Research article

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Variational formulations of steady rotational equatorial waves

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Abstract: When the vorticity is monotone with depth, we present a variational formulation for steady periodic water waves of the equatorial flow in the f-plane approximation, and show that the governing equations for this motion can be obtained by studying variations of a suitable energy functional $\mathcal{H}$ in terms of the stream function and the thermocline. We also compute the second variation of the constrained energy functional, which is related to the linear stability of steady water waves.

Keywords: Steady periodic water waves; Equatorial flows; Vorticity; Variational formulations

MSC: Primary 76B15, 35J60, 47J15, 76B03.

1 Introduction

The mathematical study of geophysical flows is currently of great interest since an in-depth understanding of the ongoing dynamics is essential in predicting features of these large-scale natural phenomena. Geophysical fluid dynamics is the study of fluid motion where the Earth's rotation plays a significant role, the Coriolis forces are incorporated into the governing Euler equations, and applies to a wide range of oceanic and atmospheric flows [10, 20, 29, 44]. Because geophysical fluid dynamics is highly complex, one usually uses the f-plane approximation of Euler equations. This approximation has been applied widely in the study of the equatorial flows [3, 4, 9, 34, 35, 41].

Because the Coriolis force vanishes along the equator, equatorial water waves exhibit particular dynamics. Besides, in this region the vertical stratification of the ocean is greater than anywhere else. Both factors facilitate the propagation of geophysical waves that either raise or lower the equatorial thermocline, which is the sharp boundary between warm and deeper cold waters. The rigorous mathematical study of equatorial water waves was initiated by [9], in which Constantin presented the model of wave-current interactions in the f-plane approximation for underlying currents of positive constant vorticity. Starting with this pioneering paper, recently some essential results on equatorial water waves have been proved in the literature. See [5, 6, 9, 10, 14, 19, 20, 32, 36, 45]. In the model constructed by Constantin in [9], the upper boundary of the centre layer is assumed to be flat, while the lower boundary is the thermocline near which the equatorial undercurrent resides. See [19, 40] for analytical results concerning the dynamics of the thermocline in the equatorial region. In recent work [4], the authors continued to study such a model by considering the general vorticity, and we proved the existence of steady two-dimensional periodic waves in the f-plane approximation by an application of the Crandall-Rabinowitz bifurcation theory. We also derived the dispersion relations for various choices of vorticity, including the negative constant vorticity and non-

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constant vorticity. For the classical gravity water waves, we refer the reader to [11–13, 25, 27] for the existence of steady periodic waves and the related properties. The study of steady periodic water waves with vorticity has received much attention since the work [25] by Constantin and Strauss. See [7] for more detailed discussions. Vorticity plays the key role in describing oceanic flows, and this aspect was very recently emphasized in thorough analytical studies [21–23].

In this paper, we obtain the variational formulation for steady equatorial waves with vorticity of the model in [4, 9]. It has a long history to study the variational formulations for steady water waves for the irrotational flows. We refer the reader to [33, 39] for a Lagrangian formulation and [33, 43, 46] for a Hamiltonian formulation. For the Hamiltonian formulation of the rotational flows, we refer the reader to [18] for the constant vorticity, [17] for the piecewise constant vorticity allowing for stratification, [15, 16, 37] for the extension of the Hamiltonian formulation to various scenarios pertaining to equatorial water flows. There are many results on variational formulations of the various classical small-amplitude long-wave approximations to the governing equations–the shallow water equations, the Boussinesq, and the Korteweg-de Vries equations all emerge from this process, see [28] and the references therein. For the steady water waves with vorticity, variational formulations have been given by Constantin, Sattinger and Strauss [24], in which they provide two variational formulations. When the vorticity varies monotonically with the depth, they provided a fundamental variational principle which can be expressed entirely in terms of the natural invariants (energy, mass, momentum and vorticity). One motivation of our research is to extend the results in [26] to the new setting presented here. Some new aspects originate from the dynamic boundary condition (2.5) below and the fact that the pressure on the thermocline is not a constant. Note that the latter is in contrast to the case of classical gravity water waves, where - in absence of surface tension effects - the pressure is given as the constant atmospheric pressure. Of course the Earth's rotation plays a significant role in our analysis.

By computing and analyzing the second variation of the constrained energy functional, we prove linear stability results of steady water waves. Remarkable progress on the linear stability and nonlinear stability properties of steady water waves with vorticity was given by Constantin and Strauss in [26]. In the literature, there are many works that deals with the stability of the full water wave equations (not their approximate models). Benjamin and Feir [1] presented a significant analysis for a small-amplitude approximation in the irrotational case, showing that there always is a sideband instability, which means that the perturbation has a different period from the steady wave. Bridges and Mielke [2] studied the existence and linear stability for the Stokes periodic wavetrain on fluids of finite depth, by the Hamiltonian structure of the water-wave problem. Zakharov [46] and Mackay and Saffman [42] discussed the linear stability for the Hamiltonian system that arises with the use of the velocity potential in the irrotational case.

2 Preliminaries

The vanishing of the Coriolis parameter along the Equator confers the flows in this part of the ocean a two-dimensional character. The vorticity equation plays an appreciable role in proving the two-dimensionality of gravity wave trains over flow with constant vorticity vector, and the boundary conditions are decisive in proving the two-dimensionality. See the rigorous analytical argument in [8]. In a rotating framework, let the x-axis be chosen horizontally due east, the y-axis horizontally due north and the z-axis vertically upward. $z = -d$ is the upper boundary of the centre layer and $z = -\eta(x,t)$ is the thermocline. In the region $-\eta(x,t) \le z \le -d$, the full governing equations in the f-plane approximation near the equator are the Euler equations

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, \\ w_t + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \end{cases} \tag{2.1}$$

together with the equation of mass conservation

$$u_x + w_z = 0, \tag{2.2}$$

where Ω is the rotation speed of the Earth, P is the pressure, g is the gravitational acceleration, ρ is the water's density. The kinematic boundary conditions are

$$w = -\eta_t - u\eta_x \quad \text{on} \quad z = -\eta(x,t), \tag{2.3}$$

and

$$w = 0 \quad \text{on } z = -d. \tag{2.4}$$

Beneath the thermocline, the motionless colder water has a slightly higher density $\rho + \Delta\rho$ (for example, for the equatorial Pacific the typical value of $\Delta\rho/\rho$ is 0.006, see the discussion in [30]). For this reason, the dynamic boundary condition

$$P = P_0 - g(\rho + \Delta\rho)z \quad \text{on} \quad z = -\eta(x,t). \tag{2.5}$$

See [9] and [4] for the details on the above equations (2.1)-(2.5).

Given $c > 0$, we are looking for the periodic waves traveling at speed c, that is, u, w, P, η have the form $(x - ct)$ and all of them are periodic with period L. In the new reference frame $(x - ct, z) \mapsto (x, z)$, we assume that there are no stagnation points of the flow, that is,

$$u < c \quad \text{for} \quad -\eta(x) \le z \le -d, \tag{2.6}$$

throughout the fluid. Due to (2.2), we can define the stream function $\psi(x, z)$ by

$$\psi_x = -w, \quad \psi_z = u, \quad \text{for} \quad -\eta(x) < z < -d.$$

Thus

$$-\Delta\psi = \omega = w_x - u_z,$$

where ω is the vorticity.

Throughout this paper, let $\mathcal{R} := \{0 < x < L, -\eta(x) < z < -d\}$ and $S := \{(x, -\eta(x)), 0 < x < L\}$ be the thermocline, $B := \{(x, -d), 0 < x < L\}$ be the upper boundary of the centre layer. Since on S the function $\psi - cz$ is constant, we can choose $\psi - cz = 0$ on S. Thus on B, $\psi - cz = m$, where[1]

$$m := \int_{-\eta(x)}^{-d} \Big(u(x,z) - c\Big)\, dz < 0$$

is the relative mass flux. It is not difficult to verify that the equations of motion (2.1)-(2.5) are expressed as

$$\begin{cases} (\psi_z - c)\psi_{xz} - \psi_x\psi_{zz} - 2\Omega\psi_x = -\frac{1}{\rho}P_x, & \text{for} \quad -\eta(x) < z < -d, \\ -(\psi_z - c)\psi_{xx} + \psi_x\psi_{xz} - 2\Omega\psi_z = -\frac{1}{\rho}P_z - g, & \text{for} \quad -\eta(x) < z < -d, \\ \psi_x = (\psi_z - c)\eta_x, & \text{on} \quad z = -\eta(x), \\ \psi - cz = 0, & \text{on} \quad z = -\eta(x), \\ \psi - cz = m, & \text{on} \quad z = -d. \end{cases} \tag{2.7}$$

As was shown in [4], the condition (2.6) ensures that there exists a C^1 vorticity function y such that

$$-\Delta\psi = \omega = y(\psi - cz).$$

From the first two equations in (2.7) we obtain in analogy with Bernoulli's law for gravity water waves [7], which states that the expression

$$E := \frac{\psi_x^2 + (\psi_z - c)^2}{2} - 2\Omega\psi + gz + \frac{P}{\rho} - \Gamma(cz - \psi)$$

1 Note that m is independent of x.

is constant throughout $\mathcal{R}$, where

$$\Gamma(p) := \int_0^p \gamma(-s)ds, \quad 0 \le p \le -m.$$

As shown in [4], the governing equations (2.1)-(2.5) are equivalent to problem

$$\begin{cases} \Delta\psi = -\gamma(\psi - cz), & \text{for } -\eta(x) < z < -d, \\ |\nabla(\psi - cz)|^2 - 2(\tilde{g} + 2\Omega c)z = Q, & \text{on } z = -\eta(x), \\ \psi - cz = 0, & \text{on } z = -\eta(x), \\ \psi - cz = m, & \text{on } z = -d, \end{cases} \tag{2.8}$$

where $Q := 2(E - \frac{P_0}{\rho})$ is the physical constant and

$$\tilde{g} := g\frac{\Delta\rho}{\rho}$$

is the reduced gravity [30].

3 Main results

3.1 Invariants

Let $\mathcal{R}(t) := \{(x, z) \in \mathbb{R}^2 : 0 < x < L, -\eta(x, t) < z < -d\}$ be a periodic cell of the fluid domain. We will obtain several invariants for the equatorial flow, which are in analogy to the well-known results in [38] for the classical gravity water waves. Two of them have to be modified due to the Earth's rotation and the non-constant density, cf. the dynamic boundary condition (2.5). First the fluid mass

$$\mathcal{M} := \iint_{\mathcal{R}(t)} dzdx$$

is invariant. Secondly, for an arbitrary C^1–function F, the integral

$$\mathcal{F} := \iint_{\mathcal{R}(t)} F(\omega)dzdx$$

is invariant[2]. In fact, as done in [24], to prove that $\mathcal{F}$ is invariant, we only need to show

$$\frac{D\omega}{Dt} = \omega_t + u\omega_x + w\omega_z = 0,$$

which is indeed a fact following from the equations (2.1). Now we consider the third invariant given as

$$\mathcal{E} := \iint_{\mathcal{R}(t)} \Big[\frac{u^2 + w^2}{2} - \tilde{g}z\Big] dzdx.$$

In fact, let C be the boundary of $\mathcal{R}(t)$, using Green's identity, the conditions (2.5) and (2.2), we obtain

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \iint_{\mathcal{R}(t)} \frac{D}{Dt}\Big(\frac{u^2 + w^2}{2}\Big)dzdx - \tilde{g}\iint_{\mathcal{R}(t)} \frac{D}{Dt}(z)dzdx \\ &= -\frac{1}{\rho}\iint_{\mathcal{R}(t)} (uP_x + wP_z)dzdx - g\iint_{\mathcal{R}(t)} wdzdx - \tilde{g}\iint_{\mathcal{R}(t)} wdzdx \end{aligned}$$

2 The freedom of choosing F will be used later on.

$$
\begin{aligned}
&= -\frac{1}{\rho}\int_C P(wdx - udz) - g\iint_{\mathcal{R}(t)} wdzdx - \tilde{g}\iint_{\mathcal{R}(t)} wdzdx \\
&= -\frac{P_0}{\rho}\int_C (wdx - udz) + \frac{g(\rho+\Delta\rho)}{\rho}\int_C z(wdx - udz) - (\tilde{g}+g)\iint_{\mathcal{R}(t)} wdzdx \\
&= -\frac{P_0}{\rho}\int_C (wdx - udz) + \frac{g(\rho+\Delta\rho)}{\rho}\iint_{\mathcal{R}(t)} wdzdx - (\tilde{g}+g)\iint_{\mathcal{R}(t)} wdzdx \\
&= -\frac{P_0}{\rho}\int_C (wdx - udz) \\
&= -\frac{cP_0}{\rho}\int_0^L \eta_x(x,t)dx \\
&= 0.
\end{aligned}
$$

Finally we consider the fourth invariant defined as

$$\mathcal{U} := \iint_{\mathcal{R}(t)} \Big(u(x,z,t) + 2\Omega z\Big)dzdx.$$

To see that $\mathcal{U}$ is invariant, by the fact $\int_C zdz = \int_S \eta(x,t)\eta_x(x,t)dx = 0$, we compute

$$
\begin{aligned}
\frac{d\mathcal{U}}{dt} &= \iint_{\mathcal{R}(t)} \frac{D}{Dt}(u)dzdx + 2\Omega\iint_{\mathcal{R}(t)} \frac{D}{Dt}(z)dzdx \\
&= \iint_{\mathcal{R}(t)} (u_t + uu_x + wu_z)dzdx + 2\Omega\iint_{\mathcal{R}(t)} wdzdx \\
&= \iint_{\mathcal{R}(t)} (-\frac{1}{\rho}P_x - 2\Omega w)dzdx + 2\Omega\iint_{\mathcal{R}(t)} wdzdx \\
&= -\frac{1}{\rho}\iint_{\mathcal{R}(t)} P_x dzdx \\
&= \frac{1}{\rho}\int_C \Big(P_0 - g(\rho+\Delta\rho)z\Big)dz \\
&= \frac{1}{\rho}\int_C P_0 dz.
\end{aligned}
$$

3.2 Variational formulation

Now we will write the above functionals in terms of ψ and η, which are defined on the function space

$$\mathbb{F} := \Big\{(\psi, -\eta) \in C^2_{per}(\mathbb{R}\times(-\infty,-d]) \times C^1_{per}(\mathbb{R}) \times \mathbb{R} : \psi_z < c\Big\},$$

where c is the travelling speed. We will restrict perturbations $(\psi_1, -\eta_1)$ of $(\psi, -\eta)$ to the subspace

$$\mathbb{D} := \Big\{(\psi, -\eta) \in \mathbb{F} : \int_B \psi_z dx = 0\Big\}.$$

We assume that $F : \mathbb{R} \to \mathbb{R}$ is a C^2–function for which F'' vanishes nowhere, that is, F is either strictly convex or strictly concave. Let us define the C^1–function y by $y = (F')^{-1}$. Obviously, the function y is monotone.

We say that $(\psi, -\eta) \in \mathbb{F}$ is a steady periodic equatorial water wave with the vorticity function y if there exist $k, Q \in \mathbb{R}$ such that

$$\begin{cases} \Delta\psi = -y(\psi - cz - k), & \text{for } -\eta(x) < z < -d, \\ |\nabla(\psi - cz)|^2 - 2(\tilde{g} + 2\Omega c)z = Q, & \text{on } z = -\eta(x), \\ \psi - cz = k, & \text{on } z = -\eta(x), \\ \psi - cz = m + k, & \text{on } z = -d, \end{cases} \tag{3.1}$$

where the constants $\tilde{g}$, Q are described above and

$$m := \int_{-\eta(0)}^{-d} (\psi_z(0, z) - c)\, dz.$$

Obviously, $(\psi - k, -\eta)$ solves the equations (2.8) if $(\psi, -\eta)$ is a steady periodic equatorial water wave with the vorticity function y.

We remark that the stream function ψ, determined up to a constant by (3.1) and the free surface profile η completely determine the steady flow.

Theorem 3.1. *Any critical point in $\mathbb{F}$ of $\mathcal{E} - \mathcal{F}$, subject to the constraints of $\mathcal{M}$ and $\mathcal{U}$, is a steady periodic equatorial water wave with the vorticity function y.*

Proof. Let $(\psi, -\eta) \in \mathbb{F}$ be a critical point of $\mathcal{E} - \mathcal{F}$. Then $(\psi, -\eta)$ satisfies the Euler-Lagrange equation

$$\delta(\mathcal{E} - \mathcal{F}) = \lambda\delta\mathcal{U} + \mu\delta\mathcal{M}, \tag{3.2}$$

where λ and μ are Lagrange multipliers. Let $(\psi_1, -\eta_1) \in \mathbb{D}$ denote a perturbation of $(\psi, -\eta)$ and set $\omega := -\Delta\psi$ and $\omega_1 := -\Delta\psi_1$. By direct computations, we obtain

$$\delta\mathcal{M}(\psi, -\eta)(\psi_1, -\eta_1) = \lim_{\varepsilon\to 0} \frac{\mathcal{M}(\psi + \varepsilon\psi_1, -\eta - \varepsilon\eta_1) - \mathcal{M}(\psi, -\eta)}{\varepsilon} = \int_S \eta_1 dx,$$

$$\delta\mathcal{U}(\psi, -\eta)(\psi_1, -\eta_1) = \iint_{\mathcal{R}} u_1 dzdx + \int_S (u + 2\Omega z)\eta_1 dx,$$

$$\delta\mathcal{F}(\psi, -\eta)(\psi_1, -\eta_1) = \iint_{\mathcal{R}} F'(\omega)\omega_1 dzdx + \int_S F(\omega)\eta_1 dx,$$

and by Green's formula

$$\begin{aligned} \delta\mathcal{E}(\psi, -\eta)(\psi_1, -\eta_1) &= \iint_{\mathcal{R}} (uu_1 + ww_1) dzdx + \int_S \Big(\frac{u^2 + w^2}{2} - \tilde{g}z\Big)\eta_1 dx \\ &= \iint_{\mathcal{R}} \nabla\psi \cdot \nabla\psi_1 dzdx + \int_S \Big(\frac{|\nabla\psi|^2}{2} - \tilde{g}z\Big)\eta_1 dx \\ &= \int_S \Big\{\Big(\frac{|\nabla\psi|^2}{2} - \tilde{g}z\Big)\eta_1 + \psi(\psi_{1z} + \eta_x\psi_{1x})\Big\} dx \\ &\quad - \iint_{\mathcal{R}} \psi\Delta\psi_1 dzdx - \int_B \psi\psi_{1z} dx. \end{aligned}$$

By definition, any critical point $(\psi, -\eta)$ satisfies (3.2), so that

$$\int_S \Big\{\frac{|\nabla\psi|^2}{2} - (\tilde{g} + 2\Omega\lambda)z - F(\omega) - \mu - \lambda\psi_z\Big\}\eta_1 dx - \int_B \psi\psi_{1z} dx$$

$$+\int_S \psi(\psi_{1z}+\eta_x\psi_{1x})dx-\iint_{\mathcal{R}}\Big\{\psi\Delta\psi_1+F'(\omega)\omega_1+\lambda\psi_{1z}\Big\}dzdx=0,$$

which can be rewritten in the following equivalent form

$$\begin{aligned}
&\int_S\Big\{\frac{|\nabla(\psi-\lambda z)|^2}{2}-\frac{\lambda^2}{2}-(\tilde g+2\Omega\lambda)z-F(\omega)-\mu\Big\}\eta_1dx-\int_B(\psi-\lambda z)\psi_{1z}dx\\
&+\int_S(\psi-\lambda z)(\psi_{1z}+\eta_x\psi_{1x})dx-\iint_{\mathcal{R}}\Big\{(\psi-\lambda z)\Delta\psi_1+F'(\omega)\omega_1\Big\}dzdx\\
&-\lambda\iint_{\mathcal{R}}(\psi_{1z}+z\Delta\psi_1)dzdx-\lambda\int_B z\psi_{1z}dx+\lambda\int_S z(\psi_{1z}+\eta_x\psi_{1x})dx=0.
\end{aligned}$$

Let n be the unit outer normal on the surface S and dl be the measure of arclength. It is easy to see that

$$\iint_{\mathcal{R}}(z\Delta\psi_1+\psi_{1z})dzdx-\int_S z\frac{\partial\psi_1}{\partial n}dl+\int_B z\psi_{1z}dx=0.$$

Therefore we obtain that

$$\begin{aligned}
&\int_S\Big\{\frac{|\nabla(\psi-\lambda z)|^2}{2}-\frac{\lambda^2}{2}-(\tilde g+2\Omega\lambda)z-F(\omega)-\mu\Big\}\eta_1dx-\int_B(\psi-\lambda z)\psi_{1z}dx\\
&+\int_S(\psi-\lambda z)\frac{\partial\psi_1}{\partial n}dl-\iint_{\mathcal{R}}\Big\{(\psi-\lambda z)\Delta\psi_1+F'(\omega)\omega_1\Big\}dzdx=0.
\end{aligned}\tag{3.3}$$

Let us choose four different types of perturbation functions.

(i) Firstly, we take $\eta_1=0$ and aim ψ_1 to be a solution of the elliptic problem

$$\begin{cases}\Delta\psi_1=-\omega_1, & \text{in } \mathcal{R},\\ \psi_{1z}=0, & \text{on } B\\ \frac{\partial\psi_1}{\partial n}=0, & \text{on } S.\end{cases}$$

Then (3.3) reduces to

$$\iint_{\mathcal{R}}\Big\{\psi-\lambda z-F'(\omega)\Big\}\omega_1dzdx=0.$$

The latter is valid for all smooth functions ω_1 with $\iint_{\mathcal{R}}\omega_1dzdx=0$, implying that

$$\psi-\lambda z=F'(\omega)+k\qquad\text{in}\quad\mathcal{R},\tag{3.4}$$

for some constant k. Therefore $\omega=\gamma(\psi-\lambda z-k)$. By taking $\lambda=c$, we obtain the first equation in (3.1).

Plugging (3.4) into (3.3), we obtain

$$\begin{aligned}
&\int_S\Big\{\frac{|\nabla(\psi-\lambda z)|^2}{2}-\frac{\lambda^2}{2}-(\tilde g+2\Omega\lambda)z-F(\omega)-\mu\Big\}\eta_1dx\\
&-\int_B(\psi-\lambda z-k)\psi_{1z}dx+\int_S(\psi-\lambda z-k)\frac{\partial\psi_1}{\partial n}dl=0.
\end{aligned}\tag{3.5}$$

(ii) Secondly, we choose $\eta_1=0$ and ψ_1 is a solution of the elliptic problem

$$\begin{cases}\Delta\psi_1=0, & \text{in } \mathcal{R},\\ \psi_{1z}=0, & \text{on } B\\ \frac{\partial\psi_1}{\partial n}=f, & \text{on } S,\end{cases}$$

where f is an arbitrary smooth function defined on S. Now (3.5) reduces to

$$\int_S (\psi - \lambda z - k) f dl = 0.$$

Thus $\psi - \lambda z - k = 0$ on S. Therefore, we get the third identity in (3.1) by taking $\lambda = c$.

(iii) Next we take $\eta_1 = 0$ and ψ_1 is a solution of the elliptic problem

$$\begin{cases} \Delta\psi_1 = 0, & \text{in } \mathcal{R}, \\ \psi_{1z} = f, & \text{on } B \\ \frac{\partial\psi_1}{\partial n} = 0, & \text{on } S, \end{cases}$$

where f is an arbitrary smooth function defined on B with $\int_B f dx = 0$. Now (3.3) reduces to

$$\int_B (\psi - \lambda z - k) f dx = 0.$$

Thus $\psi - \lambda z - k = k_B$ on B with a constant k_B. Therefore, we get the fourth identity in (3.1) by taking $\lambda = c$. Note that $k_B = m$ when $\lambda = c$.

(iv) Finally, we take $\psi_1 = 0$ throughout $\mathcal{R}$ with η_1 arbitrary, we obtain

$$\int_S \left\{ \frac{|\nabla(\psi - \lambda z)|^2}{2} - \frac{\lambda^2}{2} - (\tilde{g} + 2\Omega\lambda)z - F(\omega) - \mu \right\} \eta_1 dx = 0$$

for all smooth functions η_1. Therefore,

$$\frac{|\nabla(\psi - \lambda z)|^2}{2} - \frac{\lambda^2}{2} - (\tilde{g} + 2\Omega\lambda)z - F(\omega) - \mu = 0, \qquad \text{on} \quad S.$$

It follows from (3.4) and (ii) that $F'(\omega) = 0$ on S, and since F' is strictly monotone, this is only possible, if ω is constant on S, e.g., $\omega = \omega_0$ on S. Hence letting $\lambda = c$ we find that

$$\frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z = \frac{c^2}{2} + F(\omega_0) + \mu, \qquad \text{on} \quad S.$$

It remains to choose $Q = c^2 + 2\mu + 2F(\omega_0)$ in order to fullfil also the second equation in (3.1). □

Theorem 3.2. a) *If* $(\psi, -\eta) \in \mathbb{F}$ *is a solution of the equations* (2.8) *then*

$$\delta\mathcal{H}(\psi, -\eta)(\psi_1, -\eta_1) = 0, \tag{3.6}$$

for all $(\psi_1, -\eta_1) \in \mathbb{D}$, *where the functional* $\mathcal{H}$ *is given as*

$$\mathcal{H}(\psi, -\eta) = \iint_{\mathcal{R}} \left[\frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(-\Delta\psi) \right] dz dx$$

so that the variation of $\mathcal{H}$ *at* $(\psi, -\eta)$ *is given by* $\delta\mathcal{H} = \delta\mathcal{H}_1 + \delta\mathcal{H}_2 + \delta\mathcal{H}_3$, *where*

$$\delta\mathcal{H}_1 = \iint_{\mathcal{R}} \left\{ (\psi - cz)\omega_1 - F'(\omega)\omega_1 \right\} dz dx,$$

$$\delta\mathcal{H}_2 = \int_S \left\{ \frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(\omega) \right\} \eta_1 dx,$$

$$\delta\mathcal{H}_3 = \int_S (\psi - cz)\frac{\partial\psi_1}{\partial n} dl - \int_B (\psi - cz)\psi_{1z} dx.$$

b) *Conversely, assume that* $(\psi, -\eta) \in \mathbb{F}$ *satisfies* (3.6) *for all* $(\psi_1, -\eta_1) \in \mathbb{D}$. *Then* $(\psi, -\eta)$ *is a steady periodic water wave with the vorticity function* y.

Proof. a) We first show that (3.6) holds for any $(\psi_1, -\eta_1) \in \mathbb{D}$ if $(\psi, -\eta)$ satisfies the equations (2.8). Note that $-\Delta\psi = \omega = \gamma(\psi - cz)$ implies $F'(\omega) = \psi - cz$, so that $\delta\mathcal{H}_1 = 0$. By the later two boundary conditions of (2.8) and the fact $\int_B \psi_{1z} dx = 0$, we know that

$$\int_S (\psi - cz)\frac{\partial \psi_1}{\partial n} dl = 0$$

and

$$\int_B (\psi - cz)\psi_{1z} dx = m \int_B \psi_{1z} dx = 0,$$

and therefore $\delta\mathcal{H}_3 = 0$. By the second equation of (2.8), we obtain

$$\delta\mathcal{H}_2 = -\int_S F(\omega)\eta_1 dx = 0$$

by letting $F(\gamma(0)) = 0$ and using the fact $\omega = \gamma(0)$ on S. The choice of F satisfying $F(\gamma(0)) = 0$ does not change the vorticity function γ since $\gamma = (F')^{-1}$.

b) Let $(\psi, -\eta) \in \mathbb{F}$ be given and assume that for any $(\psi_1, -\eta_1) \in \mathbb{D}$ (3.6) holds. As in the proof of Theorem 3.1 it follows that

$$\psi - cz = F'(\omega) + k \quad \text{in} \quad \mathcal{R}$$

for some constant k. Plugging this into (3.6), we have

$$\begin{aligned} 0 = & \int_S \left\{ \frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(\omega) \right\} \eta_1 dx \\ & + \int_S (\psi - cz - k)\frac{\partial \psi_1}{\partial n} dl - \int_B (\psi - cz - k)\psi_{1z} dx. \end{aligned} \tag{3.7}$$

By choosing $\eta_1 = 0$ and

$$\psi_1(x, z) = z\chi(\frac{z + d}{\epsilon}) f(x)$$

with a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(z) = 1$ for $|z| \leq 1$ and $f \in C^1_{per}(\mathbb{R})$ with $\int_B f dx = 0$. Taking $\epsilon > 0$ small enough, we obtain

$$\int_B (\psi - cz - k) f dx = 0$$

for any $f \in C^1_{per}(\mathbb{R})$ with $\int_B f dx = 0$, so that

$$\psi - cz - k = C, \quad \text{on} \quad B \tag{3.8}$$

for some constant C. Moreover, for any $f \in C^1_{per}(\mathbb{R})$, we can construct ψ_1 satisfying $\frac{\partial \psi_1}{\partial n} = f$ on S and $\psi_1 = 0$ outside of a small neighborhood of S, so that one has

$$\psi - cz - k = 0, \quad \text{on} \quad S. \tag{3.9}$$

Now let us take $\psi_1 = 0$ throughout $\mathcal{R}$ with η_1 arbitrary, we obtain

$$\int_S \left\{ \frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(\omega) \right\} \eta_1 dx = 0.$$

Thus

$$\frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(\omega) = 0, \qquad \text{on} \quad S. \tag{3.10}$$

Since F' is strictly monotone and $F'(\omega) = \psi - cz - k = 0$ on S, we know that ω is constant on S, e.g., $\omega = \omega_0$ on S. Hence

$$\frac{|\nabla(\psi - cz)|^2}{2} - (\tilde{g} + 2\Omega c)z - \frac{Q}{2} - F(\omega_0) = 0, \qquad \text{on} \quad S.$$

Thus we obtain

$$\begin{cases} \Delta\psi = -(F')^{-1}(\psi - cz - k), & \text{for } -\eta(x) < z < -d, \\ |\nabla(\psi - cz)|^2 - 2(\tilde{g} + 2\Omega c)z = Q + 2F(\omega_0), & \text{on } z = -\eta(x), \\ \psi - cz - k = C, & \text{on } z = -\eta(x), \\ \psi - cz - k = 0, & \text{on } z = -d. \end{cases} \tag{3.11}$$

By the choice of the space $\mathbb{F}$, we know that $\psi - cz$ is strictly decreasing from the upper bound B to the surface S, and therefore $C < 0$. Thus we obtain the equations (3.1) by taking $m = C$ and $F(\omega_0) = 0$, and also obtain the equations (2.8) by adding the constant k to ψ. The choice of F satisfying $F(\omega_0) = 0$ does not change the vorticity function y since $y = (F')^{-1}$. □

Remark 3.3. (a) It is worthwhile to note that the choice of the spaces $\mathbb{F}$ and $\mathbb{D}$ is important to ensure the efficiency of Theorem 3.2. To explain this point, we first note that the restriction $\psi_z < c$ in the definition of the space $\mathbb{F}$ is quite natural in view of (2.6). However this restriction ensures that the constant C in the proof of Theorem 3.2 is negative.

(b) Next we show that the restriction in $\mathbb{D}$ is needed in Theorem 3.2. Indeed, if $\mathbb{F}$ is used as the space of perturbations, then, as in the in the proof of Theorem 3.2 we would obtain[3]

$$\psi - cz = k, \quad \text{on} \quad B,$$

and consequently also (3.9). Thus we get the equations (3.11) with $C = 0$, which is essentially the same as (2.8) with $m = 0$, and this contradicts the assumption (2.6).

3.3 Second variation

Next we calculate the second variation of $\mathcal{H}$. Beginning with a critical point $(\psi, -\eta) \in \mathbb{F}$, we denote a pair of variations of $(\psi, -\eta)$ by $(\psi_1, -\eta_1) \in \mathbb{D}$ and $(\psi_2, -\eta_2) \in \mathbb{D}$. We further let $\omega_2 = -\Delta\psi_2$.

Theorem 3.4. *Let $(\psi, -\eta) \in \mathbb{F}$ be a solution of the equations (2.8). Then the second variation of $\mathcal{H}$ is*

$$\delta^2\mathcal{H} = \int_S \left(\frac{\partial\psi}{\partial z} - c\right)\left\{\frac{\partial\psi_2}{\partial n}\eta_1 + \frac{\partial\psi_1}{\partial n}\eta_2\right\}dl + \iint_{\mathcal{R}} \left\{\nabla\psi_2 \cdot \nabla\psi_1 - F''(\omega)\omega_1\omega_2\right\}dzdx$$
$$+ \int_S \left\{(\tilde{g} + 2\Omega c) - \frac{1}{2}\frac{\partial}{\partial z}[|\nabla(\psi - cz)|^2]\right\}\eta_1\eta_2 dx.$$

Proof. We start from the formulas given in Theorem 3.2 and calculate further variation of each term. First,

$$\begin{aligned} \delta^2\mathcal{H}_1 &= \iint_{\mathcal{R}} \left\{\psi_2\omega_1 - F''(\omega)\omega_2\omega_1\right\}dzdx + \int_S \left\{(\psi - cz)\omega_1 - F'(\omega)\omega_1\right\}\eta_2 dx \\ &= \iint_{\mathcal{R}} \left\{\psi_2\omega_1 - F''(\omega)\omega_2\omega_1\right\}dzdx \\ &= \iint_{\mathcal{R}} \left\{\nabla\psi_2 \cdot \nabla\psi_1 - F''(\omega)\omega_2\omega_1\right\}dzdx - \int_S \psi_2\frac{\partial\psi_1}{\partial n}dl + \int_B \psi_2\frac{\partial\psi_1}{\partial z}dx, \end{aligned}$$

where we used the fact $\psi - cz = F'(\omega) = 0$ on S.

3 Note that $\int_B f dx = 0$ implies that $\int_B \psi_{1z} dx = 0$.

We can compute the remaining two terms as follows

$$\begin{aligned}\delta^2\mathcal{H}_2 &= \int_S \Big\{\psi_x\psi_{2x} + (\psi_z - c)\psi_{2z} - F'(\omega)\omega_2\Big\}\eta_1 dx \\ &\quad - \int_S \Big\{\psi_x\psi_{xz} + (\psi_z - c)\psi_{zz} - (\tilde{g} + 2\Omega c) - F'(\omega)\omega_z\Big\}\eta_2\eta_1 dx \\ &= \int_S \Big\{\psi_x\psi_{2x} + (\psi_z - c)\psi_{2z}\Big\}\eta_1 dx \\ &\quad - \int_S \Big\{\psi_x\psi_{xz} + (\psi_z - c)\psi_{zz} - (\tilde{g} + 2\Omega c)\Big\}\eta_2\eta_1 dx \\ &= \int_S (\psi_z - c)\frac{\partial\psi_2}{\partial n}\eta_1 dl + \int_S \Big\{(\tilde{g} + 2\Omega c) - \Big[\frac{1}{2}|\nabla(\psi - cz)|^2\Big]_z\Big\}\eta_2\eta_1 dx,\end{aligned}$$

and

$$\delta^2\mathcal{H}_3 = \int_S \Big\{\psi_2 + (\psi_z - c)\eta_2\Big\}\frac{\partial\psi_1}{\partial n} dl - \int_B \psi_2\frac{\partial\psi_1}{\partial z} dx.$$

Combining all the terms, we obtain

$$\begin{aligned}\delta^2\mathcal{H} &= \iint_{\mathcal{R}} \Big\{\nabla\psi_2\cdot\nabla\psi_1 - F''(\omega)\omega_2\omega_1\Big\}dzdx + \int_S (\psi_z - c)\frac{\partial\psi_2}{\partial n}\eta_1 dl \\ &\quad + \int_S \Big\{(\tilde{g} + 2\Omega c) - \Big[\frac{1}{2}|\nabla(\psi - cz)|^2\Big]_z\Big\}\eta_2\eta_1 dx + \int_S \Big\{(\psi_z - c)\eta_2\Big\}\frac{\partial\psi_1}{\partial n} dl.\end{aligned}$$

Now we obtain the desired equality and the proof is finished. □

The second variation of the functional $\mathcal{H}$ is related to the stability properties of steady periodic waves. In order to explain this, let us take $\psi_1 = \psi_2$ and $\eta_1 = \eta_2$ in Theorem 3.4, to obtain the quadratic form

$$\begin{aligned}\delta^2\mathcal{H} &= \iint_{\mathcal{R}} \Big\{|\nabla\psi_2|^2 - F''(\omega)|\omega_2|^2\Big\}dzdx + 2\int_S (\psi_z - c)\frac{\partial\psi_2}{\partial n}\eta_2 dl \\ &\quad + \int_S \Big\{(\tilde{g} + 2\Omega c) - \Big[\frac{1}{2}|\nabla(\psi - cz)|^2\Big]_z\Big\}|\eta_2|^2 dx.\end{aligned} \tag{3.12}$$

Definition 3.5. *The traveling wave $(\psi, -\eta)$ is linear stable if for any $(\psi_2, -\eta_2) \in \mathbb{D}$, the quadratic form $\delta^2\mathcal{H}$ is nonnegative.*

Therefore, when we try to obtain the stability results, we need to find suitable conditions under which the symmetric quadratic form (3.12) is nonnegative. First we state the following almost trivial stability result.

Theorem 3.6. *Assume that $\omega_z > 0$. Then a classical travelling wave is linearly stable if the surface is unperturbed.*[4]

Proof. The hypothesis $\omega_z > 0$ implies that $F'' < 0$. Therefore the first integral in (3.12) is nonnegative. When the surface is unperturbed, we have $\eta_2 = 0$, and thus all the rest terms in (3.12) are zero. Therefore in this case, $\delta^2\mathcal{H} \geq 0$. □

In order to analyze the cases of perturbed surface, we first prove the following result.

4 This means that the restricted form $\delta^2\mathcal{H}|_{\mathbb{F}_1}$ is nonnegative, where $\mathbb{F}_1 := \{(\psi_2, -\eta_2) \in \mathbb{D} : \eta_2 = 0\}$.

Lemma 3.7. *Assume that*

$$y(m)(c - u|_B) < \tilde{g} + 2\Omega c. \tag{3.13}$$

Then we have

$$(\tilde{g} + 2\Omega c) - \left[\frac{1}{2}|\nabla(\psi - cz)|^2\right]_z \geq 0$$

on the thermocline S.

Proof. We calculate

$$\begin{aligned} & \left\{\frac{1}{2}|\nabla(\psi - cz)|^2\right\}_z - (\tilde{g} + 2\Omega c) \\ = \ & \psi_x\psi_{xz} + (\psi_z - c)\psi_{zz} - (\tilde{g} + 2\Omega c) \\ = \ & \left\{-\frac{P}{\rho} + 2\Omega(\psi - cz) + \Gamma(cz - \psi) - (g + \tilde{g})z\right\}_z. \end{aligned}$$

It follows from (2.7) that

$$\frac{\Delta P}{\rho} = 2\Omega(\psi_{xx} + \psi_{zz}) + 2\psi_{xx}\psi_{zz} - 2\psi_{xz}^2.$$

Since $\Delta\psi = -y(\psi - cz)$, we obtain

$$\begin{aligned} \Delta\Gamma(cz - \psi) &= -y'(\psi - cz)|\nabla(\psi - cz)|^2 - y(\psi - cz)(\psi_{xx} + \psi_{zz}) \\ &= -y'(\psi - cz)|\nabla(\psi - cz)|^2 + (\Delta\psi)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \Delta\left\{-\frac{P}{\rho} + 2\Omega(\psi - cz) + \Gamma(cz - \psi) - (g + \tilde{g})z\right\} \\ = \ & -2\psi_{xx}\psi_{zz} + 2\psi_{xz}^2 - y'(\psi - cz)|\nabla(\psi - cz)|^2 + (\Delta\psi)^2 \\ = \ & 2\psi_{xz}^2 + \psi_{xx}^2 + \psi_{zz}^2 - y'(\psi - cz)|\nabla(\psi - cz)|^2 \geq 0 \end{aligned}$$

because $y' < 0$. Thus

$$\frac{P}{\rho} - 2\Omega(\psi - cz) - \Gamma(cz - \psi) + (g + \tilde{g})z$$

is superharmonic and by the maximum principle [31], its minimum can only be attained on the thermocline S or on the upper boundary B of the centre layer unless it is a constant.

On the upper boundary $z = -d$ of the centre layer, we have $\psi_x = w = 0$ and $\psi - cz = m$, thus

$$\begin{aligned} & \left\{-\frac{P}{\rho} + 2\Omega(\psi - cz) + \Gamma(cz - \psi) - (g + \tilde{g})z\right\}_z \\ = \ & \psi_x\psi_{xz} - (\psi_z - c)[\psi_{xx} + y(\psi - cz)] - (\tilde{g} + 2\Omega c) \\ = \ & -(\psi_z - c)y(\psi - cz) - (\tilde{g} + 2\Omega c) \\ = \ & y(m)(c - u|_B) - (\tilde{g} + 2\Omega c) < 0 \end{aligned}$$

by using the condition (3.13). Thus the minimum of

$$\frac{P}{\rho} - 2\Omega(\psi - cz) - \Gamma(cz - \psi) + (g + \tilde{g})z$$

must be attained on the thermocline S.

However, on S we know that the function is constant because

$$\frac{P}{\rho} - 2\Omega(\psi - cz) - \Gamma(cz - \psi) + (g + \tilde{g})z = \frac{P_0}{\rho} - 2\Omega(\psi - cz) - \Gamma(cz - \psi) = \frac{P_0}{\rho},$$

by the condition (2.5). Thus it is minimized at every point of S. Therefore

$$0 < \left\{\frac{P}{\rho} - 2\Omega(\psi - cz) - \Gamma(cz - \psi) + (g + \tilde{g})z\right\}_z\Big|_S$$

$$= -\left[\frac{1}{2}|\nabla(\psi - cz)|^2\right]_z + (\tilde{g} + 2\Omega c)$$

by the Hopf maximum principle [31]. □

Theorem 3.8. *Assume that $\omega_z > 0$ and that* (3.13) *is satisfied. Then a classical travelling wave is linearly stable if the surface is perturbed only normally.*

Proof. The hypothesis $\omega_z > 0$ is equivalent to $F'' < 0$. By Lemma 3.7 the first and third integral in (3.12) are nonnegative.

For the velocity on the surface to be perturbed only normally, it means that the tangential component of the velocity perturbation vanishes. But this means that $\partial\psi_2/\partial n = 0$ on S. Therefore the second term in (3.12) vanishes. Therefore, $\delta^2\mathcal{H} \geq 0$ and the proof is finished. □

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References

[1] T. J. Benjamin, J. E. Feir, Disintegration of wave trains on deep water, *J. Fluid. Mech.* 27 (1967), 417-437.
[2] T. J. Bridges, A. Mielke, A proof of the Benjamin-Feir instability, *Arch. Rational Mech. Anal.* 133 (1995), 145-198.
[3] J. Chu, Q. Ding, Y. Yang, Steady periodic waves and formal stability for fixed-depth rotational equatorial flows, *J. Differential Equations*, 269 (2020), 4192-4214.
[4] J. Chu, J. Escher, Steady periodic equatorial water waves with vorticity, *Discrete Cont. Dyn. Syst.* 39 (2019), 4713-4729.
[5] J. Chu, D. Ionescu-Kruse, Y. Yang, Exact solution and instability for geophysical waves with centripetal forces and at arbitrary latitude, *J. Math. Fluid Mech.* 21 (2019), no. 2, 21:19.
[6] J. Chu, Y. Yang, Constant vorticity water flows in the equatorial β-plane approximation with centripetal forces, *J. Differential Equations*, 269 (2020), 9336-9347.
[7] A. Constantin, Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis, CBMS-NSF Conference Series in Applied Mathematics, vol. 81, SIAM, Philadelphia, 2011.
[8] A. Constantin, Two-dimensionality of gravity water flows of constant nonzero vorticity beneath a surface wave train, *Eur. J. Mech. B Fluids* 30 (2011), 12-16.
[9] A. Constantin, On the modelling of equatorial waves, *Geophys. Res. Lett.* 39 (2012), L05602.
[10] A. Constantin, Some three-dimensional nonlinear equatorial flows, *J. Phys. Oceanogr.* 43 (2013), 165-175.
[11] A. Constantin, M. Ehrnström, E. Wahlén, Symmetry of steady periodic gravity water waves with vorticity, *Duke Math. J.* 140 (2007), 591-603.
[12] A. Constantin, J. Escher, Symmetry of steady periodic surface water waves with vorticity, *J. Fluid Mech.* 498 (2004), 171-181.
[13] A. Constantin, J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math. (2)* 173 (2011), 559-568.
[14] A. Constantin, P. Germain, Instability of some equatorially trapped waves, *J. Geophys. Res. Oceans* 118 (2013), 2802-2810.
[15] A. Constantin, R.-I. Ivanov, A Hamiltonian approach to wave-current interactions in two-layer fluids, *Physics of Fluids*, 27 (2015), pp. 086603.
[16] A. Constantin, R.-I. Ivanov, Equatorial wave-current interactions, *Comm. Math. Phys.* 370 (2019), 1-48.
[17] A. Constantin, R.-I. Ivanov, C.-I. Martin, Hamiltonian formulation for wave-current interactions in stratified rotational flows, *Arch. Ration. Mech. Anal.* 221 (2016), 1417-1447.
[18] A. Constantin, R.-I. Ivanov, E.-M. Prodanov, Nearly-Hamiltonian structure for water waves with constant vorticity, *J. Math. Fluid Mech.* 10 (2008), 224-237.
[19] A. Constantin, R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.* 109 (2015), 311-358.
[20] A. Constantin, R. S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, *J. Phys. Oceanogr.* 46 (2016), 1935-1945.

[21] A. Constantin, R. S. Johnson, A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific Equatorial Undercurrent and thermocline, *Physics of Fluids*, 29 (2017), 056604.
[22] A. Constantin, R. S. Johnson, Large gyres as a shallow-water asymptotic solution of Euler's equation in spherical coordinates, *Proc. Roy. Soc. London A.* 473 (2017), 20170063.
[23] A. Constantin, R.S. Johnson, Steady large-scale ocean flows in spherical coordinates, *Oceanography* 31 (2018), 42-50.
[24] A. Constantin, D. Sattinger, W. Strauss, Variational formulations for steady water waves with vorticity, *J. Fluid Mech.* 548 (2006), 151-163.
[25] A. Constantin, W. Strauss, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.* 57 (2004), 481-527.
[26] A. Constantin, W. Strauss, Stability properties of steady water waves with vorticity, *Comm. Pure Appl. Math.* 60 (2007), 911-950.
[27] A. Constantin, W. Strauss, E. Varvaruca, Global bifurcation of steady gravity water waves with critical layers, *Acta Math.* 217 (2016), 195-262.
[28] W. Craig, M. D. Groves, Hamiltonian long-wave approximations to the water-wave problem, *Wave Motion*, 19 (1994), 367-389.
[29] B. Cushman-Roisin, J. M. Beckers, Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects, Academic, Waltham, Mass., 2011.
[30] A. V. Fedorov, J. N. Brown, Equatorial waves, in: J. Steele (Ed.), Encyclopedia of Ocean Sciences, Academic, San Diego, Calif., 2009, 3679-3695.
[31] L. E. Fraenkel, An introduction to maximum principles and symmetry in elliptic problems, Cambridge Tracts in Mathematics, 128. Cambridge University Press, Cambridge, 2000.
[32] F. Genoud, D. Henry, Instability of equatorial water waves with an underlying current, *J. Math. Fluid Mech*. 16 (2014), 661-667.
[33] M. D. Groves, J. T. Toland, On variational formulations for steady water waves, *Arch. Ration. Mech. Anal.* 137 (1997), 203-226.
[34] D. Henry, Exact equatorial water waves in the f-plane, *Nonlinear Anal. Real World Appl*. 28 (2016), 284-289.
[35] D. Henry, H.-C. Hsu, Instability of Equatorial water waves in the f-plane, *Discrete Contin. Dyn. Syst.* 35 (2015), 909-916.
[36] D. Henry, H.-C. Hsu, Instability of internal equatorial water waves, *J. Differential Equations*, 258 (2015), 1015-1024.
[37] D. Ionescu-Kruse, C.-I. Martin, Periodic equatorial water flows from a Hamiltonian perspective, *J. Differential Equations*, 262 (2017), 4451-4474.
[38] M. S. Longuet-Higgins, On integrals and invariants for inviscid, irrotational flow under gravity, *J. Fluid Mech.* 134 (1983), 155-159.
[39] J. C. Luke, A variational principle for a fluid with a free surface, *J. Fluid. Mech*. 27 (1967), 395-397.
[40] C. I. Martin, Dynamics of the thermocline in the equatorial region of the Pacific ocean, *J. Nonlinear Math. Phys*. 22 (2015), 516-522.
[41] C. I. Martin, On periodic geophysical water flows with discontinuous vorticity in the equatorial f-plane approximation, *Philos. Trans. Roy. Soc. A* 376 (2018), 20170096, 23 pp.
[42] R. S. MacKay, P. G. Saffman, Stability of water waves, *Proc. Roy. Soc. London Ser. A* 406 (1986), 115-125.
[43] J. W. Miles, On Hamilton's principle for water waves, *J. Fluid. Mech.* 83 (1977), 153-158.
[44] J. Pedlosky, Geophysical fluid dynamics, Springer, New York, 1979.
[45] R. Quirchmayr, On irrotational flows beneath periodic traveling equatorial waves, *J. Math. Fluid Mech*. 19 (2017), 283-304.
[46] V. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* 9 (1968), 190-194.