(s)] + [T_2(s), T_3(s)]\right)\Big)\mathrm{d}s - \int_I \operatorname{Tr}\left(\rho(s)\mathrm{d}\frac{\mathrm{d}T_1(s)}{\mathrm{d}s}\right)\mathrm{d}s \\
=& \int_I \operatorname{Tr}\left(\rho(s)\cdot \mathrm{d}\left([T_0(s), T_1(s)] + [T_2(s), T_3(s)] - \frac{\mathrm{d}T_1(s)}{\mathrm{d}s}\right)\right)\mathrm{d}s \\
=& \langle \rho(s), \mathrm{d}\mu_{1,\text{Nahm}}\rangle.
\end{aligned}$$

Thus, the moment map is exactly the left hand side of Nahm equations and the set $\mu^{-1}(0)$ is given by functions T_0, T_1, T_2, T_3 satisfying Nahm equations

$$\begin{aligned}\frac{\mathrm{d}T_1(s)}{\mathrm{d}s} &= [T_0(s), T_1(s)] + [T_2(s), T_3(s)] \\ \frac{\mathrm{d}T_2(s)}{\mathrm{d}s} &= [T_0(s), T_2(s)] + [T_3(s), T_1(s)] \\ \frac{\mathrm{d}T_3(s)}{\mathrm{d}s} &= [T_0(s), T_3(s)] + [T_1(s), T_2(s)].\end{aligned} \tag{3.5}$$

Observe that there is a point in every orbit of the $\mathcal{G}_0$ action with $T_0 = 0$. As the gauge group is the group of $U(n)$-valued smooth functions, its Lie algebra are $\mathfrak{u}(n)$-valued smooth functions. So choose $\rho \in \mathfrak{g}$ such that $\frac{\mathrm{d}\rho(s)}{\mathrm{d}s} = -T_0(s)$. As T_0 is analytic, this is possible. Then the action (3.3) with $g(s) = \exp \rho(s)$ sends $T_0(s)$ to $\tilde{T}_0(s) \equiv 0$. Thus, we have a description for the hyperkähler quotient

$$\mathcal{M}_0 = \mathrm{Dat}(\mathcal{R}) /\!/\!/ \mathcal{G}_0 = \mu^{-1}_{\mathrm{Nahm}}(0)/\mathcal{G}_0$$

as the space $\mathrm{Dat}(\mathcal{R})$, where $T_0 \equiv 0$ and the remaining Nahm data satisfies the (reduced) Nahm equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} T_1(s) =& [T_2(s), T_3(s)] \\ \frac{\mathrm{d}}{\mathrm{d}s} T_2(s) =& [T_3(s), T_1(s)] \\ \frac{\mathrm{d}}{\mathrm{d}s} T_3(s) =& [T_1(s), T_2(s)].\end{aligned} \tag{3.6}$$

Now we can do the second step and build the hyperkähler quotient of $\mathcal{M}_0$ with $\mathcal{G}/\mathcal{G}_0 = U(n) \times U(n) \times U(n)$.

As we now know that the T_i satisfy Nahm equations, we have

$$\begin{aligned}\iota_{X^\#_{\rho(s)}} \omega_{1,\mathrm{Nahm}} &= \int_I \mathrm{Tr}\left(\rho(s) \cdot \mathrm{d}\left([T_0(s), T_1(s)] + [T_2(s), T_3(s)]\right)\right) \mathrm{d}s + \int_I \mathrm{Tr}\left(\frac{\mathrm{d}\rho(s)}{\mathrm{d}s} \mathrm{d}T_1(s)\right) \mathrm{d}s \\ &= \int_I \mathrm{Tr}\left(\rho(s) \cdot \mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}s} T_1(s)\right)\right) \mathrm{d}s + \int_I \mathrm{Tr}\left(\frac{\mathrm{d}\rho(s)}{\mathrm{d}s} \mathrm{d}T_1(s)\right) \mathrm{d}s \\ &= \mathrm{Tr}\left(\rho(s) \cdot \mathrm{d}T_1(s)\right)\Big|_{\partial I} - \int_I \mathrm{Tr}\left(\frac{\mathrm{d}\rho(s)}{\mathrm{d}s} \mathrm{d}T_1(s)\right) \mathrm{d}s + \int_I \mathrm{Tr}\left(\frac{\mathrm{d}\rho(s)}{\mathrm{d}s} \mathrm{d}T_1(s)\right) \\ &= \mathrm{Tr}\left(\rho(s) \cdot \mathrm{d}T_1(s)\right)\Big|_{\partial I} \\ &= \langle \rho(s), \mathrm{d}\mu_{1,\mathrm{Nahm}}(s) \rangle\big|_{\partial I},\end{aligned}$$

where we used integration by parts now for the first integral. As we want to obtain the moment maps at the special points $s = p_L, \lambda, p_R$, we have to choose the intervals

$I_L = [p_L, \lambda]$ and $I_R = [\lambda, p_R]$. The moment maps for the Nahm data at the boundary points are therefore ($i = 1, 2, 3$)

$$\begin{aligned}\mu_{i,\mathrm{Nahm}}(p_R) =& T_i^R(p_R),\\ \mu_{i,\mathrm{Nahm}}(\lambda^+) =& -T_i^R(\lambda^+),\\ \mu_{i,\mathrm{Nahm}}(\lambda^-) =& T_i^L(\lambda^-),\\ \mu_{i,\mathrm{Nahm}}(p_L) =& -T_i^L(p_L),\end{aligned} \tag{3.7}$$

where the T^L denote the Nahm data on the inteval I_L and T^R the ones on I_R.

Analogously, we obtain for the bifundamental data a Kähler form

$$\begin{aligned}\omega_{\mathrm{Arrow}} =& \frac{i}{2}\operatorname{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}B - \mathrm{d}A \wedge \mathrm{d}A^\dagger\right)e_1 + \frac{1}{2}\operatorname{Tr}\left(\mathrm{d}A \wedge \mathrm{d}B - \mathrm{d}B^\dagger \wedge \mathrm{d}A^\dagger\right)e_2\\ &- \frac{i}{2}\operatorname{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}A^\dagger + \mathrm{d}A \wedge \mathrm{d}B\right)e_3.\end{aligned}$$

Together with the fundamental vector fields for the actions at $s = p_R$ and $s = p_L$

$$X_\rho^\#(p_R) = \begin{pmatrix} -\rho A \\ A^\dagger \rho \\ B\rho \\ -\rho B^\dagger \end{pmatrix} \qquad \text{and} \qquad X_\rho^\#(p_L) = \begin{pmatrix} A\rho \\ -\rho A^\dagger \\ -\rho B \\ B^\dagger \rho \end{pmatrix}$$

we obtain moment maps

$$\begin{aligned}\mu_{1_{\mathrm{Bifund}}}(p_R) =& \frac{i}{2}\left(AA^\dagger - B^\dagger B\right),\ \mu_{2_{\mathrm{Bifund}}}(p_R) = \frac{1}{2}\left(B^\dagger A^\dagger - AB\right),\ \mu_{3_{\mathrm{Bifund}}}(p_R) = \frac{i}{2}\left(B^\dagger A^\dagger + AB\right)\\ \mu_{1_{\mathrm{Bifund}}}(p_L) =& \frac{i}{2}\left(BB^\dagger - A^\dagger A\right),\ \mu_{2_{\mathrm{Bifund}}}(p_L) = \frac{1}{2}\left(BA - A^\dagger B^\dagger\right),\ \mu_{3_{\mathrm{Bifund}}}(p_L) = -\frac{i}{2}\left(A^\dagger B^\dagger + BA\right).\end{aligned} \tag{3.8}$$

Finally, for the fundamental data we obtain completely analogous

$$\mu_{1,\mathrm{Fund}}(\lambda) = \frac{i}{2}\left(II^\dagger - J^\dagger J\right),\ \mu_{2,\mathrm{Fund}}(\lambda) = \frac{1}{2}\left(J^\dagger I^\dagger - IJ\right),\ \mu_{3,\mathrm{Fund}}(\lambda) = \frac{i}{2}\left(J^\dagger I^\dagger + IJ\right). \tag{3.9}$$

Now (3.7), (3.8) and (3.9) give us boundary conditions for all special points namely the λ-points and the ends of the interval. For performing the hyperkähler reduction we need to consider the space $\mu^{-1}(c)$ with $c = c_1e_1 + c_2e_2 + c_3e_3$. For a λ-point we obtain

$$\begin{aligned}T_1^R(\lambda_+) - T_1(\lambda_-) =& \frac{i}{2}\left(J^\dagger J - II^\dagger\right) + c_1\mathrm{Id}\\ T_2^R(\lambda_+) - T_2^L(\lambda_-) =& \frac{1}{2}\left(IJ - J^\dagger I^\dagger\right) + c_2\mathrm{Id},\\ T_3^R(\lambda_+) - T_3^L(\lambda_-) =& -\frac{i}{2}\left(J^\dagger I^\dagger + IJ\right) + c_3\mathrm{Id},\end{aligned} \tag{3.10}$$

while we have for the left end of the interval we have

$$\begin{aligned} T_1^L(p_L) &= -\frac{i}{2}\left(A^\dagger A - BB^\dagger\right) - c_1\mathrm{Id} \\ T_2^L(p_L) &= \frac{1}{2}\left(BA - A^\dagger B^\dagger\right) - c_2\mathrm{Id} \\ T_3^L(p_L) &= -\frac{i}{2}\left(A^\dagger B^\dagger + BA\right) - c_3\mathrm{Id} \end{aligned} \tag{3.11}$$

and on the right end

$$\begin{aligned} T_1^R(p_R) &= -\frac{i}{2}\left(AA^\dagger - B^\dagger B\right) + c_1\mathrm{Id} \\ T_2^R(p_R) &= -\frac{1}{2}\left(B^\dagger A^\dagger - AB\right) + c_2\mathrm{Id} \\ T_3^R(p_R) &= -\frac{i}{2}\left(B^\dagger A^\dagger + AB\right) + c_3\mathrm{Id}. \end{aligned} \tag{3.12}$$

These are the boundary conditions the data have to satisfy. They will extend to more complicated bows and have exactly the same form for different λ-points and for the ends of different intervals with one exception: If there are more than one arrow starting or ending at an interval, we will have data A_i and B_i for every arrow and the boundary condition will be the sum over all those data.

Therefore, the bow variety will be the quotient of the space $\mathrm{Dat}(\mathcal{R})$, where the Nahm data satisfy Nahm equations (3.6) and the fundamental and bifundamental data satisfy the boundary conditions (3.10)-(3.12) by the k-th product of $U(n)$ with itself $U(n)^k$ where $k = \#\lambda\text{-points} + 2\,\#\,\text{intervals}$.

3.1.3 Bow varieties and Yang-Mills instantons

To finish this section, we want to work out the relation between bow varieties and Yang-Mills instantons. Therefore, we follow chapter 6 of [Che11]. First, we have the following definition:

Definition 3.5. Let M be a 4-dimensional Riemannian manifold and $E \to M$ a hermitian vector bundle over M. A connection $\mathrm{d} + A$ is called a *Yang-Mills instanton*, if its curvature form $F = \mathrm{d}A + A \wedge A$ is self dual and the Chern number of the bundle is finite $\int_M \mathrm{Tr}(F \wedge F) \leq \infty$.

The idea is now to define a differential operator whose kernel can be written as a hermitian bundle over the bow variety such that the induced connection is self dual. This operator is built out of the boundary conditions for the bow variety (3.10) - (3.12).

Let $\mathcal{D} : \Gamma(S \otimes E) \to \Gamma(S \otimes E) \oplus W \oplus E_L \oplus E_R$ be the Dirac operator given by

$$\mathcal{D} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}s} + i\mathbf{T}^* \\ Q^\dagger \\ (B^-)^\dagger \\ (B^+)^\dagger \end{pmatrix}, \tag{3.13}$$

where we use the quaternionic representation of the bow data. Its conjugate operator is $\mathcal{D}^\dagger : \Gamma(S \otimes E) \oplus W \oplus E_L \oplus E_R \to \Gamma(S \otimes E)$ given by

$$\mathcal{D}^\dagger = \frac{\mathrm{d}}{\mathrm{d}s} - i\mathbf{T} + \sum_{\lambda \in \Lambda_0} \delta(s-\lambda) Q_\lambda + \sum_{e \in \mathcal{E}} (\delta(s-t(e)) B_e^- + \delta(s-h(e)) B_e^+).$$

It is an easy calculation to see that the moment map is of the form

$$\mu(Q, B, \mathbf{T}) = \mathrm{Im}(-i)\mathcal{D}^\dagger \mathcal{D}. \tag{3.14}$$

Cherkis showed in [Che11] that the bow varieties of representations without any λ-points and with all bundles being of rank 1 for some special bow diagrams can be identified with A_k-ALF spaces, i.e. with (Multi)-Taub-NUT spaces. This will also be demonstrated in section 3.2. We call this representation a small representation and denote all the corresponding data with a subscript $\mathfrak{s}$. We now define a large representation of the same bow diagram that includes λ-points and vector bundles of higher rank. The data of that representation are denoted with a subscript $\mathfrak{L}$. Every such large representation of the same bow defines an instanton over the underlying ALF space, as the following construction shows.

We choose the level sets of the moment maps to be $\mu_{\mathfrak{L}}^{-1}(-c)$ and $\mu_{\mathfrak{s}}^{-1}(c)$ and define the twisted dirac operator

$$\mathcal{D}_t = \mathcal{D}_{\mathfrak{L}} \otimes \mathrm{Id}_{E_{\mathfrak{s}}} + \mathrm{Id}_{E_{\mathfrak{L}}} \otimes \mathcal{D}_{\mathfrak{s}}, \tag{3.15}$$

where $\mathcal{D}_{\mathfrak{s}}$ and $\mathcal{D}_{\mathfrak{L}}$ are of the form (3.13) and the moment maps are of the form (3.14). With our choice of the moment maps, we can directly calculate that the operator $\mathcal{D}_t^\dagger \mathcal{D}_t$ is purely real, i.e. $\mathrm{Im}\mathcal{D}_t^\dagger \mathcal{D}_t = 0$. We have even more: Away from a degenerate locus which does not lie in $\mu_{\mathfrak{L}}^{-1}(-c) \times \mu_{\mathfrak{s}}^{-1}(c)$ (cf. [Che11]) the operator is strictly positive and therefore the kernel $\ker(\mathcal{D}_t) = \emptyset$. Again according to [Che11], $\mathcal{D}_t$ is a Fredholm operator, which means the cokernel of $\mathcal{D}_t$ i.e. the kernel of $\mathcal{D}_t^\dagger$ is of finite dimension.

This means, if we fix a bow solution $\mu_{\mathfrak{L}}(-c)$ we obtain a hermitian vector bundle $\ker(\mathcal{D}_t^\dagger) \to \mu_{\mathfrak{s}}^{-1}(c)$. We can view this bundle as a subbundle of the trivial bundle, where the fibres are given by the spaces of sections of $S \otimes E_{\mathfrak{L}} \otimes E_{\mathfrak{s}} \to \mu_{\mathfrak{s}}^{-1}(c)$. The trivial connection induces now a connection on the subbundle $\ker(\mathcal{D}_t^\dagger)$.

The gauge group $\mathcal{G}$ acts on the base $\mu_{\mathfrak{s}}^{-1}(c)$ as described above and simultaneously on the bundle mapping the corresponding fibres into each other. If we trivialize the bundle $\ker(\mathcal{D}_t^\dagger)$ along the orbits of this action we can take the quotient to obtain the bow variety $\mathcal{M} = \mu_{\mathfrak{s}}^{-1}(c)/\mathcal{G}$ and the connection descends to the quotient. Cherkis shows in [Che11] that this connection is self dual and therefore defines a Yang-Mills instanton over the underlying ALF space given by $\mathcal{M}$.

Some techniques used for the construction of bow varieties (and later for the asymptotic metric) are similar to the ones that are used for describing monopoles. Therefore, also the boundary conditions we obtain are similar. This is not surprising, because in both cases Nahm equations play an important role and it is well known that these equations can be obtained from Bogomolny equations for monopoles. For more details about that correspondence see e.g. [Nah81] or [Hit83] for the case of $SU(2)$ monopoles. For $SU(n)$ monopoles things become more complicated, compare also [HM89].

3.2 Nahm equations

In this section we want to discuss Nahm equations. There are 2 basic known solutions (namely solutions on a line for rank 1 and 2), which we will present as they are a basis for the instantons we will consider later. Finally, we will present two different forms of Nahm equations, which will be needed later in this work. These are a complex form and the Basu-Harvey-Terashima equations, which are a lift of Nahm equations on the double cover. At the end of this section, we will show as an application, that bow representations of higher rank but without any λ-points of the special bow diagrams, that lead to A_k-ALF spaces, do not contain any new information, but are just the multiple product of the ALF spaces with itself.

3.2.1 Solutions to Nahm equations of rank 1 and 2 and the Taub-NUT space

In this section we will recall some well known solutions for Nahm equations of low ranks, especially for rank 1 and 2. Rank 1 only allows the trivial solution, as we will see, but it is important nonetheless as we can write down bow diagrams of rank 1 such that the corresponding bow varieties are the Taub-NUT space and higher A_k-ALF spaces (i.e. multi Taub-NUT spaces). The rank 2 Nahm equations are therefore the first case that has a non-trivial solution which was constructed in e.g. [Dan93].

Starting with the rank 1 case, the Nahm matrices are $\mathfrak{u}(1)$-valued functions and as such they commute. Therefore, the Nahm equations tell us that the T_i, $i = 1, 2, 3$ have to be constant.

Let us consider a bow consisting of one interval $I = [p_L, p_R]$ with Nahm data of rank 1 on it and an arrow leading from p_R to p_L. As we have seen above we can choose a gauge such that T_0 is constant. As the gauge group consists of $U(1)$ valued functions, we can choose the constant $T_0 \in S^1$. Nahm equations $\frac{\mathrm{d}}{\mathrm{d}s} T_i(s) = \frac{1}{2}\epsilon_{ijk}[T_j(s), T_k(s)]$ are trivial now, because the commutator of rank 1 matrices always vanishes and the only solution is $T_i = \text{const}$ for $i = 1, 2, 3$. Hence, the Nahm data are just given by $\mathbb{R}^3 \times S^1$. The fibres E_L and E_R of the hermitian vector bundle $E \to I$ over the end points p_L and p_R of the interval I are just identified with $\mathbb{C}$ and therefore, the bifundamental data is given by two homomorphisms $A, B : \mathbb{C} \to \mathbb{C}$, i.e. by two 1-dimensional complex matrices. Thus, the bifundamental data can be identified with $\mathbb{R}^4$. The remaining gauge group is now $U(1) \times U(1)$ acting on the boundary points, but as e.g. $g(p_L)Bg^{-1}(p_R) = e^{i\phi_L}Be^{-i\phi_R} = Be^{i(\phi_L - \phi_R)}$ the action reduces to the action of a single

$U(1)$. Therefore the bow variety is

$$\mathbb{R}^3 \times S^1 \times \mathbb{R}^4 /\!/\!/ U(1).$$

This is the Taub-NUT space with scale parameter given by the length of the interval $l = p_R - p_L$. Observe, that the identification $\mathbb{H} \simeq \mathbb{R}^4$ and the Riemannian submersion $\mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$ whose fibres are S^1 induced by the moment map $\mu = \frac{1}{2} q i \bar{q}$ corresponding to the one parameter family of right multiplications on $\mathbb{H}$ (cf. [GRG97]) leads to the same construction of the Taub-NUT space as in section 2.1.4.

In a similar way, one obtains n-centred multi Taub-NUT spaces as the bow varieties corresponding to bow diagrams consisting of n intervals with rank 1 Nahm data and arrows between the intervals such that the diagrams forms a circle (cf. [Che11],[GRG97]).

Let us now consider the first non-trivial case of Nahm equations on an interval of rank 2. The following construction can be found in [Dan93]. We consider the space C of quadruplets (T_0, T_1, T_2, T_3) of analytic $\mathfrak{u}(2)$-valued function on the interval $I = [p_L, p_R]$ with T_1, T_2 and T_3 having simple poles at $s = p_L$ such that the residues form an irreducible representation of $\mathfrak{su}(2)$, i.e. the residues being given as $-\frac{1}{2}\sigma_i$ where σ_i are the Pauli spin matrices. The group $\mathcal{G}$ of analytic $U(2)$-valued functions acts on C as in (3.3). We consider the 12-dimensional hyperkähler quotient $M_{12} = C /\!/\!/ \mathcal{G}_0^0$ where $\mathcal{G}_0^0$ denotes the subgroup of $\mathcal{G}$ consisting of transformation that are equal to the identity at p_L and p_R. This quotient is given by the T_i satisfying Nahm equations as we have already seen.

There are 3 group actions on M_{12}:

- a $U(2) = \mathcal{G}_0 / \mathcal{G}_0^0$ action where $\mathcal{G}_0$ is the subgroup of transformations equal to identity at p_L,
- a $\mathbb{R}^3$ action $T_j \mapsto T_j - i\lambda_j \mathrm{Id}$ for $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}$,
- a $\mathrm{Spin}(3)$ action as follows: Let $P \in \mathrm{Spin}(3)$ with $P \mapsto A \in SO(3)$ and let $g \in \mathcal{G}$ satisfy $g(p_L) = P$ and $g(p_R) = 0$. Then the mapping

$$\begin{aligned} T_0 \mapsto & g T_0 g^{-1} - \frac{\mathrm{d}g}{\mathrm{d}s} g^{-1} \\ T_i \mapsto & g \left(\sum_j a_{ij} T_j \right) g^{-1} \end{aligned}$$

  defines an action on M_{12}.

We obtain a 5 dimensional space N_5 by taking quotients with respect to $\mathbb{R}^3 \times U(1)$ and with respect to $SU(2)$. This can be realised by making the T_i tracefree and choosing $T_0 = 0$. We have the $SO(3)$ action $T_i \mapsto \sum_j a_{ij} T_j$ descending to the quotient. Details can be found in [Dan93].

On N_5 we have the quantities $\alpha_1, ..., \alpha_5$ of (2.6) being constant in s and can be taken as coordinates. We have an $SO(3)$ equivariant homeomorphism

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto \begin{pmatrix} \frac{1}{3}(\alpha_1 + \alpha_2) & \alpha_3 & \alpha_4 \\ \alpha_3 & \frac{1}{3}(2\alpha_2 - \alpha_1) & \alpha_5 \\ \alpha_5 & \alpha_5 & \frac{1}{3}(\alpha_1 - 2\alpha_2) \end{pmatrix}$$

from N_5 to the space of traceless real symmetric 3x3 matrices. This can be diagonalized by an $SO(3)$ matrix such that the eigenvalues are in increasing order. Therefore, we find that in every $SO(3)$ orbit there is an element with $\langle T_1, T_2\rangle = \langle T_1, T_3\rangle = \langle T_2, T_3\rangle = 0$ and $\langle T_1, T_1\rangle \leq \langle T_2, T_2\rangle \leq \langle T_3, T_3\rangle$. This means the Nahm matrices are of the form

$$T_0 = 0, \qquad T_i = \frac{i}{2} f_i \sigma_i$$

where σ_i are the Pauli spin matrices and the f_i are real functions analytic on $(p_l, p_R]$ with simple poles at p_L satisfying $f_1^2 \leq f_2^2 \leq f_3^2$ and

$$\begin{aligned} \dot{f}_1 &= f_2 f_3, \\ \dot{f}_2 &= f_1 f_3, \\ \dot{f}_3 &= f_1 f_2. \end{aligned} \tag{3.16}$$

We can write down an explicit solution to these equations in terms of Jacobi elliptic function. This solution is the following:

$$\begin{aligned} f_1(s) &= -\frac{D\mathrm{cn}_k(Ds)}{\mathrm{sn}_k(Ds)}, \\ f_2(s) &= -\frac{D\mathrm{dn}_k(Ds)}{\mathrm{sn}_k(Ds)}, \\ f_3(s) &= -\frac{D}{\mathrm{sn}_k(Ds)}. \end{aligned} \tag{3.17}$$

Note that this solution can not be used immediately for describing bow varieties as we have different boundary conditions there. In particular, the boundary conditions for bow varieties do not admit the above $\mathbb{R}^3$ action. Nevertheless, it is worth pointing out that solutions to rank 2 Nahm equations are in terms of Jacobi elliptic functions.

The next question would be, how to use this solution to Nahm equations on an interval to write down the bow variety of a representation consisting of only one interval together with a single arrow but without any λ-points of rank 2 (i.e. the same bow representation that gives rise to the Taub-NUT space but with a rank 2 bundle instead of a rank 1 bundle). This example (even for arbitrary rank n) is discussed at the end of this chapter.

3.2.2 Complex form for Nahm equations

There are several ways to express Nahm equations. A first one have already been studied in chapter 2, when we constructed the spectral curve for rank 2 Nahm equations. It was the equivalence between Nahm equations and Lax pair equations. Another one is the complex form due to Donaldson [Don84]. We can choose any of the complex structures and find a complexification with respect to it. With the representation of the complex structures we have chosen in terms of Pauli matrices, it seems natural to demonstrate the complexification with respect to I represented by e_1. We then define

$$\begin{aligned}\alpha =& T_0 + iT_1 \\ \beta =& T_2 + iT_3.\end{aligned} \tag{3.18}$$

These are now functions taking values in the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. For the case of Nahm matrices T_i take values in $\mathfrak{u}(n)$ the new functions α and β take values in $\mathfrak{gl}(n,\mathbb{C})$. In this context "with respect to the complex structure I" means that the action of I in $\mathfrak{g}^{\mathbb{C}}$ corresponds to multiplication with i.

We obtain an action of the complexified gauge group $\mathcal{G}^{\mathbb{C}}$ on the data defined by

$$\begin{aligned}\alpha &\mapsto g\alpha g^{-1} + \frac{\mathrm{d}g}{\mathrm{d}s}g^{-1} \\ \beta &\mapsto g\beta g^{-1}\end{aligned} \tag{3.19}$$

and we obtain as before moment map conditions which we read as boundary conditions on the special points and as complexified Nahm equations on the interior of the intervals. These equations are

$$\frac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^\dagger) = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] \tag{3.20}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = [\alpha, \beta]. \tag{3.21}$$

The first equation is real while the second is complex. These two equations are equivalent to Nahm equations in their standard form (in particular, in this form the equation for α is equivalent to the one for T_1 while the equation for β is equivalent to the ones for T_2 and T_3. This is because we haven the complexification with respect to the complex structure I). For the complex equation we have

$$\frac{\mathrm{d}T_2}{\mathrm{d}s} + i\frac{\mathrm{d}T_3}{\mathrm{d}s} = \frac{\mathrm{d}\beta}{\mathrm{d}s} = [\alpha, \beta] = [T_0, T_2] + [T_3, T_1] + i([T_0, T_3] + [T_1, T_2])$$

which yields Nahm equations for T_2 and T_3, whereas for the real equation we have

$$2i\frac{\mathrm{d}T_1}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^\dagger) = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] = 2i([T_0, T_1] + [T_2, T_3]),$$

which simply is the Nahm equation for T_1. The boundary conditions we obtain now are the following:

$$\begin{aligned}\beta^R(\lambda_+) - \beta^L(\lambda_-) =& IJ + c_+\mathrm{Id}, \\ \alpha^R(\lambda_+) - \alpha(\lambda_-) =& \frac{i}{2}\left(II^\dagger - J^\dagger J\right) - ic_1\mathrm{Id},\end{aligned} \tag{3.22}$$

for a λ-point,

$$\begin{aligned}\beta^L(p_L) =& BA - c_+\mathrm{Id} \\ \alpha^L(p_L) =& \frac{1}{2}\left(A^\dagger A - BB^\dagger\right) - ic_1\mathrm{Id}\end{aligned} \tag{3.23}$$

for the left end and

$$\begin{aligned}\beta^R(p_R) =& AB + c_+\mathrm{Id} \\ \alpha^R(p_R) =& \frac{1}{2}\left(AA^\dagger - B^\dagger B\right) + ic_1\mathrm{Id}\end{aligned} \tag{3.24}$$

for the right end of the interval. This is exactly the identification

$$\mathcal{M} = \mathrm{Dat}(\mathcal{R})/\!/\!/\mathcal{G} = \mu^{-1}(0)/\mathcal{G} = (\mu^{\mathbb{C}})^{-1}(0)/\mathcal{G}^{\mathbb{C}} = \mathrm{Dat}(\mathcal{R})/\!/\mathcal{G}^{\mathbb{C}}.$$

A great and widely discussed advantage of this form of Nahm equations is that (as above) we can find a local gauge such that $\alpha \mapsto 0$ and therefore the complex equation is $\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}s} = 0$ (with $g : \beta \mapsto \tilde{\beta}$) with can simply be integrated (c.f. [Don84],[Kro04]). We have the following theorem:

Theorem 3.6. *[Don84] Let (α, β) be a local solution of the complex Nahm equation (3.21). Then there is a gauge $g \in \mathcal{G}^{\mathbb{C}}$ such that $g(\alpha, \beta)$ also satisfies the real equation (3.20).*

Thus, one is left with finding the right gauge, i.e. solving (only) the real equation. The proof of this theorem uses variational methods. But here lies the problem why this straight forward method of solving Nahms equations does not work for bow varieties: The solution obtained via variational methods heavily uses Dirichlet boundary conditions, while the boundary conditions given for bow varieties are of Robin type, i.e. a linear combination of the value and the first derivative of g at the boundary points.

Nevertheless it is worth discussing the complex form for another reason. The form of the boundary conditions (3.22)-(3.24) justifies the consideration of another form of Nahm equations, the so called Basu-Harvey-Terashima equations, which will be discussed in the next section.

3.2.3 The Basu-Harvey-Terashima equations

In this section we discuss the Basu-Harvey-Tershima equations (BHT equations) which turn out to be the lift of Nahm equation to the double cover. These equations are very useful because they are constructed in a way that the boundary conditions with the bifundamental data at the ends of the interval (3.23) and (3.24) are automatically satisfied. In this section we follow the argumentation in [Bie15].

We start with the $U(n) \times U(n)$ action on $\mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ given by

$$(g,h) : (A,B) \mapsto (gAh^{-1}, hBg^{-1}).$$

This is essentially the action of the gauge group on the bifundamental data in the construction of a bow variety and we already computed the moment maps. For the first $U(n)$ it is

$$i\mu_1(A,B) = \frac{1}{2}(AA^\dagger - B^\dagger B), \quad (\mu_2 + i\mu_3)(A,B) = AB$$

and for the second $U(n)$ we obtain

$$i\nu_1(A,B) = \frac{1}{2}(A^\dagger A - BB^\dagger), \quad (\nu_2 + i\mu_3)(A,B) = BA.$$

The important observation here is that the function

$$F = ||\mu_i(A,B)||^2 - ||\nu_i(A,B)||^2 = \frac{1}{2}\,\mathrm{Tr}(A^\dagger ABB^\dagger - B^\dagger BAA^\dagger) \tag{3.25}$$

is independent of $i = 1,2,3$. This leads to the following theorem from [Bie15]:

Theorem 3.7. *Let $m(s) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ be the curve of the gradient flow of the function $F = \frac{1}{2}(||\mu_1(A,B)||^2 - ||\nu_1(A,B)||^2)$. Then the $U(n)$-valued functions $T_i(s) = \mu_i(m(s))$ and $S_i(s) = \nu_i(m(s))$ satisfy Nahm equations (3.6).*

Proof. This can be proven by using the fact that the function F can also be obtained by using μ_i and ν_i for an arbitrary $i = 1,2,3$. This means that the gradient vector field we obtain in all 3 cases must be the same. Since we have

$$g(\nabla\mu^X, Y) = \mathrm{d}\mu^X(Y) = \omega(X,Y) = g(IX,Y)$$

the gradient field is given by

$$I_1X_{\mu_1} - I_1X_{\nu_1} = I_2X_{\mu_2} - I_2X_{\nu_2} = I_3X_{\mu_3} - I_3X_{\nu_3}$$

and therefore Nahm equations can easily be calculated. Using $\mathrm{d}\mu_1(I_2X_{\nu_2}) = 0$ (because X_{ν_2} is constant for $\mathrm{d}\mu_3$ as the actions commute), $\mathrm{d}\mu_1(I_2X_{\mu_2}) = \mathrm{d}\mu_3(X_{\mu_2})$ and the standard identification of the Lie algebra with its dual we have

$$\begin{aligned}\dot{T}_1(s) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}(\mu_1(m(s))) = \mathrm{d}\mu_1(\nabla F) = \mathrm{d}\mu_1(I_2X_{\mu_2} - I_2X_{\nu_2}) \\ &= \mathrm{d}\mu_3(X_{\mu_2}) = [\mu_2,\mu_3](m(s)) = [T_2(s), T_3(s)]\end{aligned}$$

and similarly for cyclic permutations and for the S_i). □

The gradient flow equations of F, which are the BHT equations are given by

$$\begin{aligned}\dot{A}(s) =& \frac{1}{2}(A(s)B(s)B^\dagger(s) - B^\dagger(s)B(s)A(s))\\ \dot{B}(s) =& \frac{1}{2}(A^\dagger(s)A(s)B(s) - B(s)A(s)A^\dagger(s)).\end{aligned} \tag{3.26}$$

Defining

$$iT_1(s) := \frac{1}{2}(A(s)A^\dagger(s) - B^\dagger(s)B(s)), \quad T_2(s) + iT_3(s) := A(s)B(s) \tag{3.27}$$

and

$$iS_1(s) := \frac{1}{2}(A^\dagger(s)A(s) - B(s)B^\dagger(s)), \quad S_2(s) + iS_3(s) := B(s)A(s) \tag{3.28}$$

one can directly compute that for a solution to (3.26) the T_i and S_i both satisfy Nahms equations. All in all we obtain the following result [Bie15]:

Corollary 3.8. *The Basu-Harvey-Terashima equations (3.26) are equivalent to Nahms equations for the T_i (3.27). In addition they imply Nahms equations for the S_i (3.28).*

Of course the role of the T_i and the S_i is interchangeable in this corollary. In other words, a solution to Nahm equations lifts to a solution to BHT equations and a solution to BHT equations can be pushed down to two copies of solutions to Nahm equations. This is, why we can view BHT equations as the double cover of Nahm equations.

This has two benefits now. First, we see that setting $A(s_0) = A$ and $B(s_0) = B$ for some $s_0 \in \mathbb{R}$ the definition of the S_i and T_i automatically satisfy the boundary conditions (3.23) and (3.24) in the complex form for the level set of 0 at that point. Second, we carried the parameter dependency over to the A and B i.e. to the bifundamental data. As the development of A and B along the gradient flow lines of F is equivalent to a solution of Nahm equations we can now take a starting point, where the boundary conditions are satisfied and follow that flow to obtain a solution to Nahm equation to get a description of a bow variety.

3.2.4 Application: Bow varieties of higher rank without λ-points

To finish the chapter about Nahm equations we want to give an application of the correspondence between Nahm and BHT equations. We have already seen that the bow variety of a single interval $[p_L, p_R]$ with Nahm data of rank 1 together with a single arrow leading from p_R to p_L is the Taub-NUT space (for convenience we call a pair of arrows, one leading from the right end to a left end of and interval and the other connecting the same points but with reverse orientation, just as a single arrow from now on). We now take the same bow diagram but with Nahm data of arbitrary rank n. We will show that this setup also has only a trivial solution. Moreover, we show that the solution is n times a copy of the solution of the rank 1 case. To do so, we need one more ingredient from a later chapter which we anticipate here, as this calculation in mainly an application of BHT equations.

For convenience we choose the interval to be $[0,1]$. For the level set $\mu^{-1}(0)$ we have boundary conditions (3.23) at $s=0$ and (3.24) at $s=1$ with $c_1 = 0 = c_+$. First observe that the setup for this bow variety is independent of the choice of the level set $\mu^{-1}(c)$. Taking a level set different from $\mu^{-1}(0)$ such that $c_1, c_+ \neq 0$ we have the following. Let (α,β) be a solution to Nahm equations with the given boundary conditions. Then $(\tilde{\alpha},\tilde{\beta})$ with $\tilde{\alpha}(s) = \alpha(s) + ic_1\mathrm{Id}$ and $\tilde{\beta}(s) = \beta(s) + c_+\mathrm{Id}$ is a solution to Nahm equations (as $ic_1\mathrm{Id}$ and $c_+\mathrm{Id}$ are constant and central in $\mathfrak{u}(n)$) that satisfies boundary conditions with $c_1 = 0 = c_+$. Therefore we only need to consider the case $\mu^{-1}(0)$.

So starting with the described bow diagram and lifting it to the double cover we obtain 2 bow diagrams as follows.

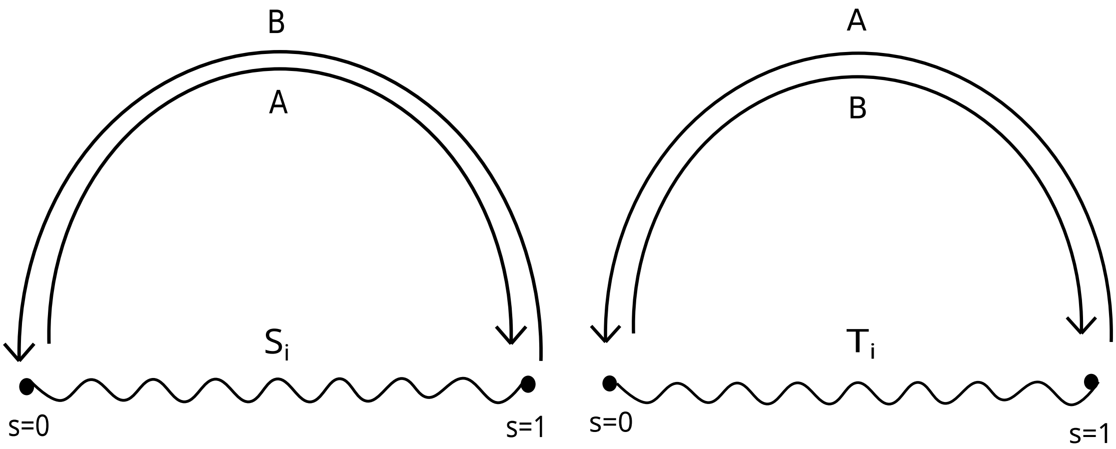

Figure 3.2: Two copies of the same bow corresponding to Basu-Harvey-Terashima equations

Taking $A(0)$ and $B(0)$ from the BHT-equations and setting $A(0) = A$ and $B(0) = B$ with A and B from the diagram we obtain new conditions at $s=1$ for both the T_i and the S_i. To understand this compare the defining equations for BHT formalism (3.27) and (3.28) with the matching conditions of Nahm data with the bifundamental data at the boundary points of a bow diagram in their real (3.11), (3.12) and complex form (3.23) and (3.24). For the T_i we get

$$A(1)B(1) = B(0)A(0), \qquad A(1)A^\dagger(1) - B^\dagger(1)B(1) = A^\dagger(0)A(0) - B(0)B^\dagger(0)$$

and for the S_i we have

$$B(1)A(1) = A(0)B(0), \qquad A^\dagger(1)A(1) - B(1)B^\dagger(1) = A(0)A^\dagger(0) - B^\dagger(0)B(0).$$

In particular, we obtain even more.

Lemma 3.9. *If we just follow the flow of BHT equations we get*

$$(AB)(s+1) = (BA)(s), \qquad \Big(AA^\dagger - B^\dagger B\Big)(s+1) = \Big(A^\dagger A - BB^\dagger\Big)(s)$$

and

$$(BA)(s+1) = (AB)(s), \qquad \Big(A^\dagger A - BB^\dagger\Big)(s+1) = \Big(AA^\dagger - B^\dagger B\Big)(s)$$

for all $s \in \mathbb{R}$.

Let us first discuss the consequences before proving the lemma. If we insert one of these line into the definition of the function F defining BHT equations (3.25) we obtain $F(s+1) = -F(s)$. Further inserting both of the above conditions into each other and then into F we obtain $F(s+2) = F(s)$ for all s. Thus, F is periodic (with period equal to 2 times the length of the interval) and changes sign. Since $\dot{F}(s)$ cannot be negative along a gradient flow curve, and F cannot have any poles as it is a polynomial in the entries of $A(s)$ and $B(s)$ (which are given as solutions to Nahms equations and are therefore analytic), F must be constant and equal to 0 along a gradient flow curve. Thus we also obtain $\dot{F} = 0$ along a curve.

On the other hand, we can use BHT equation (3.26) to obtain

$$\dot{F}(s) = 2\,\mathrm{Tr}(\dot{A}(s)\dot{A}^\dagger(s) + \dot{B}(s)\dot{B}^\dagger(s)).$$

This means $\dot{F} = 0$ if and only if $\dot{A} = 0$ and $\dot{B} = 0$, i.e. we have a stationary solution to BHT equations. The boundary conditions in the above lemma then reduce to

$$AB = BA, \qquad A^\dagger A - BB^\dagger = AA^\dagger - B^\dagger B. \tag{3.29}$$

Let us now prove lemma 3.9:

Proof. Let $A(\xi, s) := A(s) - \xi B^\dagger(s)$ and $B(\xi, s) := B(s) + \xi A^\dagger(s)$. Then we have

$$AB(\xi, s) := A(\xi, s) \cdot B(\xi, s) = A(s)B(s) + \xi(A(s)A^\dagger(s) - B^\dagger(s)B(s)) - \xi^2 B^\dagger(s)A^\dagger(s).$$

Observe that the polynomial part of $\frac{AB(\xi,s)}{\xi}$ is

$$\left(\frac{AB(\xi, s)}{\xi}\right)^+ = A(s)A^\dagger(s) - B^\dagger(s)B(s) - \xi B^\dagger(s)A^\dagger(s).$$

Therefore define

$$AB_\#(\xi, s) := \frac{1}{2}(A(s)A^\dagger(s) - B^\dagger(s)B(s)) - \xi B^\dagger(s)A^\dagger(s).$$

Further observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}AB(\xi,s) =& -\frac{1}{2}\left[A(s)B(s), A(s)A^\dagger(s) - B^\dagger(s)B(s)\right] + \xi\left[A(s)B(s), B^\dagger(s)A^\dagger(s)\right] \\ &+ \frac{1}{2}\xi^2\left[A(s)B(s), A(s)A^\dagger(s) - B^\dagger(s)B(s)\right]^\dagger \\ =& -\left[AB(\xi,s), AB_\#(\xi,s)\right],\end{aligned}$$

so we have a Lax pair equation. The same holds for

$$\frac{\mathrm{d}}{\mathrm{d}s}BA(\xi,s) = -\left[BA(\xi,s), BA_\#(\xi,s)\right]$$

with

$$BA(\xi,s) := B(\xi,s)\cdot A(\xi,s), \quad BA_\#(\xi,s) := \frac{1}{2}(A^\dagger(s)A(s) - B(s)B^\dagger(s)) - \xi A^\dagger(s)B^\dagger(s).$$

Now this is the point where we need to anticipate that there is a 1:1 correspondence between these matrix polynomials satisfying Lax pair equations and linear flows on the Jacobian of the spectral curve determined by these matrix polynomials. The details of this correspondence will be discussed in the next chapter. All we need to know for this proof is the existence of the correspondence, that the flows are linear and that the direction of the flows is given by $AB_\#$ and $BA_\#$ respectively. Note that the spectral curves for both polynomials are the same as they are cut out by $\det(\eta\mathrm{Id} - X(\xi,s)) = 0$, where $X(\xi,s) = AB(\xi,s)$ or $X(\xi,s) = BA(\xi,s)$ respectively and we have $\det(\eta\mathrm{Id} - AB(\xi,s)) = \det(\eta\mathrm{Id} - BA(\xi,s))$.

The boundary conditions at $s = 1$ now turn into

$$AB(\xi, s=1) = BA(\xi, s=0), \qquad AB_\#(\xi, s=1) = BA_\#(\xi, s=0)$$

and

$$BA(\xi, s=1) = AB(\xi, s=0), \qquad BA_\#(\xi, s=1) = AB_\#(\xi, s=0).$$

Both conditions mean that the flows intersect in a point with the same direction but at different times. As the flows are linear this means they are the same but time shifted where the shift is equal to the length of the interval. Further, the flow must be cyclic due to the second intersection point with inverse time shift. All in all we obtain

$$AB(\xi, s+1) = BA(\xi,s), \qquad AB_\#(\xi, s+1) = BA_\#(\xi,s),$$

and also the other way around which directly translates into the conditions we wanted to show. □

Now taking a stationary solution with the tightened boundary conditions we can finally show that every such orbit contains a gauge in which the solution is trivial. The remaining gauge group is a single $U(n)$ acting on both A and B via conjugation.

For a stationary solution the complex form of Nahm equations reduce to (setting $\alpha = A^\dagger A - BB^\dagger$ and $\beta = BA$)

$$\begin{aligned}[\alpha, \beta] =& [A^\dagger A - BB^\dagger, BA] = 0 \\ [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] =& [BA, (BA)^\dagger] = 0.\end{aligned}$$

The first equation says that α and β commute, the second one means that β is normal. α is hermitian via definition and thus also normal. Therefore, both α and β are simultaneously unitary diagonalizable.

We can write

$$\begin{aligned}BA &= \mathrm{diag}(\rho_1, ..., \rho_n) \\ A^\dagger A - BB^\dagger &= \mathrm{diag}(\sigma_1, ..., \sigma_n)\end{aligned}$$

with non of the ρ_i and σ_i equal to zero (this is true since α and β are supposed to be solutions to rank n Nahm equations). Therefore, A and B must be invertible and we can write $B = A^{-1} \cdot \mathrm{diag}(\rho_1, ..., \rho_n)$. It is easy to check that $[BB^\dagger, \alpha] = 0 = [BB^\dagger, \beta]$ and obviously the same holds for $A^\dagger A$. $A^\dagger A$ and $BB^\dagger$ are both hermitian and thus normal and therefore also diagonal in this gauge. Putting this into the second boundary condition we obtain $BB^\dagger = B^\dagger B$ and similar for A and therefore A and B are also normal and as they commute, they are also simultaneously unitary diagonalizable.

So every orbit in $\mu^{-1}(0)$ contains a pair of diagonal matrices (A, B) and therefore the bow variety is just the n-th direct sum of the solution of the rank one case with itself, i.e. the n-th direct sum of the Taub-NUT space with itself.

This method can be generalized to a *circle diagram* below.

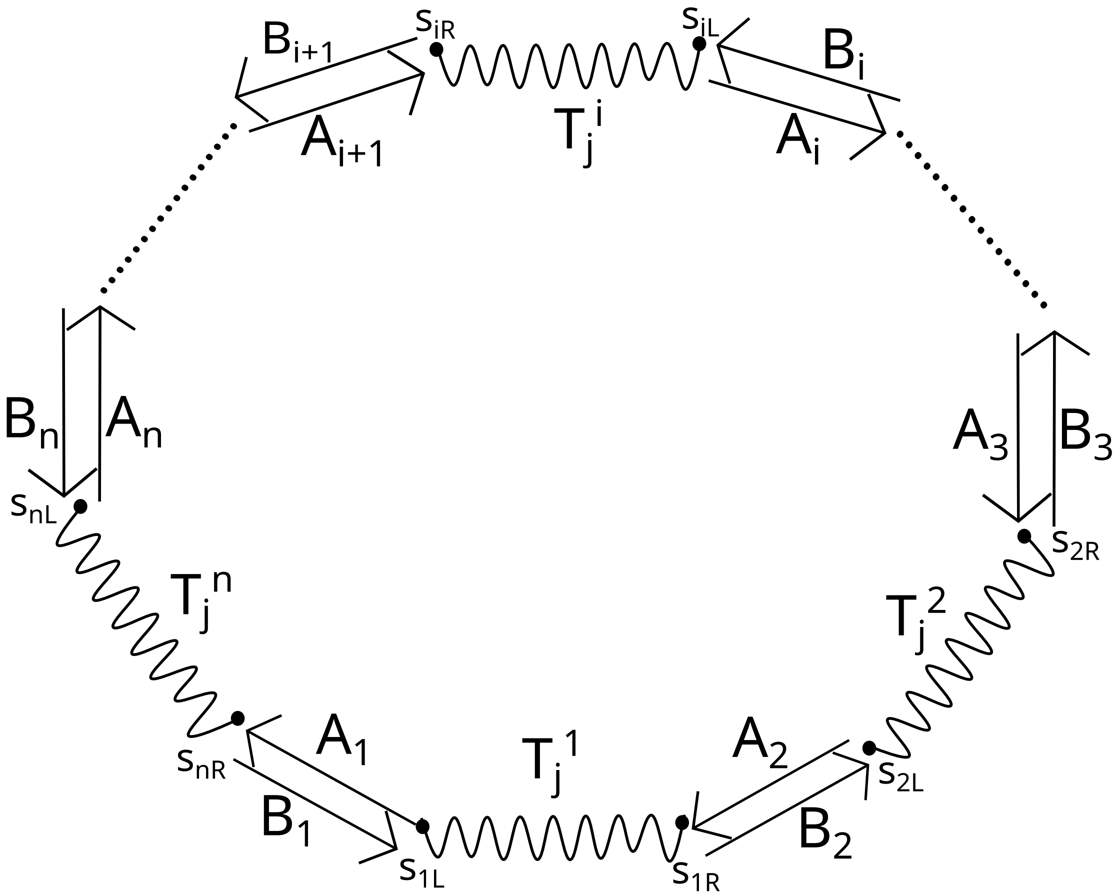

Figure 3.3: Cyclic bow diagram

This diagram consists of n intervals where we parametrise the i-th interval $I_i = [s_{i_L}, s_{i_R}]$ such that all intervals are disjoint. The diagram consists further of n arrows such that the arrow e_i leads from the right end of the Interval I_{i-1} to the left end of the interval I_i. For the diagram to become a circle we have $e_{n+1} = e_1$ leading from the interval I_n to I_1. Cherkis showed in [Che11] that a bow representation of rank 1 corresponds to an n centered multi Taub-NUT space, i.e. to an $A_{n-1}-$ALF space. We consider here a representation such that Nahm data are of rank k on every interval. In our notation the data on the interval I_i are given by Nahm matrices T^i_j with $j = 1, 2, 3$, while the data on the arrow e_i leading from I_{i-1} to I_i is given by the two homomorphisms $A_i : E_i\big|_{s_{i_L}} \to E_{i-1}\big|_{s_{i-1_R}}$ and $B_i : E_{i-1}\big|_{s_{i-1_R}} \to E_i\big|_{s_{i_L}}$. We again lift this bow diagram to its double cover to obtain BHT-equation which we can project to a second bow diagram. We call the original bow as the *first copy* and set

$$\begin{aligned} iT_1^i(s) =& \frac{1}{2}(A_i^\dagger(s)A_i(s) - B_i(s)B_i^\dagger(s)) \\ T_2^i(s) + iT_3^i(s) =& B_i(s)A_i(s) \end{aligned}$$

where we define $A_i(s_{i_L}) := A_i$ and $B_i(s_{i_L}) := B_i$ for $i = 1, ..., n$. With this definition the boundary conditions on the left ends of each interval are automatically satisfied and on the right end we have (here for the interval I_i):

$$\begin{aligned} A_i^\dagger(s_{i_R})A_i(s_{i_R}) - B_i(s_{i_R})B_i^\dagger(s_{i_R}) =& A_{i+1}(s_{i+1_L})A_{i+1}^\dagger(s_{i+1_L}) - B_{i+1}^\dagger(s_{i+1_L})B_{i+1}(s_{i+1_L}) \\ B_i(s_{i_R})A_i(s_{i_R}) =& A_{i+1}(s_{i+1_L})B_{i+1}(s_{i+1_L}). \end{aligned} \tag{3.30}$$

The *second copy* is the same bow diagram but with reverse orientation.

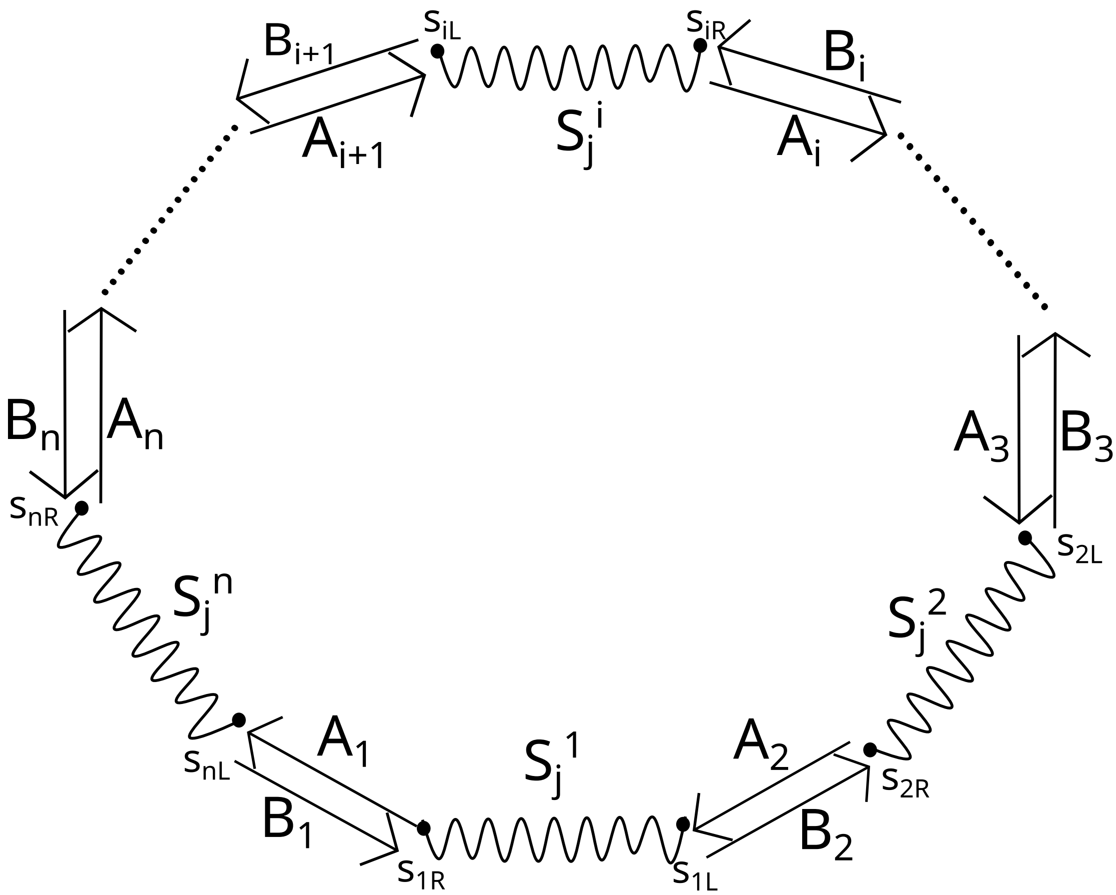

Figure 3.4: Second copy of the cyclic bow diagram

This means the interval $I_i = [s_{i_L}, s_{i_R}]$ still carries Nahm data S_j^i but the arrow e_i now leads from the left end of the interval I_{i-1} to the right end of the interval I_i. This means it is now represented by the following homomorphisms: $A_i : E_i|_{s_{i_R}} \to E_{i-1}|_{s_{i-1_L}}$ and $B_i : E_{i-1}|_{s_{i-1_L}} \to E_i|_{s_{i_R}}$. As the projection of BHT-equations we have

$$\begin{aligned} iS_1^i(s) =& \frac{1}{2}(A_i(s)A_i^\dagger(s) - B_i^\dagger(s)B_i(s)) \\ S_2^i(s) + iS_3^i(s) =& A_i(s)B_i(s) \end{aligned}$$

and we obtain boundary conditions

$$\begin{aligned} A_i(s_{i_R})A_i^\dagger(s_{i_R}) - B_i^\dagger(s_{i_R})B_i(s_{i_R}) =& A_i^\dagger(s_{i_L})A_i(s_{i_L}) - B_i(s_{i_L})B_i^\dagger(s_{i_L}) \\ A_i(s_{i_R})B_i(s_{i_R}) =& B_i(s_{i_L})A_i(s_{i_L}) \end{aligned} \tag{3.31}$$

at the upper end s_{i_R} of the interval I_i and

$$\begin{aligned} A_i(s_{i_L})A_i^\dagger(s_{i_L}) - B_i^\dagger(s_{i_L})B_i(s_{i_L}) =& A_{i+1}(s_{i+1_L})A_{i+1}^\dagger(s_{i+1_L}) - B_{i+1}^\dagger(s_{i+1_L})B_{i+1}(s_{i+1_L}) \\ A_i(s_{i_L})B_i(s_{i_L}) =& A_{i+1}(s_{i+1_L})B_{i+1}(s_{i+1_L}) \end{aligned} \tag{3.32}$$

at the lower end s_{i_L} of the interval I_i. Here, as the $A_i(s)$ and $B_i(s)$ come from BHT-equations, note that they are the same and therefore we still have the identification $A_i(s_{i_L}) := A_i$ and $B_i(s_{i_L}) := B_i$. Combining the boundary conditions (3.30) with (3.32) we can get rid of the A_{i+1} and B_{i+1} in the conditions and together with (3.31) we are left with

$$\begin{aligned} A_i(s_{i_R})A_i^\dagger(s_{i_R}) - B_i^\dagger(s_{i_R})B_i(s_{i_R}) =& A_i^\dagger(s_{i_L})A_i(s_{i_L}) - B_i(s_{i_L})B_i^\dagger(s_{i_L}) \\ A_i(s_{i_L})A_i^\dagger(s_{i_L}) - B_i^\dagger(s_{i_L})B_i(s_{i_L}) =& A_i^\dagger(s_{i_R})A_i(s_{i_R}) - B_i(s_{i_R})B_i^\dagger(s_{i_R}) \\ A_i(s_{i_R})B_i(s_{i_R}) =& B_i(s_{i_L})A_i(s_{i_L}) \\ A_i(s_{i_L})B_i(s_{i_L}) =& B_i(s_{i_R})A_i(s_{i_R}) \end{aligned}$$

which are exactly the conditions we had for lemma 3.9 so we can individually follow the BHT flow on every interval and obtain a stationary solution and observe that there is a diagonal solution in each orbit as before.

Now, boundary equations (3.30) and (3.32) must still be satisfied individually. First, we observe that our remaining gauge freedom reduces to single $U(k)$ action and this means that all the A_i and B_i are simultaneously diagonalized by the same gauge transformation.

Now, if all A_i and B_i are diagonal we can make a second observation. The second equation of say (3.32) gives

$$A_{i+1}^{jj} = \lambda_i^j A_i^{jj}, \qquad B_{i+1}^{jj} = \frac{1}{\lambda_i^j} B_i^{jj}$$

for each $j = 1, ..., k$ where A^{jj} denotes the jj-entry of the matrix A and the $\lambda_i^j \in \mathbb{C}\backslash\{0\}$. With this the first equation of (3.32) reads

$$\lambda_i^j \bar{\lambda}_i^j A_i^{jj} \bar{A}_i^{jj} - \frac{1}{\lambda_i^j \bar{\lambda}_i^j} B_i^{jj} \bar{B}_i^{jj} = A_i^{jj} \bar{A}_i^{jj} - B_i^{jj} \bar{B}_i^{jj}$$

for each $j = 1, ..., k$. This equation has a solutions $\lambda_i^j \bar{\lambda}_i^j = 1$ or $\lambda_i^j \bar{\lambda}_i^j = -\frac{B_i^{jj} \bar{B}_i^{jj}}{A_i^{jj} \bar{A}_i^{jj}} < 0$. As $\lambda_i^j \bar{\lambda}_i^j > 0$ we have a unique solution $\lambda_i^j = \exp(i\varphi_i^j)$. Thus, we have k individual solutions to rank 1 Nahm equations on every interval and the solutions differ only by phases from interval to interval.

4 The spectral picture

In this chapter we want to give a description of bow varieties in the language of spectral data. This includes two steps. First, we will give the equivalent procedure for solving the differential equations on the intervals of a bow diagram in the spectral picture, which will turn out to be way easier to handle. In the second part we will discuss the boundary conditions at a λ-point and their counterparts. Further, We have observed that solutions to Nahm equations satisfy a reality condition. We will find a translation to the spectral curve for this, too.

4.1 Nahm equations and flows on the Jacobian

In this section we finally want to explain the correspondence between solutions to Nahm equations and flows on the Jacobian of the spectral curve as it was already mentioned in earlier chapters. This is due to Arnaud Beauville and was constructed in [Bea90]. In particular, the correspondence is between regular matrix valued polynomials and linear flows on the Jacobian without the canonical theta divisor of the spectral curve of the given matrix polynomial.

4.1.1 The Beauville correspondence

We now want to give the crucial Beauville correspondence and as it is so important for the rest of this work, we will also give a proof for it. To do so, we need one more ingredient, namely Grauerts direct image theorem [GPR94]:

Proposition 4.1. *Let L be an invertible sheaf on a spectral curve $(S, \mathcal{O}_S)$ and denote by $\pi_* L$ the direct image sheaf (as defined in 2.24). Then, $\pi_* L$ is locally free and the cohomology groups are isomorphic as $\mathbb{C}$-vector spaces, i.e. for $p \geq 0$ we have*

$$\check{H}^p(S, L) \simeq \check{H}^p(\mathbb{CP}^1, \pi_* L). \tag{4.1}$$

We have the following theorem [Bea90]:

Theorem 4.2. *Let $P(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + ... + a_k(\zeta)$ be a polynomial with $a_j(\zeta)$ being polynomials of degree $\deg(a_j) = jd$ in ζ, where $d \in \mathbb{N}$ is some fixed positive integer. Let $S \in |\mathcal{O}(d)|$ be the spectral curve given by the equation $P(\eta, \zeta) = 0$ and let*

$$M(P) = \{A(\zeta) \in \mathfrak{gl}_k(\mathbb{C})[\zeta] | \deg(A(\zeta)) = d, \det(\eta \mathrm{Id}_k - A(\zeta)) = P(\eta, \zeta)\}$$

be the variety of square matrices of rank k with coefficients in the space of complex polynomials in ζ of degree d, whose characteristic polynomial is equal to P, such that the group $PGL_k(\mathbb{C})$ acts freely and properly on $M(P)$ via conjugation.

Then, every line bundle $L \in J^{g-1}(S)\backslash\Theta$ has a free resolution of the form

$$0 \longrightarrow \mathcal{O}_{|\mathcal{O}(2)|}(-d-1)^{\oplus k} \xrightarrow{\eta \mathrm{Id} - A(\zeta)} \mathcal{O}_{|\mathcal{O}(2)|}(-1)^{\oplus k} \longrightarrow L \longrightarrow 0 \tag{4.2}$$

with $A(\zeta) \in M(P)$. This in fact gives a 1-1 correspondence between the quotient $M(P)/PGL_k(\mathbb{C})$ and the Jacobian variety $J^{g-1}(S)\backslash\Theta$ of the spectral curve S without the canonical theta divisor.

Proof. Let $L \in J^{g-1}(S)\backslash\Theta$ be a line bundle. First, we show that the cohomology of L and hence the one of π_*L is zero. Observe, that the condition that L does not belong to the theta divisor means that $\check{H}^0(S,L) = 0$. With the Riemann-Roch theorem 2.25 we then have

$$-\dim \check{H}^1(S,L) = \dim \check{H}^0(S,L) - \dim \check{H}^1(S,L) = \deg(L) + 1 - g = 0,$$

hence $\check{H}^1(S,L) = 0$ as well. With Grauerts direct image theorem 4.1 we obtain $\check{H}^0(\mathbb{CP}^1, \pi_*L) = \check{H}^0(S,L) = 0$ and $\check{H}^1(\mathbb{CP}^1, \pi_*L) = \check{H}^1(S,L) = 0$. This now gives us a description of π_*L the following way. We have

$$\begin{aligned}
\deg(L) + 1 - g &= \dim \check{H}^0(S,L) - \dim \check{H}^1(S,L) \\
&= \dim \check{H}^0(\mathbb{CP}^1, \pi_*L) - \dim \check{H}^1(\mathbb{CP}^1, \pi_*L) \\
&= \deg(\pi_*L) + \mathrm{rank}(\pi_*L)(1 - g_{\mathbb{CP}^1}) \\
&= \deg(\pi_*L) + k
\end{aligned}$$

again by using Grauerts direct image theorem and twice the Riemann-Roch theorem. As $\deg(L) = g-1$ we have $\deg(\pi_*L) = -k$. Thus, in this case rank k bundles with degree $-k$ are all isomorphic to $\mathcal{O}(-1)^{\oplus k}$ (compare with the argument on p.22), hence we have an isomorphism

$$\varphi : \mathcal{O}(-1)^{\oplus k} \to \pi_*L. \tag{4.3}$$

This isomorphism has a representation matrix $B(\zeta)$. In a basis $\mathcal{B}$ of $H^0(S, L(1))$ it is the matrix that represents the multiplication as it was explained in section 2.3.3.

Now, π_*L is also a module over $\pi_*\mathcal{O}_S$ and therefore we have an additional structure given by a morphism of algebras $U : \pi_*\mathcal{O}_S \to \mathrm{End}(\pi_*L)$, i.e. by a homomorphism $A : \pi_*L \to \pi_*L(d)$ satisfying $P(\zeta, A(\zeta)) = 0$ [RNB89]. The matrix $A(\zeta)$ is regular for each $\zeta \in \mathbb{CP}^1$ since L is a line bundle and thus $P(\zeta,\eta)$ is the characteristic polynomial of $A(\zeta)$. Hence, $A(\zeta) \in M(P)$. We obtain the same data $A(\zeta) \in M(P)$ when we take the conjugation $\varphi^{-1}A\varphi : \mathcal{O}(-1)^{\oplus k} \to \mathcal{O}(d-1)^{\oplus k}$ and as any isomorphism $\mathcal{O}(-1)^{\oplus k} \to \pi_*L$ is given by conjugating φ with an automorphism $g \in GL_k(\mathbb{C})$ of $\mathcal{O}(-1)^{\oplus k}$ we have a map

$$\mathrm{Jac}^{g-1}(S)\backslash\Theta \to M(P)/PGL_k(\mathbb{C}).$$

The inverse mapping is given by mapping a conjugation class of matrix polynomials $A(\zeta) \in M(P)/PGL_k(\mathbb{C})$ to the line bundle $L(-1)$ where L is the bundle given by the exact sequence (4.2). For more details of this see [Bea90] or [AHH90]. □

4.1.2 Flows on the Jacobian variety

We now have a correspondence between matrix polynomials on the one hand and line bundles on the spectral curve on the other hand. For the purpose of describing bow varieties we want to consider flows of the matrix polynomials. In particular, we want to consider flows in $M(P)$ such that Nahm equations are satisfied. We will find an expression for this flow in the spectral picture, which will be shifting the variation to a local frame in which the Lax pair equations are satisfied.

In order to do so and to describe flows on the Jacobian, we need a better understanding of the line bundles involved, i.e. of the cohomology group $\check{H}^1(S, \mathcal{O})$.

We have already seen that all spaces of isomorphisms classes of holomorphic line bundles of a fixed degree are isomorphic to each other and to the Jacobian. We have also seen that any two holomorphic line bundles of the same degree d differ only by a line bundle of degree zero. This means, if $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are of degree d with transition functions c and $\tilde{c}$ respectively, we find a line bundle L of degree zero with transition function f such that $\tilde{\mathcal{F}} = \mathcal{F} \otimes L$ with their transition functions satisfying $\tilde{c} = cf$. Therefore, to obtain a flow on the space of isomorphism classes of holomorphic line bundles of degree d and thus a flow on the Jacobian, we only need to vary the degree zero line bundle. Thus, the flow will be

$$\mathcal{F} \mapsto \mathcal{F}_t := \mathcal{F} \otimes L^t$$

with transition function $c(t) = cf(t)$ for c the transition function of $\mathcal{F}$ and $f(t)$ the transition function of L^t. To do an explicit construction, we need to now the transition function $f(t)$ of L^t. In the following we will construct this function. Recall that they are given as classes in $H^1(S, \mathcal{O}^*)$.

We have a short exact sequence of sheaves, which is the basis of this. In our case it is the following sequence

$$0 \to \mathcal{O}_{|\mathcal{O}(2)|}(-2k) \overset{\det(\eta Id_k - A(\zeta))}{\to} \mathcal{O}_{|\mathcal{O}(2)|} \to \mathcal{O}_S \to 0.$$

The key point is that we can identify

$$\mathcal{O}_{|\mathcal{O}(2)|} \cong \mathcal{O}_{|\mathcal{O}(2)|}(-2k) \otimes \mathcal{O}_{|\mathcal{O}(2)|}(2k),$$

because a section s on the tensor product will be just the product of a section s^- on the left factor and a section s^+ on the right factor. As s^- and s^+ are sections of the named specific bundles, they satisfy $s_1^-(\tilde{\zeta}, \tilde{\eta}) = \zeta^{2k} s_0^-(\zeta, \eta)$ and $s_1^+(\tilde{\zeta}, \tilde{\eta}) = \zeta^{-2k} s_0^+(\zeta, \eta)$ respectively on the overlap, so the product $s = s^- s^+$ is a holomorphic function. With this given, the sequence above is just the cokernel sequence. As $\det(\eta Id_k - A(\zeta))$ is a section of $\mathcal{O}_{|\mathcal{O}(2)|}(2k)$, the image of the first map are the holomorphic functions f on $|\mathcal{O}(2)|$ that can be written as a product of a section of $\mathcal{O}_{|\mathcal{O}(2)|}(-2k)$ with this special section $\det(\eta Id_k - A(\zeta))$. As the sequence is the cokernel sequence the sheaf on the right must be the cokernel

$$\frac{\mathcal{O}_{|\mathcal{O}(2)|}}{\alpha(\mathcal{O}_{|\mathcal{O}(2)|}(-2k))}$$

where α is this map det. The sequence is exact by construction. If we have a look at the image of β, we see, that for a holomorphic function f on $\mathcal{O}_{|\mathcal{O}(2)|}$ we can always take the quotient $\frac{f}{\det(\eta Id_k - A(\zeta))}$ and obtain a section of $\mathcal{O}_{|\mathcal{O}(2)|}(-2k)$, such that f lies in the kernel as long as $\det(\eta Id_k - A(\zeta))$ is not zero. Therefore the cokernel can be identified with the space of holomophic functions on $|\mathcal{O}(2)|$ which have support on the subset of points (ζ, η) that satisfy $\det(\eta Id_k - A(\zeta)) = 0$ and this is exactly the spectral curve. Thus, we have

$$\frac{\mathcal{O}_{|\mathcal{O}(2)|}}{\alpha(\mathcal{O}_{|\mathcal{O}(2)|}(-2k))} \cong \mathcal{O}_S.$$

From the short exact sequence of sheaves we obtain the long exact sequence of cohomology groups. This looks as follows

$$... \to H^1(|\mathcal{O}(2)|, \mathcal{O}(-2k)) \overset{\det(\eta Id_k - A(\zeta))}{\to} H^1(|\mathcal{O}(2)|, \mathcal{O}) \overset{\beta}{\to} H^1(S, \mathcal{O}) \to H^2(|\mathcal{O}(2)|, \mathcal{O}(-2k)) \to ...$$

We are interested in the group $H^1(S, \mathcal{O})$. Therefore, we note that $H^2(|\mathcal{O}(2)|, \mathcal{O}(-2k)) = 0$, because the standard cover (U_0, U_1) of the total space $|\mathcal{O}(2)|$ is a Leray cover, so the cohomology group of the total space is the same as the one of this cover and this vanishes as there are no threefold intersections, because the cover consists only of two open sets. Since the sequence is exact, we can describe $H^1(S, \mathcal{O})$ as $H^1(|\mathcal{O}(2)|, \mathcal{O})$ modulo the kernel of β, which is exactly the image of $\det(\eta Id_k - A(\zeta))$.

Now, from the definition of sheaf cohomology we have a description of the group $H^1(|\mathcal{O}(2)|, \mathcal{O})$ namely as the set of holomorphic functions on the intersection $U_0 \cap U_1 \cong \mathbb{C}^* \times \mathbb{C}$ (which is also the kernel of the map to the space of holomorphic functions on triple intersection as there are no triple intersections) modulo the image of $\delta : \mathcal{O}(U_0) \oplus \mathcal{O}(U_1) \to \mathcal{O}(U_0 \cap U_1)$, which are holomorphic functions on the intersection that extend to U_0 and U_1 respectively. Using the series expansions for holomorphic functions, we obtain

$$f(\zeta, \eta) = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} \eta^i \zeta^j$$

for holomorphic functions on $\mathbb{C}^* \times \mathbb{C}$. Holomorphic functions on $U_0 \cong \mathbb{C}^2$ have a Taylor expansion in ζ and η, so we have

$$f(\zeta, \eta) = \sum_{i,j=0}^{\infty} a_{ij} \eta^i \zeta^j$$

and the quotient contains functions with a Laurent series of the form

$$f(\zeta,\eta) = \sum_{i=0}^{\infty}\sum_{j=-\infty}^{-1} a_{ij}\eta^i\zeta^j.$$

Analogously, for holomorphic functions on $U_1 \cong \mathbb{C}^2$ with Taylor expansion in $\tilde{\zeta}$ and $\tilde{\eta}$

$$f(\tilde{\zeta},\tilde{\eta}) = \sum_{i,j=0}^{\infty}\tilde{a}_{ij}\tilde{\eta}^i\tilde{\zeta}^j = \sum_{i,j=0}^{\infty}\tilde{a}_{ij}(\frac{\eta}{\zeta^2})^i(\frac{1}{\zeta})^j = \sum_{i,j=0}^{\infty}\tilde{a}_{ij}\eta^i\zeta^{-j-2i},$$

we obtain for the quotient functions with a Laurent expansion of the form

$$f(\zeta,\eta) = \sum_{i=0}^{\infty}\sum_{j=-2i+1}^{-1} a_{ij}\eta^i\zeta^j.$$

This is the description of an arbitrary function in $H^1(|\mathcal{O}(2)|,\mathcal{O})$. To obtain $H^1(S,\mathcal{O})$ we have to take this modulo $\det(\eta Id_k - A(\zeta)) = \eta^k + a_1(\zeta)\eta^{k-1} + ... + a_k(\zeta) = 0$, where the a_i are polynomials in ζ. This means, whenever the parameter i in the sum above reaches k so that we have a term proportional to η^k we identify it with $-a_1(\zeta)\eta^{k-1} - ... - a_k(\zeta)$ so that we can write

$$f(\zeta,\eta) = \sum_{i=1}^{k-1}\frac{p_i[\zeta]\eta^i}{\zeta^N}$$

for some fixed $N \in \mathbb{N}$ and polynomials $p[\zeta]$ which actual forms depend on the polynomials $a_i(\zeta)$ in the determinant. This in now representation of a class in $H^1(S,\mathcal{O})$. As we are interested in $H^1(S,\mathcal{O}^*)$ we can just take the exponential of it and obtain $\exp f(\zeta,\eta)$. As this does not have any zero, it corresponds to a degree zero line bundle L, which is, what we wanted to construct. A linear variation of the element in $H^1(S,\mathcal{O})$ gives us a family of degree zero line bundles L^t with transition functions $\exp(tf(\zeta,\eta))$.

Thus, the flow on the Jacobian is the flow of line bundles with fixed degree d $\mathcal{F} \mapsto \mathcal{F}\otimes L^t$ with transition function $c\exp(tf(\zeta,\eta))$ where c is the transition function of our starting point $\mathcal{F}$.

4.1.3 Linearity and relation to Lax pair equations

In this section, we will show the equivalence of a solution to Nahm equations and a linear flow on the Jacobian variety of the corresponding spectral curve without the theta divisor following [HSW99]. This will be done by writing Nahm equations as Lax pair equations (this was shown in chapter 2) and showing that we can obtain a solution by locally varying the frame, i.e. shifting the evolution to the base.

We start with a family of sections each of them being in $\check{H}^0(S,\mathcal{F}\otimes L^t)$. Such a section has a local representation $(s(t),\tilde{s}(t))$ on U_0 and U_1 respectively. Let $\sigma_1(t),...,\sigma_n(t)$ be a holomorphically varying local frame for L^t. The section is then represented by functions $(s_i(t), \tilde{s}_i(t))$ satisfying

$$\tilde{s}_i(t) = \exp(tf(\zeta,\eta))cs_i(t)$$

on $U_0 \cap U_1$. Taking the derivative with respect to t we have

$$\begin{aligned}
\frac{\partial \tilde{s}_i(t)}{\partial t} &= f(\zeta,\eta)\exp(tf(\zeta,\eta))cs_i(t) + \exp(tf(\zeta,\eta))c\frac{\partial s_i(t)}{\partial t} \\
&= f(\zeta,\eta)\tilde{s}_i(t) + \exp(tf(\zeta,\eta))c\frac{\partial s_i(t)}{\partial t} \\
&= \sum_j f(\zeta, A(\zeta))_{ji}\tilde{s}_j(t) + \exp(tf(\zeta,\eta))c\frac{\partial s_i(t)}{\partial t}
\end{aligned}$$

using $\eta s_i = \sum_j A_{ji}s_j$ as in (2.3). We can write the Laurent expansion for the term $f(\zeta, A(\zeta))$ and split it into a polynomial part in ζ and one in ζ^{-1} to obtain a global section of L^t. Thus, writing $f = f^+ + f^-$ where the constant term belongs to f^+ we can arrange the last expression as follows:

$$\frac{\partial s_i}{\partial t} - \sum_j f^+_{ji}s_j = \exp(tf(\zeta,\eta))c\left(\frac{\partial \tilde{s}_i}{\partial t} + \sum_j f^-_{ji}\tilde{s}_j\right).$$

The left hand side is holomorphic in ζ while the right hand side is holomorphic in ζ^{-1} and thus the pair

$$\left(\frac{\partial s_i}{\partial t} - \sum_j f^+_{ji}s_j, \frac{\partial \tilde{s}_i}{\partial t} + \sum_j f^-_{ji}\tilde{s}_j\right)$$

on U_0 and U_1 respectively defines a section $\tau_i(t)$ of L^t.

Now the σ_i have been defined to be a frame, so there is a matrix representation of this section corresponding to that basis, i.e. we have

$$\tau_i(t) = \sum_j M_{ji}(t)\sigma_j(t)$$

where M depends holomorphically on t and satisfies

$$\frac{\partial s_i}{\partial t} - \sum_j f^+_{ji}s_j = \sum_j M_{ji}s_j$$

or equivalently

$$\frac{\partial s_i}{\partial t} = \sum_j \left(f^+_{ji} + M_{ji}\right)s_j. \tag{4.4}$$

Now, recall that we have (2.3) for the matrix $A(t,\zeta)$ defining the spectral curve (now with a frame varying in t). Differentiating this equation with respect to t and inserting on both sides the condition (4.4) for the derivative of a section we just calculated we obtain

$$\begin{aligned}\sum_{j,k}\left(f^+_{ji}+M_{ji}\right)A_{kj}s_k &= \sum_j\left(f^+_{ji}+M_{ji}\right)\eta s_j\\ &= \eta\frac{\partial s_i}{\partial t}\\ &= \sum_j\left(\frac{\partial A}{\partial t}\right)_{ji}s_j+\sum_j A_{ji}\frac{\partial s_j}{\partial t}\\ &= \sum_j\left(\frac{\partial A}{\partial t}\right)_{ji}s_j+\sum_{j,k} A_{ji}\left(f^+_{kj}+M_{kj}\right)s_k.\end{aligned}$$

This means

$$\sum_j\left(\frac{\partial A}{\partial t}\right)_{ji}s_j=\sum_{j,k}\left(\left(f^+_{ji}+M_{ji}\right)A_{kj}-A_{ji}\left(f^+_{kj}+M_{kj}\right)\right)s_k$$

and hence

$$\frac{\partial A(t,\zeta)}{\partial t}=[f^+(\zeta,A(t,\zeta))+M(t),A(t,\zeta)]. \tag{4.5}$$

This is now almost a Lax pair equation. All we have left to do is to conjugate $A(t,\zeta)$ in the following way. Conjugating A means changing the moving frame, we have chosen arbitrary to the frame in which the evolution of the matrix A is in Lax form. Let us choose a holomorphic $GL_k(\mathbb{C})$-valued function P in the parameter t such that

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}=-M(t)P(t). \tag{4.6}$$

Then $\tilde{A}=P^{-1}AP$ satisfies the equation

$$\frac{\mathrm{d}\tilde{A}(t,\zeta)}{\mathrm{d}t}=[f^+(\zeta,\tilde{A}(t,\zeta)),\tilde{A}(t,\zeta)], \tag{4.7}$$

which is a Lax pair equation. Finding the correct moving frame is done by solving the linear system of ODEs (4.6). Hence, we have shown that a linear flow on the Jacobian of the spectral curve S defines a moving frame for $|\mathcal{O}(2)|$ such that the matrix $A(t,\zeta)$, which defines the spectral curve S, satisfies a Lax pair equation and thus Nahm equations.

Here, the direction of the flow was given by the transition function

$$f(\zeta,\eta)=\sum_{i=1}^{k-1}\frac{p_i[\zeta]\eta^i}{\zeta^N}$$

as we have shown above. Then, we second matrix occurring in the Lax pair equation is exactly the polynomial part the this direction. This is of course not unique, as we can add arbitrary powers of A to this expression and still satisfy the Lax pair equation.

4.2 The real structure

For a triple (T_1, T_2, T_3) satisfying Nahm equations we have already seen that the matrix polynomial

$$A(\zeta) = \underbrace{T_2 + iT_3}_{A_0} + \underbrace{2iT_1}_{A_1}\zeta + \underbrace{(T_2 - iT_3)}_{A_2}\zeta^2$$

defines a spectral curve via $S = \{\det(\eta \mathrm{Id} - A(\zeta)) = 0\}$. The function A_0, A_1 and A_2 satisfy a so called reality condition given by $A_0^\dagger = -A_2$ and $A_1^\dagger = A_1$. If S is real (see below), the corresponding invertible sheaf $\mathcal{F} \in \mathrm{Jac}^{g-1}(S)$ satisfies also a reality condition [Hit83] and [Bie07]. In this section we want to work out some properties of this reality condition.

The antipodal map on $\mathbb{CP}^1$ induces an anti-holomorphic involution on the total space $|\mathcal{O}_{\mathbb{CP}^1}(2)|$ given by

$$\begin{aligned} \tau : |\mathcal{O}_{\mathbb{CP}^1}(2)| &\to |\mathcal{O}_{\mathbb{CP}^1}(2)| \\ (\zeta, \eta) &\mapsto \left(-\frac{1}{\bar{\zeta}}, -\frac{\bar{\eta}}{\bar{\zeta}^2}\right) \end{aligned}$$

called the real strucure. It is obviously anti-holomorphic and satisfies the condition $\tau^2 = \mathrm{Id}$.

For Nahm equations the spectral curve S is real in the sense that it is invariant under the real structure. Observe therefore that in the standard open cover (U_0, U_1) of S we have $\tau|_{U_0}(U_0) = U_1$ and $\tau|_{U_1}(U_1) = U_0$ and thus $\tau|_{U_1 \cap U_0}(U_1 \cap U_0) = U_0 \cap U_1$.

The real structure induces an anti-holomorphic involution σ on the Jacobian as follows (cf. [Bie07]): Let L be a line bundle on S that is trivialized by an open cover $\{U_\alpha\}$ with transition function $g_{\alpha\beta}$ from U_α to U_β. Then $\sigma(L)$ is trivialized by the open cover $\{\tau(U_\alpha)\}$ with transition functions $g_{\alpha\beta}(\tau(\zeta, \eta))$. In particular this means the following: Let s be a local section of L on an open set U. In the standard cover s is represented by the tuple (s_1, s_0) of functions satisfying

$$s_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = g_{10}(\zeta, \eta) s_0(\zeta, \eta).$$

The real structure now gives us a new transition function defined by

$$\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) := \overline{g_{10}(\tau(\zeta, \eta))},$$

where the local section is given by functions $(\tilde{s}_0, \tilde{s}_1)$ satisfying

$$\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right)\tilde{s}_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \tilde{s}_0(\zeta, \eta).$$

Taking in particular the bundle $\mathcal{O}(k)$ for some fixed k we have a transition function $g_{10}(\zeta, \eta) = \frac{1}{\zeta^k}$. This induces the transition function $\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \zeta^k$ on $\sigma(\mathcal{O}(k))$. This means $\sigma(\mathcal{O}(k)) \cong \mathcal{O}(k)$ and hence, the real structure preserves the degree of line bundles.

The real structure also preserves $\mathrm{Jac}^{g-1}(S)\backslash\Theta$ as the following argument shows. Writing $\sigma(s) = \overline{\tau^* s}$, a local section $s = (s_1, s_0)$ on L induces a local section on $\sigma(L)$ given by $(\overline{\tau^* s_0}, \overline{\tau^* s_1})$. Hence, if L does not have any non-trivial global section, then $\sigma(L)$ also does not have any. We have the following definition and proposition from [Bie07]:

Definition 4.3. A line bundle L of degree $d = ik$ on S, $i \in \mathbb{Z}$ is called *real*, if $L \cong \sigma(L)^* \otimes \mathcal{O}_S(2i)$. We write $\mathrm{Jac}^{g-1}_{\mathbb{R}}(S)$ for the corresponding subspace of $\mathrm{Jac}^{g-1}(S)$ and $\Theta_{\mathbb{R}}$ for $\Theta \cap \mathrm{Jac}^{g-1}_{\mathbb{R}}(S)$.

Proposition 4.4. *There is a 1-1 correspondence between* $\mathrm{Jac}^{g-1}_{\mathbb{R}}(S)\backslash\Theta_{\mathbb{R}}$ *and conjugacy classes of* $\mathfrak{gl}_k(\mathbb{C})$*-valued polynomials* $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ *such that* $A(\zeta)$ *is regular for every* ζ*, the characteristic polynomial of* $A(\zeta)$ *is*

$$P(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + \dots + a_k(\zeta)$$

where $a_i(\zeta)$ *is a polynomial in* ζ *of degree* $2i$ *and such that there exists a hermitian matrix* $h \in GL_k(\mathbb{C})$ *with*

$$hA_0h^{-1} = -A_2^\dagger, \qquad hA_1h^{-1} = A_1^\dagger, \qquad hA_2h^{-1} = -A_0^\dagger.$$

Proof. $\mathrm{Jac}^{g-1}(S)\backslash\Theta$ is real if and only if $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ is conjugated to $-A_2^\dagger + A_1^\dagger\zeta - A_0^\dagger\zeta^2$. Let h be the matrix that realizes this conjugation. Then $(h^\dagger)^{-1}h$ centralizes $A(\zeta)$. If $(h^\dagger)^{-1}h$ does not belong to the centre of $GL_k(\mathbb{C})$, the $A(\zeta)$ belongs to the reductive subalgebra $Z((h^\dagger)^{-1}h)$ and $\ker(\eta\mathrm{Id} - A(\zeta))$ is not everywhere 1-dimensional. This contradicts the assumption that $A(\zeta)$ belongs to a line bundle. Hence, h is hermitian up to a central factor (cf. [Bie07]). $\square$

4.3 Boundary conditions

We now know how Nahm equations induce a linear flow on the Jacobian, but we still have to investigate what happens to the boundary conditions, when the evolution of Nahm equations passes a λ-point. These are the rank 1 jumps over each λ-point inside the interval. This was treated by Hurtubise and Murray [HM89] and we follow the argumentation of their chapter 3.

We start with two sets of Nahm data, one being T_i^{p-1} on the interval $[\lambda_{p-1}, \lambda_p]$ and another one T_i^p on $[\lambda_p, \lambda_{p+1}]$ $(i = 1, 2, 3)$. Let us write

$$A_p(t, \zeta) = T_2^p(t) + iT_3^p(t) + 2iT_1^p(t)\zeta + (T_2^p(t) - iT_3^p(t))\zeta^2 \tag{4.8}$$

Both Nahm data shall be of the same rank k and the boundary condition is the one of a rank 1 jump, i.e. both limits

$$A^+(\zeta) := \lim_{t \nearrow \lambda_p} A_{p-1}(t, \zeta)$$

and

$$A^-(\zeta) := \lim_{t \searrow \lambda_p} A_p(t,\zeta)$$

exist and we have

$$A^+(\zeta) - A^-(\zeta) = (u_0 - \zeta u_1)(\bar{u}_1 + \zeta \bar{u}_0)^T =: u(\zeta) u^*(\zeta)^T \tag{4.9}$$

for all ζ. Here u_0, $u_1 \in \mathbb{C}^k$ are column vectors. Observe that this condition is equivalent to (3.10) when we set $I = u_0$ and $J = s_1^\dagger$.

Then we have $\det(\eta \mathrm{Id} - A^+(\zeta)) = 0$ defining the spectral curve S_{p-1} which corresponds to the Nahm data on $[\lambda_{p-1}, \lambda_p]$ and analogously $\det(\eta \mathrm{Id} - A^-(\zeta)) = 0$ defining the spectral curve S_p corresponding to the other Nahm data on $[\lambda_p, \lambda_{p+1}]$ (recall that the spectral curves are independent on the parameter t as Nahm equations are isospectral). Let us assume that the Nahm data are generic, meaning that $S_p \cap S_{p-1}$ consists of $2k^2$ distinct points.

We define the following sheaf on S_p:

$$K_t = \operatorname{coker}(\eta \mathrm{Id} - A_p(t,\zeta)) : \mathcal{O}(-2)^{\oplus k} \to \mathcal{O}^{\oplus k}$$

and its dual

$$K_t^* = \ker(\eta \mathrm{Id} - A_p(t,\zeta))^T : \mathcal{O}^{\oplus k} \to \mathcal{O}(2)^{\oplus k}.$$

Let the subscript adj denote the adjoint, such that

$$(\eta \mathrm{Id} - A_p(t,\zeta))^T (\eta \mathrm{Id} - A_p(t,\zeta))^T_{\mathrm{adj}} = \det(\eta \mathrm{Id} - A_p(t,\zeta)) \mathrm{Id} \tag{4.10}$$

holds and we have

$$\operatorname{Im}(\eta \mathrm{Id} - A_p(t,\zeta))^T_{\mathrm{adj}} \subset K_t^*$$

over S_p with equality if $(\eta \mathrm{Id} - A_p(t,\zeta))$ is of corank one, meaning that $(\eta \mathrm{Id} - A_p(t,\zeta))^T_{\mathrm{adj}}$ is of rank one. In this case K_t is a line bundle. If the corank of $(\eta \mathrm{Id} - A_p(t,\zeta))$ is greater than one, the adjoint vanishes.

K_{λ_p} can be defined on both S_p and S_{p-1}. Our first aim is to show that K_{λ_p} is a line bundle on both spectral curves. Using (4.9) and the matrix determinant lemma (stating $\det(A + xy^T) = \det(A) + y^T A_{\mathrm{adj}} x$ where A is a square matrix and x, y are column vectors) we have

$$\begin{aligned} \det(\eta \mathrm{Id} - A^+(\zeta))^T &= \det(\eta \mathrm{Id} - A^-(\zeta)^T - u^*(\zeta) u(\zeta)^T) \\ &= \det(\eta \mathrm{Id} - A^-(\zeta)^T) - u(\zeta)^T (\eta \mathrm{Id} - A^-(\zeta))^T u^*(\zeta). \end{aligned} \tag{4.11}$$

Over S_p

$$(\eta \mathrm{Id} - A^-(\zeta))^T (\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) = \det(\eta \mathrm{Id} - A^-(\zeta))^T u^*(\zeta) = 0 \tag{4.12}$$

and over S_{p-1}

$$\begin{aligned}
(\eta \mathrm{Id} - A^+(\zeta))^T & (\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) \\
&= \left(\eta \mathrm{Id} - (A^-(\zeta) - u^*(\zeta)u(\zeta)^T)\right)(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) \\
&= (\eta \mathrm{Id} - A^-(\zeta))^T (\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) - u^*(\zeta)u(\zeta)^T(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) \\
&= \det(\eta \mathrm{Id} - A^-(\zeta))u^*(\zeta) - u^*(\zeta)u(\zeta)^T(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta) \\
&= u^*(\zeta)\det(\eta \mathrm{Id} - A^+(\zeta))^T \\
&= 0
\end{aligned} \tag{4.13}$$

by using (4.9) - (4.11). Hence, $(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta)$ defines a section of $K^*_{\lambda_p}$ over both S_p and S_{p-1}. $(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}}$ is non-zero on S_p away from the intersection $S_p \cap S_{p-1}$ (cf. (4.11)) and $(\eta \mathrm{Id} - A^+(\zeta))^T_{\mathrm{adj}}$ is non-zero on S_{p-1} away from the intersection for symmetry reasons. The genericity condition of the Nahm data now implies that these both expressions are also non-zero on the intersection. Therefore, K_{λ_p} is a line bundle on both S_p and S_{p-1}.

We need the following two lemmas from [HM89]:

Lemma 4.5. *Let E be the divisor on S_p cut out by S_{p-1} and let f be given by*

$$f^*(\eta, \zeta) = \zeta^{2k-1}\overline{f(-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})}.$$

Then there is a divisor D such that f vanishes on D and $D + \sigma(D) = E$ if and only if $f(f^*)^T$ vanishes on E.

Proof. Let $f(f^*)^T$ vanish on E. Then also $f_i f_j^*$ vanishes on E for all i, j. E being real means that we can write $E = \sum m_k(p_k + \tau(p_k))$ where p_k are points of S. Let f_i vanish at p_k with multiplicity $g_{k,i}$ and at $\tau(p_k)$ with multiplicity $h_{k,i}$. We have $g_{k,i} + h_{k,j} \geq m_k$ for all i, j, as $f_i f_j^*$ vanishes on E. Let us set $g_k = \min_i g_{k,i}$ and $h_k = m_k - g_k$. Then f_i vanishes at $D = \sum(g_k p_k + h_k \tau(p_k))$ for all i and $D + \sigma(D) = E$. □

With this we can prove the following lemma:

Lemma 4.6. *There is a partition of $S_p \cap S_{p-1}$ into divisors D and $\sigma(D)$ such that over S_p and over S_{p-1} we have $K_{\lambda_p} \cong \mathcal{O}(2k-1)[-D]$.*

Proof. $(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta)$ is of degree $2k - 1$ and hence we have a map

$$K_{\lambda_p} \to \mathcal{O}(2k-1).$$

Using the last lemma to show that this map vanishes on the appropriate divisor D, it is sufficient to show that $(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}} u^*(\zeta)u(\zeta)^T(\eta \mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}}$ vanishes on $S_p \cap$

S_{p-1}. From (4.11) we know that on $S_p \cap S_{p-1}$ we have $u(\zeta)^T(\eta\mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}}u^*(\zeta) = 0$. Now, $(\eta\mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}}$ is of rank 1 over S_{p-1} and thus can be written as an outer product vw^T, where v and W are column vectors. Hence we get $u^T vw^T u^* = 0$, so either $u^T v = 0$ or $w^T u^* = 0$. Then, $(\eta\mathrm{Id} - A^-(\zeta))^T_{\mathrm{adj}}u^*(\zeta) = 0$ or $u(\zeta)^T(\eta\mathrm{Id} - A^-(\zeta))^T = 0$. For more details why the map only vanishes on D see [HM89]. □

This leads directly to the behaviour of the flows of Nahm data at such boundary points.

Proposition 4.7. *At a rank 1 jump of solutions to Nahm equations we have*

$$S_p \cap S_{p-1} = S_{p,p-1} \cup S_{p-1,p}$$

where $\tau(S_{p,p-1}) = S_{p-1,p}$ *and the line bundles at* λ_p *are equal to*

$$\mathcal{O}_{S_p}(2k-1)[-S_{p-1,p}] \in \mathrm{Jac}^{g-1}(S_p)$$

and

$$\mathcal{O}_{S_{p-1}}(2k-1)[-S_{p-1,p}] \in \mathrm{Jac}^{g-1}(S_{p-1}).$$

Considering the flow of K_t for $t \in [\lambda_p, \lambda_{p+1}]$ we have

$$K_{\lambda_p} \cong \mathcal{O}(2k-1)[-S_{p-1,p}]$$

at $t = \lambda_p$ and

$$K_{\lambda_{p+1}} \cong \mathcal{O}(2k-1)[-S_{p,p+1}]$$

at $t = \lambda_{p+1}$.

5 Bow varieties over an interval

In this chapter we describe precisely the bow variety of the following bow diagram.

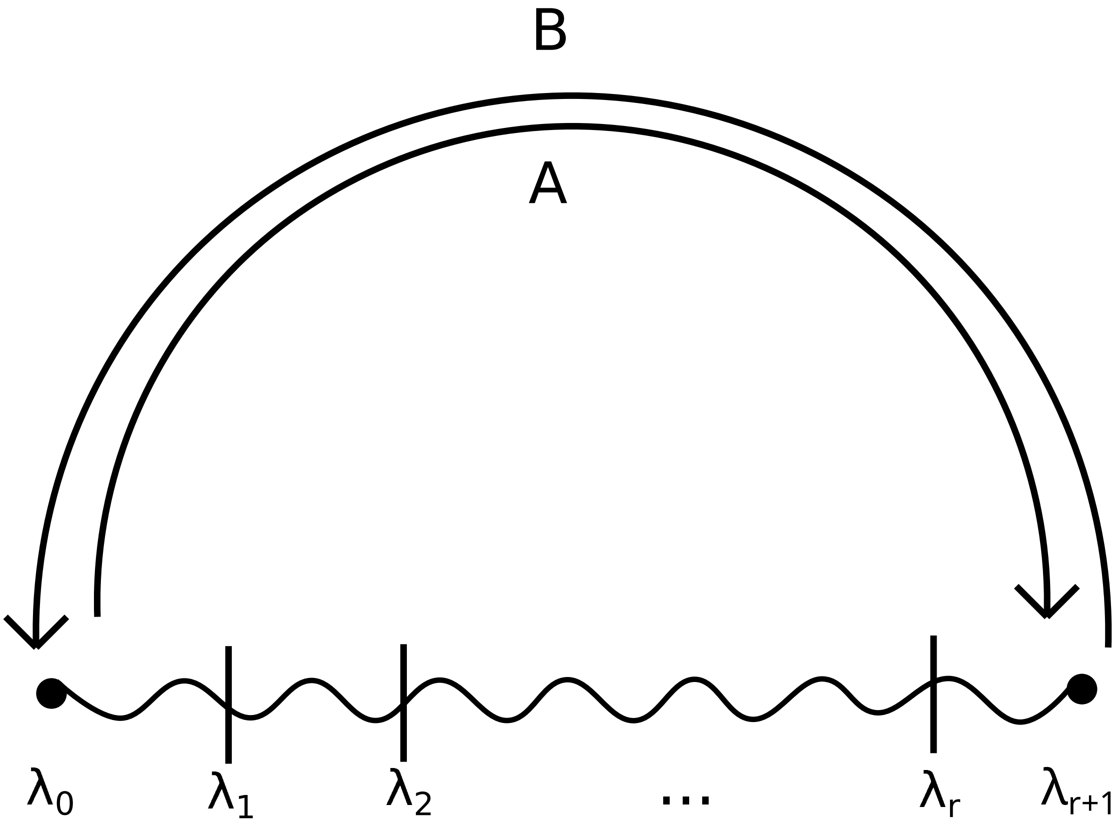

Figure 5.1: Bow diagram with r λ-points

It consists of a single interval $[\lambda_0, \lambda_{r+1}]$ with r λ-points located at $\lambda_1, ..., \lambda_r$. Further we have a pair of arrows denoted by A and B connecting the endpoints of the interval. The rank of the solution to Nahms equations is equal to k on every subinterval. This means $\mathcal{M}$ is the moduli space of $\mathfrak{u}(k)$-valued solutions to Nahm equations on the interval $[\lambda_0, \lambda_{r+1}]$ which have a rank 1 jump at each λ_i for $i = 1, ..., r$ given by

$$T_2^{i+1}(\lambda_i) + iT_3^{i+1}(\lambda_i) - (T_2^i(\lambda_i) + iT_3^i(\lambda_i)) = I_i J_i^\dagger$$
$$iT_1^{i+1}(\lambda_i) - iT_1^i(\lambda_i) = \frac{1}{2}(I_i I_i^\dagger - J_i^\dagger J_i)$$

and boundary conditions

$$T_2^1 + iT_3^1 + 2iT_1^1\zeta + (T_2^1 - iT_3^1)\zeta^2 = (B + A^\dagger\zeta)(A - B^\dagger\zeta) + c_L(\zeta)\mathrm{Id}$$

at λ_0 and

$$T_2^{r+1} + iT_3^{r+1} + 2iT_1^{r+1}\zeta + (T_2^{r+1} - iT_3^{r+1})\zeta^2 = (A - B^\dagger\zeta)(B + A^\dagger\zeta) + c_R(\zeta)\mathrm{Id}$$

at λ_{r+1}. Here T_j^i, $i = 1, ..., r+1$ denotes the solution T_j to Nahm equations on the i-th interval $[\lambda_{i-1}, \lambda_i]$, $A, B \in \mathrm{Mat}_{k,k}(\mathbb{C})$ and c_L, c_R are quadratic polynomials in ζ satisfying the reality condition.

Observe that if we let the length of all intervals $[\lambda_{i-1}, \lambda_i]$ go to zero, then $\mathcal{M}$ is the quotient of the set of solutions of

$$[A - B^\dagger\zeta, B - A^\dagger\zeta] = \sum_{i=1}^{r}(I_i - J_i^\dagger\zeta)(J_i - I_i^\dagger\zeta) + (c_L(\zeta) - c_R(\zeta)) \tag{5.1}$$

by $U(k)$. If $c_L(\zeta) - c_R(\zeta) = a\zeta$, then $\mathcal{M}$ with the complex structure corresponding to $\zeta = 0$ is biholomorphic to the moduli space of framed torsion-free sheaves on $\mathbb{CP}^2$ with rank r and $c_2 = k$ [Nak99]. For arbitrary $(c_L(\zeta) - c_R(\zeta))$, $\mathcal{M}$ with $\lambda_0 = ... = \lambda_{r+1}$ has been interpreted as a moduli space of instantons on a non-commutative $\mathbb{R}^4$ [Nek98]. Therefore, for arbitrary λ_i and arbitrary c_L, c_R we call $\mathcal{M}$ a *deformed instanton moduli space*. For $r = 1$ these moduli spaces have been investigated in [Tak15].

In the first part of this chapter we will describe the moduli space in in the language of spectral curves as it was introduced in the previous chapter, whereas in the second part we will find an asymptotic metric for the bow variety.

5.1 The bow variety

5.1.1 Factorization of matrix polynomials

In order to describe the bow variety of the above bow diagram in terms of spectral curves and line bundles we need some results about matrix polynomials first (also cf. [BBC22]).

We start with the flat hyperkähler manifold $T^*\mathrm{Mat}_{k,k}(\mathbb{C}) \cong \mathrm{Mat}_{k,k}(\mathbb{C}) \oplus \mathrm{Mat}_{k,k}(\mathbb{C})$ with its standard tri-hamiltonian $U(k) \times U(k)$ action as it has been discussed in chapter 3.2.4. We can view the moment maps as sections of $\mathcal{O}(2) \otimes \mathfrak{gl}_k(\mathbb{C})$ over $\mathbb{CP}^1$ and write them as quadratic polynomials

$$\begin{aligned} \mu(\zeta) &= (A - B^\dagger\zeta)(B + A^\dagger\zeta), \\ \nu(\zeta) &= -(B + A^\dagger\zeta)(A - B^\dagger\zeta). \end{aligned} \tag{5.2}$$

As explained in the last section $\mu(\zeta)$ and $-\nu(\zeta)$ define 1-dimensional real, analytic sheaves $\mathcal{F}$ and $\mathcal{F}'$ via the Beauville correspondence, which are both supported on the same spectral curve S, because $\det(\eta\mathrm{Id} - A(\zeta)B(\zeta)) = \det(\eta\mathrm{Id} - B(\zeta)A(\zeta))$. This means, we can relate $\mathcal{F}$ to $\mathcal{F}'$.

Let us assume that the roots of $\det A(\zeta)$ are disjoint from the roots of $\det B(\zeta)$ and let Δ_A (resp. Δ_B) be the Cartier divisors on S given by $\eta = 0$ on the open subset $\det B(\zeta) \neq 0$ (resp. on the open subset $\det A(\zeta) \neq 0$).

Then we have the following statement.

Proposition 5.1. *With the above definitions $\mathcal{F}' = \mathcal{F}(1)[-\Delta_A]$.*

Proof. First, observe that $[\Delta_A + \Delta_B] \simeq \mathcal{O}_S(2)$. As the roots of $\det A(\zeta)$ are disjoint from the roots of $\det B(\zeta)$ the open subsets $U_1 = \{\det B(\zeta) \neq 0\}$ and $U_2 = \{\det A(\zeta) \neq 0\}$ are a locally finite open cover of S. Observe, that $\eta = 0$ is equivalent to $\det A(\zeta) \det B(\zeta) = 0$ on S. Hence, on U_1 $\eta = 0 \Leftrightarrow \det A(\zeta) = 0$ and on U_2 $\eta = 0 \Leftrightarrow \det B(\zeta) = 0$. On $U_1 \cap U_2$ we have $\det A(\zeta), B(\zeta) \neq 0$ and thus we can take $\det A(\zeta) / \det B(\zeta)$ as a transition function of the bundle $[\Delta_A + \Delta_B]$.

Now, let $C = \mathrm{coker} B(\zeta)$. Then we have short exact sequences that we can assemble into the following diagram such that sub diagram consisting of the rectangle commutes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{T\mathbb{CP}^1}(-3)^{\oplus k} \arrow[r, "\eta - A(\zeta)B(\zeta)"] \arrow[d, "B(\zeta)"] & \mathcal{O}_{T\mathbb{CP}^1}(-1)^{\oplus k} \arrow[r] \arrow[d, "B(\zeta)"] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{T\mathbb{CP}^1}(-2)^{\oplus k} \arrow[r, "\eta - B(\zeta)A(\zeta)"] \arrow[d] & \mathcal{O}_{T\mathbb{CP}^1}^{\oplus k} \arrow[r] \arrow[d] & \mathcal{F}'(1) \arrow[r] & 0 \\
 & C \arrow[d] & C(2) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Here, the two sequences in the columns are Beauville correspondences. We naturally have $\eta : C \to C(2)$ such that the following diagram commutes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{T\mathbb{CP}^1}(-3)^{\oplus k} \arrow[r, "\eta - A(\zeta)B(\zeta)"] \arrow[d, "B(\zeta)"] & \mathcal{O}_{T\mathbb{CP}^1}(-1)^{\oplus k} \arrow[r] \arrow[d, "B(\zeta)"] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{T\mathbb{CP}^1}(-2)^{\oplus k} \arrow[r, "\eta - B(\zeta)A(\zeta)"] \arrow[d] & \mathcal{O}_{T\mathbb{CP}^1}^{\oplus k} \arrow[r] \arrow[d] & \mathcal{F}'(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r, "\eta"] \arrow[d] & C(2) \arrow[r] \arrow[d] & \mathcal{O}_{\Delta_B} \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This means $\mathcal{F}'(1) = \mathcal{F}[\Delta_B]$ and the claim follows because $[\Delta_A + \Delta_B] \simeq \mathcal{O}_S(2)$. $\square$

We now ask whether a given quadratic polynomial $T(\zeta)$, corresponding to a real, analytic sheaf, can be factorized as in (5.2). Generically, the answer is yes:

Proposition 5.2. *Let $T(\zeta)$ be of form (4.8) and suppose that*

1. *the polynomial $\det T(\zeta)$ has $2n$ distinct zeros $\zeta_1, ..., \zeta_{2n}$,*
2. *the corresponding eigenvectors $v_i \in \ker T(\zeta_i)$, $i = 1, ..., 2n$ are in general position, i.e. $v_{i_1}, ..., v_{i_n}$ are linearly independent for any choice $i_1 < ... < i_n \in 1, ..., 2n$.*

Then $T(\zeta)$ can be factorized as $(A - B^*\zeta)(B + A^*\zeta)$.

Proof. After rotating $\mathbb{CP}^1$ we can assume that $\zeta = \infty$ is not a root of $\det T(\zeta)$. Let $\Delta \cup \sigma(\Delta)$ be a decomposition of the set of zeros of $\det T(\zeta)$. Theorem 1 in [Mal82] implies that there is a decomposition $T(\zeta) = (C_1 + D_1\zeta)(C_2 + D_2\zeta)$ such that Δ is the set of roots of $\det(C_2 + D_2\zeta)$. Applying the real structure shows that $(D_2^\dagger - C_2^\dagger\zeta)(-D_1^\dagger + C_1^\dagger\zeta)$ is also a factorization of $T(\zeta)$. We can rewrite these factorizations as

$$T(\zeta) = (C_1 D_1^{-1} + \zeta)(D_1 C_2 + D_1 D_2 \zeta) = (-D_2^\dagger (C_2^\dagger)^{-1} + \zeta)(C_2^\dagger D_1^\dagger - C_2^\dagger C_1^\dagger \zeta).$$

Theorem 2 in [Mal82] implies now that $C_1 D_1^{-1} = -D_2^\dagger (C_2^\dagger)^{-1}$, i.e. $D_1^{-1} C_2^\dagger = -C_1^{-1} D_2^\dagger$. In addition, comparing the constant coefficients of the two factorizations, we have $C_1 C_2 = -D_2^\dagger D_1^\dagger$. Hence

$$(D_1^{-1} C_2^\dagger)^\dagger = C_2 (D_1^\dagger)^{-1} = -C_1^{-1} D_2^\dagger D_1^\dagger (D_1^\dagger)^{-1} = -C_1^{-1} D_2^\dagger = D_1^{-1} C_2^\dagger.$$

Therefore $D_1^{-1} C_2^\dagger$ is hermitian (and invertible). We can write it as $-gdg^\dagger$, where g is invertible and d is diagonal with diagonal entries equal ± 1. Then:

$$T(\zeta) = (C_1 + D_1\zeta) g g^{-1} (C_2 + D_2\zeta) = (C_1 g + D_1 g \zeta) d (-g^\dagger D_1^\dagger + g^\dagger C_1^\dagger \zeta). \tag{5.3}$$

The uniqueness of monic factors of $T(\zeta)$ implies that the map $\Delta \mapsto d$ is injective. Since both sets have the same cardinality (equal to 2^k), this map is surjective, i.e. there is a choice of Δ such that the corresponding d is the identity matrix, and (5.3) becomes the desired factorisation. □

5.1.2 Bow variety as deformed instanton moduli space

In this section we want to describe the bow variety in terms of spectral data. For every subinterval $I_0 = [\lambda_0, \lambda_1], ..., I_r = [\lambda_r, \lambda_{r+1}]$ we have a linear flow on the Jacobian of the corresponding spectral curve $S_0, ..., S_r$. For the spectral curves $S_1, ..., S_{r-1}$ the boundary conditions on both ends of the corresponding subintervals are rank 1 jumps. The translation of these kind of boundary conditions where discussed in the last chapter (following [HM89]) and we found that on S_p we have

$$\mathcal{L}_{\lambda_{p+1}} = \mathcal{O}_{S_p}(2k-1)[-D_{p,p+1}] \tag{5.4}$$

at $t = \lambda_{p+1}$ and

$$\mathcal{L}_{\lambda_p} = \mathcal{O}_{S_p}(2k-1)[-D_{p-1,p}] \tag{5.5}$$

at $t = \lambda_p$. On the other hand, we know that

$$\mathcal{L}_{\lambda_{p+1}} \simeq L^{\lambda_{p+1}-\lambda_p} \otimes \mathcal{L}_{\lambda_p}. \tag{5.6}$$

Putting (5.4) and (5.5) into (5.6) we obtain

$$L^{\lambda_{p+1}-\lambda_p}[D_{p,p+1} - D_{p-1,p}] \simeq \mathcal{O}_{S_p}. \tag{5.7}$$

This is true for all spectral curves S_p for $p = 1, ..., r-1$.

It remains to identify the conditions on S_0 and S_r. They both have a rank one jump on one end of the interval, i.e. following [HM89] we have $\mathcal{L}_{\lambda_r} = \mathcal{O}_{S_r}(2k-1)[-D_{r-1,r}]$ and $\mathcal{L}_{\lambda_1} = \mathcal{O}_{S_0}(2k-1)[-D_{1,2}]$. Using the linear flow (5.6) we obtain bundles

$$\mathcal{L}_{\lambda_0} = L^{-(\lambda_1-\lambda_0)}(2k-1)[-D_{1,2}]$$

at the left end $t = \lambda_0$ and

$$\mathcal{L}_{\lambda_{r+1}} = L^{\lambda_{r+1}-\lambda_r}(2k-1)[-D_{r-1,r}]$$

at the right end $t = \lambda_{r+1}$ of the interval.

Now, the boundary conditions on the left and the right ends of the interval are given by (3.11) and (3.12). Hence, the spectral curves S_0 and S_r can be identified with each other. Therefore, for any quadratic polynomial $c = c(\zeta)$ denote by ϕ_c the automorphism $\eta \mapsto \eta + c(\zeta)$ on $T\mathbb{CP}^1$. We denote by S_c the image of S_0 under ϕ_{c_L} and equivalently the image of S_r under ϕ_{c_R}. It follows that on $\mathrm{Jac}^{g-1}(S_c)$, $B(\zeta)A(\zeta)$ represents the line bundle $L_{S_c}^{\lambda_0-\lambda_1}(2k-1)[-\phi_{c_L}(D_{1,2})]$ and $A(\zeta)B(\zeta)$ represents the line bundle $L_{S_c}^{\lambda_{r+1}-\lambda_r}(2k-1)[-\phi_{c_R}(D_{r-1,r})]$. We can view these line bundles as $\mathcal{F}$ and $\mathcal{F}'$ in proposition 5.1. This means, we obtain the condition

$$L_{S_c}^{\lambda_0-\lambda_1}(2k-1)[-\phi_{c_L}(D_{1,2})] \simeq L_{S_c}^{\lambda_{r+1}-\lambda_r}(2k-1)[-\phi_{c_R}(D_{r-1,r})] \otimes \mathcal{O}_{S_c}(1)[-\Delta_A],$$

i.e.

$$L_{S_c}^{\lambda_{r+1}-\lambda_r+\lambda_1-\lambda_0}(1)[\phi_{c_L}(D_{1,2}) - \phi_{c_R}(D_{r-1,r}) - \Delta_A] \simeq \mathcal{O}_{S_c}, \tag{5.8}$$

where Δ_A is the divisor cut out by $\det(A(\zeta)) = 0$ and $\eta = 0$. In addition, the spectral curves $S_c, S_1, ..., S_{r-1}$ satisfy some non-degeneracy conditions meaning that the linear flow of line bundles on the Jacobian of each of them does not meet the theta divisor. Conversely, given generic spectral curves $S_c, S_1, ..., S_{r-1}$ satisfying these conditions together with trivializations (5.7) and (5.8) we obtain a unique gauge equivalence class of solutions to Nahm equations in $\mathcal{M}$ using chapter 2 of [HM89] and proposition 5.2.

5.1.3 The bifundamental data

In this section we show how to treat the bifundamental data belonging to the homomorphisms A and B of the arrows in the bow diagram. Let $\tilde{\mathcal{M}}$ be the moduli space of solutions to Nahm equations on an interval with r λ-points. We have $\mathcal{M} \cong \tilde{\mathcal{M}} \times T^*\mathrm{Mat}_{k,k}(\mathbb{C})/\!/\!/(U(k) \times U(k))$ as hyperkähler quotients, where $\mathcal{M}$ is the bow variety. We will use this to show that $\mathcal{M}$ is isometric to an instanton moduli space as complex symplectic manifolds. To do so, we view both sides as complex symplectic quotients first. Then, the hyperkähler quotient is the Kähler quotient of the zero-set of the complex symplectic quotient of $\tilde{\mathcal{M}} \times T^*\mathrm{Mat}_{k,k}(\mathbb{C})$ by the complexified Lie group $GL_k(\mathbb{C}) \times GL_k(\mathbb{C})$ with the compact Lie group $U(k) \times U(k)$. The aim is now to relate these two kinds of quotients to each other. Therefore, we need a better description of the complex symplectic quotient. The idea is to identify the complex symplectic quotient with a geometric invariant theory quotient (GIT quotient) by using a Kempf-Ness theorem [KN79]. We follow chapter 3.1 in [Nak99].

Let X be a complex affine variety cut out by polynomials $f_1, ..., f_k$ and let $A(X)$ be its coordinate ring. Let G be a compact Lie group acting on X and let $G^{\mathbb{C}}$ be its complexification. The $G^{\mathbb{C}}$ action on X induces a $G^{\mathbb{C}}$ action on $A(X)$ and we denote by $A(X)^{G^{\mathbb{C}}}$ the ring of invariants of this action. Then the *affine algebro-geometric quotient* $X/\!/G^{\mathbb{C}}$ of X by $G^{\mathbb{C}}$ is defined as $\mathrm{Spec}(A(X)^{G^{\mathbb{C}}})$ and can by identified with the space of orbits of stable points (i.e. closed orbits such that the stabilizer of every point is finite). This quotient can be identified with a symplectic quotient $\mu^{-1}(0)/G$ via a Kempf-Ness theorem (for details on this see e.g. [Nak99]). However, this construction needs the value of the moment map to be equal to zero. In our case this is the value of the moment map μ_1 given by the real part of $c_L(\zeta) - c_R(\zeta)$, which is not zero. Fortunately, we can generalize this concept to the GIT quotient by lifting the problem to a holomorphic hermitian line bundle over X. This can be done by choosing a (unitary) character

$\chi^{\mathbb{R}} : G \to U(1)$ and its complexification $\chi : G^{\mathbb{C}} \to \mathbb{C}^*$. With this we can lift the $G^{\mathbb{C}}$ action the following way:

$$g \cdot (x,z) = (g \cdot x, \chi(g)^{-1} \cdot z) \tag{5.9}$$

in a local trivialization. As we can simply choose the trivial bundle $X \times \mathbb{C}$ for our purposes, let us restrict to that case now. Then, the space of $G^{\mathbb{C}}$ invariant sections of $X \times \mathbb{C}$ can be identified with the sum over spaces $A(X)^{G^{\mathbb{C}},\chi^n}$ of polynomials satisfying

$$f(g \cdot x) = \chi(g)^n f(x),$$

since $f \in A(X)^{G^{\mathbb{C}},\chi^n}$ implies $\tilde{f}(x,z) := f(x)z^n \in A(L)^{G^{\mathbb{C}},\chi^n}$.

Definition 5.3. Let X be a complex affine variety with coordinate ring $A(X)$. Let G act on X as above and let χ be the above defined character. The *geometric invariant theory quotient (GIT quotient)* is given as

$$X /\!\!/_{\chi} G^{\mathbb{C}} = \operatorname{Proj}\left(\bigoplus_{n \geq 0} A(X)^{G^{\mathbb{C}},\chi^n} \right).$$

We have the following geometric interpretation of a GIT quotient as the space of equivalence classes of semistable points in X where equivalence is defined by the intersection of orbits (cf. [Nak99]).

Definition 5.4. A point $x \in X$ is called χ*-semistable*, if there is a function $f \in A(X)^{G^{\mathbb{C}},\chi^n}$ for some $n \geq 1$ such that $f(x) \neq 0$. We denote the set of χ-semistable points by $X^{ss}(\chi)$.

In the trivialization above this happens precisely, if the closure of the orbit $\overline{\mathrm{Orb}_{G^{\mathbb{C}}}(x,z)}$ does not intersect with $X \times \{0\}$ for all points (x,z) with $z \neq 0$. On $X^{ss}(\chi)$ we can define an equivalence relation by $x \sim y$ if and only if $\overline{\mathrm{Orb}_{G^{\mathbb{C}}}(x)} \cap \overline{\mathrm{Orb}_{G^{\mathbb{C}}}(y)} \subset X^{ss}(\chi)$ and the intersection is not empty. Then,

$$X /\!\!/_{\chi} G^{\mathbb{C}} \cong X^{ss}(\chi) / \sim .$$

To see this, take a representative x in $X^{ss}/\sim$ such that the orbit $\mathrm{Orb}_{G^{\mathbb{C}}}(x,z)$ is closed for $z \neq 0$ for the lifted $G^{\mathbb{C}}$ action. For such a representative $\mathrm{Orb}_{G^{\mathbb{C}}}(x)$ is closed in $X^{ss}(\chi)$. Hence, the quotient $X^{ss}(\chi)/\sim$ is bijective to the set of $G^{\mathbb{C}}$ orbits in X such that $\mathrm{Orb}_{G^{\mathbb{C}}}(x,z)$ of the lifted action is closed for $z \neq 0$.

We have two important theorems now. One is again a Kempf-Ness theorem, which yields an equivalence between the zeroes of the moment map on X and the critical points of a certain function on the lifted orbits as in the case of an algebro-geometric quotient. In fact, the critical points of that (modified) functions lie precisely over the zeroes of the moment map (also cf. [DK90]).

For the other theorem, note that we obtain different lifted $G^{\mathbb{C}}$ action on the line bundle depending on how we choose the character χ. If the action on the bundle is *linear* (for a definition see below), this uniquely defines a moment map on X and vice versa, which we shell denote by μ_χ.

In our case, the affine variety X will be the zero set of the complex moment map and we will obtain identifications

$$\left(\mu_{\mathbb{R}}^{-1}(\mathrm{Re}(c_L(\zeta)-c_R(\zeta)))\cap\mu_{\mathbb{C}}^{-1}(0)\right)\Big/G=\left(\mu_\chi^{-1}(0)\cap\mu_{\mathbb{C}}^{-1}(0)\right)\Big/G=\mu_{\mathbb{C}}^{-1}(0)/\!\!/_\chi G.$$

So, the advantage of this construction is, that different choices of the character can represent different values of the real moment map, which we need to cover our situation. We have the following definition and theorem from [Bry97]

Definition 5.5. Let (X,ω) be a Kähler manifold and $G^{\mathbb{C}}$ acting on X as above. A *linearisation* of this action is a holomorphic action of $G^{\mathbb{C}}$ on a holomorphic, hermitian line bundle $\pi: L\to X$ covering the action on X and such that G acts unitarily on the fibres of L.

Observe, that we obtain a linearisation when we lift the $G^{\mathbb{C}}$ action as in (5.9). Then we have:

Theorem 5.6. *A choice of a linearisation uniquely determines a moment map on X and conversely, a choice of a moment map uniquely determines a linearisation.*

For a bow variety consisting of only one interval and one arrow (as in our case) the moment map μ_χ has been calculated in detail by Takayama [Tak15]. He used only one λ-point, but the calculation for of them is completely analogous. At the boundary points λ_0 and λ_{r+1} we obtain values $\mu_\chi=\mu_{\mathbb{R}}-2c_\pm\sqrt{i}\mathrm{Id}$, where $c_\pm$ are constants. Hence, there is a choice of the lift i.e. a choice of the term $\chi^{-1}(g)$ in (5.9) for which we have

$$\left(\mu_{\mathbb{R}}^{-1}(\mathrm{Re}(c_L(\zeta)-c_R(\zeta)))\cap\mu_{\mathbb{C}}^{-1}(0)\right)\Big/G=\left(\mu_\chi^{-1}(0)\cap\mu_{\mathbb{C}}^{-1}(0)\right)\Big/G.$$

The final step is the identification with the GIT quotient by a Kempf-Ness theorem. Therefore, we define a modified function defined on the lifted orbits. It is given as follows:

$$\begin{aligned}p_{(x,z)}:G^{\mathbb{C}}&\to\mathbb{R}_+\\ g&\mapsto\log(||(x,z)||g\cdot(x,z))\end{aligned}\tag{5.10}$$

For this function, we have a Kempf-Ness theorem, i.e. an identification of the critical points of p with the zeroes of the moment map μ_χ (cf. [Nak99] or [DK90]). In particular, if we take a path $g=exp(t(\zeta_1+i\zeta_2))$ in $G^{\mathbb{C}}$, we can take the derivative of $p_{(x,z)}$ in that direction and obtain, that a critical point $p'_{(x,z)}=0$ is obtained if and only if on the base $\mu_\chi=0$. On the other hand $p_{(x,z)}$ attains a critical point if and only if the orbit $\mathrm{Orb}_{G^{\mathbb{C}}}(x,z)$ is closed in $X\times\mathbb{C}$.

Now, we return to our problem. We have already seen that (considering the complex structure I corresponding to $0 \in \mathbb{CP}^1$) we can write Nahm equations in a complex form with $\beta(s) = T_2(s) + iT_3(s)$ and $\alpha(s) = iT_1(s)$ on each interval with rank 1 jumps at each λ-point $\lambda_1, ..., \lambda_r$. $\tilde{M}$ is then biholomorphic to the quotient of the space of solutions to this the complexified gauge group, i.e. by the group of $GL_k(\mathbb{C})$-valued gauge transformations, which are equal to identity at $s = \lambda_0$ and $s = \lambda_{r+1}$ and match at the $\lambda_1, ..., \lambda_r$ (also cf. [Don84] and [Hur89]). This biholomorphism preserves the complex symplectic form.

We can apply a gauge transformation on each subinterval to make $\alpha \equiv 0$ and $\beta = \text{const}$. Recall therefore, that the action on α is $\alpha \mapsto g\alpha g^{-1} + \dot{g}g^{-1}$, which we can make identically zero. In this gauge the complex Nahm equation $\dot{\beta} = [\beta, \alpha]$ is easily solved by $\beta = \text{const}$. Hence, we obtain constant β_i for each $[\lambda_{i-1}, \lambda_i]$, $i = 1, ..., r+1$ and $\beta_{i+1} - \beta_i = I_i J_i$ for $I_i \in \mathbb{C}^k$ and $J_i \in (\mathbb{C}^k)^*$. Therefore, we obtain a complex symplectic isomorphism

$$\begin{aligned} \Psi : \tilde{M} &\to T^*GL_k(\mathbb{C}) \times T^*\mathrm{Mat}_{k,r}(\mathbb{C}) \\ \mathrm{pt} &\mapsto (\beta(\lambda_0), g(\lambda_{r+1}), I, J) \end{aligned}$$

with $I = [I_1, ..., I_r]$ and $J = [J_1, ..., J_r]^T$ (cf. [Bie97] and [Kro04]).

Now, the complex symplectic quotient of $T^*GL_k(\mathbb{C}) \times T^*\mathrm{Mat}_{k,r}(\mathbb{C}) \times T^*\mathrm{Mat}_{k,k}(\mathbb{C})$ by $GL_k(\mathbb{C}) \times GL_k(\mathbb{C})$ can be performed in two stages. First, we take the quotient of the left copy of $GL_k(\mathbb{C})$, which acts trivially on I and J and we obtain $T^*\mathrm{Mat}_{k,r}(\mathbb{C}) \times T^*\mathrm{Mat}_{k,k}(\mathbb{C})$. The remaining symplectic quotient by the right copy of $GL_k(\mathbb{C})$ is then the one for the case $\lambda_0 = ... = \lambda_{r+1}$. But these spaces have been interpreted as instanton moduli spaces on a non-commutative $\mathbb{R}^4$ by [Nek98] as long as $c_L(\zeta) - c_R(\zeta) \neq 0$. Hence, if this condition is satisfied we have the same stability conditions and therefore $\mathcal{M}$ is isomorphic as a complex symplectic manifold to the corresponding space of instantons.

5.2 The asymptotic metric

In this section we will finally calculate the asymptotic metric for the bow variety. In this context asymptotic means that we let the length of the subintervals go to infinity. The structure will be the following. We will describe some building blocks given by the space of solutions to Nahm equations on the whole real line with a rank 1 jump at $s = 0$. Via a twistor construction we will then find a metric for this building blocks, which will work as follows: We have theorem 2.29 (Thm 3.3 from [HKLR87]), which gives us an identification of the parameter space of real sections of some $2n+1$ complex dimensional manifold Z with certain properties with the (real) $4n$ dimensional hyperkähler manifold M whose twistor space is equal to Z.

Therefore, the strategy is to construct the twistor space and its real sections of the building block and observe that they are the same as for the Gibbons-Manton manifold.

From this we conclude with the above theorem that the metric on the building block is the Gibbons-Manton metric. To obtain the asymptotic metric of the bow variety we will take multiple copies of these building blocks and glue them together by taking the hyperkähler quotient. This will represent the subintervals together with the λ-points. We will need one additional building block representing the arrows which will be built in the same way. We will therefore obtain a metric on a bow variety with r λ-points on a single interval with the length of the $r+1$ subintervals going to infinity. Lastly, we will proof that the constructed metric is indeed the asymptotic metric of the bow variety.

5.2.1 The Gibbons-Manton metric

As we will find the metric on each of the building blocks by comparison with the well known Gibbons-Manton metric [GM95], we will start this section with the construction of that metric.

It was shown by Pederson and Poon [PP88] that every hyperkähler metric can be written locally in the form

$$g = \Phi \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} + \Phi^{-1}(\mathrm{d}t + A)^2$$

with local coordinates $(t, \mathbf{x})$, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ with $\mathbf{x}_i = (x_i^1, ..., x_i^k)$ and $t = (t^1, ..., t^k)$ where A is a 1-form satisfying certain PDE's but depending only on $\mathbf{x}$. Hence, the metric depends only on the matrix Φ. For the Gibbons-Manton metric this matrix is given by

$$\Phi_{ij} = \begin{cases} -\frac{1}{\nu} - \sum_{n \neq i} \frac{1}{||x_i - x_n||} & \text{if } i = j, \\ \frac{1}{||x_i - x_j||} & \text{if } i \neq j \end{cases}, \tag{5.11}$$

where ν is the mass parameter.

Starting with a flat hyperkähler manifold $M = \mathbb{H}^{\frac{1}{2}k(k-1)} \times (S^1 \times \mathbb{R}^3)^k$ we can achieve the Gibbons-Manton manifold as a hyperkähler quotient following [GRG97]. Taking the flat metric on M to be $g = g_1 - g_2$ where g_1 is the metric on $M_1 := \mathbb{H}^{\frac{1}{2}k(k-1)}$ and g_2 is the one on $M_2 := (S^1 \times \mathbb{R}^3)^k$, we choose coordinates q_{ij}, $i = 1, ..., k; i < j$ on M_1 and $(t_i, \mathbf{x}_i)$ on M_2. The metric g_1 is invariant under the left diagonal action of $T^{k(k-1)/2}$ given by $q_{ij} \mapsto e^{t_{ij}} q_{ij}$ and the metric g_2 is invariant under translations of T^k.

Using the homomorphism $\phi : T^{k(k-1)/2} \to T^k$ given by (cf. [Bie98])

$$(t_{ij})_{i<j} \mapsto \left(\prod_{j=i+1}^{k} t_{ij} \prod_{j=1}^{i-1} t_{ji}^{-1} \right)_{i=1,...,k}$$

we obtain a $T^{k(k-1)/2}$ action on the product by $t \cdot (m_1, m_2) = (t \cdot m_1, \phi(t) \cdot m_2)$. With the above coordinates, we obtain a moment map condition of the form

$$\frac{1}{2} q_{ij} i \overline{q}_{ij} = \mathbf{x}_i - \mathbf{x}_j$$

and in [GRG97] it is shown that the hyperkähler quotient M_{GM} of (M, g) with this action is the Gibbons-Manton metric. The T^k action on M_2 induces a free tri-Hamiltonian action on M_{GM} for which the moment map is $(\mathbf{x}_1, ..., \mathbf{x}_k)$ (cf. [Bie98]). This makes M_{GM} into a T^k bundle over the configuration space $\tilde{C}_k(\mathbb{R}^3)$ of k distinct points in $\mathbb{R}^3$.

5.2.2 The building blocks

In this section we will define the building blocks for the bow variety and find coordinates for them.

We recall from [Bie99] the moduli spaces $\tilde{F}_{k,k}(c, c')$ and their metrics. These can be viewed as spaces of solutions of rank k to Nahm equations on $\mathbb{R}$ with a jump of rank 1 at $t = 0$.

Let us write $\tilde{F}_k(k, c)$ for the moduli space of solutions to Nahm equations on $\mathbb{R} > 0$. These solutions satisfy the following conditions [Bie99]:

- The T_i are analytic at $t = 0$ and approach a diagonal limit as $t \to +\infty$ exponentially fast with $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$ being a regular triple,
- The gauge group has a Lie algebra consisting of functions $\rho : [0, +\infty) \to \mathfrak{u}(k)$ such that:
    1. $\rho(0) = 0$ and $\dot{\rho}$ has a diagonal limit at $t \to +\infty$,
    2. $(\dot{\rho} - \dot{\rho}(+\infty))$ and $[\rho, \tau]$ decay exponentially fast for any regular diagonal matrix $\tau \in \mathfrak{u}(k)$,
    3. $c\dot{\rho}(+\infty) + \lim_{t\to+\infty}(\rho(t) - t\dot{\rho}(+\infty)) = 0$,
- $\tilde{F}_k(k, c)$ has a tri-hamiltonian action of $U(k) \times T^k$, where the $U(k)$ action is given by a gauge transformation with an arbitrary value at $t = 0$ and the T^k action is given by gauge transformations which are asymptotic to $\exp(-th + \lambda h)$ for a diagonal h and $\lambda \in \mathbb{R}$; the moment maps for these actions are $(T_1(0), T_2(0), T_3(0))$ for the $U(k)$ action and $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$ for the T^k action.

In fact, the assumptions on the Lie algebra translate to a gauge group consisting of functions $g : [0, +\infty) \to U(n)$ such that

1. $g(0) = 1$ and $\dot{g}(t)g^{-1}(t)$ has a diagonal limit, say $s(g)$,
2. $\dot{g}(t)g^{-1}(t) - s(g)$ and $\tau - \mathrm{Ad}(g(t))\tau$ both decay exponentially fast,
3. $\exp(cs(g)) \lim_{t\to+\infty} (g(t) \exp(-ts(g))) = 1$.

The values 0 and $+\infty$ now give the tri-hamiltonian actions of $U(k)$ and T^k, where the 3. assumption is just that $g(t)$ is asymptotic to $\exp(-th + \lambda h)$ as stated above. With this definition we can identify the moduli space $\tilde{F}_{k,k}(c, c')$ with the hyperkähler quotient $\tilde{F}_k(k, c) \times \tilde{F}_k(k, c') \times \mathbb{H}^k$ by the diagonal action of $U(k)$.

If we write T_i^+ for a solution to Nahm equations on $\mathbb{R} > 0$ and T_i^- for a solution on $\mathbb{R} < 0$ we have the following boundary condition at $t = 0$: There exist vectors $I, J^\dagger \in \mathbb{C}^k$ such that

$$\begin{aligned}(T_2^+ + iT_3^+)(0_+) - (T_2^- + iT_3^-)(0_-) &= IJ,\\ T_1^+(0_+) - T_1^-(0_-) &= \frac{1}{2}(II^\dagger - J^\dagger J).\end{aligned} \tag{5.12}$$

Using the results of Biquard [Biq96] we have the following theorem from [Bie98] and [Bie99] to find a biholomorphic identification for $\tilde{F}_{k,k}(c, c')$:

Proposition 5.7. *Let $\mathfrak{n}$ be a nilpotent algebra containing the diagonal matrices, $\mathfrak{d}$ the algebra of diagonal matrices and $\mathfrak{b} = \mathfrak{d} + \mathfrak{n}$. Further let $N = \exp(\mathfrak{n})$. Then there is a biholomorphism between an open dense subset of*

$$GL_k(\mathbb{C}) \times_N \mathfrak{b}$$

and an open dense subset $F(\mathfrak{n})$ of $\tilde{F}_{k,k}(c, c')$ containing of all the solutions to $(\alpha, \beta) = (T_0 + iT_1, T_2 + iT_3)$ to Nahm equations such that the intersections of the sum of all positive eigenvalues of $\mathrm{ad}(iT_1(+\infty))$ with the centralizer of $\beta(+\infty)$ is contained in $\mathfrak{n}$.

The charts are then given by $[g, d + n] \sim [g', d' + n']$ if and only if $n \in \mathfrak{n}$, $n' \in \mathfrak{n}'$ and either $\mathfrak{n}' \subset \mathfrak{n}$ and $\exists m \in N$ such that $gm^{-1} = g'$ and $\mathrm{Ad}(m)(d + \mathfrak{n}) = d' + \mathfrak{n}'$ or vice versa.

Here, the element $g \in GL_k(\mathbb{C})$ is the value of the gauge transformation $g(t)$ at $t = 0$ which sends $(0, \beta(+\infty) + n)$ to (α, β).

For $\tilde{F}_{k,k}(c, c')$ we obtain via the same construction a biholomorphism to $\mathfrak{b} \times (GL_k(\mathbb{C}) \times_{N'} \mathfrak{b}') \times \mathbb{C}^{2k}$, so let us write $(b_-, (g, b_+), (IJ))$ as local coordinates with (cf. [Bie99])

$$gb_+g^{-1} = b_- + IJ^T. \tag{5.13}$$

Observe that we only have one $GL_k(\mathbb{C})$ in the product space as $\tilde{F}_{k,k}(c, c')$ was defined as a hyperkähler quotient by a diagonal action.

Further, recall that we have an open dense subset where $\beta(+\infty)$ is regular. This corresponds to $\mathfrak{n} = 0$ and hence to $\mathfrak{b} = \mathfrak{b}' = \mathfrak{d}$, i.e. on that subset we have a representation in which b_- and b_+ are given by diagonal matrices $\beta_d^- = (\beta_1^-, ..., \beta_n^-)$ and $\beta_d^+ = (\beta_1^+, ..., \beta_n^+)$ with $\beta_i^\sigma = \beta_j^\tau$ if and only if $i = j$ and $\sigma = \tau$ for $i, j \in \{1, ..., k\}$ and $\sigma, \tau \in \{+, -\}$. This means there can be at most one of the $\beta_i^\pm$ being zero and hence at least one of the matrices β_d^-, or β_d^+ must be invertible (say β_d^-). Further, IJ^T must have full rank, because otherwise β_d^- and β_d^+ would share an eigenvalue (because of (5.13)) which contradicts $\beta_i^\sigma \neq \beta_j^\tau$ for $i \neq j$ or $\sigma \neq \tau$. Therefore, non of the components of J can be zero and hence J is a cyclic vector for β_d^-. (To see this take $(\beta_d^-)^n J = \lambda_{n-1}(\beta_d^-)^{n-1}J + ... + \lambda_1(\beta_d^-)J + \lambda_0 J$ for some $n \leq k$. Induction on n leads to the contradiction $\beta_d^- = (\beta_d^-)^{-1}$.)

Knowing this we have two results from linear algebra [HJ12]. First, the columns of the matrix

$$B = (\beta_d^-)^{k-1}J, ..., \beta_d^- J, J)^T$$

are an ordered basis for $\mathbb{C}^k$ and the representation matrix of β_d^- in that basis is in form of a companion matrix

$$\beta^- = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & -a_0 \\ 1 & \ddots & & & \vdots & \vdots \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & 1 & -a_{k-1} \end{pmatrix}. \tag{5.14}$$

In that chart b_+ must also be of the form (5.14), hence we have a description of the chart as pairs $((g_-, \beta_d^-), (g_+, \beta_d^+))$ with $g_-\beta_d^- g_-^{-1}$ and $g_+\beta_d^+ g_+^{-1}$ both are of the form (5.14). But we also know that the transition matrix to that basis is the matrix B defined above. Therefore, we obtain (together with (5.13))

$$g_- = Bd, \qquad g_+ = Bdg$$

with d being some diagonal matrix that is equal to g_- in the chart where b_- is diagonal.

The second result is the following: Let $f(x) = x^k + a_{k-1}x^{k-1} + \dots + a_1 x + a_0$ be a polynomial with companion matrix $A(f)$ as above such that the entries on the last column are the coefficients of the monomials. Then f has exactly k pairwise distinct zeros $\beta_1, \dots, \beta_k$ if and only if $A(f)$ is diagonalizable with $VA(f)V^{-1} = \mathrm{diag}(\beta_1, \dots, \beta_k)$ where V is the Vandermonde matrix

$$V = V(\beta_1, \dots, \beta_k) = \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \dots & \beta_1^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta_k & \beta_k^2 & \dots & \beta_k^{k-1} \end{pmatrix}.$$

This transformation is not unique. In fact, we obtain a family of transformations given by right multiplication of the Vandermonde matrix with diagonal matrices. We end up with

$$g_\pm = V(\beta_1^\pm, \dots, \beta_k^\pm)^{-1}\mathrm{diag}(u_1^\pm, \dots, u_k^\pm)$$

where all of the $u_i^\pm$ are non zero. As the Vandermonde matrix consists only of the $\beta_i^\pm$ we can choose the coordinates in that chart to be $(\beta_i^\pm, u_i^\pm)$. Comparing these two formulae we obtain

$$\begin{aligned} \mathrm{diag}(u_1^+, \dots, u_k^+) &= V(\beta_1^+, \dots, \beta_k^+)Bdg, \\ \mathrm{diag}(u_1^-, \dots, u_k^-) &= V(\beta_1^-, \dots, \beta_k^-)Bd. \end{aligned} \tag{5.15}$$

5.2.3 Twistor space of the building blocks

To obtain a metric for this building block we have to calculate the twistor space. Let ζ be the affine coordinate on $\mathbb{CP}^1$. For every hyperkähler manifold admitting an

$SU(2)$-action rotating the complex structures the twistor space can be trivialized using two charts $\{\zeta \neq \infty\}$ and $\{\zeta \neq 0\}$ with $\tilde{\zeta} = 1/\zeta$ on the overlap.

Proposition 5.8. *Let Z denote the twistor space of $\tilde{F}_{k,k}(c,c')$. In the above trivialization we have*

$$\begin{aligned}\tilde{\beta}_i^{\pm} =& \frac{\beta_i^{\pm}}{\zeta^2},\\ \tilde{u}_i^{\pm} =& \zeta^{1-2k}\exp(-c^{\pm}\beta_i^{\pm}/\zeta)u_i^{\pm}.\end{aligned}$$

Proof. For a moduli space of solutions to Nahm equations we can achieve the above trivialization by putting (cf. [Dan94])

$$\begin{aligned}\eta^{\pm} =& \beta^{\pm} + \zeta(\alpha^{\pm} + (\alpha^{\pm})^{\dagger}) - \zeta^2(\beta^{\pm})^{\dagger},\\ u^{\pm} =& \alpha^{\pm} - \zeta(\beta^{\pm})^{\dagger}\end{aligned}$$

over $\zeta \neq \infty$ and

$$\begin{aligned}\tilde{\eta}^{\pm} &= \frac{\beta^{\pm}}{\zeta^2} + \frac{1}{\zeta}(\alpha^{\pm} + (\alpha^{\pm})^{\dagger}) - (\beta^{\pm})^{\dagger},\\ \tilde{u}^{\pm} &= -(\alpha^{\pm})^{\dagger} - \frac{\beta^{\pm}}{\zeta}\end{aligned}$$

over $\zeta \neq 0$. Then on the overlap $\zeta \neq 0, \infty$ we have

$$\begin{aligned}\tilde{\eta}^{\pm} &= \frac{\eta^{\pm}}{\zeta^2},\\ \tilde{u}^{\pm} &= u^{\pm} - \frac{\eta^{\pm}}{\zeta}.\end{aligned}$$

We obtain immediately that $\beta_i^{\pm} \mapsto \beta_i^{\pm}/\zeta^2$.

For the $u_i^{\pm}$ observe that (writing $B = B(\beta_d^-, J)$) the product

$$V(\tilde{\beta}_1^{\pm}, ..., \tilde{\beta}_k^{\pm})B(\tilde{\beta}_d^-, \tilde{J}) = \zeta^{1-2k}V(\beta_1^{\pm}, ..., \beta_k^{\pm})B(\beta_d^-, J),$$

thus with $\tilde{d} = d\exp(-c'\beta_d^-/\zeta)$ and $\tilde{g} = \exp(c'\beta_d^-/\zeta)g\exp(-c\beta_d^+/\zeta)$ (cf. [Bie99]) we obtain the formulae for $u_i^{\pm}$. $\square$

The real structure is given by

$$\begin{aligned}\zeta &\mapsto \frac{1}{\bar{\zeta}},\\ \beta_i^{\pm} &\mapsto -\frac{\overline{\beta_i^{\pm}}}{\bar{\zeta}^2},\\ u_i^{\pm} &\mapsto \frac{1}{\overline{u_i^{\pm}}}\left(\frac{1}{\bar{\zeta}}\right)^{2k-1}\exp(c^{\pm}\overline{\beta_i^{\pm}}/\bar{\zeta})\prod_{j\neq i}\left(\overline{\beta_i^{\pm}} - \overline{\beta_j^{\pm}}\right)\prod_{j=1}^{n}\left(\overline{\beta_i^{\pm}} - \overline{\beta_j^{\mp}}\right)\end{aligned} \tag{5.16}$$

and can be obtained from the real structure of the moduli space of solutions to Nahm equations

$$\zeta \mapsto \frac{1}{\bar{\zeta}}, \qquad \eta \mapsto -\frac{\eta^\dagger}{\bar{\zeta}}, \qquad u \mapsto -u^\dagger + \frac{\eta^\dagger}{\bar{\zeta}}$$

as it was calculated in [Bie98].

The next step is to find a family of real sections. Following [HKLR87] they are given as

$$\beta_i^\pm = z_i^\pm + 2x_i^\pm \zeta - \bar{z}_i^\pm \zeta^2 \tag{5.17}$$

for $i = 1, ..., k$ and where $p_i^\pm := (z_i^\pm, x_i^\pm) \in \mathbb{C} \times \mathbb{R}$ satisfies $p_i^\sigma = p_j^\tau$ if and only if $i = j$ and $\sigma = \tau$ for $\sigma, \tau \in \{+, -\}$. These curves will be denoted by $S_i^\pm$ and can be thought of as spectral curves of individual monopoles (cf. [Bie99]). We need to know the behaviour of the $u_i^\pm$ along those sections. We have Prop. 5.3, i.e. the $u_i^\pm$ behave like sections of the bundle $\mathcal{O}(2k-1) \otimes L^{c^\pm}$ with transition function $\frac{1}{\zeta^{2k-1}} \exp(-c^\pm \eta/\zeta)$. This is true as long as the curves S_i do not intersect. Hence, we need to know what happens at the intersection points. Real sections intersect in distinct points ([HKLR87]), hence we have 3 cases to consider: the intersections $S_i^+ \cap S_j^+$, $S_i^- \cap S_j^-$, which can be treated analogously to the first case, and finally $S_i^+ \cap S_j^-$ for $i \neq j$.

In all 3 cases setting $\beta_i^\sigma(\zeta) = \beta_j^\tau(\zeta)$ we have an algebraic equations of second order which has the two solutions $\zeta = a_{ij}^{\sigma\tau}, a_{ji}^{\sigma\tau}$, where $a_{ij}^{\sigma\tau}$ is given by

$$a_{ij}^{\sigma\tau} = \frac{x_i^\sigma - x_j^\tau + r_{ij}^{\sigma\tau}}{\bar{z}_i^\sigma - \bar{z}_j^\tau} \tag{5.18}$$

with

$$r_{ij}^{\sigma\tau} = r_{ji}^{\tau\sigma} = \sqrt{(x_i^\sigma - x_j^\tau)^2 + |z_i^\sigma - z_j^\tau|^2} \tag{5.19}$$

and $\sigma, \tau \in \{+, -\}$ labelling the case of intersection we are working with (i.e. $++$, $+-$, $-+$ and $--$).

For convenience, we will from now on suppress the upper indices $\sigma\tau$ and reintroduce them when they are needed for calculating the metric.

So let us start with the intersection of $S_i^- \cap S_j^-$ or equivalently $S_i^+ \cap S_j^+$. With the argumentation in the beginning of this section, β_d^- and β_d^+ have pairwise distinct eigenvalues and thus J is cyclic for β_d^-. Hence, B and therefore also $g_\pm$ are invertible at the points a_{ij} and a_{ji}. We follow the lines in [Bie98] to observe that $u_i^\pm$ has a single zero at a_{ji} and is nonzero at a_{ij} and that all other $u_s^\pm$ with $s \neq i, j$ are nonzero at both a_{ij} and a_{ji}.

The i, j-entry of $V^{-1}(\beta_1^\pm, ..., \beta_k^\pm)$ the inverse Vandermonde matrix is given by [Bie98]

$$W_{ij} = (-1)^{k-i} S_{k-i}(\beta_1^\pm, ..., \hat{\beta}_j^\pm, ..., \beta_k^\pm) \Big/ \prod_{n \neq j} (\beta_j^\pm - \beta_n^\pm)$$

where $S_i(x_1, ..., x_k)$ denotes the i-th elementary symmetric polynomial with $S_0 = 1$. The product $\mathrm{diag}(u_1^\pm, ..., u_k^\pm)\exp(\mathfrak{n})$ for $\mathfrak{n}$ being generated only by a single matrix, which has every entry equal to zero except the i, j-entry, is of the form

$$\begin{pmatrix} u_1^\pm(\zeta) & & & & \\ & \ddots & & p(\zeta) & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & u_k^\pm(\zeta) \end{pmatrix}$$

where the only non-zero off diagonal entry is the i, j-entry $p(\zeta)$. In order to obtain an invertible limit, we need the expressions $W_{ni}(\zeta)u_i(\zeta)$ and $W_{nj}(\zeta)u_j(\zeta) + W_{ni}(\zeta)p(\zeta)$ to have finite limits at $\zeta = a_{ji}$ for all $n = 1, ..., k$. For $n = k$ we find $W_{ki} = 1 \Big/ \prod_{n\neq i}(\beta_i^\pm - \beta_n^\pm)$, which has a single pole because of the $n = j$ term in the product, as a_{ji} is an intersection point of β_i with β_j. Hence, we obtain $u_i(a_{ji}) = 0$. Further, the i-th column may not be equal to zero to obtain an invertible limit. Since the entries of that column are of the form $W_{ni}u_i$ for $n = 1, ..., k$ and u_i has a zero at $\zeta = a_{ji}$, this can only be achieved by W_{ni} having a pole cancelling out the zero. The only poles arising in any of the W_{ni} though are, whenever the product contains a term $\beta_i^\pm - \beta_j^\pm$ which is a single pole. Hence, the zero of u_i has to be a single zero.

All other W_{nl}, $n = 1, ..., k$ and $l \neq i, j$ do not contain terms of the form $1 \Big/ (\beta_i - \beta_j)$ and thus non of the $u_l(\zeta)$ may have a zero at $\zeta = a_{ji}$.

Finally, for u_j we have that $W_{kj}(\zeta)u_j(\zeta) + W_{ki}(\zeta)p(\zeta)$ needs to have a finite limit. This we can write as $\frac{f(\zeta)u_j(\zeta)+g(\zeta)p(\zeta)}{\beta_i(\zeta)-\beta_j(\zeta)}$ with f and g having a finite limit at $\zeta = A_{ji}$. Then, for $n = k - 1$ we find

$$\begin{aligned} W_{k-1,j}(\zeta)u_j(\zeta) + W_{k-1,i}(\zeta)p(\zeta) &= -\frac{f(\zeta)u_j(\zeta)\sum_{l\neq j}\beta_l(\zeta) + g(\zeta)p(\zeta)\sum_{l\neq i}\beta_l(\zeta)}{\beta_i(\zeta) - \beta_j(\zeta)} \\ &= -f(\zeta)u_j(\zeta) - \frac{f(\zeta)u_j(\zeta) + g(\zeta)p(\zeta)}{\beta_i(\zeta) - \beta_j(\zeta)}\sum_{l\neq i}\beta_l(\zeta) \end{aligned}$$

where the latter term has a finite limit because of the W_{kj} term and so the first term and thus $u_j(\zeta)$ must also have a finite limit at $\zeta = a_{ji}$. Because of the symmetry of u_i and u_j when exchanging the intersection points a_{ji} and a_{ij} this means that both u_i and u_j have no other poles and equivalently no other zeroes as the one of u_i at a_{ji} and u_j at a_{ij}.

For more than two curves intersecting in the same point, we need to take a larger generator of the algebra $\mathfrak{n}$ which makes the computations more complicated but the structure of them remains the same. For each $j \neq i$ the point $\zeta = a_{ji}$ contributes a single zero of u_i such that u_i has exactly $k - 1$ zeroes (counting multiplicities).

Now, we have to consider the case $S_i^+ \cap S_j^-$ (or vice versa). Observe, that for $I, J \in \mathbb{C}^k$ the matrix IJ^T has rank one and hence all $n \times n$ minors vanish for $n > 1$. Thus, for a diagonal matrix d we have the formula (cf. [Bie99])

$$\det(d + IJ^T) = \prod_n d_n + \sum_n \left(I_n J_n \prod_{l \neq n} d_l \right).$$

We had (5.13) in the chart, where β^- is diagonal and hence we have $\det(\beta_d^- - \beta_i^+ \mathrm{Id} + IJ^T) = 0$. Defining $d = \beta_d^- - \beta_i^+ \mathrm{Id}$ we have on the intersection $S_i^+ \cap S_j^-$ that $d_j = 0$ and $d_n \neq 0$ for all $n \neq j$. The formula for the determinant now is $\det(d + IJ^T) = I_j J_j \prod_{l \neq j} d_l$ which has to vanish, thus we obtain $I_j J_j = 0$ at both a_{ij} and a_{ji}. As I_j and J_j are sections of $\mathcal{O}(1)$ they have exactly one zero, hence I_j vanishes on a_{ij} and only there and so does J_j on a_{ji} or vice versa. If we take $d = \beta_d^- - \beta_l^- \mathrm{Id}$ instead, we obtain from the non vanishing of the determinant $\det(d + IJ^T)$ that $I_l J_l$ does not vanish at a_{ij} or a_{ji} for $l \neq j$. This means that exactly the j-th column of the matrix $B = ((\beta_d^-)^{k-1} J, ..., \beta_d^- J, J)^T$ vanishes at either a_{ij} or a_{ji}. With $g_- = Bd$ and $g_+ = Bdg$ the same is true for $g_\pm$ and with $g_\pm = V(\beta_1^\pm, ..., \beta_k^\pm)^{-1} \mathrm{diag}(u_1^\pm, ..., u_k^\pm)$ we find that both u_j^+ and u_j^- vanish at either a_{ij} or a_{ji} and no other $u_l^\pm$ vanishes at that point. As before, each $j = 1, ..., k$ contributes a single zero c_{ij} which is either a_{ij} or a_{ji}.

Thus, we finally obtain a formula for the $u_i^\pm$ for $i = 1, ..., k$, where we need to introduce the upper indices $\sigma\tau \in \{+, -\}$ again:

$$u_i^\pm(\zeta) = A_i^\pm \prod_{\substack{j \neq i \\ j \leq k}} (\zeta - a_{ji}^{\pm\pm}) \prod_{j \leq k} (\zeta - c_{ij}^\pm) e^{c^\pm (x_i - \bar{z}_i \zeta)}. \tag{5.20}$$

The $c_{ij}^\pm$ can be either $a_{ij}^{\pm\mp}$ or $a_{ji}^{\pm\mp}$. The coefficients $A_i^\pm$ satisfy the condition

$$A_i^\pm \overline{A_i}^\pm = \prod_{\substack{j \neq i \\ j \leq k}} (x_i^\pm - x_j^\pm + r_{ij}^{\pm\pm}) \prod_{j \leq k} (\sigma(x_i^\pm - x_j^\mp) + r_{ij}^{\pm\mp})$$

which is induced by the real structure (cf. [Bie98] and [Bie99]), where $\sigma = +1$, if $c_{ij}^\pm = a_{ji}^{\pm\mp}$ and $\sigma = -1$ if $c_{ij} = a_{ij}^{\pm\mp}$.

5.2.4 Topology and metric of the building blocks

Before calculating the metric we need to have some topological considerations. We have the following theorem [Bie98]

Proposition 5.9. *The space $\tilde{F}_{k,k}(c^-, c^+)$ is topologically a torus bundle over $\tilde{C}_k(\mathbb{R}^3) \times \tilde{C}_k(\mathbb{R}^3)$, where $\tilde{C}_k(\mathbb{R}^3)$ denotes the configuration space of k distinguishable points in $\mathbb{R}^3$.*

The tri-hamiltonian action of $T^k \times T^k$ on $\tilde{F}_{k,k}(c^-, c^+)$ defines a moment map on $\mathbb{R}^3 \times \mathbb{R}^{2k}$ given by

$$\Big(\big(T_1(+\infty), T_1(-\infty)\big), \big(T_2(+\infty), T_2(-\infty)\big), \big(T_3(+\infty), T_3(-\infty)\big) \Big).$$

The proof was done in [Bie98] by identifying the hyperkähler quotient of $M(\tau_1^+, \tau_2^+, \tau_3^+) \times \mathbb{H}^k \times M(\tau_1^-, \tau_2^-, \tau_3^-)$ by the diagonal action of $U(k)$ with the space of $T^k \times T^k$ orbits that are mapped to (τ^+, τ^-) and showing that the hyperkähler quotient is a single point. This can be done by showing that the action of the complexified group $GL_k(\mathbb{C})$ on the zero set of the complex moment map has closed orbits of the form $GL_k(\mathbb{C})/T^{\mathbb{C}}$ for some subtorus T in $U(k)$ and then showing that the hyperkähler quotient and the complex symplectic quotient coincide.

Writing $\mathbf{x}_i^{\pm}$ for the i-th diagonal entry of $(T_1(\pm\infty), T_2(\pm\infty), T_3(\pm\infty))$ and $\mathbf{x}_\nu \in \mathbb{R}^{2k}$, $\nu = 1, 2, 3$ for the vector of ν-coordinates of the $(\mathbf{x}_i^+, \mathbf{x}_i^-)$, we have the hyperkähler moment map of the $T^k \times T^k$ action given by $\mathbf{x}_i^-, \mathbf{x}_i^+$ and we can make a Gibbons-Hawking ansatz for the metric and write

$$g = \sum_{\nu=1}^{3} \mathrm{d}\mathbf{x}_\nu^T \Phi \mathrm{d}\mathbf{x}_\nu + (\mathrm{d}t + A)^T \Phi^{-1} (\mathrm{d}t + A), \tag{5.21}$$

where $\mathrm{d}t$ is the diagonal matrix of 1-forms dual to Killing fields and A is a connection 1-form. Hence, the only thing left to do is to calculate the matrix Φ. This is done by comparison with the Gibbons-Manton metric, which is also of the above form and has some specific matrix Φ.

To achieve this the last ingredient is the complex symplectic form on the twistor space. It is given by [Bie98]:

$$\omega = \sum_{i=1}^{k} \frac{\mathrm{d}u_i^+}{u_i^+} \wedge \mathrm{d}\beta_i^+ - \sum_{i<j} \frac{\mathrm{d}\beta_i^+ \wedge \mathrm{d}\beta_j^+}{\beta_i^+ - \beta_i^+} + \sum_{i=1}^{k} \frac{\mathrm{d}u_i^-}{u_i^-} \wedge \mathrm{d}\beta_i^- - \sum_{i<j} \frac{\mathrm{d}\beta_i^- \wedge \mathrm{d}\beta_j^-}{\beta_i^- - \beta_i^-}.$$

Because of the product rule, we conclude that every factor occurring in the $u_i^{\pm}$ contributes separately to Φ.

Now we can compare the real section computed here with the one of the Gibbons-Manton metric, which are computed in [Bie99]. The factors $(\zeta - a_{ji}^{\pm}) e^{c^{\pm}(x_i - \bar{z}_i \zeta)}$ in the $u_i^{\pm}$ are exactly the ones of the Gibbons-Manton metric (5.11), hence their contribution to Φ

is the following

$$
\begin{aligned}
&c^+ - \sum_{\substack{j\le k\\ j\ne i}} \frac{1}{r_{ij}^{++}} && \text{for } \Phi_{ii}, i \le k\\
&\frac{1}{r_{ij}^{++}} && \text{for } \Phi_{ij}, i \ne j; i,j \le k\\
&c^- - \sum_{\substack{j\le k\\ j\ne i}} \frac{1}{r_{ij}^{--}} && \text{for } \Phi_{ii}, i > k\\
&\frac{1}{r_{ij}^{--}} && \text{for } \Phi_{ij}, i \ne j; i,j > k
\end{aligned}
$$

The remaining terms also contribute terms of the above type but with a sign ambiguity [Bie98]. Thus, we obtain a $2k \times 2k$ matrix Φ for $\tilde{F}_{k,k}(c^-, c^+)$ of the form

$$
\Phi_{ij} = \begin{cases} c_i + \sum_{n\ne i} \frac{s_{in}}{r_{in}} & \text{if } i = j\\ -\frac{s_{ij}}{r_{ij}} & \text{if } i \ne j \end{cases} \tag{5.22}
$$

with

$$
c_i = \begin{cases} c^+ & \text{for } i \le k\\ c^- & \text{for } i > k \end{cases}
$$

and

$$
s_{ij} = \begin{cases} -1 & \text{for } i,j \le k \text{ or } i,j > k\\ (-1)^{\epsilon_{ij}} & \text{for } i \le k, j > k \text{ or } i > k, j \le k \end{cases}
$$

where $\epsilon_{ij} = 0$ if $c_{ij}^{\pm} = a_{ij}^{\pm\mp}$ and $\epsilon_{ij} = 1$ if $c_{ij}^{\pm} = a_{ji}^{\pm\mp}$. The r_{ij} are the $r_{ij}^{\pm\pm}$ depending on whether the index i (resp. j) is $\le k$ or $> k$ ($+$ is for $\le k$).

There is one more building block corresponding to the matrices A and B in the description of the bow variety that represent the arrows. In the asymptotic region, where the T_i approach a diagonal limit exponentially fast, these matrices have to be also almost diagonal. Therefore, this building block will be $\mathbb{H}^k$ with its standard flat metric.

5.2.5 Asymptotic metric

Now, that we have a good description of all the building blocks, we are in the position to write down an asymptotic metric for the bow variety by gluing together the metrics of the building blocks, which means by performing the hyperkähler quotient of

$$
\prod_{i=1}^{r} \tilde{F}_{k,k}(c_i^-, c_i^+) \times \mathbb{H}^k
$$

by the torus. This product has a tri-hamiltonian action $T^k \times T^k$ on the first r factors and a T^k action on the last factor. Let us denote the action of $T^k \times T^k$ on the i-th factor $\tilde{F}_{k,k}(c_i^-, c_i^+)$ by $T_i^+ \times T_i^-$ given by gauge transformations asymptotic to $\exp(\lambda h - th)$ as $t \to \pm\infty$ with $\lambda \in \mathbb{R}$ and $h \in \mathfrak{u}(k)$ being diagonal. Further let T_0^+ denote the standard action of T^k on $\mathbb{H}^k$ $(t, q) \mapsto \phi(t, q)$ and T_{r+1}^- the T^k action $(t, q) \mapsto \phi(t^{-1}, q)$ on $\mathbb{H}^k$. Now we want to take the hyperkähler quotient with respect to the torus action. Gluing the building blocks together means we need appropriate matching conditions. We want to glue together the space $\tilde{F}_{k,k}(c_i^-, c_i^+)$ at $+\infty$ with $\tilde{F}_{k,k}(c_{i+1}^-, c_{i+1}^+)$ at $-\infty$ for each $i = 1, ..., r-1$. This means the i-th factor in $(T^k)^{r+1}$ is embedded diagonally into $T_i^+ \times T_{i+1}^-$, such that the level set of the hyperkähler moment map is equal to zero for all copies of T^k except from the first and the last ($r+1$-th copy). There we have level sets equal to $(c_L, ..., c_L)$ for the first copy and $(c_R, ..., c_R)$ for the last copy, where c_L and c_R are points in $\mathbb{R}^3$ determined by the quadratic polynomials $c_L(\zeta)$, $c_R(\zeta)$ defining the bow variety (5.8).

As before, writing $\mathbf{x}_i^+$ for the i-th entry of $(T_1^i(+\infty), T_2^i(+\infty), T_3^i(+\infty))$ and $\mathbf{x}_i^-$ for the corresponding value at $-\infty$ on $\tilde{F}_{k,k}(c_i^-, c_i^+)$ we obtain now k points for every of the $r-1$ middle intervals (instead of $2k$ points as calculated for the building blocks) because of the matching condition. If we write $\mathbf{x}_{ij}$ with $j = 1, ..., k$ for the points on the i-th interval $[\lambda_i, \lambda_{i+1}]$, then each $\mathbf{x}_{ij}$ has to be equal to $\mathbf{x}_j^+$ for $\tilde{F}_{k,k}(c_i^-, c_i^+)$ but also equal to $\mathbf{x}_j^-$ for $\tilde{F}_{k,k}(c_{i+1}^-, c_{i+1}^+)$. This gives us $k(r-1)$ points in $\mathbb{R}^3$. We get another k points from the factor $\mathbb{H}^k$. If we write $\mathbb{H} = S^1 \times \mathbb{R}^3$ we get k points $\mathbf{y}_1, ..., \mathbf{y}_k \in \mathbb{R}^3$ given by the hyperkähler moment map. The matching conditions on the first and the last intervals are then $\mathbf{x}_j^- = \mathbf{y}_j + c_L$ for the $\mathbf{x}_j^-$ for $\tilde{F}_{k,k}(c_1^-, c_1^+)$ and $\mathbf{x}_j^+ = \mathbf{y}_j + c_R$ for the $\mathbf{x}_j^+$ for $\tilde{F}_{k,k}(c_r^-, c_r^+)$ and $j = 1, ..., k$. Therefore, in the end we obtain kr points in $\mathbb{R}^3$.

On this glued spaces we want to introduce the notation of intervals which will be needed later. Consider the gluing of $\tilde{F}_{k,k}(c_i^-, c_i^+)$ with $\tilde{F}_{k,k}(c_{i+1}^-, c_{i+1}^+)$ and denote the gluing point $+\infty$ in $\tilde{F}_{k,k}(c_i^-, c_i^+)$ respectively $-\infty$ in $\tilde{F}_{k,k}(c_{i+1}^-, c_{i+1}^+)$ with ∞_i. We find that $\infty_i \in [\lambda_i, \lambda_{i+1}]$ in this glued space, so we need a new definition and a new notation for this to make clear that this is not an interval in $\mathbb{R}$. We set

$$[[\lambda_i, \lambda_{i+1}]] = [\lambda_i, \infty_i) \cup (\infty_i, \lambda_{i+1}] \tag{5.23}$$

with the matching conditions described above (meaning that solutions to Nahm equations are continuous at ∞_i and gauge transformations are exponentially close to $\exp(a_i h_i - t h_i)$ with $a_i \in \mathbb{R}$ and $h_i \in \mathfrak{u}(k)$). We use the same notation for each connected subset of $[[\lambda_i, \lambda_{i+1}]]$, whenever ∞_i is included in that subset.

With the discussion in the previous sections we can write down the metric for every block. The metric on $\mathbb{H}$ is given by the Gibbons Hawking ansatz (5.21) with $\Phi = ||\mathbf{y}||^{-1}$, while the formula for each of the $\tilde{F}_{k,k}(c_i^-, c_i^+)$ can be described by the $2k \times 2k$ matrix Φ as in (5.22) with all $\epsilon_{ij} = 0$. The latter statement will be shown later.

The asymptotic metric on the hyperkähler quotient is then given by the $kr \times kr$ matrix Φ where all the $2k \times 2k$ matrices for the building blocks are added together viewed as submatrices of bigger $kr \times kr$ matrices. Taking into account the matching conditions this is done the following way: For each $i = 1, ..., r$ we label the 4 $k \times k$ blocks of the matrix Φ as

$$\Phi = \left(\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{21} & \Phi_{22} \end{array}\right)$$

and define the $kr \times kr$ matrix Ψ^i as a block matrix consisting of r^2 $k \times k$ blocks

$$\Psi^i_{(a,b)} = \left(\begin{array}{c|c|c} \Psi^i_{11} & \cdots & \Psi^i_{1r} \\ \hline \vdots & & \vdots \\ \hline \Psi_{r1} & \cdots & \Psi_{rr} \end{array}\right)$$

by

$$\Psi^i_{(a,b)} = \begin{cases} \Phi_{st} & \text{if } a = i + s - 1 \text{ and } b = i + t - 1 \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, ..., r-1$. For $i = r$ we define Ψ^r as $\Psi^r_{rr} = \Phi_{11}$, $\Psi^r_{r1} = \Phi_{12}$, $\Psi^r_{1r} = \Phi_{21}$ and $\Psi^r_{11} = \Phi_{11}$ and all other Ψ^r_{ab} equal to zero. Finally, for Ψ^0 we define the (1,1)-block as $\mathrm{diag}(||\mathbf{y}_1||^{-1}, ..., ||\mathbf{y}_k||^{-1})$ and all other blocks equal to zero. The asymptotic metric is then defined by the new matrix $\Phi = \sum_{i=0}^{r} \Psi^i$.

There are two steps left. First, we need to determine signs ϵ_{ij} in the matrices Φ of (5.22). Secondly, we have to show that the model metric is asymptotically close to the metric on the bow variety. For both of these steps we need some estimates on the solutions and on the tangent vectors. To obtain them we need a mapping between both spaces.

Theorem 5.10. *Let M be the model space described above with metric $\tilde{g}$ and $\mathcal{M}$ the bow variety with metric g. There is a complex-symplectic isomorphism*

$$\phi : M \to \mathcal{M}.$$

Proof. First, observe that we have shown in chapter 5.1.3 that the bow variety $\mathcal{M}$ can be split off into a part that contains the information about the Nahm data and the fundamental data and a part that contain the information about the bifundamental data given by $T^*\mathrm{Mat}_{k,k}(\mathbb{C})$ which we could identify as a complex symplectic quotient with an instanton moduli space, because the stability conditions are the same. Therefore, we can restrict ourselves to $\tilde{M}$ here and still know that we can solve the real Nahm equation with the boundary condition of $\mathcal{M}$, i.e. with the boundary condition induced by the bifundamental data being satisfied.

We need to introduce an intermediate space M_I as the space of solutions (α, β) to the complex Nahm equation (3.21), which are constant and diagonal on each

$$[[\lambda_i + c, \lambda_{i+1} - c]],$$

for some $c < \min\{\frac{\lambda_{i+1}-\lambda_i}{2}, i = 1,...,r-1\}$ (cf. (5.23)) and satisfy the boundary condition (5.12), modulo gauge transformations satisfying the matching conditions $g(t) = \exp(h_i t - p_i)$ in a neighbourhood of ∞_i for some diagonal matrices h_i and p_i. Notice, that the real Nahm equation (3.20) does not need to be satisfied on M_I.

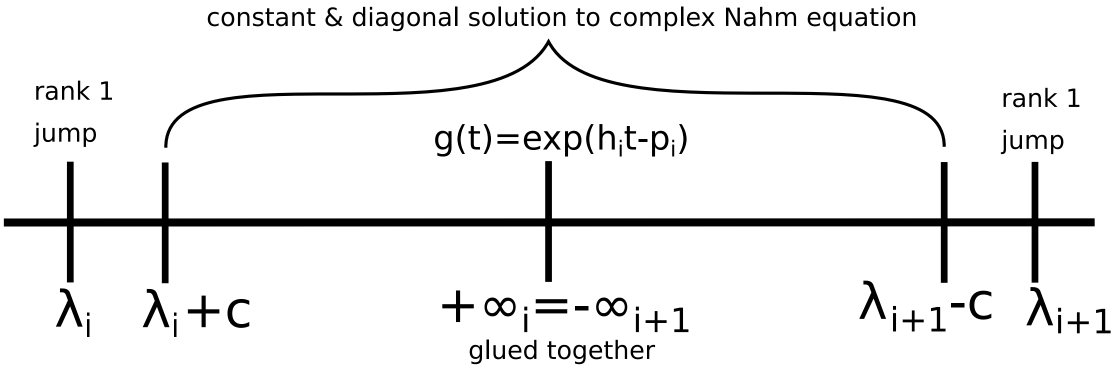

Figure 5.2: The intermediate space M_I

The map ϕ will be the composition $\phi = \phi_2\phi_1$ where $\phi_1 : M \to M_I$ and $\phi_2 : M_I \to \mathcal{M}$. We want to define these two maps and give their inverse mapping. The map ϕ_1 is given by a complex gauge transformation as in Prop. 5.7 sending a solution (α, β) of Nahm equations to a constant and diagonal solution of the complex equation on $[[\lambda_i + c, \lambda_{i+1} - c]]$ with $g(\infty_i) = 1$ and $g(t) = 1$ on each $[\lambda_i, \lambda_i + \frac{c}{2}] \cup [\lambda_{i+1} - \frac{c}{2}, \lambda_{i+1}]$. The inverse mapping is given by solving the real Nahm equation via a complex gauge transformation that is exponentially close to $\exp(h_i t - p_i)$ near ∞_i for some diagonal matrices h_i and p_i. For this we first choose a bounded gauge g_i on each $(-\infty_{i-1}, \lambda_i) \cup (\lambda_i, +\infty_i)$ satisfying the matching condition at λ_i to solve the real equation there. The T^k action extends to a global $(\mathbb{C}^*)^k$ action (using corollary 8.2 of [Bie99]), which allows us to replace the g_i on the subinterval with a global gauge transformation $g(s)$, which solves the real equation and satisfies the condition near the ∞_i.

The map $\phi_2 : M_I \to \mathcal{M}$ is given by restricting the solutions to the complex equation to the union of $[[\lambda_i, \frac{\lambda_i+\lambda_{i+1}}{2}]] \cup [[\frac{\lambda_i+\lambda_{i+1}}{2}, \lambda_{i+1}]]$ viewing them as solutions to the complex equations on $[\lambda_1, \lambda_{k-1}]$ and then solving the real equation as in Hurtubise [Hur89] and knowing that the boundary conditions belonging to the ends of the intervals are satisfied. The inverse mapping is given as follows: We take an element (α, β) in $\mathcal{M}$ and use a complex gauge to make it constant and diagonal on each $[\lambda_i + c, \lambda_{i+1} - c]$. We cut this off at the center of each interval and extend the solution to the complex equation trivially to $[[\lambda_i, \lambda_{i+1}]]$.

The map ϕ respects complex symplectic forms as it is a composition of complex gauge transformations with restrictions and extensions of constant solutions which both respect complex symplectic forms. Hence, ϕ is a complex symplectic isomorphism. □

We can now obtain estimates on the solutions to Nahm equations. As in both the bow variety $\mathcal{M}$ and the model space M the bifundamental data are represented by a quaternionic space that carries the standard flat hyperkähler metric, we only need estimates for the other part, which carries the information about the Nahm data. We start with the model space and consider a regular solution (α, β) to Nahm equations on the half line $[x, +\infty)$. Regular means that that the eigenvalues of β at $+\infty$ are all distinct, i.e. $\min\{|\beta_{ii}(+\infty) - \beta_{jj}(+\infty)|; i \neq j\} \geq \delta > 0$. For every $\epsilon > 0$ we can conjugate α and β into lower triangular matrices on $[x+\epsilon, +\infty)$ and obtain a priori estimates

$$|\alpha_{ij}(t)| + |\beta_{ij}(t)| \leq Me^{-\lambda R_{ij} t} \tag{5.24}$$

for $i > j$, $t \geq x + \epsilon$, $R_{ij} = |\mathrm{Re}\alpha_{ii}(+\infty) - \mathrm{Re}\alpha_{jj}(+\infty)| + |\beta_{ii}(+\infty) - \beta_{jj}(+\infty)|$ and $M, \lambda > 0$ being constants depending only on δ and ϵ and

$$|\mathrm{Re}\alpha_{ii}(t) - \mathrm{Re}\alpha_{ii}(+\infty)| \leq K \tag{5.25}$$

for all i, $t \geq x + \epsilon$ and $K > 0$ being a constant depending only on δ and ϵ. This has been shown in [Bie96]. The real Nahm equation then yields

$$\begin{aligned}
2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\alpha_{ii} &= [\alpha^\dagger, \alpha]_{ii} + [\beta^\dagger, \beta]_{ii} = \sum_{j=1}^{n}(\alpha_{ji}\overline{\alpha}_{ji} - \alpha_{ij}\overline{\alpha}_{ij}) + \sum_{j=1}^{n}(\beta_{ji}\overline{\beta}_{ji} - \beta_{ij}\overline{\beta}_{ij}) \\
&= \sum_{j=1}^{n}\left(\underbrace{\left(|\alpha_{ji}|^2 + |\beta_{ji}|^2\right) - \left(|\alpha_{ij}|^2 + |\beta_{ij}|^2\right)}_{\alpha,\beta \text{ lower triangular} \Rightarrow \text{ one term } = 0\ \forall j}\right) \\
&= \sum_{j=1}^{i-1} - \left(|\alpha_{ij}|^2 + |\beta_{ij}|^2\right) + \sum_{j=i+1}^{n}\left(|\alpha_{ji}|^2 + |\beta_{ji}|^2\right) \\
&\leq \sum_{j=i+1}^{n}\left(|\alpha_{ji}|^2 + |\beta_{ji}|^2\right) \\
&\leq \sum_{j=i+1}^{n}\left(|\alpha_{ji}| + |\beta_{ji}|\right)^2 \\
&\leq \sum_{j=i+1}^{n}\left(Me^{-\lambda R_{ij} t}\right)^2
\end{aligned}$$

Integrating this expressing from t to $+\infty$ and redefining the constant M yields

$$|\mathrm{Re}\alpha_{ii}(t) - \mathrm{Re}\alpha_{ii}(+\infty)| \leq \frac{M}{R}e^{-2\lambda R t}, \tag{5.26}$$

where $R = \min R_{ij}$. We have the gauge freedom to make $\mathrm{Im}\alpha_{ii}$ constant on $[x+\epsilon, +\infty)$. Further, we can use ϕ_1 to obtain estimates on the intermediate space. Let $(\tilde{\alpha}, \tilde{\beta}) =$

$\phi_1(\alpha,\beta)$, then $\tilde{\alpha}$ is constant and diagonal on $[[\lambda_i+c,\lambda_{i+1}-c]]$ and equal to $\alpha(\infty_i)$ because of the condition $g(\infty_i)=1$. Hence we obtain the following estimate:

$$|\tilde{\alpha}(t)-\alpha(t)| = \begin{cases} 0 & \text{if } t\in[\lambda_i,\lambda_i+\frac{c}{2}]\cup[\lambda_{i+1}-\frac{c}{2},\lambda_{i+1}], \\ \mathcal{O}(\frac{1}{R_i}\exp(-\lambda R_i t)) & \text{if } t\in[[\lambda_i+c,\lambda_{i+1}-c]]. \end{cases} \tag{5.27}$$

As the commutator of two lower triangular matrices is strictly lower triangular, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\beta(t)_{ii}=0$ so even stronger estimates for β. Hence, using ϕ_2 we obtain (after cutting off solutions) a solution $(\hat{\alpha},\hat{\beta})$ to the complex equation satisfying

$$F(\hat{\alpha},\hat{\beta}) := \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\alpha}+\hat{\alpha}^\dagger)+[\hat{\alpha},\hat{\alpha}^\dagger]+[\hat{\beta},\hat{\beta}^\dagger] = \mathcal{O}(\exp(-\lambda R)) \tag{5.28}$$

for the expression in the real equation. The construction in [Don84] and chapter 2 of [Hur89] shows that there is a unique element of the quotient $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ such that every element $g(t)$ in the orbit sends $(\hat{\alpha},\hat{\beta})$ to a solution of the real equation satisfying the boundary condition (i.e. the rank one jump) (5.12) and hence to an element of $\mathcal{M}$. The above estimates yields the following estimates on this gauge transformation:

$$|g^\dagger g-1| = \mathcal{O}(e^{-\lambda R}) \tag{5.29}$$

uniformly on $[\lambda_1,\lambda_k]$. To show this we start with a convexity theorem (Lemma 2.10 in [Don84]), where we have estimates (5.28). Following the construction we obtain a solution to the real Nahm equation on the interval $[\lambda_i,\lambda_{i+1}]$ via a complex gauge transformation $g_i(t)$ with $g_i(\lambda_i)=g_i(\lambda_{i+1})=1$, $g_i^\dagger g_i$ uniformly bounded by $\mathcal{O}(e^{-\lambda R})$ and $|g_i^\dagger(t)g_i(t)-1|\leq(t-\mu_i)ce^{-\lambda R}$ near λ_i and near λ_{i+1} analogously. This solution will not satisfy the boundary conditions, but the last inequality shows that the jump at the boundary points is of order$(e^{-\lambda R})$. Hurtubise shows in Lemma 2.19 - Prop 2.21 of [Hur89] that there is a unique complex gauge g' such that the boundary conditions are satisfied. In this construction the transformation g' is measured by the logarithm of the biggest and the smallest eigenvalue ϕ and θ of $g'^\dagger g'$ as in Lemma 2.10 of [Don84]. The construction yields estimates on the jumps of the derivatives of ϕ and θ (i.e. $\Delta\dot{\phi}$ and $\Delta\dot{\theta}$ are of order $e^{-\lambda R}$) at the boundary points and on ϕ and θ themselves too, which are given by $\phi(\mu_i)=c\Delta\dot{\phi}+(e^{-\lambda R})$ and similar for θ. Hence, ϕ and θ are both of order $e^{-\lambda R}$ and so is $|g^\dagger g-1|$.

These are the estimates we need on the solutions. We continue with estimates on the tangent vectors. Writing $a=t_0+it_1$ and $b=t_2+it_3$ for a tangent vector (t_0,t_1,t_2,t_3), then a and b have to satisfy

$$\begin{aligned} \dot{a} &= [\alpha^\dagger,a]+[\beta^\dagger,b] \\ \dot{b} &= [\beta,a]+[b\alpha] \end{aligned} \tag{5.30}$$

as well as the matching conditions at the boundary points. As above, taking $\mathbf{x}_j^\pm =$

$(x_j^\pm, z_j^\pm)$ on every factor $\tilde{F}_{k,k}(c_i^-, c_i^+)$ and $\mathbf{y}_j$ on $\mathbb{H}^k$, we define

$$\begin{aligned} R^\pm &= \min |x_i^\pm - x_j^\pm| + |z_i^\pm - z_j^\pm|;\ i \neq j, \\ Z^\pm &= \min |z_i^\pm - z_j^\pm|;\ i \neq j, \\ S &= \min |x_i^- - x_j^+| + |z_i^- - z_j^+| \end{aligned}$$

and analogously for the $\mathbb{H}^k$ factor. Then we have the following proposition from [Bie99]:

Proposition 5.11. *For any $\delta, \epsilon, \nu > 0$ there exist constants $M, C, \lambda > 0$ depending only on k, ϵ, δ, ν with the following properties:*

Let $((\alpha^-, \beta^-), (\alpha^+\beta^+))$ be a representative of $\tilde{F}_{k,k}(0,0)$ with $Z^\pm \geq \delta > 0$, $S \geq \nu > 0$ and $R^+, R^- \geq C$ as above and $((a^-, b^-), (a^+b^+))$ a tangent vector at $((\alpha^-, \beta^-), (\alpha^+\beta^+))$. Further, let

$$A^2 = |a^-(-\infty)|^2 + |b^-(-\infty)|^2 + |a^+(+\infty)|^2 + |b^+(+\infty)|^2.$$

Then for all $t \geq \epsilon$ the following estimates hold:

$$|a^-(t) - a^-(-\infty)| + |b^-(t) - b^-(-\infty)| \leq Me^{-\lambda R^- t} A, \tag{5.31}$$

$$|a^+(t) - a^+(+\infty)| + |b^+(t) - b^+(+\infty)| \leq Me^{-\lambda R^+ t} A. \tag{5.32}$$

Proof. The proof is as in [Bie99]. As the proofs are completely analogous, we only consider (α^+, β^+) and omit the superscript $+$. We only need to consider the normalized case $A = 1$ and the can again assume that α and β are lower triangular for $|t| \geq \epsilon$. Let us choose C such that the right hand side of (5.24) is small compared to R_{ij}^{-1} and the right hand side of (5.26) is small compared to R^{-1}. Then, writing y for the diagonal components of a and b and x for the off-diagonal components, the tangent equations (5.30) yield

$$\dot{y} = A(t)x, \qquad |A(t)| \leq Me^{-\lambda Rt}, \tag{5.33}$$

by using the estimates on α and β and the fact that the tangent equations are linear in a and b. Differentiating them again we have

$$\begin{aligned} \ddot{a} &= [\alpha^\dagger, [\alpha^\dagger, a] + [\beta^\dagger, b]] + [\beta^\dagger, [\beta, a] + [b, \alpha]], \\ \ddot{b} &= [\beta, [\alpha^\dagger, a] + [\beta^\dagger, b]] + [[\beta, a] + [b, \alpha], \alpha] \end{aligned}$$

This is again linear in a and b and we obtain by direct computation

$$\ddot{x} = D(t)x + B(t)y, \quad |B(t)| \leq Me^{-\lambda Rt}, \quad \exists s > 0 \forall z : \mathrm{Re}(D(t)z, z) \geq s^2 R^2 |z|^2. \tag{5.34}$$

This is true, because the diagonal entries of D are of the form

$$2\mathrm{Re}((\alpha_{ii} - \alpha_{jj})^2) + 2|\beta_{ii} - \beta_{jj}|^2 + \sum_{n=1}^{j-1} \beta_{nj}\overline{\beta}_{nj} + \sum_{n=j+1}^{k} \beta_{jn}\overline{\beta}_{jn} + \sum_{n=1}^{i-1} \beta_{in}\overline{\beta}_{in} + \sum_{n=i+1}^{k} \beta_{ni}\overline{\beta}_{ni},$$

where the last four terms are exponentially small (cf. (5.24)) and the off diagonal terms look more complicated but are also exponentially small. As R was defined as the minimum over terms of the form of the remaining first two ones here, the second inequality follows. The first inequality follows directly from the computation of the coefficients of the diagonal terms in a and b.

Let $t_0 \in [\epsilon, +\infty]$ be the first point for which $|x(t_0)|^2 + |y(t_0)|^2 \leq |a(+\infty)|^2 + |b(+\infty)|^2 \leq 1$. Then $X := \sup\{x(t)|t \in [\epsilon, +\infty]\}$ and $Y := \sup\{y(t)|t \in [\epsilon, +\infty]\}$ are finite and we can integrate (5.33) to find

$$|y(t) - y(t_0)| \leq \frac{MX}{R}. \tag{5.35}$$

Now, consider the solution to (5.34) with $D(t) = \text{const}$ such that we have

$$\ddot{\tilde{x}} = \lambda^2 R^2 \tilde{x} + B(t)y.$$

With the above discussion for y and $|B(t)|$ we can make the last term as small as we wish and obtain a stable solution $\tilde{x} \sim e^{-\lambda R t}$. With an appropriate choice of the multiplicative constant, we can use the comparison theorem (Lemma 1.8 of [Bie96]) to obtain

$$|x(t)| \leq |x(t_0)| MY e^{-\lambda R(t-t_0)}. \tag{5.36}$$

Therefore, the estimate (5.32) holds for $t \geq t_0$ (taking larger R if necessary). For the proposition to hold we need to show the estimates for $t \in [\epsilon, t_0]$. To see this we need to calculate the length of the vector $((a^-, b^-), (a^+, b^+))$ in the metric of $\tilde{F}_{k,k}(-\epsilon, \epsilon)$. Let $s_0 \in [-\infty, -\epsilon]$ be the number with the same properties for $\alpha^-, \beta^-, a^-, b^-$ as t_0 for $\alpha^+ \beta^+, a^+, b^+$. We have

$$\begin{aligned} L^2 \geq &\int_{-\infty}^{s_0} (|a^-(t)|^2 + |b^-(t)|^2 - |a^-(-\infty)|^2 - |b^-(-\infty)|^2) \mathrm{d}t \\ &+ \int_{s_0}^{-\epsilon} (|a^-(t)|^2 + |b^-(t)|^2 - |a^-(-\infty)|^2 - |b^-(-\infty)|^2) \mathrm{d}t \\ &+ \int_{-\epsilon}^{0} (|a^-(t)|^2 + |b^-(t)|^2) \mathrm{d}t + \int_{0}^{\epsilon} (|a^+(t)|^2 + |b^+(t)|^2) \mathrm{d}t \\ &+ \int_{\epsilon}^{t_0} (|a^+(t)|^2 + |b^+(t)|^2 - |a^+(+\infty)|^2 - |b^+(+\infty)|^2) \mathrm{d}t \\ &+ \int_{t_0}^{+\infty} (|a^+(t)|^2 + |b^+(t)|^2 - |a^+(+\infty)|^2 - |b^+(+\infty)|^2) \mathrm{d}t. \end{aligned} \tag{5.37}$$

The first and the last line are bounded, because we already showed that the estimates hold for $t \geq t_0$ (and analogously for $s \leq s_0$). This can be seen by replacing $|a^{\pm}(t)|^2$ with $|a^{\pm}(\pm\infty) + a^{\pm}(t) - a^{\pm}(\pm\infty)|^2$ and similar for $|b^{\pm}(t)|^2$. Hence we have (here the calculation is for the + case suppressing the superscript)

$$
\begin{aligned}
&\int_{t_0}^{\infty} (|a(t)|^2 + |b(t)|^2 - |a(\infty)|^2 - |b(\infty)|^2)\mathrm{d}t \\
&\quad = \int_{t_0}^{\infty} (|a(\infty) + a(t) - a(\infty)|^2 + |b(\infty) + b(t) - b(\infty)|^2 - |a(\infty)|^2 - |b(\infty)|^2)\mathrm{d}t \\
&\quad \leq \int_{t_0}^{\infty} ((|a(\infty)| + |a(t) - a(\infty)|)^2 + (|b(\infty)| + |b(t) - b(\infty)|)^2 - |a(\infty)|^2 - |b(\infty)|^2)\mathrm{d}t \\
&\quad = \int_{t_0}^{\infty} (2|a(\infty)||a(t) - a(\infty)| + |a(t) - a(\infty)|^2 + 2|b(\infty)||b(t) - b(\infty)| + |b(t) - b(\infty)|^2)\mathrm{d}t \\
&\quad \leq \int_{t_0}^{\infty} (2(|a(\infty)| + |b(\infty)|)(|a(t) - a(\infty)| + |b(t) - b(\infty)|) \\
&\quad + (|a(t) - a(\infty)| + |b(t) - |b(\infty)|)^2)\mathrm{d}t \\
&\quad \leq \int_{t_0}^{\infty} (2(|a(\infty)| + |b(\infty)|)Me^{-\lambda R t}A + (Me^{-\lambda R t}A)^2)\mathrm{d}t \\
&\quad = 2(|a(\infty)| + |b(\infty)|)\frac{MA}{\lambda R}e^{-\lambda R t_0} + \frac{M^2A^2}{2\lambda R}e^{-2\lambda R t_0},
\end{aligned}
$$

which is bounded. Hence, we can write

$$\int_{t_0}^{+\infty} (|a^+(t)|^2 + |b^+(t)|^2 - |a^+(+\infty)|^2 - |b^+(+\infty)|^2)\mathrm{d}t \leq \rho(k, \delta, \epsilon)$$

which can be made arbitrary small by changing C (i.e. taking larger R). The expressions in the three middle lines are non-negative because of the definition of the points t_0 and s_0. Writing T for the sum of the first and the last line we have $T \leq L^2 + 2\rho$. From the actual form (5.22) we find a bound $L^2 \leq P$ for some $P > 0$ depending only on k, ϵ, δ, ν. Hence, $T \leq P'$. If both t_0 and $-s_0$ are smaller than 2ϵ we are done (by replacing ϵ with 2ϵ), thus let us take $t_0 \geq 2\epsilon$. Since the integrands in the second and fourth line are non-negative there must be a point $t_1 \in [\epsilon, 2\epsilon]$ with $|a^+(t_1)|^2 + |b^+(t_1)|^2 \leq P'/\epsilon$. Repeating the argumentation in the beginning of this proof with t_1 replacing t_0 shows that the estimate (5.32) holds for $t \geq 2\epsilon$. A similar argument works for $s_0 \leq -2\epsilon$. □

We can strengthen the last results to the following [Bie99]:

Lemma 5.12. *We the same assumptions as above we obtain estimates (with $R_{ij}^{\pm} := |x_i^{\pm} - x_j^{\pm}| + |z_i^{\pm} - z_j^{\pm}|$)*

$$
\begin{aligned}
|a_{ij}^-(-t)| + |b_{ij}^-(-t)| &\leq Me^{-\lambda R_{ij}^- t}A, \\
|a_{ij}^+(t)| + |b_{ij}^+(t)| &\leq Me^{-\lambda R_{ij}^+ t}A
\end{aligned}
\tag{5.38}
$$

for all $i \neq j$ and $t \geq \epsilon$.

Now we have all estimates we need for determining the signs in (5.22) and showing that the model metric is asymptotically close to the metric on the bow variety.

Lemma 5.13. *All ϵ_{ij} in the formula (5.22) are equal to zero.*

Proof. Using the form (5.37) for the norm of a tangent vector to $\tilde{F}_{k,k}(c^-,c^+)$ with $\epsilon = t_0 = -s_0 = 1$ we find by the same calculation that for sufficiently large R we have $||v|| \geq \rho A^2$.

On the other hand, if we take (5.22) we can calculate the length of a tangent vector to a point $\mathbf{x} \in \tilde{F}_{k,k}(c^-,c^+)$ explicitly. We choose the point $\mathbf{x}$ such that $S(\mathbf{x}) \geq \nu > 0$ as before and R arbitrary large. Then all terms in the metric proportional to $1/R$ are arbitrary small, therefore only the terms proportional to $-\frac{(-1)^{\epsilon_{ij}}}{||x_i - x_j||}$ contribute where $i \leq k,\ j > k$ or $i > k,\ j \leq k$. Now, we find a tangent vector v at $\mathbf{x}$ such that if any $\epsilon_{ij} = 1$, the corresponding term above contributes to the length of the tangent vector as $\leq cA^2/\nu$ because $||x_i - x_j|| \geq S(\mathbf{x}) \geq \nu$ for such a pair x_i and x_j. Hence, we have $||v|| \leq cA^2/\nu$ and this is a contradiction. □

Let

$$R = \min\{||\mathbf{y}_a - \mathbf{y}_b||, ||\mathbf{x}_{ia} - \mathbf{x}_{ib}|| :\ i = 1, \ldots, r-1;\ a, b = 1, \ldots, k;\ a \neq b\}$$

and

$$Z = \min\{||w_a - w_b||, ||z_{ia} - z_{ib}|| :\ i = 1, \ldots, r-1;\ a, b = 1, \ldots, k;\ a \neq b\}$$

where $\mathbf{x}_{ia} = (x_{ia}, \mathrm{Re}\ z_{ia}, \mathrm{Im}\ z_{ia})$ and $\mathbf{y}_a = (y_a, \mathrm{Re}\ w_a, \mathrm{Im}\ w_a)$. Observe that for $R \to \infty$ the spectral curve degenerates to a union of lines, since we obtain diagonal limits for the Nahm data. We have the following theorem:

Theorem 5.14. *Let*

$$U(\gamma, \delta, C) = \{\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\} | Z(\mathbf{z}) \geq \delta > 0,\ R(\mathbf{z}) \geq C,\ \zeta^T \Phi \zeta \geq \gamma |\zeta|^2\ \forall \zeta \in \mathbb{R}^{kr}\},$$

where $C = C(\gamma, \delta)$ is given in a way such that all above estimates hold. Then,

$$\phi^* g \leq (1 + Me^{-\lambda R})\tilde{g} \tag{5.39}$$

on $U(\gamma, \delta, C)$ for some $\lambda, M > 0$ depending only on δ and γ. Here, the map ϕ is given as in 5.10.

Proof. First, observe that the spaces $T^*\mathrm{Mat}_{k,k}(\mathbb{C})$ and $\mathbb{H}^k$ representing the bifundamental data both carry the standard flat hyperkähler metric (which can be written in appropriate coordinates in the Gibbon-Manton form with $\Phi = ||\mathbf{y}||^{-1}$ as above). As we have seen, the map ϕ guarantees that the boundary conditions of the solutions of Nahm equations with the bifundamental data are satisfied. Therefore, we can again restrict ourselves to the Nahm data. The proof is the same as the one of theorem 9.2 in

[Bie99]. As $\gamma > 0$ the metric is positive definite and quasi isometric to the flat metric. Let (a, b) be a tangent vector to $U(\gamma, \delta, C)$ of norm equal to 1 in this metric. Then

$$\sum_i (|a(\infty_i)|^2 + |b(\infty_i)|^2) \leq B,$$

where B depends only on γ. By the same argument as above we obtain estimates of the form (5.31) and (5.32). For $t \in [[\lambda_i + \epsilon, \lambda_{i+1} - \epsilon]]$ we have

$$|a(t) - a(\infty_i)| + |b(t) - b(\infty_i)| \leq Me^{-\lambda R_i t} B. \tag{5.40}$$

We also have the stronger estimates from lemma 5.12. Writing the metric $\tilde{g}$ of the model space as in (5.37) with $\epsilon = t_0 = -s_0 = c$ we obtain by the same calculation as before

$$\sum_{i=1}^{r} \int_{\lambda_i - c}^{\lambda_i + c} (|a(t)|^2 + |b(t)|^2) \mathrm{d}t \leq MB. \tag{5.41}$$

We now use the map $\phi = \phi_1 \phi_2$ to compare with the metric of the bow variety. The map ϕ_1 was given by an infinitesimal gauge $p(t)$ which can be uniformly estimated by $\mathcal{O}(e^{-\lambda R})$. Conjugating with such a gauge transformation pap^{-1} and pbp^{-1} still satisfy the estimate (5.40) and further $\big| ||(pap^{-1}, pbp^{-1})|| - 1 \big| = \mathcal{O}(e^{-\lambda R})$. This is not yet the tangent vector $\mathrm{d}\phi_1(a, b)$ to the intermediate space, because this has to be constant and diagonal on each $[[\lambda_i + c, \lambda_{i+1} - c]]$. To obtain this tangent vector, we use an infinitesimal gauge transformation ρ_1 with $\rho_1(\infty_i) = 1$. This changes the norm of $p(a, b)p^{-1}$ by something of order $e^{-\lambda R}$ [Bie99] and the estimate (5.41) still holds.

As in the construction of ϕ we continue by restricting $\mathrm{d}\phi_1(a, b)$ to $[\lambda_1, ..., \lambda_r]$. Now, the norm of $\mathrm{d}\phi_1(a, b)$ in the metric $\tilde{g}$ of the model space is the same as the norm of the restriction $(\hat{a}, \hat{b})$ of the tangent vector to $\mathcal{M}$ in the metric g of the bow variety, because $\mathrm{d}\phi_1(a, b)$ is constant and diagonal on the union of $[[\lambda_i + c, \lambda_{i+1} - c]]$. To obtain a tangent vector to the bow variety, we need conjugate by the complex gauge transformation $g(t)$ as shown above such that the real Nahm equation is satisfied. Using (5.41) and the estimate on the gauge transformation (5.29) we find

$$\big| ||g\hat{a}g^{-1}, g\hat{b}g^{-1}|| - 1 \big| \leq Me^{-\lambda R} \tag{5.42}$$

for some $M, \lambda > 0$ depending only on γ and δ. The vector $g(\hat{a}, \hat{b})g^{-1}$ needs to satisfy the conditions (5.30) to be the tangent vector $\mathrm{d}\phi(a, b)$ to the bow variety. This can be achieved by a complex infinitesimal orthogonal gauge transformation [Bie99] and therefore, the norm does not increase. □

5.2.6 Example

To finish this chapter, we want to consider an explicit example of an asymptotic metric for this type of bow variety. We choose a bow diagram as above and consider a

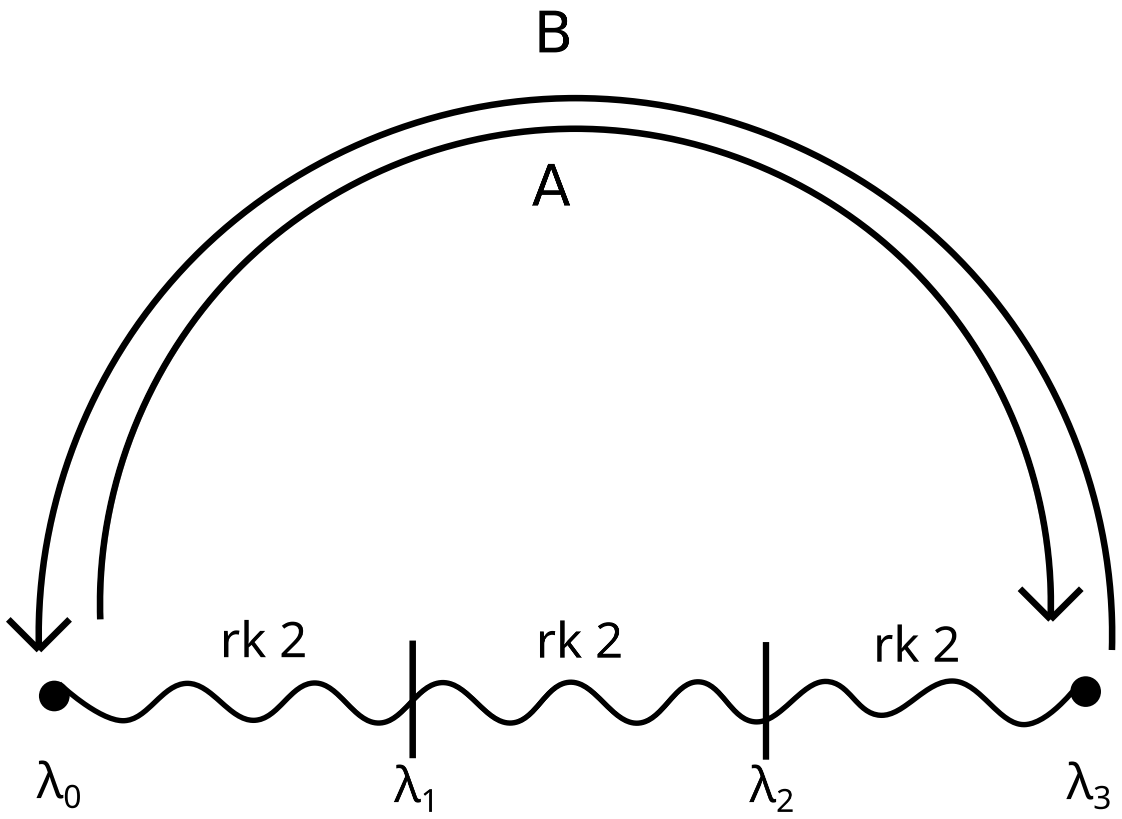

Figure 5.3: Bow diagram for a single interval with two λ-points

representation consisting of two λ-points and all hermitian vector bundles are of rank 2. Hence, we have the following bow diagram:

This means the model space is the hyperkähler quotient of

$$\tilde{F}_{2,2}(c_1^-, c_1^+) \times \tilde{F}_{2,2}(c_2^-, c_2^+) \times \mathbb{H}^2$$

by the torus action. On each of the $\tilde{F}_{2,2}(c_i^- c_i^+)$ we have coordinates $({}^i\mathbf{x}_1^-, {}^i\mathbf{x}_2^-, {}^i\mathbf{x}_1^+, {}^i\mathbf{x}_2^+)$ with $i = 1, 2$. The moment map condition for the level set of zero at $+\infty$ for $\tilde{F}_{2,2}(c_1^-, c_1^+)$ and $-\infty$ for $\tilde{F}_{2,2}(c_2^-, c_2^+)$ which we glue together yields

$$({}^1\mathbf{x}_1^+, {}^1\mathbf{x}_2^+) = ({}^2\mathbf{x}_1^-, {}^2\mathbf{x}_1^+) =: (\mathbf{x}_1, \mathbf{x}_2).$$

These coordinates $\mathbf{x}_j$ are the same as the $\mathbf{x}_{1,j}$ for the middle interval $[\lambda_1, \lambda_2]$ as above. Since we have only one middle interval, we can drop the first index in the subscript. We obtain another two points $\mathbf{y}_1, \mathbf{y}_2$ for the factor $\mathbb{H}^2$. The level sets of the moment map at the boundary points $-\infty$ for $\tilde{F}_{2,2}(c_1^-, c_1^+)$ is c_L and the one at $+\infty$ for $\tilde{F}_{2,2}(c_2^- c_2^+)$ is c_R as in (5.8). The moment map conditions are then

$$({}^1\mathbf{x}_1^-, {}^1\mathbf{x}_2^-) = (\mathbf{y}_1 + c_L, \mathbf{y}_2 + c_L)$$

and
$$({}^2\mathbf{x}_1^+, {}^2\mathbf{x}_2^+) = (\mathbf{y}_1 + c_R, \mathbf{y}_2 + c_R).$$
Hence, the $rk = 2 \cdot 2$ points we obtain as coordinates are $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$. We have to calculate the matrix Φ in the Gibbons-Manton metric to obtain the asymptotic metric for this bow variety. For the two building blocks $\tilde{F}_{2,2}(c_i^-, c_i^+)$ we obtain matrices
$$^i\Phi = \left(\begin{array}{c|c} ^i\Phi_{11} & ^i\Phi_{12} \\ \hline ^i\Phi_{21} & ^i\Phi_{22} \end{array}\right)$$
which are assembled together with a contribution $\mathrm{diag}(||\mathbf{y}_1||^{-1}, ||\mathbf{y}_2||^{-1})$ from the $\mathbb{H}^2$ part to a matrix
$$\Phi = \left(\begin{array}{c|c} ^1\Phi_{11} + {}^2\Phi_{22} + \mathrm{diag}(||\mathbf{y}_1||^{-1}, ||\mathbf{y}_2||^{-1}) & ^1\Phi_{12} + {}^2\Phi_{21} \\ \hline ^1\Phi_{21} + {}^2\Phi_{12} & ^1\Phi_{22} + {}^2\Phi_{11} \end{array}\right).$$
The two matrices of the building blocks are the following. For $\tilde{F}_{2,2}(c_1^-, c_1^+)$ we have coordinates $({}^1\mathbf{x}_1^-, {}^1\mathbf{x}_2^-, {}^1\mathbf{x}_1^+, {}^1\mathbf{x}_2^+) = (\mathbf{y}_1 + c_L, \mathbf{y}_2 + c_L, \mathbf{x}_1, \mathbf{x}_2)$ and therefore, we obtain the contribution
$$^1\Phi = \left(\begin{array}{cc|cc} ^1a_{11} & \frac{1}{||\mathbf{y}_1-\mathbf{y}_2||} & -\frac{1}{||\mathbf{y}_1+c_L-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_1+c_L-\mathbf{x}_2||} \\ \frac{1}{||\mathbf{y}_1-\mathbf{y}_2||} & ^1a_{22} & -\frac{1}{||\mathbf{y}_2+c_L-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_2+c_L-\mathbf{x}_2||} \\ \hline -\frac{1}{||\mathbf{y}_1+c_L-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_2+c_L-\mathbf{x}_1||} & ^1a_{33} & \frac{1}{||\mathbf{x}_1-\mathbf{x}_2||} \\ -\frac{1}{||\mathbf{y}_1+c_L-\mathbf{x}_2||} & -\frac{1}{||\mathbf{y}_2+c_L-\mathbf{x}_2||} & \frac{1}{||\mathbf{x}_1-\mathbf{x}_2||} & ^1a_{44} \end{array}\right)$$
with diagonal entries
$$\begin{aligned} ^1a_{11} &= c_1^- - \frac{1}{||\mathbf{y}_1 - \mathbf{y}_2||} + \frac{1}{||\mathbf{y}_1 + c_L - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_1 + c_L - \mathbf{x}_2||}, \\ ^1a_{22} &= c_1^- - \frac{1}{||\mathbf{y}_1 - \mathbf{y}_2||} + \frac{1}{||\mathbf{y}_2 + c_L - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_2 + c_L - \mathbf{x}_2||}, \\ ^1a_{33} &= c_1^+ - \frac{1}{||\mathbf{x}_1 - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_1 + c_L - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_2 + c_L - \mathbf{x}_1||}, \\ ^1a_{44} &= c_1^+ - \frac{1}{||\mathbf{x}_1 - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_1 + c_L - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_2 + c_L - \mathbf{x}_2||}. \end{aligned}$$
For the second space $\tilde{F}_{2,2}(c_2^-, c_2^+)$ the coordinates are $({}^2\mathbf{x}_1^-, {}^2\mathbf{x}_2^-, {}^2\mathbf{x}_1^+, {}^2\mathbf{x}_2^+)$ with the identification $({}^2\mathbf{x}_1^-, {}^2\mathbf{x}_2^-, {}^2\mathbf{x}_1^+, {}^2\mathbf{x}_2^+) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1 + c_R, \mathbf{y}_2 + c_R)$. Hence, the contribution to Φ is the matrix
$$^2\Phi = \left(\begin{array}{cc|cc} ^2a_{11} & \frac{1}{||\mathbf{x}_1-\mathbf{x}_2||} & -\frac{1}{||\mathbf{y}_1+c_R-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_2+c_R-\mathbf{x}_1||} \\ \frac{1}{||\mathbf{x}_1-\mathbf{x}_2||} & ^2a_{22} & -\frac{1}{||\mathbf{y}_1+c_R-\mathbf{x}_2||} & -\frac{1}{||\mathbf{y}_2+c_R-\mathbf{x}_2||} \\ \hline -\frac{1}{||\mathbf{y}_1+c_R-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_1+c_R-\mathbf{x}_2||} & ^2a_{33} & \frac{1}{||\mathbf{y}_1-\mathbf{y}_2||} \\ -\frac{1}{||\mathbf{y}_2+c_R-\mathbf{x}_1||} & -\frac{1}{||\mathbf{y}_2+c_R-\mathbf{x}_2||} & \frac{1}{||\mathbf{y}_1-\mathbf{y}_2||} & ^2a_{44} \end{array}\right)$$

with diagonal entries

$$
\begin{aligned}
{}^2a_{11} &= c_2^- - \frac{1}{||\mathbf{x}_1 - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_1 + c_R - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_2 + c_R - \mathbf{x}_1||} \\
{}^2a_{22} &= c_2^- - \frac{1}{||\mathbf{x}_1 - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_1 + c_R - \mathbf{x}_2||} + \frac{1}{||\mathbf{y}_2 + c_R - \mathbf{x}_2||} \\
{}^2a_{33} &= c_2^+ - \frac{1}{||\mathbf{y}_1 - \mathbf{y}_2||} + \frac{1}{||\mathbf{y}_1 + c_R - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_1 + c_R - \mathbf{x}_2||} \\
{}^2a_{44} &= c_2^+ - \frac{1}{||\mathbf{y}_1 - \mathbf{y}_2||} + \frac{1}{||\mathbf{y}_2 + c_R - \mathbf{x}_1||} + \frac{1}{||\mathbf{y}_2 + c_R - \mathbf{x}_2||}.
\end{aligned}
$$

Here, c_L and c_R denote vectors having every component equal to c_L or c_R respectively.

6 Outlook

In this thesis, we described bow varieties as hyperkähler quotients of the solution space of Nahm equations with certain boundary conditions and we also gave an algebraic description in terms of flows on the Jacobian variety of some spectral curve defined by the Nahm equations using the Beauville correspondence.

This gave us enough information to explicitly construct the bow variety consisting of one interval with one arrow and a representation with r λ-points and Nahm data of rank k on each subinterval (i.e. we described Nahm data with r rank 1 jumps) in chapter 5. We also found an asymptotic metric for it, which gave us the interpretation of a deformed instanton moduli space over the Taub-NUT space.

There are several possible generalizations of this which are worth to consider. One would be taking the same bow, but choosing a representation with different ranks of the Nahm data on the different subintervals. The inner boundary conditions would not stay the simple rank 1 jump, but Hurtubise and Murray also discussed that case in [HM89] for the conditions on the spectral curve. Bielawskis construction of the asymptotic metric does have methods for different ranks, too, so it seems that our constructions should be adoptable to that case.

Another generalization would be to discuss deformed instanton moduli spaces over higher A_k-ALF spaces. Our construction should also work in that case and should produce similar results, because the boundary conditions stay more or less the same. On the inner of each interval, we still have rank 1 jumps and the boundary conditions corresponding to the arrows also remain the same, as we still get the same spectral curve on the subintervals the arrows connect. We simply have more of these kinds of conditions to consider.

Of course, we could always choose more complicated underlying quivers and try to adapt our construction, especially the ones Cheris identified with D_k-ALF spaces. But these more complicated bow varieties contain more difficult boundary conditions on the arrows and more work is needed to find a suitable expression for those cases.

Finally, one could expect that our bow variety $\mathcal{M}$ is isometric to the moduli space of instantons on a non-commutative Taub-NUT, because in a recent but not yet finished series of papers ([CLHS21b]) Cherkis shows that for $c_L = c_R$ the bow variety is isometric to an instanton moduli space on a (commutative) Taub-NUT space.

Bibliography

[AHH90] M. R. Adams, J. Harnad, and J. Hurtubise. Isospectral Hamiltonian flows in finite and infinite dimensions. II. Integration of flows. *Communications in Mathematical Physics*, 134(3):555 – 585, 1990.

[BBC22] R. Bielawski, Y. Borchard, and S. A. Cherkis. Deformations of instanton metrics [arXiv:2208.14936 [math.DG]]. 2022.

[Bea90] A. Beauville. Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables. *Acta Mathematica*, 164:211–235, 1990.

[Bie96] R. Bielawski. Monopoles, particles and rational functions. *Annals of Globals Analysis and Geometry*, 14(2):123–145, 1996.

[Bie97] R. Bielawski. Hyperkähler Structures and Group Actions. *Journal of the London Mathematical Society*, 55(2):400–414, 1997.

[Bie98] R. Bielawski. Monopoles and the Gibbons–Manton Metric. *Communications in Mathematical Physics*, 194(2):297–321, 1998.

[Bie99] R. Bielawski. Asymptotic metrics for SU(N)-monopoles. *Communications in Mathematical Physics*, 199(2):297–325, 1999.

[Bie07] R. Bielawski. Reducible spectral curves and the Hyperkähler geometry of adjoint orbits. *Journal of the London Mathematical Society*, 76(3):719–738, 2007.

[Bie15] R. Bielawski. Nahm's, Basu-Harvey-Terashima's equations and Lie superalgebras, [arXiv:1503.03779 [math-ph]], 2015.

[Biq96] O. Biquard. Sur les équations de nahm et la structure de poisson des algèbres de lie semi-simples complexes. *Mathematische Annalen*, 304:253–276, 1996.

[Bry97] J. Bryan. Symplectic geometry and the relative Donaldson invariants of $\overline{\mathbb{CP}}^2$. *Forum mathematicum*, 9(3):325–366, 1997.

[CH19] S. A. Cherkis and J. Hurtubise. Monads for Instantons and Bows. *Adv. Theor. Math. Phys.*, 23(1):167–251, 2019.

[CH21] Sergey A Cherkis and Jacques Hurtubise. Instantons and Bows for the Classical Groups. *The Quarterly Journal of Mathematics*, 72(1-2):339–386, 11 2021.

[Che09] S. A. Cherkis. Moduli Spaces of Instantons on the Taub-NUT Space. *Communications in Mathematical Physics*, 290(2):719–736, 2009.

[Che10] S. A. Cherkis. Instantons on the Taub-NUT Space. *Advances in Theoretical and Mathematical Physics*, 14(2):609–642, 2010.

[Che11] S. A. Cherkis. Instantons on gravitons. *Communications in Mathematical Physics*, 306(2):449–483, 2011.

[CLHS21a] S. A. Cherkis, A. Larraín-Hubach, and M. Stern. Instantons on multi-Taub-NUT spaces I: Asymptotic form and index theorem. *Journal of Differential Geometry*, 119(1):1 – 72, 2021.

[CLHS21b] S. A. Cherkis, A. Larraín-Hubach, and M. Stern. Instantons on multi-Taub-NUT-Spaces II: Bow construction, [arXiv:2103.12754 [math.DG]]. 2021.

[Dan93] A. S. Dancer. Nahm's equations and Hyperkähler geometry. *Communications in Mathematical Physics*, 158(3):545–568, 1993.

[Dan94] A. S. Dancer. A Family of Hyperkähler manifolds. *The Quarterly Journal of Mathematics*, 45(4):463–478, 1994.

[DK90] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford University Press, 1990.

[Don84] S. K. Donaldson. Nahm's equations and the classification of monopoles. *Communications in Mathematical Physics*, 96(3):387–407, 1984.

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley, 1978.

[GM95] G.W. Gibbons and N.S. Manton. The moduli space metric for well-separated BPS monopoles. *Physics Letters B*, 356(1):32–38, 1995.

[GPR94] H. Grauert, T. Peternell, and R. Remmert. *Several Complex Variables VII*. Springer Berlin, Heidelberg, 1994.

[GRG97] G.W. Gibbons, P. Rychenkova, and R. Goto. HyperKähler Quotient Construction of BPS Monopole Moduli Spaces. *Communications in Mathematical Physics*, 186(3):581–599, 1997.

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, 1977.

[Hit83] N. J. Hitchin. On the construction of monopoles. *Communications in Mathematical Physics*, 89(2):145–190, 1983.

[HJ12] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2012.

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyperkähler metrics and supersymmetry. *Communications in Mathematical Physics*, 108(4):535–589, 1987.

[HM89] J. Hurtubise and M. K. Murray. On the construction of monopoles for the classical groups. *Communications in Mathematical Physics*, 122(1):35–89, 1989.

[HSW99] N. J. Hitchin, G. B. Segal, and R. S. Ward. *Integrable systems: twistors, loop groups and Riemann surfaces*. Clarendon Press, 1999.

[Hur89] J. Hurtubise. The classification of monopoles for the classical groups. *Communications in Mathematical Physics*, 120(4):613–641, 1989.

[KN79] G. Kempf and L. Ness. The length of vectors in representation spaces. In Knud Lønsted, editor, *Algebraic Geometry*, pages 233–243. Springer Berlin Heidelberg, 1979.

[Kro04] P. B. Kronheimer. A hyperkahler structure on the cotangent bundle of a complex Lie group, [arXiv:0409253v1 [math.DG]], 2004.

[Mal82] A. N. Malyshev. Factorization of matrix polynomials. *Siberian Mathematical Journal*, 23(3):399–408, 1982.

[Nah81] W. Nahm. All self-dual multimonopoles for arbitrary gauge groups. *NATO ASI, B*, 82:301–310, 1981.

[Nak99] H. Nakajima. *Lectures on Hilbert Schemes of Points on Surfaces*. American Mathematical Society, 1999.

[Nek98] A. Nekrasov, N. Schwarz. Instantons on Noncommutative $\mathbb{R}^4$, and (2,0) Superconformal Six Dimensional Theory. *Communications in Mathematical Physics*, 198(3):689–703, 1998.

[NT17] H. Nakajima and Y. Takayama. Cherkis bow varieties and coulomb branches of quiver gauge theories of affine type a. *Selecta Mathematica*, 23(4):2553–2633, 2017.

[PP88] H. Pedersen and Y. S. Poon. Hyperkähler metrics and a generalization of the Bogomolny equations. *Communications in Mathematical Physics*, 117(4):569–580, 1988.

[RNB89] S. Ramanan, M. S. Narasimhan, and A. Beauville. Spectral curves and the generalised theta divisor. *Journal für die reine und angewandte Mathematik*, 398:169–179, 1989.

[Tak15] Y. Takayama. Bow varieties and ALF spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 158(1):37–82, 2015.

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