Differential Inclusion using Matrix Exponential

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Introduction

On $\mathbb{R}^n$, we consider the differential inclusion problem defined as:

$$\dot{x} = f(x, u) \quad \text{where } u \in [u] \tag{1}$$

where f is differentiable and u can take any value in a box $[u]$ at any time. From a set of initial states (at $t = 0$), our goal is to get an overapproximation of the possible states at time t.

Contribution

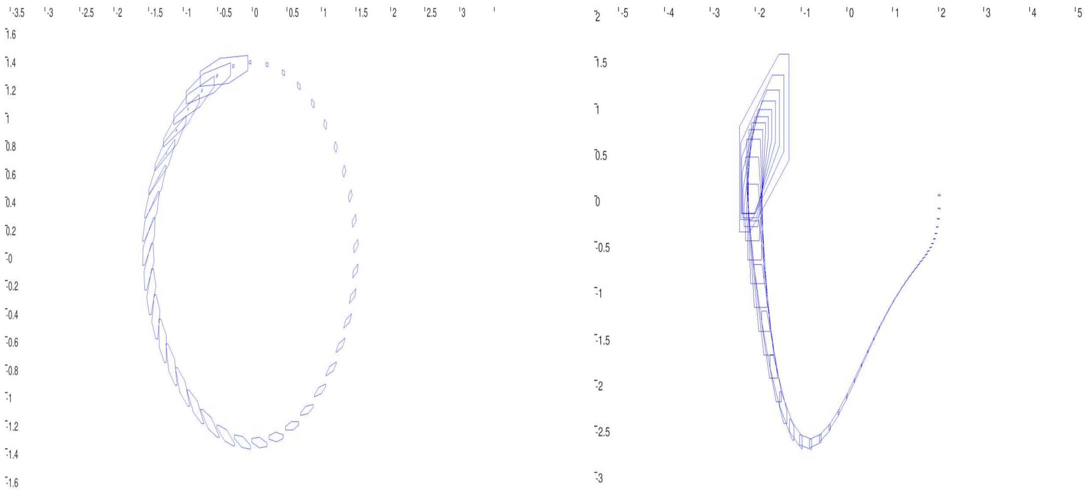
On $[0, t]$, we can express f as:

$$f(x, u) = C + A(x - x_m) + \phi(x, t) \tag{2}$$

where C is an vector of intervals, A is a matrix of intervals and $\phi(x, t) \in [\Phi]$ where Φ is a zero-centered box.

In this case, if $x(0) = x_0$, the solution of the differential equation (for a given ϕ) is:

$$x(t) = x_m + e^{tA}(x_0 - x_m) + \int_0^t e^{(t-\tau)A} d\tau C + \int_0^t e^{(t-\tau)A} \phi(x(\tau), \tau) d\tau \tag{3}$$



(A) Pendulum with perturbation

(B) Van der Pol oscillator

Figure 1: Solutions of two differential inclusion. (A) A pendulum with uncertainties. (B) A Van der Pol oscillator. We represent sets as intersections of parallepipeds.

Following previous works on exponentiation of interval matrices[1], we compute precise and safe overapproximations of e^{tA} and $\int_0^t e^{(t-\tau)A} d\tau$ using Taylor developments as well as scaling and squaring techniques.

We show that bounding $\int_0^t e^{(t-\tau)A} \phi(x(\tau), \tau) d\tau$ can be done by bounding $I(A, t) = \int_0^t |e^{\tau A}| d\tau$ (which $|V|$ being the component-wise absolute value of V). This is done by computing $[K]$ such that $e^{\tau A} \in \mathrm{Id} + \tau[K]$ and bounding $I(A, t)$ from the components of $[K]$.

Fig 1 graphically shows the evolution of the solutions for a few classical examples. We compared our approach with CAPD [2] on a Van der Pol oscillator with a small perturbation:

$$(\dot{x}; \dot{y}) = (y + [-10^{-4}, 10^{-4}]; (1 - x^2) * y - x + [-10^{-4}, 10^{-4}])$$

Fig 2 gives the enclosing boxes for $t = 1$, for CAPD and our approach. The precision depends heavily on the number of time steps, but these results indicate the interest of our approach.

Initial state	Our approach	CAPD (CW method)
(2;0)	$[1.507982, 1.508306]$ $\times[-0.780351, -0.780088]$	$[1.508005, 1.508283]$ $\times[-0.780311, -0.780126]$
(2;3)	$[2.300337, 2.300655]$ $\times[-0.479899, -0.479744]$	$[2.300371, 2.300625]$ $\times[-0.479863, -0.479778]$

Figure 2: Comparaison of our approach and CAPD on a simple example.

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References

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[2] T. Kapela, M. Mrozek, D. Wilczak, and P. Zgliczyński. CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, Volume 101, 2021, 105578, ISSN 1007-5704, https://doi.org/10.1016/j.cnsns.2020.105578.